Singular solutions in some QFT models

V.V. Belokurov

Lomonosov Moscow State University and
Institute for Nuclear Research of Russian Academy of Sciences
vvbelokurov@yandex.ru

V.V. Belokurov and E.T. Shavgulidze

arXiv: 1112.3899v2 [hep-th];
arXiv: 1303.3523 [math-ph];
arXiv: 1303.5027 [hep-th];
arXiv: 1403.7464 [math-ph]

1. One-dimensional quantum oscillator with a singular potential.

First let us consider the quantum oscillator with the Hamiltonian

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 2x^{-2}\right) . \tag{1}$$

The wave functions regular at $x = 0$ are well known. They are expressed in terms of degenerate hypergeometric functions and form a basis in the Hilbert space of functions on the semi-axis $0 < x < +\infty$ bounded at $x = 0$.

However, there is another class of solutions. To find them we represent the Hamiltonian as

$$H = a_\alpha^+ \, a_\alpha^- \, + \frac{1}{2} - \alpha \, , \tag{2}$$

where

$$a_\alpha^\pm = \frac{1}{\sqrt{2}} \left(\mp \frac{d}{dx} + x + \alpha x^{-1} \right) \tag{3}$$

and $\alpha = -2$, or $\alpha = 1$.

Although a^+ and a^- are formally conjugate to each other, they cannot be considered as the increasing and the decreasing operators because of the "wrong" commutators with H .

Nevertheless, there are operators A^+ and A^- that satisfy

$$[A^{\pm},\, H] = \mp 2A^{\pm} \tag{4}$$

increasing or decreasing the energy by 2. Namely,

$$A^+ = a_\alpha^+\, a_{-\alpha}^+ = \frac{1}{2}\left(\frac{d^2}{dx^2} + x^2 - 2x^{-2} - 2x\frac{d}{dx} - 1\right) \tag{5}$$

and

$$A^- = a_{-\alpha}^-\, a_\alpha^- = \frac{1}{2}\left(\frac{d^2}{dx^2} + x^2 - 2x^{-2} + 2x\frac{d}{dx} + 1\right)\,. \tag{6}$$

Now we can find the "vacuum" states $\Psi_0^{(\alpha)}(x)$ as the solutions of the differential equations

$$a_\alpha^- \, \Psi_0^{(\alpha)}(x) = 0 \tag{7}$$

on the semi-axis $0 < x < +\infty$.

For $\alpha = -2$, it is

$$\Psi_0^{(-2)}(x) = x^2 \exp\left(-\frac{x^2}{2}\right) , \qquad (8)$$

with the "vacuum" energy $E_0^{(-2)} = \frac{5}{2}$.

(The normalizing factors of wave functions are dropped in what follows.)

The successive actions of operator A^+ give the states $\Psi_n^{(-2)} = (A^+)^n \, \Psi_0^{(-2)}$ with the energies $E_n^{(-2)} = \frac{5}{2} + 2n$. They have the form

$$\Psi_n^{(-2)}(x) = P_{2n+2}(x) \exp\left(-\frac{x^2}{2}\right) ,$$

where $P_{2n+2}(x)$ is a polynomial with only even powers of x and $P_{2n+2}(0) = 0$. Thus, for $\alpha = -2$ we get the above-mentioned regular at $x = 0$ solutions that give a basis in $L_2(0, +\infty)$.

For $\alpha = 1$, the wave functions are singular at $x = 0$. The solution of eq. (7) is

$$\Psi_0^{(1)}(x) = x^{-1} \exp\left(-\frac{x^2}{2}\right) , \tag{9}$$

with the "vacuum" energy $E_0^{(1)} = -\frac{1}{2}$.

And

$$\Psi_n^{(1)}(x) = \left(A^+\right)^n \Psi_0^{(1)}(x) = \left(x^{-1} + P_{2n-1}(x)\right) \exp\left(-\frac{x^2}{2}\right) , \tag{10}$$

with $E_n^{(1)} = -\frac{1}{2} + 2n$. Here, $P_{2n-1}(x)$ is a polynomial with only odd powers of x.

From the equation

$$\lim_{\varepsilon\to 0} \frac{\int_a^b \left|\Psi_n^{(1)}(x)\right|^2 dx}{\int_\varepsilon^b \left|\Psi_n^{(1)}(x)\right|^2 dx} = 0\,, \quad a > 0\,, \tag{11}$$

it follows that the new wave functions are concentrated in the infinitesimal vicinity of $x = 0$.

In spite of the divergency of the integral

$$\int_0^{+\infty} x^{-2} \exp(-x^2) dx \,,$$

it is possible to give a consistent meaning to the norm of the state $\Psi_0^{(1)}(x)$.

Note that the norms of the states $\Psi_n^{(-2)}(x)$ are expressed in terms of $\Gamma-$functions, say

$$\left(\Psi_0^{(-2)}(x)\,,\ \Psi_0^{(-2)}(x)\right) = \int_0^{+\infty} x^4\,\exp(-x^2)dx = \frac{1}{2}\Gamma\left(\frac{5}{2}\right)\,.$$

According to the definition of $\Gamma-$function at negative arguments, we regularize the divergent integral as

$$\int_0^{+\infty} x^{-2}\,\exp(-x^2)dx = \frac{1}{2}\int_0^{+\infty} t^{-\frac{3}{2}}\,\exp(-t)dt = \frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$

and get the norm

$$\left(\Psi_0^{(1)}(x)\,,\ \Psi_0^{(1)}(x)\right) = \frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = -\sqrt{\pi}\,. \tag{12}$$

In this case, one can easily verify the orthogonality of the functions $\Psi_n^{(1)}$

$$\Big(\Psi_n^{(1)}(x)\,,\ \Psi_m^{(1)}(x)\Big) = 0\,, \quad n, m = 0, 1, \ldots\,, \quad n \neq m\,,$$

and the self-adjointness of the operator H .

Although the norm of $\Psi_0^{(1)}$ is negative and the vacuum state is unphysical, the other functions $\Psi_n^{(1)}$ have positive norms

$$\Big(\Psi_n^{(1)}(x)\,,\ \Psi_n^{(1)}(x)\Big) > 0\,, \quad n \geq 1\,, \tag{13}$$

and correspond to physical states.

2. The system of two quantum harmonic oscillators.

Now, consider the quantum system of two harmonic oscillators with the Hamiltonian

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x^2 + y^2\right) = \frac{1}{2}\left(-\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} z\right) . \qquad (14)$$

Here, as usual,

$$z = x + \imath y \,, \;\; \bar{z} = x - \imath y \,, \;\; \frac{\partial}{\partial z} = \frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y} \,, \;\; \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y} \,,$$

and

$$\frac{\partial}{\partial z} z = \frac{\partial}{\partial \bar{z}} \bar{z} = 2 \,.$$

Now, introduce two pairs of relatively conjugate operators $b_+^- = (b_+^+)^*$ and $b_-^- = (b_-^+)^*$

$$b_+^+ = \frac{1}{2}\left(-\frac{\partial}{\partial \bar{z}} + z\right), \quad b_+^- = \frac{1}{2}\left(\frac{\partial}{\partial z} + \bar{z}\right), \tag{15}$$

$$[b_+^-,\, b_+^+] = 1\,,$$

$$b_-^+ = \frac{1}{2}\left(-\frac{\partial}{\partial z} + \bar{z}\right), \quad b_-^- = \frac{1}{2}\left(\frac{\partial}{\partial \bar{z}} + z\right), \tag{16}$$

$$[b_-^-,\, b_-^+] = 1\,.$$

These pairs of operators act independently from each other:

$$[b_-^+,\, b_+^+] = [b_-^-,\, b_+^+] = [b_-^+,\, b_+^-] = [b_-^-,\, b_+^-] = 0\,. \tag{17}$$

The Hamiltonian can be written in the form

$$H = \frac{1}{2}\,(b_+^+ b_+^- + b_-^+ b_-^-) + 1\,. \tag{18}$$

It is convenient to introduce the charge operator Q

$$Q = \frac{1}{2}\left(-\bar{z}\frac{\partial}{\partial \bar{z}} + z\frac{\partial}{\partial z}\right) = \frac{1}{2}\left(b_+^+ b_+^- - b_-^+ b_-^-\right) \tag{19}$$

that commutes with H

$$[Q,\, H] = 0\,. \tag{20}$$

Due to the following commutators

$$
\begin{aligned}
&[b_+^+,\, H] = -b_+^+, \quad [b_+^-,\, H] = b_+^-, \\
&[b_-^+,\, H] = -b_-^+, \quad [b_-^-,\, H] = b_-^-\,, \qquad (21) \\
&[b_+^+,\, Q] = -b_+^+, \quad [b_+^-,\, Q] = b_+^-, \\
&[b_-^+,\, Q] = b_-^+, \quad [b_-^-,\, Q] = -b_-^-\,, \qquad (22)
\end{aligned}
$$

operator $b_+^+\,(b_+^-)$ is the creation (annihilation) operator of

a particle with the energy 1 and the charge 1

and $b_-^+\,(b_-^-)$ is the creation (annihilation) operator of its antiparticle, that is,

a particle with the energy 1 and the charge -1.

The ordinary vacuum state

$$\Psi_0 = \exp(-\frac{\bar{z}z}{2}) \tag{23}$$

satisfies the equations

$$b_+^- \Psi_0 = b_+^- \Psi_0 = 0\,. \tag{24}$$

In this state, the charge $Q = 0$ and the energy $E = 1$ (the sum of minimal energies of two quantum harmonic oscillators).

Other eigen-states of the operators H and Q can be generated from the state Ψ_0 in the usual way

$$(b_+^+)^k\,(b_-^+)^n\,\Psi_0\,.$$

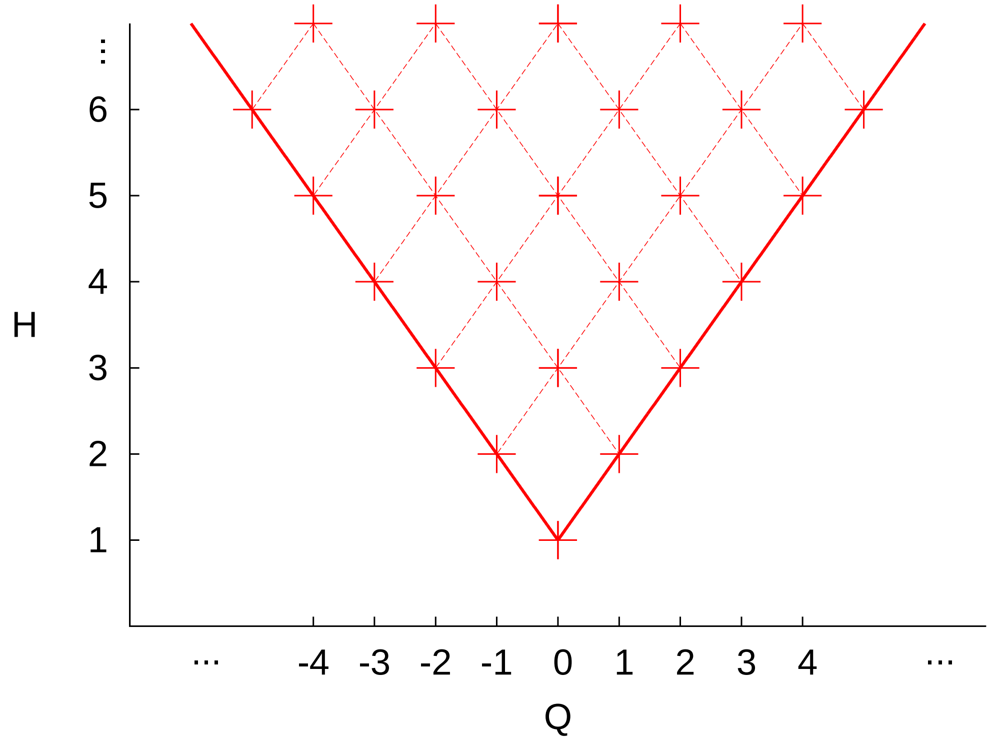

Figure 1: The eigen-states of the operators H and Q generated from the state Ψ_0.

The eigen-states of H and Q are represented as crosses at the verticies of the lattice.

The edges of the lattice correspond to the actions of the operators b_+^+, b_-^+ and b_+^-, b_-^-.

However, these states do not exhaust the set of all eigen-states of the operators H and Q .

Consider the functions

$$\Omega_{\lambda\mu} = \bar{z}^{\lambda} z^{\mu} \exp(-\frac{\bar{z}z}{2})\,. \tag{25}$$

They are the eigen-functions of the charge operator

$$Q\Omega_{\lambda\mu} = (-\lambda + \mu)\Omega_{\lambda\mu}\,. \tag{26}$$

Moreover, $\Omega_{\lambda 0}$ and $\Omega_{0\mu}$ are the eigen-functions of the Hamiltonian H

$$H\Omega_{\lambda 0} = (\lambda + 1)\Omega_{\lambda 0}\,, \quad H\Omega_{0\mu} = (\mu + 1)\Omega_{0\mu}\,. \tag{27}$$

They satisfy the following equations

$$b_+^-\Omega_{\lambda 0} = 0\,, \quad b_-^-\Omega_{0\mu} = 0 \tag{28}$$

and can be considered as

the states without positively charged particles ($\Omega_{\lambda 0}$)

and the states without negatively charged particles ($\Omega_{0\mu}$) .

Now, if we permit vacuum states to have fractional charges we can consider $(\Omega_{\lambda 0})$ with a noninteger λ as the "initial" state. The states generated from the state $\Omega_{\frac{1}{2}0}$ by the operators b_+^+, b_-^+ and b_-^- are represented at Figure 2.

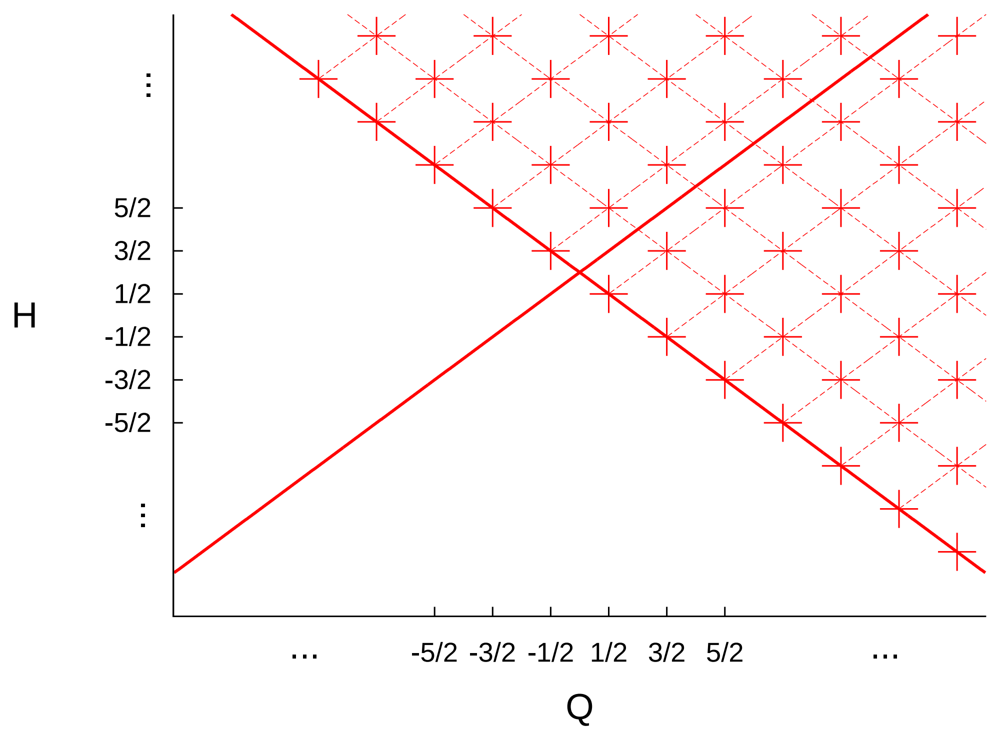

Figure 2: The eigen-states of the operators H and Q generated from the state $\Omega_{\frac{1}{2}0}$.

The same class of states is obtained if one starts from other "initial" state $\Omega_{\left(\frac{1}{2}+k\right)0}$, $\quad k = \pm 1, \pm 2, \dots$.

In this class of eigen-states, there is no state with minimal energy.

The corresponding functional space has an indefinite metric. For example, the norms of the states $\Omega_{(-\frac{1}{2}-n)0}\,, \quad n = 1, 2, \ldots$ have alternating signs.

$$\left(\Omega_{-\frac{3}{2}0}\,,\ \Omega_{-\frac{3}{2}0}\right) = \pi\Gamma\left(-\frac{1}{2}\right) = -2\pi^{\frac{3}{2}}\,, \tag{29}$$

$$\left(\Omega_{-\frac{5}{2}0}\,,\ \Omega_{-\frac{5}{2}0}\right) > 0\,, \quad \left(\Omega_{-\frac{7}{2}0}\,,\ \Omega_{-\frac{7}{2}0}\right) < 0\,,$$

and so on.

Here, the divergent integrals are regularized in the same way as in the section 1, that is,

$$\int\int (\bar{z}z)^{-\frac{3}{2}} \exp(-\bar{z}z) d\bar{z}dz = \pi\Gamma\left(-\frac{1}{2}\right)\,, \quad \ldots\,. \tag{30}$$

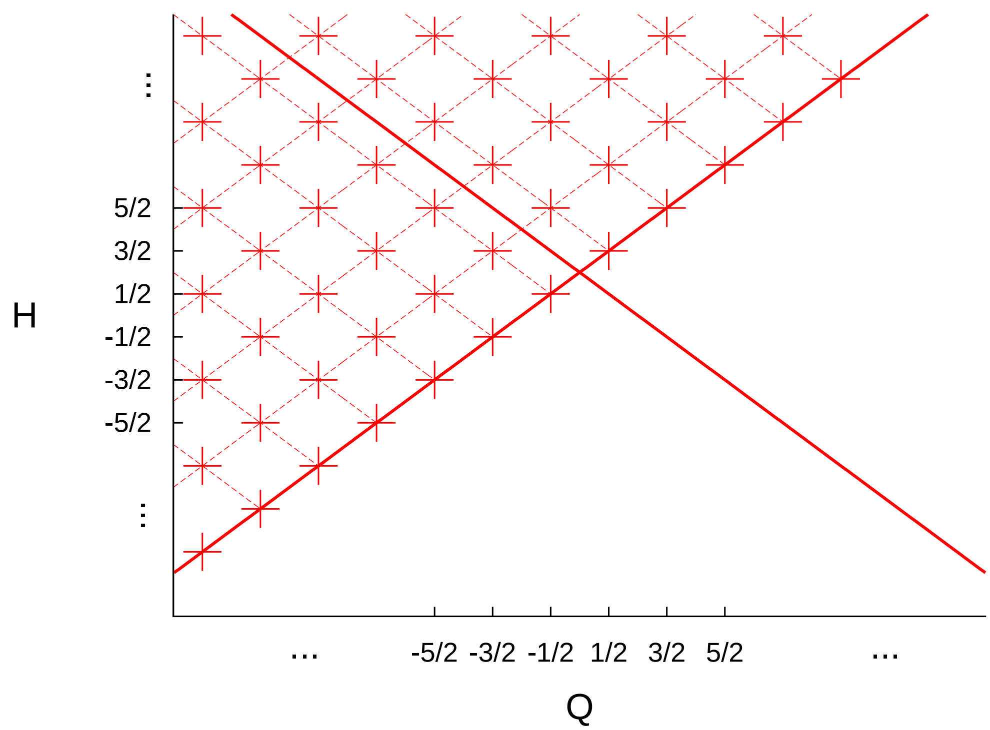

Figure 3: The eigen-states of the operators H and Q generated from the state $\Omega_{0\frac{1}{2}}$.

The symmetric picture (Fig. 3) is obtained for eigen-states generated from $\Omega_{0\mu}$.

They have the same properties as those represented at Fig. 2.

Note that the sets of states depicted at Fig. 1, Fig. 2 and Fig. 3 have no common states.

And all the states are orthogonal to each other.

3. A comment concerning the singularity x^{-2} of the Hamiltonian.

The singularities of the type r^{-2} appear in Hamiltonians of multi-dimensional problems written in spherical coordinates. The reason is rather trivial: Laplacian is the differential operator of the second order.

If we write the Hamiltonian of the section 1 in polar coordinates $(r,\,\varphi)$

$$-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}-r^2\right) \tag{31}$$

and use the substitution for wave-functions

$$\Psi(r,\,\varphi)=r^{-\frac{1}{2}}\exp\left\{\imath\left(\frac{1}{2}+\alpha\right)\varphi\right\}\Phi(r)\,, \tag{32}$$

then, for $\alpha=-2$ and $\alpha=1$, we get Schrödinger equation with the Hamiltonian considered in the section 1.

The solutions obtained there correspond to the states with $Q = -\frac{3}{2}$ and $Q = \frac{3}{2}$ depicted at Fig. 2.

Now, the meaning of the increasing (A^+) and the decreasing (A^-) operators is obvious

$$A^+ = b_+^+ b_-^+ = b_-^+ b_+^+ \,, \quad A^- = b_+^- b_-^- = b_-^- b_+^- \,.$$

4. The states generated from Ω_{-n0} and Ω_{0-n}.

Now, consider the states generated from Ω_{-n0} $(n = 1, 2, \ldots)$. It is convenient to consider them as the limit of the states Ψ_ε generated from $\Omega_{(-n+\varepsilon)0}$ that are represented by the picture similar to Fig. 2.

Some of them tend to the states of Fig. 1 as $\varepsilon \to 0$. Let us call them "ordinary" states.

The limit of the other states is represented at the right-hand side of Fig. 4.

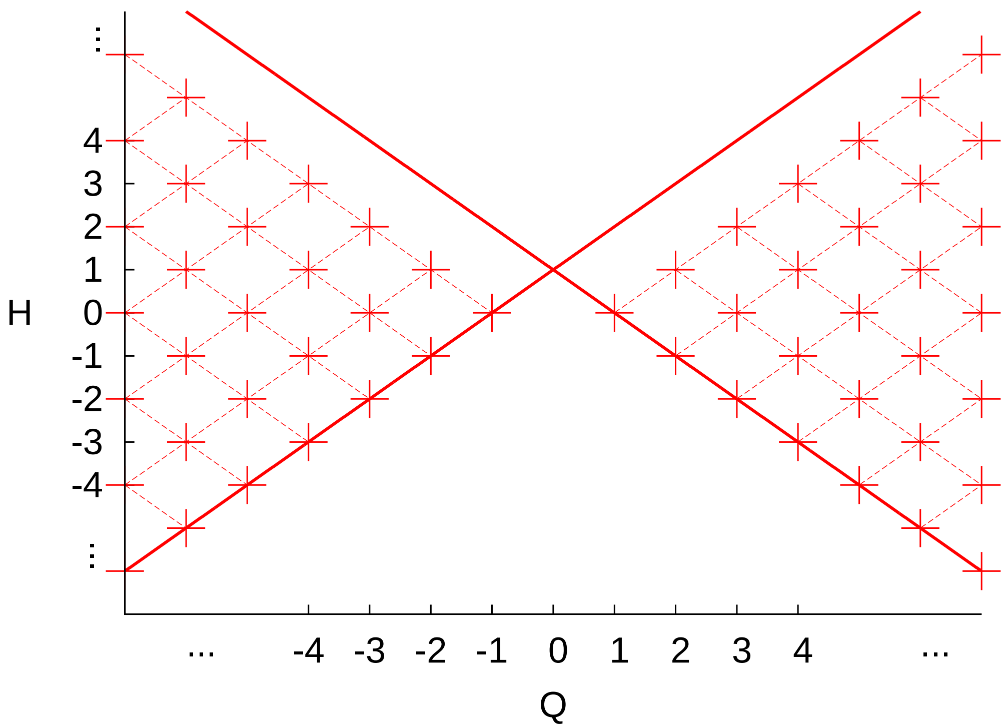

Figure 4: The states generated from Ω_{-n0} and Ω_{0-n} .

In the same way as it is done in equation (11), one can verify that these states (not the "ordinary" ones) are localized in an infinitesimal vicinity of the origin $z = 0$,

$$\lim_{\varepsilon \to 0} \frac{\int\int_{a<\bar{z}z<b} \left|\Psi(\bar{z}z)\right|^2 d\bar{z}z}{\int\int_{\varepsilon<\bar{z}z} \left|\Psi(\bar{z}z)\right|^2 d\bar{z}z} = 0\,, \quad a > 0\,. \tag{33}$$

Their norms defined according to (29), (30) tend to infinity as $\varepsilon^{-1},.$

However, we can renormalize the states $\Psi_\varepsilon \to \sqrt{\varepsilon}\Psi_\varepsilon$, or redefine the norm of the states as

$$\langle \Psi\,,\;\Psi \rangle = \lim_{\varepsilon\to 0} \varepsilon\left(\Psi_\varepsilon\,,\;\Psi_\varepsilon\right)\;. \tag{34}$$

The new norm is finite for all states. For example, for the "initial" states it is

$$\langle \Omega_{-n0}\,,\ \Omega_{-n0} \rangle = \lim_{\varepsilon \to 0} \varepsilon \left(\Omega_{(-n+\varepsilon)0}\,,\ \Omega_{(-n+\varepsilon)0} \right)$$
$$= \pi\, res\, \Gamma(-n+1)\,. \tag{35}$$

The metric of the space of states is indefinite and degenerate.

The norms of the "ordinary" states are equal to zero. So, we can factorize the space of states (identify the "ordinary" states with zero).

All the same is true for the states generated from Ω_{0-n} $(n = 1, 2, \ldots)$.

They are represented at the left-hand side of Fig. 4.

Suppose that vacuum states have integer charges. Then we get three different sets of states represented at Fig.1 and at both sides of Fig.4.

Although the states from the different sets are formed by the same particles, they have different properties because of the different vacuum states.

For example, the particles in the states from Fig. 4 are localized at origin.

All the states are orthogonal to each other.

And the transfer between these different sets cannot be realized as a result of any interaction of the (polynomial) form $H_{int} = P\left(b^{+}_{+},\, b^{+}_{-},\, b^{-}_{+},\, b^{-}_{-}\right)$.

So, from "the point of view" of an "ordinary" state at Fig. 1, the states at Fig. 4 prove to be "dark" ones.

5. Non-linear non-local substitutions in functional integrals.

Consider the Wiener measure

$$\exp\left\{-\frac{1}{2}\int_{T_0}^{T_1}(\dot{y}(t))^2 dt\right\}\ dy \equiv w(dy)\,. \qquad (36)$$

(Since $y(t)$ is a stochastic (Wiener) process the derivative is understood in a generalized sense.)

Under the non-linear non-local substitution

$$y(t) = x(t) + \int\limits_{T_0}^{t} f\left(x(\tau)\right)\, d\tau \tag{37}$$

the measure (36) turns into

$$\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1} \dot{x}^2(t)dt - \frac{1}{2}\int\limits_{T_0}^{T_1} f^2\left(x(t)\right)dt - \int\limits_{t=T_0}^{t=T_1} f\left(x(t)\right)dx\right\}\ dx\ . \tag{38}$$

The last term in the exponent is the Ito stochastic integral

$$\int_{t=T_0}^{t=T_1} f\left(x(t)\right) dx \,.$$

In addition to the boundary terms $(B.T.)$

$$B.T. = \Phi\left(x(T_1)\right) - \Phi\left(x(T_0)\right) \,, \quad \Phi(u) = \int^u f(v)dv \,,$$

it gives also the Ito term

$$-\frac{1}{2}\int_{T_0}^{T_1} f'\left(x(t)\right) dt \,.$$

It can be easily verified using the discrete version of the integral

$$\int_{T_0}^{T_1} (\dot{y}(t))^2 dt \approx \frac{1}{\Delta t} \sum_{k=1}^{n} (y(t_k) - y(t_{k-1}))^2$$

that after the substitution

$$y(t_k) = x(t_k) + \Delta t \sum_{i=0}^{k-1} f(x(t_i)) \, ,$$

looks like

$$\frac{1}{\Delta t}\sum_{k=1}^{n}(x_k - x_{k-1})^2 + \Delta t\sum_{k=1}^{n} f^2(x_{k-1}) + \sum_{k=1}^{n}(x_k - x_{k-1})f(x_{k-1})\,.$$

Taking into account that for Wiener process

$$(x_k - x_{k-1})^2 \sim \Delta t$$

we get the result given above.

Thus we get the formal equality of the measures

$$w(dy) \equiv \exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1} (\dot{y}(t))^2 dt\right\}\, dy =$$

$$\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1} \Big(\,\dot{x}^2(t) + f^2\,(x(t)) - f'\,(x(t))\,\Big)\, dt - B.T.\right\}\, dx\,. \quad (39)$$

(The connection between the theory with interaction and the free one.)

Note that because of the Ito term it is possible to factorize the Hamiltonian

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + f^2\left(x\right) - f'\left(x\right)\right) = a_f^+ \, a_f^- \,, \tag{40}$$

where

$$a_f^{\pm} = \frac{1}{\sqrt{2}}\left(\mp\frac{d}{dx} + f(x)\right) \,. \tag{41}$$

If the substitution (37) is invertible (that is, if we can determine the function $x(y)$) then

$$\int\limits_X F(x)\,\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1}\Big(\,\dot{x}^2(t)+f^2\,(x(t))-f'\,(x(t))\,\Big)\,dt-B.T.\right\}\,dx=$$

$$\int\limits_Y F(x(y))\,w(dy)\,. \tag{42}$$

So far, we have not specified the functional spaces X and Y . In the following examples we will see that they are different.

6. Liouville-type theory.

Consider a model of self-interacting scalar field with the action

$$\tilde{A}(\varphi) = \frac{1}{2} \int_0^T \left\{ (\dot{\varphi}(t))^2 dt + \lambda^2 \, e^{2\alpha\varphi(t)} - \alpha\lambda \, e^{\alpha\varphi(t)} \right\} dt \,. \tag{43}$$

Here the last term has the form of Ito term.

(For technical reasons we suppose that $\alpha < 0$ and $\lambda < 0$.)

By the substitution

$$\xi(t) = \varphi(t) - \varphi(0) + \lambda \int\limits_0^t e^{\alpha\varphi(\tau)} d\tau \tag{44}$$

we turn the measure defined by the action $\tilde{A}(\varphi)$ into Wiener measure $w(d\xi)$. (Actually, it is a definition of the functional measure on the space X).

$$\exp\left\{-\tilde{A}(\varphi)\right\} \, d\varphi \; = \; \exp\left\{-\frac{1}{2}\int\limits_0^T (\dot{\xi}(t))^2 dt\right\} \, d\xi \; \equiv \; w(d\xi)\,. \tag{45}$$

(Because of the singularity at $t = 0$ the substitution (44) is understood as the limit of a regularized equation.)

If we inverse the substitution $\xi(\varphi)$ we get the explicit form of the function $\varphi(\xi)$

$$\varphi(t) = \xi(t) - \frac{1}{\alpha} \ln\left(\alpha\lambda \int\limits_0^t e^{\alpha\xi(\tau)} d\tau\right) . \qquad (46)$$

Now we can evaluate the functional integrals over the space of functions $\varphi(t)$.

$$\int\limits_X F(\varphi)\,\exp\left\{-\tilde{A}(\varphi)\right\}\,d\varphi = \int\limits_{C[0,T]} F(\varphi(\xi))\,w(d\xi)\,. \tag{47}$$

Here, X is the space of functions that have a singularity of the form $\varphi \sim -\frac{1}{\alpha}\ln t$ at the origin $t = 0$.

Consider this theory as a toy model of **quantum** scalar field cosmology where the scale factor in conformal coordinates for the FLRW metric is determined by the **quantum** scalar field

$$g(t) \equiv a^2(t) = e^{2\varphi(t)} \,. \tag{48}$$

We have imposed the boundary condition corresponding to initial singularity

$$g(0) = 0 \,, \qquad \varphi(0) = -\infty \,. \tag{49}$$

The scale factor $a(t)$ in terms of $\xi(t)$ looks like

$$a(t) = e^{\varphi(t)} = e^{\xi(t)} \left(\alpha\lambda \int\limits_0^t e^{\alpha\xi(\tau)} d\tau \right)^{-\frac{1}{\alpha}} . \tag{50}$$

Now, in the quantum theory given by the action $\tilde{A}(\varphi)$ we can evaluate the mean value of the scale factor, moments and other quantities connected with it.

For the models with

$$\alpha = -\frac{1}{n}, \qquad n = 1, 2, \dots \ ,$$

the evaluation can be performed explicitly (Wiener integrals are reduced to iterated Gaussian integrals).

Let e.g. $\alpha = -1$. The mean value of the scale factor is

$$< a(t) >= -\lambda \int\limits_{C[0,1]} e^{\xi(t)} \int\limits_0^t e^{\alpha\xi(\tau)} d\tau \, w(d\xi) = \tag{51}$$

$$= (-2\lambda) \left(e^{\frac{t}{2}} - 1 \right) .$$

And the dispersion is

$$\Delta_a^2(t) =< a^2(t) > - < a(t) >^2=$$

$$\lambda^2 \int\limits_{C[0,1]} e^{2\xi(t)} \left(\int\limits_0^t e^{\alpha\xi(\tau)} d\tau \right)^2 w(d\xi) - 4\lambda^2 \left(e^{\frac{t}{2}} - 1 \right)^2 =$$

$$= \frac{1}{3}\lambda^2 \left(2e^{2t} - 12e^t + 16e^{\frac{t}{2}} - 6 \right). \qquad (52)$$

At small t the mean value and the dispersion are

$$< a(t) >= -\lambda t \;\; (\lambda < 0) \, , \quad \Delta_a^2(t) = \frac{1}{3}\lambda^2 \, t^3 \, . \tag{53}$$

As $\Delta_a \sim t^{\frac{3}{2}} \ll t$ at t small enough, there is a region where the mean value $< a(t) >$ can be considered as the exact value of the scale factor.

7. φ^4 interacting quantum field theory in one-dimensional "Euclidean" space-time.

In this case the substitution is

$$\chi(t) = \varphi(t) + \int\limits_{T_0}^{t} \varphi^2(\tau)d\tau\,. \tag{54}$$

So we have

$$\int\limits_X F(\varphi) \exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1}\Big((\dot{\varphi}(t))^2+\varphi^4(t)-2\varphi(t)\Big)\,dt - B.T.\right\} d\varphi$$

$$= \int\limits_{C[T_0,T_1]} F(\varphi(\xi))\,w(d\xi)\,. \tag{55}$$

Here

$$B.T. = \frac{1}{3}\left[\varphi^3(1)-\varphi^3(0)\right]\,.$$

Now, let us study the structure of the space X.

It can be easily verified that the function $\varphi(t)$ can have singularities at some points $t = t^*$ of the form

$$\varphi(t) \sim (t - t^*)^{-1} .$$

To make this result more clear, we notice that for the bounded function $\chi(t)$ there can be some regions where $|\varphi(t)|$ becomes large enough.

And in these regions the behavior of the function $\varphi(t)$ is prescribed by the equation $\dot{\varphi} = -\varphi^2$.

As the function $\chi(t)$ is bounded on $[T_0, T_1]$, there is a finite interval $[t_1, t_2] \subset [T_0, T_1]$ where the function $\varphi(t)$ has the only singularity $(t - t^*)^{-1}$.

Depending on the form of the function $\chi(t)$ there can be other finite intervals $[t_3, t_4], \ldots$ where $\varphi(t)$ has singularities of the same type $(t - t_j^*)^{-1}$.

The behavior of the function $\varphi(t)$ in the vicinity of the point t_j^* is given by the equation

$$\varphi(t) = \frac{1}{t - t_j^*} + \left[\chi(t) - \chi(t_j^*)\right]$$

$$-2(t - t_j^*)^{-2} \int\limits_{t_j^*}^{t} \left[\chi(\tau) - \chi(t_j^*)\right](\tau - t_j^*) d\tau + o((t - t_j^*)) \,. \qquad (56)$$

The second and the third terms are stochastic ones and behave as $(t - t_j^*)^{\varepsilon}$, $\varepsilon < 1$.

Now, the space X can be represented in the form

$$X = X_0 \cup X_1 \cup X_2 \cup \ldots X_n \cup \ldots ,$$

where $X_0 = C[T_0, T_1]$ and X_n is the space of functions with n singularities.

The equation (55) looks like

$$\sum_{n=o}^{\infty} \int_X F(\varphi) \exp\left\{ -\frac{1}{2} \int_{T_0}^{T_1} \Big((\dot{\varphi}(t))^2 + \varphi^4(t) - 2\varphi(t) \Big)\, dt - B.T. \right\} \, d\varphi$$

$$= \int_{C[T_0,T_1]} F(\varphi(\xi))\, w(d\xi)\,. \tag{57}$$

Thus we prove that the functional integrals of the free field theory evaluated over the space of continuous functions are equal to the functional integrals of the interacting field theory evaluated over a set of spaces containing the spaces of discontinuous functions.

For continuous field functions ($\varphi(t) \in C[T_0, T_1]$) , the integral (55) is the integral over the Wiener measure

$$\exp\{-\frac{1}{2}\int_0^1 (\dot{\varphi}(t))^2 dt\}\; d\varphi = w(d\varphi)\,,$$

the integrand

$$\exp\{-\frac{1}{2}\int_0^1 \left(\varphi^4(t) + \ldots\right) dt\}$$

being a bounded functional.

However, the functional integrals of the form

$$\int\limits_{X_n} P(\varphi)\,\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1}(\dot{\varphi}(t))^2 dt\right\}\,d\varphi$$

where $P(\varphi)$ is a polynomial, do not exist for $n \geq 1$.

$$\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1}(\dot{\varphi}(t))^2 dt\right\}\,d\varphi$$

is not a measure on X_n, $(n \geq 1)$.

And only

$$\exp\left\{-\frac{1}{2}\int\limits_{T_0}^{T_1}\Big((\dot{\varphi}(t))^2 + \varphi^4(t) - 2\varphi(t)\Big)\,dt - B.T.\right\}\,d\varphi$$

can be considered as a measure on X_n, $(n \geq 1)$.

In realistic quantum field theory (with the dimension of space-time $d = 4$) the interacting fields are not the continuous functions but distributions. That is why, we have to consider functional integrals over the functional spaces more complicated than C .

(Haag theorem)

8. Quantum restoration of broken symmetries.

The non-linear non-local substitution considered above and the equality of corresponding functional integrals lead to an interesting effect.

The classical action and the classical limit of the corresponding quantum theory turn out to be different.

Let us consider the action

$$A = \frac{1}{2} \int\limits_{-T}^{+T} (\dot{\varphi}(t))^2 dt + \frac{a^2}{2} \int\limits_{-T}^{+T} \left(\varphi^2(t) - \beta\right)^2 dt \,. \tag{58}$$

$$(\beta \equiv \frac{b}{2a}\,; \; a > 0, \; b > 0\,.)$$

The potential

$$V(\varphi(t)) = \frac{1}{2} a^2 \left(\varphi^2(t) - \beta^2\right)^2 \tag{59}$$

has two degenerate minimae at $\varphi = \pm\beta$, a local (unstable) maximum at $\varphi = 0$, and is symmetric: $V(-\varphi) = V(\varphi)$.

The Euler-Lagrange equation has the form

$$\ddot{\varphi}(t)) - 2a^2\varphi\left(\varphi^2(t) - \beta^2\right) = 0\,. \tag{60}$$

So, the classical system given by the action A is symmetrical under the substitution

$$\varphi \to -\varphi\,.$$

For the classical system given by the action

$$A_+ = A - a \int\limits_{-T}^{+T} \varphi(t)dt + \tag{61}$$

$$\frac{a}{3}\left[\varphi^3(+T) - \varphi^3(-T)\right] - a\beta^2\left[\varphi(+T) - \varphi(-T)\right]$$

the symmetry is broken because of the term linear in φ and the boundary terms.

(They appear from the Ito integral.)

The action A_+ leads to the Euler-Lagrange equation

$$\ddot{\varphi}(t)) - 2a^2\varphi\left(\varphi^2(t) - \beta^2\right) + a = 0 \tag{62}$$

and the boundary conditions

$$\dot{\varphi}(\pm T) + a\left(\varphi^2(\pm T) - \beta^2\right) = 0\,. \tag{63}$$

The corresponding quantum theory deals with the functional measure

$$\int \exp\{-A_+(\varphi)\} d\varphi \, .$$

Now the substitution

$$\chi(t) = \varphi(t) + a \int\limits_{-T}^{t} \left(\varphi^2(\tau) - \beta^2\right) d\tau \tag{64}$$

results in the equality of the functional integrals

$$\int_{X^+} F(\varphi) \exp\{-A_+(\varphi)\} d\varphi = \tag{65}$$

$$\int_{C[-T,+T]} F(\varphi(\chi)) \exp\left\{-\frac{1}{2}\int_{-T}^{+T} (\dot{\chi}(t))^2 dt\right\} d\chi \,.$$

The functional space X^+ is the space of functions that can have singularities on the interval $[-T, +T]$.

Now consider the classical limit of eq. (65)

$$F(\tilde{\varphi}) = \lim_{\hbar\to 0} \int\limits_{X^+} F(\varphi)\, \exp\{-\frac{1}{\hbar}\, A_+(\varphi)\} d\varphi =$$

$$\lim_{\hbar\to 0} \int\limits_{C[-T,+T]} F(\varphi(\chi))\, \exp\left\{-\frac{1}{\hbar}\,\frac{1}{2} \int\limits_{-T}^{+T} (\dot{\chi}(t))^2 dt\right\}\, d\chi\,. \qquad (66)$$

In the right hand side of eq. (66) the classical limit yields the equation

$$\ddot{\chi}(t) = 0\,,$$

or

$$\dot{\chi}(t) = const\,.$$

Actually, $const = 0$ as $\chi(+T)$ is not fixed. (It follows from the boundary conditions (10) as well.)

In terms of the function $\varphi(t)$ the equation looks like

$$\dot{\varphi}(t) + a\left(\varphi^2(t) - \beta^2\right) = 0\,. \tag{67}$$

Note that eq. (60) can be represented in the form

$$\dot{\varphi} = \pm\sqrt{f(\varphi)}$$

and has two branches of solutions corresponding to the different signs.

It can be easily seen that eq. (67) is the certain branch of eq. (60) with the proper integration constant.

Thus, a solution of eq. (67)

$$\tilde{\varphi}^{+}(t)$$

is a solution of eq. (60) but not one of eq. (62) **!**

In this sense, quantum theory restores the symmetry broken in classical theory **!**

The crucial point here is the integration over the functional space X^+ containing singular functions.

If it were the space C we would get the ordinary result obtained after integration over the Wiener measure

$$\exp\left\{-\frac{1}{2}\int_0^1 (\dot{\varphi}(t))^2 dt\right\} d\varphi\,.$$

The explicit form of the solution $\tilde{\varphi}^+(t)$ can be easily found. Depending on the bound value

$$\tilde{\varphi}^+(-T) \equiv -\alpha$$

it is

$$\tilde{\varphi}^+_\alpha(t) = \beta \tanh(bt + c)\,, \quad for\ -\alpha > -\beta\,, \tag{68}$$

or

$$\tilde{\varphi}^+_\alpha(t) = \beta \coth(bt + c)\,, \quad for\ -\alpha < -\beta\,. \tag{69}$$

If we set the integration constant c to be equal to zero, the solution is the odd function

$$\tilde{\varphi}^+(-t) = -\tilde{\varphi}^+(t)\,.$$

In this case the values of the action functionals at classical solution $\tilde{\varphi}^+$ are the same

$$A_+\left(\tilde{\varphi}^+\right) = A\left(\tilde{\varphi}^+\right)\,.$$

So far, we have considered the theory given by the action A_+ . However, the similar picture holds for the "mirror" action

$$A_- = A + a \int\limits_{-T}^{+T} \varphi(t)dt \tag{70}$$

$$-\frac{a}{3}\left[\varphi^3(+T) - \varphi^3(-T)\right] + a\beta^2 \left[\varphi(+T) - \varphi(-T)\right] \, .$$

The corresponding substitution is

$$\chi(t) = \varphi(t) - a \int\limits_{-T}^{t} \left(\varphi^2(\tau) - \beta^2\right) d\tau \, . \tag{71}$$

And we have the equality of the functional integrals

$$\int\limits_{X^-} F(\varphi)\, \exp\{-A_-(\varphi)\} d\varphi = \tag{72}$$

$$\int\limits_{C[-T,+T]} F(\varphi(\chi))\, \exp\left\{-\frac{1}{2}\int\limits_{-T}^{+T} (\dot{\chi}(t))^2 dt\right\}\, d\chi\,.$$

The structure of the space X^- is the same as of the space X^+. But the functions from X^+ and the functions from X^- have singularities at different points.

The same effect takes place in the "Liouville" theory considered above. The classical action $\tilde{A}(\varphi)$ and the classical limit $A(\varphi)$ of the corresponding quantum theory turn out to be different.

The true classical equation of motion is nothing else than the Euler-Lagrange equation of the action $A(\varphi)$ but not that of the action $\tilde{A}(\varphi)$.

And the classical field $\varphi_c(t)$ is, therefore,

$$\varphi_c(t) = -\frac{1}{\alpha} \ln(\alpha\lambda t) \,, \quad \alpha < 0, \ \lambda < 0 \,. \tag{73}$$

Note that singularities are used to cause obstacles in many conventional classical theories. It is due to the singularity the classical action $\tilde{A}$ is inappropriate to describe the dynamics of the universe.

In our approach the existence of singularities is essential for the adequate formulation of quantum theory. In this way we can take into account not only quantum effects but modify the classical theory.