A democratic suggestion

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Flavour space

The fermion masses live in "flavour space". Families, mixing and CP-violation are all flavour space phenomena.

The structure of flavour space is determined by the mass matrices, i.e. by the form that the mass matrices take in the weak interaction basis in flavour space where mixed fermion states interact weakly, in contrast to the mass bases, where the mass matrices are diagonal.

The information content of a $N \times N$ matrix M is contained in its N matrix invariants, which are the sums and products of the eigenvalues λ_j, such as $trace M$, $det M$, i.e. more generally

$$I_1 = \sum_j \lambda_j = \lambda_1 + \lambda_2 + \lambda_3 \ldots$$

$$I_2 = \sum_{jk} \lambda_j \lambda_k = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \ldots$$

$$I_3 = \sum_{jkl} \lambda_j \lambda_k \lambda_l = \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \ldots$$

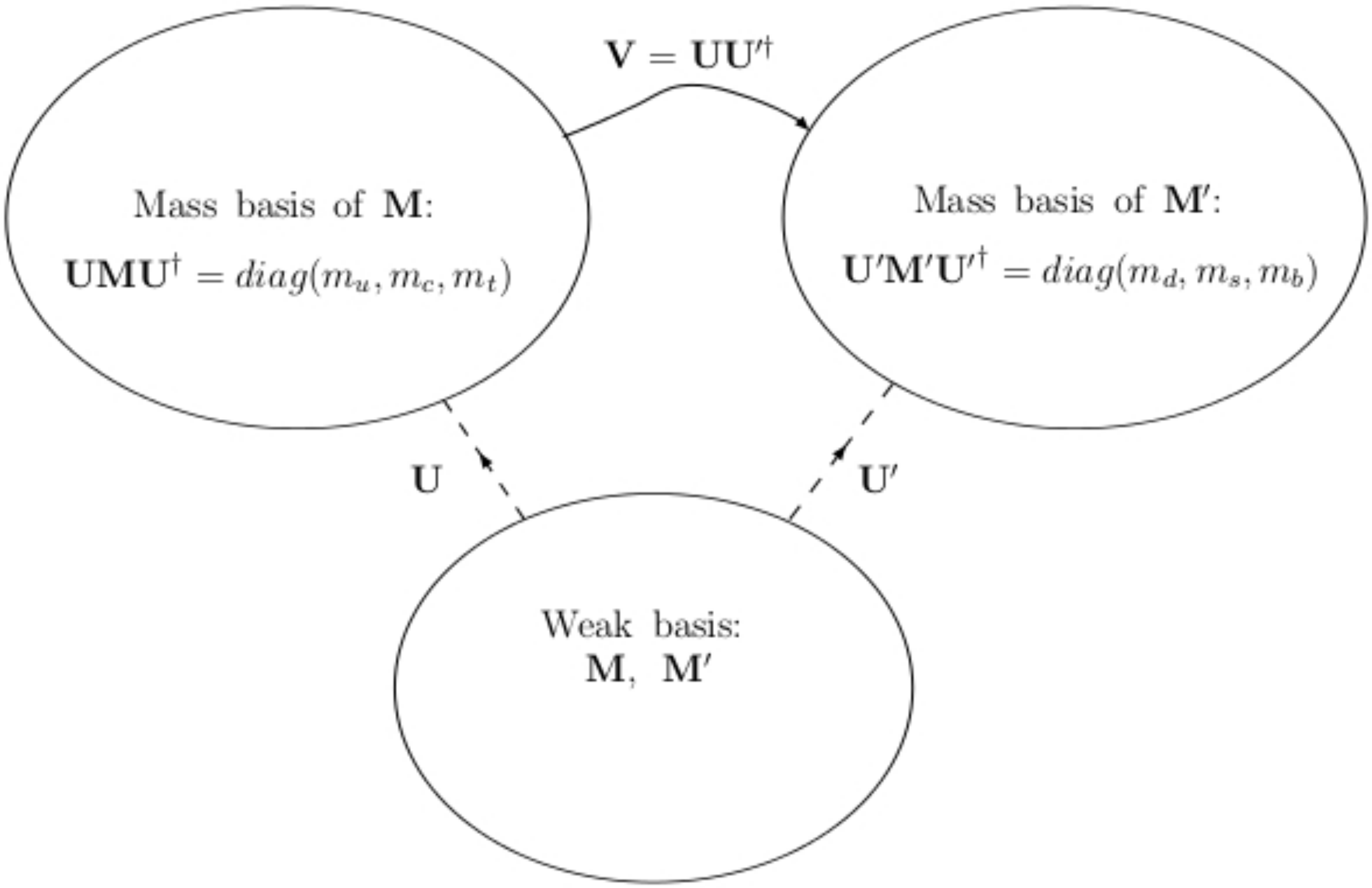
and so on, with $j, k, l, \ldots = 1, 2, \ldots N$. These expressions are invariant under permutations of the eigenvalues, which in the context of mass matrices means that they are flavour symmetric, i.e. invariant under permutations of the flavours.

So the information content of a maxtrix is in principle contained in its invariants, but the form that the mass matrices take in the weak interaction basis is also considered to be important, the hope being that it can give some hint about the origin of the unruly fermion masses, ranging from the tiny electron mass to the about 10^5 times larger top mass.

In the hope to find a mass texture that could shed some light on the origin of the mass spectra, an industry of different mass matrix ansätze has emerged. As long as we have so few guidelines, this basically amounts to guessing what structure the matrix has in the weak interaction basis in flavour space.

Bases in flavour space

Let M and M' be the mass matrices of the $2/3-$ and $-1/3$-sectors, respectively. We go from the weak interaction basis to the mass bases by rotating the matrices in flavour space, with the unitary matrices U and U'. For the 2/3-sector, $M \rightarrow UMU^\dagger = D = diag(m_u, m_c, m_t)$ and similarly for the $-1/3$-sector, $M' \rightarrow U'M'U'^\dagger = D' = diag(m_d, m_s, m_b)$, the mass bases being related by the weak mixing matrix $V = UU'^{\dagger}$,



"Democratic" mass matrices

To make a mass matrix ansatz thus amounts to guessing the form of the mass matrices $M(2/3)$ and $M'(-1/3)$ in the weak interaction basis. The guesswork relies on phenomenology, the only obvious principle being the family pattern. This is the background of the democratic assumption that in the weak interaction basis the fermion mass matrices have a structure close to the "democratic", maximally $S(3)_L \times S(3)_R$-symmetric (Nambu) matrix

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

with eigenvalues $(0, 0, 3)$, reflecting the observed family pattern.

The underlying philosophy is that in the Standard Model, where the fermions get their masses from the Yukawa couplings by the Higgs mechanism, there is no reason why there should be a different Yukawa coupling for each fermion.

The most natural assumption is then that all the couplings should be the same, and in the democratic scheme this absence of any distinguishing principle is taken at face value and (assuming three families) a zeroth order 3×3 mass matrix is postulated, of the form $M_0 = k\mathbf{N}$, where k has dimension mass, and $\mathbf{N}$ is the Nambu matrix.

The model with mass degenerate quarks introduces the idea that particles that are physically indistiguishable could in some sense be distinct, in conflict with Leibniz' principle of the identity of indiscernables, which states that no two distinct things can be exactly alike. It is however clear that actual fermions do not display a democratic permutation symmetry, so Nature does indeed obey Leibniz.

So the initial assumption is that the Yukawa couplings of the fermion fields of a given charge are equal. The corresponding mass spectrum $(m_1, m_2, m_3) \sim (0, 0, 3k)$ reflects the charged fermion mass spectra with two light families and a third much heavier family.

In order to obtain realistic mass spectra with non-zero masses, the $S(3)_L \times S(3)_R$ symmetry must be broken, and the different "democratic" matrix ansätze correspond to different symmetry breaking schemes.

The problem is the lack of more guiding principles -

The democratic basis

We define the democratic basis as the flavour space basis where the mass matrix M is diagonalized by the matrix

$$U_{dem} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

and the mass Lagrangian is symmetric under permutations of the fields $(\varphi_1, \varphi_2, \varphi_3)$,

$$\mathcal{L}_m = \bar{\varphi} M \varphi = k \sum_{ij} \bar{\varphi}_i \varphi_j$$

With primary up- and down-sector mass matrices that are both democratic, at the level of this "zeroth order approach" the mixing matrix is $V = UU'^{\dagger} = U_{dem} U_{dem}^{\dagger} = \mathbf{1}$.

Matrices with a democratic texture

We may perform a first perturbation of the Nambu matrix by just adding a unit matrix $diag(a,a,a)$, $M_0 \to M_1$,

$$M_1 = k\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} k+a & k & k \\ k & k+a & k \\ k & k & k+a \end{pmatrix}$$

corresponding to the mass spectrum $(a, a, 3a+3k)$.

This matrix has a democratic texture: it is diagonalized by U_{dem}, and in the democratic basis its mass Lagrangian is invariant under permutations of the quark fields, since

$$\mathcal{L}_{M_1} = (k+a)\sum_j \bar{\varphi}_j \varphi_j + k\sum_{i\neq j} \bar{\varphi}_i \varphi_j$$

If $M(2/3)$ and $M'(-1/3)$ both are of the form M_1, there is no CP-violation, it will appear only once the "democracy" is broken.

We can repeat this scheme, with a mass spectrum $(a, a, \ldots, a, k)$ for an undefined number n of families where the fermion mass matrices again are proportional to the $S(n)_L \times S(n)_R$ symmetric matrix

$$\mathbf{N} = \begin{pmatrix} 1 & 1 \ldots & 1 \\ 1 & 1 \ldots & 1 \\ & \ldots\ldots & \\ 1 & 1 \ldots & 1 \end{pmatrix}$$

which is diagonalized by a unitary matrix analogous to $U_{dem}(3 \times 3)$. For example, in the case of n=4, the diagonalizing matrix is

$$\mathbf{U} = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{6} & -\sqrt{6} & 0 & 0 \\ \sqrt{2} & \sqrt{2} & -2\sqrt{2} & 0 \\ 1 & 1 & 1 & -3 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \end{pmatrix}$$

We can again add a $n \times n$-dimensional diagonal matrix $diag(a, a, \ldots, a)$, and get the $n \times n$-dimensional mass matrix

$$\mathbf{M} = k\mathbf{N} + a\mathbf{I}$$

with a mass spectrum $(a, a, .., a, nk + a)$ with n massive states, of which $n - 1$ are degenerate. We still have a mass matrix with a democratic texture, and there is still no CP-violation.

Breaking the democratic symmetry

In order to obtain mass matrices for the physical flavours (ψ_1, ψ_2, ψ_3), the permutation symmetry of the democratic fermion fields $(\varphi_1, \varphi_2, \varphi_3)$ must be broken, thus lifting the mass degeneracy.

A matrix ansatz is usually obtained by introducing some explicit scheme for breaking the $S(3)_L \times S(3)_R$ symmetry, for example by modifying the M_{33}-term by

$$M = \begin{pmatrix} k+a & k & k \\ k & k+a & k \\ k & k & k+a \end{pmatrix} \rightarrow \begin{pmatrix} k+a & k & k \\ k & k+a & k \\ k & k & W \end{pmatrix}$$

whereby the (ψ_1, ψ_2)-degeneracy is broken.
This is pure ansatz, since there is so little information to work with.

Another breaking scheme

Instead of following the usual procedure of breaking the democratic symmetry by modifying the mass matrices, we "perturb" the diagonalizing matrices.
In this way we "derive" the unitary matrices which diagonalize the mass matrices. Using the "standard" parametrization of the Cabbibo-Kobayashi-Maskawa weak mixing matrix $V = UU'^{\dagger}$, we define the diagonalization matrix for the 2/3-sector and -1/3-sector as $U = \tilde{U}U_{dem}$, and $\tilde{U}'U_{dem}$, correspondingly; ensuring that the weak interaction basis remains close to the democratic basis. This is done by writing the mixing matrix as

$$\mathbf{V} = \begin{pmatrix} c_\beta c_{2\theta} & s_\beta c_{2\theta} & s_{2\theta} e^{-i2\gamma} \\ -c_\beta s_\alpha s_{2\theta} e^{i2\gamma} - s_\beta c_\alpha & -s_\beta s_\alpha s_{2\theta} e^{i2\gamma} + c_\beta s_\alpha & s_\alpha c_{2\theta} \\ -c_\beta c_\alpha c_{2\theta} e^{i2\gamma} + s_\beta s_\alpha & -s_\beta c_\alpha s_{2\theta} e^{i2\gamma} - c_\beta s_\alpha & c_\alpha c_{2\theta} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \times$$

$$\times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} e^{i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\gamma} \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and taking diagonalizing matrices $\mathbf{U}$ and $\mathbf{U}'$ for the 2/3- and -1/3-sectors to be

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

and

$$\mathbf{U}' = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

where α, β, θ and γ correspond to the parameters in the standard parametrization of the Cabbibo-Kobayashi-Maskawa mixing matrix, $\gamma = \delta/2$, $\delta = 1.2 \pm 0.08$ rad, and $\theta = c_{13}/2 \approx 0.1^\circ$, while $\alpha = c_{23} \approx 2.38^\circ$, and $\beta = c_{12} \approx 13.04^\circ$.

With the matrices $\mathbf{U}$ and $\mathbf{U}'$ we get the mass matrices $M(2/3) = \mathbf{U}^\dagger diag(m_u, m_c, m_t)\mathbf{U}$ and $M(-1/3) = \mathbf{U}'^\dagger diag(m_d, m_s, m_b)\mathbf{U}'$, in a next to democratic weak interaction basis.

Using the fermion masses (Matthias Jamin)

1. $m_d(M_Z)$ = 3 MeV, $m_s(M_Z)$ = 57 MeV, $m_b(M_Z)$ = 2878 MeV
2. $m_u(M_Z)$ = 1.7 MeV, $m_s(M_Z)$ = 620 MeV, $m_b(M_Z)$ = 171000 MeV

where the Georgi-Jarlskog relation from SO(10)

$$\frac{m_b}{m_s} = \frac{m_\tau}{m_\mu} \approx 50.5,$$

as well as the relations

$$\frac{m_u}{m_d} = 0.553 \pm 0.043 \text{ and } \frac{m_s}{m_d} = 18.9 \pm 0.8$$

from chiral perturbation theory are taken into account; we get the following mass matrices, with a (next to) democratic texture:

$$M(2/3) = K_u \begin{pmatrix} 1 & 1.0004 & 1.069e^{-i\eta} \\ 1.0004 & 1.0009 & 1.069e^{-i\eta} \\ 1.069e^{i\eta} & 1.069e^{i\eta} & 1.1608 \end{pmatrix}$$

with $K_u \approx \sum(m_u, m_c, m_t)/3$ and $\eta = 2.786^\circ$, and

$$M(-1/3) = K_d \begin{pmatrix} 1.046 & 1.029e^{i\mu} & e^{-i\mu} \\ 1.029e^{-i\mu} & 1.025 & 1.008e^{i\mu} \\ e^{i\mu} & 1.008e^{-i\mu} & 1.064 \end{pmatrix}$$

with $K_d \approx \sum(m_d, m_s, m_b)/3$ and $\mu = 0.23^\circ$. Both sectors are nearly democratic, with a phase structure

$$\begin{pmatrix} \bullet & \bullet & e^{-i\eta} \\ \bullet & \bullet & e^{-i\eta} \\ e^{i\eta} & e^{-i\eta} & \bullet \end{pmatrix}, \quad \begin{pmatrix} \bullet & e^{i\mu} & e^{-i\mu} \\ e^{-i\mu} & \bullet & e^{i\mu} \\ e^{i\mu} & e^{-i\mu} & \bullet \end{pmatrix},$$

respectively.

Calculable matrices

In the mass matrix literature, there is an emphasis on "calculability", i.e. that the fermion mass matrices in the weak interaction basis should have a handable form, making it possible to calculate the eigenvalues and so on.

There is however nothing that forces nature to be so accomodating as to serve us a user-friendly formalism. In the democratic scheme, the philosophy being that the mass matrices remain next to democratic, it is on the contrary conceivable that the mass textures are not of a "calculable" kind. It is however tempting to look for relations between the elements presumeably that could make the matrices more "calculable".

Attempting to find matrices with a texture with at least an approximate "calculability", we use the form $M_1 = k\mathbf{N} + a\mathbf{1}$ as a point of departure, and maintaining the phase structure in the "experimental" $M(2/3)$ and $M(-1/3)$, which gives for the up-sector

$$M(2/3) \approx K_u \begin{pmatrix} A & B & Be^{-i\eta} \\ B & A & Be^{-i\eta} \\ Be^{i\eta} & Be^{i\eta} & C \end{pmatrix},$$

displaying the approximately conserved (φ_1, φ_2)-permutation symmetry.
Inserting the 2/3-mass values in

$$A = B + m_1$$
$$C = m_2 + m_3 - 2B$$
$$B = [m_2 + m_3 - m_1 +$$
$$+\sqrt{(m_2 + m_3 - m_1)^2 + 6(m_1 m_2 + m_1 m_3 - m_2 m_3)}]/6,$$

we get the nearly democratic texture:
$(A, B, C) = B(1.00003, 1, 1.016)$.

In $M(-1/3)$ the (φ_1, φ_2)-permutation symmetry is however totally broken, and the form of the matrix is extremely "unhandable". The phase structure is so abominable that the phase μ even enters the matrix invariants:

$$\begin{pmatrix} \bullet & e^{i\mu} & e^{-i\mu} \\ e^{-i\mu} & \bullet & e^{i\mu} \\ e^{i\mu} & e^{-i\mu} & \bullet \end{pmatrix}$$

This unpleasant $M(-1/3)$ comes from defining the diagonalizing matrices U and U' as 'evenly' as possible, we can however choose other ways to define the diagonalizing matrices and thereby $M(-1/3)$ and $M(2/3)$.

With the choice

$$\mathbf{U}' = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

and

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos 2\theta & 0 & \sin 2\theta \, e^{-i2\gamma} \\ 0 & 1 & 0 \\ -\sin 2\theta \, e^{-i2\gamma} & 0 & \cos 2\theta \end{pmatrix} \times$$

$$\times U_{dem}$$

we get the mass matrices in the weak interaction basis

$$M(-1/3) = K_d \begin{pmatrix} 1.04 & 1.03 & 1 \\ 1.03 & 1.03 & 1.015 \\ 1 & 1.015 & 1.06 \end{pmatrix}$$

and

$$M(2/3) = K_u \begin{pmatrix} 1 & 1.003\, e^{i2\rho} & 1.08\, e^{i\rho} \\ 1.003\, e^{-i2\rho} & 1.006 & 1.08\, e^{-i\rho} \\ 1.08\, e^{-i\rho} & 1.08\, e^{i\rho} & 1.19 \end{pmatrix}$$

with $\rho = 0.23^\circ$, $K_d \approx \sum(m_d, m_s, m_b)/3$ and $K_u \approx \sum(m_u, m_c, m_t)/3$.

Using this phase structure, we can imagine an ansatz à la

$$M(2/3) \approx K_u \begin{pmatrix} A & B\, e^{i2\rho} & Be^{i\rho} \\ B\, e^{-i2\rho} & A & Be^{-i\rho} \\ Be^{-i\rho} & Be^{i\rho} & C \end{pmatrix},$$

and

$$M(-1/3) \approx K_d \begin{pmatrix} X & Y & Y \\ Y & X & Y \\ Y & Y & W \end{pmatrix},$$

with $(A, B, C) = B(1.00003, 1, 1.016)$ and $(X, Y, W) = Y(1.003, 1, 1, 08)$.

The most extreme version of this scheme is to take the diagonalizing matrix for the -1/3-sector to simply be $U' = U_{dem}$, dumping the whole mixing into the 2/3-sector. This scheme retains the same phase structure as the one above, and is in this sense its equivalent.

The object of this approach is to squeeze out as much information as possible from existing data, to guide us in breaking the democratic permutation symmetry. But what parametrization of the mixing matrix one uses, and how it is inserted in the expressions for the diagonalizing matrices for the -1/3- and 2/3-sectors, of course (and unfortunately) remains a matter of choice - or guesswork!