Continuing the hunt (on mass matrices)

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The weak mixing matrix

Last year I talked about a way of digging into the quark mass matrices by factorizing the Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix which appears in the charged current Lagrangian

$$\mathcal{L}_{cc} = -\frac{g}{2\sqrt{2}}\,\bar{f}_L \gamma^\mu V f'_L W_\mu + h.c.$$

where f and f' are quark fields with charges Q and $Q-1$, correspondingly.

From the weak interaction-perspective, $\mathcal{L}_{cc}$ describes an interaction between (left-handed) flavour states.

From the point of view of all other interactions, the interaction takes place between particle mixings. Our dilemma is how to understand the relation between ``physical particles'', i.e. mass states, and ``flavour states'', which are the states that interact weakly with each other.

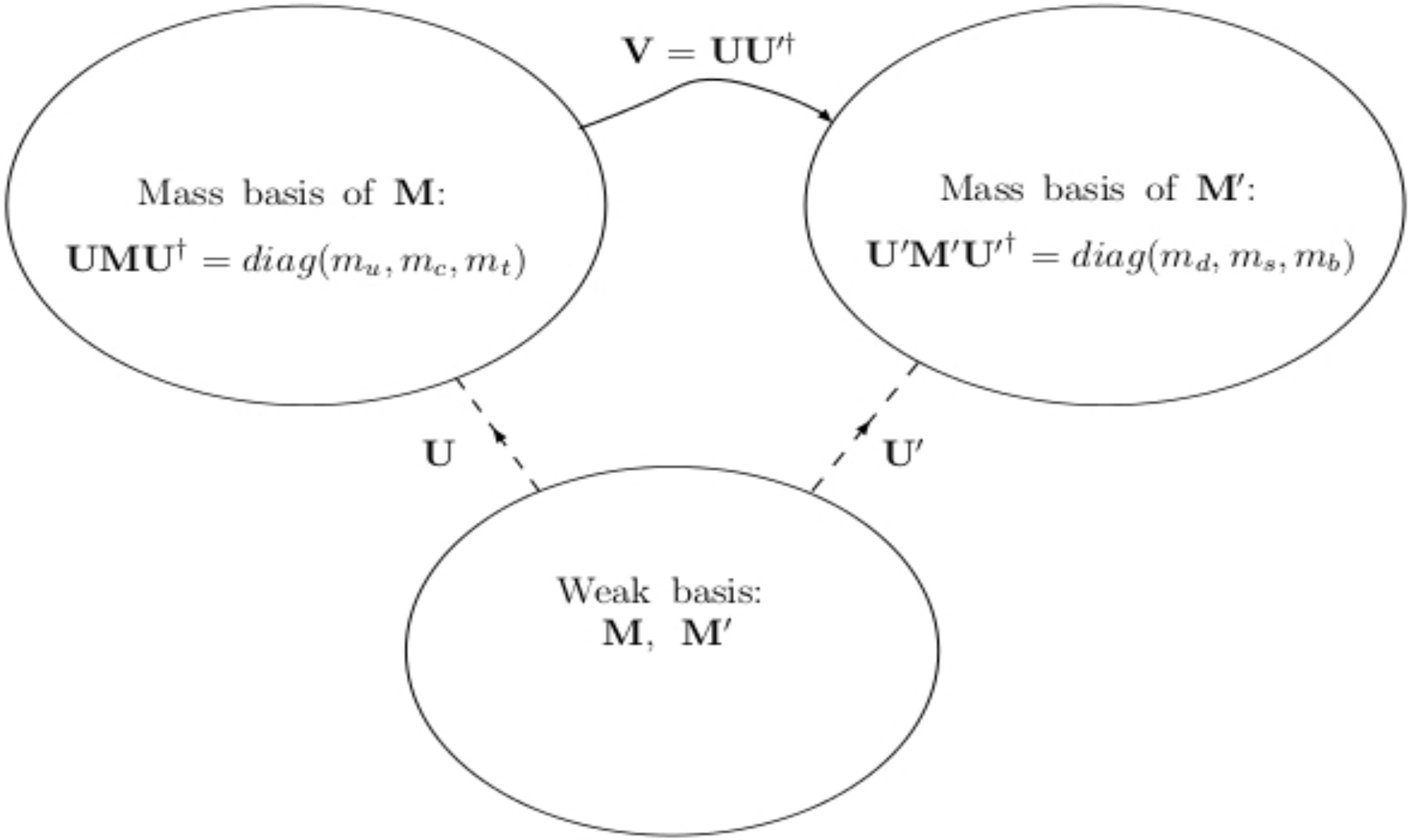
We imagine the flavour states living in the weak basis in flavour space, related to the mass bases (where the usual particles live) by the unitary rotation matrices U and U', factors of the CKM-matrix, $V = UU'^{\dagger}$.
$V \neq \mathbf{1} \rightarrow$ the up-sector mass basis is different from the down-sector mass basis, the CKM matrix thus bridges the two mass bases.

In the mass Lagrangian $\mathcal{L}_{mass} = \bar{\varphi} M \varphi = \bar{\psi} D \psi$, φ are flavour states, and ψ are mass states, and M and D are the mass matrices in the flavour basis and mass basis, respectively.

We go between the bases by rotating by U and U',

$$M \rightarrow UMU^{\dagger} = D = diag(m_u, m_c, m_t)$$

$$M' \rightarrow U'M'U'^{\dagger} = D' = diag(m_d, m_s, m_b)$$



Mass states and flavour states

The quark mass eigenstates, corresponding to particles with definite masses are usually perceived as ``physical''; then the weakly interacting flavour states are percieved as mixings of physical particles.

The lepton sector is different:neutrino mass eigenstates don't ever appear in interactions, they merely propagate in free space. In the realm of neutral leptons it is actually the flavour states ν_e, ν_μ, ν_τ that we perceive as ``physical'', since they are the only neutrinos that we ``see'', as they appear together with the charged leptons.

The charged leptons e, μ, τ are assumed to be both weak eigenstates and mass eigenstates.
The only mixing matrix that appears in the lepton sector is the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix U which only operates on neutrino states,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U_{(PMNS)} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

where (ν_1, ν_2, ν_3) are mass eigenstates, and $(\nu_e, \nu_\mu, \nu_\tau)$ are the weakly interacting ``flavour states''. In the lepton sector, the charged currents are thus interpreted as charged lepton flavours $(e, \mu, \tau)$ interacting with the neutrino ``flavour states'' $(\nu_e, \nu_\mu, \nu_\tau)$.

Factorizing the mixing matrix

Usual procedure in establishing an ansatz for the quark mass matrices: start from some argument or model.

Instead: look for a factorization of the Cabbibo-Kobayashi-Maskawa mixing matrix that could lead us to the 'right' mass matrices.

The CKM matrix $V = UU'^\dagger$ can of course be parametrized and factorized in many different ways, and different factorizations correspond to different rotation matrices U and U', and correspondingly to different mass matrices M and M'.

The most obvious and ``symmetric'' factorization of the CKM mixing matrix is, following the standard parametrization with three Euler angles α, β, 2θ,

$$V = \begin{pmatrix} c_\beta c_{2\theta} & s_\beta c_{2\theta} & s_{2\theta} e^{-i\delta} \\ -c_\beta s_\alpha s_{2\theta} e^{i\delta} - s_\beta c_\alpha & -s_\beta s_\alpha s_{2\theta} e^{i\delta} + c_\beta c_\alpha & s_\alpha c_{2\theta} \\ -c_\beta c_\alpha s_{2\theta} e^{i\delta} + s_\beta s_\alpha & -s_\beta c_\alpha s_{2\theta} e^{i\delta} - c_\beta s_\alpha & c_\alpha c_{2\theta} \end{pmatrix} = UU'^\dagger$$

with the diagonalizing rotation matrices for the up- and down-sectors

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

and

$$U' = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & & \\ & 1 & \\ & & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$\gamma = \delta/2$, $\delta = 1.2 \pm 0.08$ rad, and $2\theta = 0.201 \pm 0.011^\circ$, while $\alpha = 2.38 \pm 0.06^\circ$ and $\beta = 13.04 \pm 0.05^\circ$.

In this factorization scheme, α and β are rotation angles operating in the up-sector and the down-sector, respectively.

With U and U', we obtain the the up- and down-sector mass matrices

$$M = U^\dagger diag(m_u, m_c, m_t)U \text{ and } \mathrm{M}' = \mathrm{U}'^\dagger \mathrm{diag}(\mathrm{m_d}, \mathrm{m_s}, \mathrm{m_b})\mathrm{U}',$$

such that

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} Xc_\theta^2 + Ys_\theta^2 & Zs_\theta\, e^{-i\gamma} & (X - \\ Zs_\theta\, e^{i\gamma} & Y - 2Z\cot 2\alpha & -Z \\ (X-Y)c_\theta s_\theta & -Zc_\theta\, e^{-i\gamma} & Xs_\theta^2 \end{pmatrix}$$

where $X = m_u$, $Z = (m_t - m_c)\sin\alpha\cos\alpha$ and $Y = m_t - Z\tan\alpha = m_c\sin^2\alpha + m_t\cos^2\alpha$, and

$$M' = \begin{pmatrix} M'_{11} & M'_{12} & M'_{13} \\ M'_{21} & M'_{22} & M'_{23} \\ M'_{31} & M'_{32} & M'_{33} \end{pmatrix} = \left(\begin{matrix} X' s_\theta^2 + Y' c_\theta^2 & Z' c_\theta \, e^{i\gamma} & (\\ Z' c_\theta \, e^{-i\gamma} & Y' + 2Z' \cot 2\beta & \\ (X' - Y') c_\theta s_\theta & -Z' s_\theta \, e^{i\gamma} & \end{matrix} \right.$$

where $X' = m_b$, $Z' = (m_s - m_d) \sin\beta \cos\beta$ and $Y' = m_d + Z' \tan\beta = m_d \cos^2\beta + m_s \sin^2\beta$.

The two mass matrices thus have similar textures, or forms, and there is even a relational equality,

$$M_{32}/M_{12} = M_{23}/M_{21} = M'_{12}/M'_{32} = M'_{21}/M'_{23} = -\cot\theta$$

which is independent of the quark masses.

From $Y = m_c \sin^2\alpha + m_t \cos^2\alpha$, $Z = (m_t - m_c) \sin\alpha \cos\alpha$, $Y' = m_d \cos^2\beta + m_s \sin^2\beta$ and $Z' = (m_s - m_d) \sin\beta \cos\beta$, we moreover have

$$m_u = X,\ m_c = Y - Z\cot\alpha,\ m_t = Y + Z\tan\alpha$$
$$m_d = Y' - Z'\tan\beta,\ m_s = Y' + Z'\cot\beta,\ m_b = X'$$

Relating the mass matrices of the up- and down-sectors

This way of factorizing the CKM matrix implies that m_u and m_b are independent of α and β.

It also implies that in the limit $\alpha = \beta$ the two rotation matrices can be transformed into each other by the unitary permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

in the sense that

$$U' = PU^*P\big|_{\alpha=\beta=\xi}$$

This relation carries over to the mass matrices, so from a structural point of view, in the limit $\alpha \to \beta$

$$M' = PM^*P$$

which implies that in this limit the up- and down-sector mass spectra are equal/proportional to each other.

In this limit, $\alpha = \beta = \xi$, the mass eigenvalues go to

$m_u = X$, $m_c = Y - Z\cot\xi$, $m_t = Y + Z\tan\xi$

$m_d = Y' - Z'\tan\xi$, $m_s = Y' + Z'\cot\xi$, $m_b = X'$

$\Rightarrow (m_t - Y)(m_s - Y') = (Y - m_c)(Y' - m_d)$

$Z^2 = (m_t - Y)(Y - m_c)$,

$$Z'^2 = (m_s - Y')(Y' - m_d),$$

$$Z/Z' = (m_t - Y)/(Y' - m_d) = (Y - m_c)/(m_s - Y')$$

Numerical matrices

Using the numerical values $\beta = 13.04^o$, $\alpha = 2.38^o$, $\delta = 1.2 \pm 0.08$ rad, and $2\theta = 0.201 \pm 0.011^\circ$ for the the angles, and using the mass values (Jamin 2014) for the up- and down-sectors,

$$m_u(M_Z) = 1.24 MeV \quad m_c(M_Z) = 624 MeV \quad m_t(M_Z) = 171550 Me$$

$$m_d(M_Z) = 2.69 MeV \quad m_s(M_Z) = 53.8 MeV \quad m_b(M_Z) = 2850 Me$$

we get the numerical values for the mass matrices M and M'

$$M = \begin{pmatrix} 1.767 & 12.439e^{-i\gamma} & -300.389 \\ 12.439e^{i\gamma} & 918.759 & -7091.892e^{i\gamma} \\ -300.389 & -7091.892e^{-i\gamma} & 171254.714 \end{pmatrix} MeV$$

and

$$M' = \begin{pmatrix} 5.299 & 11.23e^{i\gamma} & 4.99 \\ 11.23e^{-i\gamma} & 51.18 & -0.0197e^{-i\gamma} \\ 4.99 & -0.0197e^{i\gamma} & 2849.99, \end{pmatrix} MeV$$

Traces of a democratic structure

Our factorization of the Cabbibo-Kobayashi-Maskawa mixing matrix is only one of many possible choices, since we can sandwich any number of unitary matrices between U and U',

$$V = UU'^{\dagger} = UO_1O_1^{\dagger}U'^{\dagger} = UO_1O_2O_2^{\dagger}O_1^{\dagger}U'^{\dagger} = \ldots$$

where O_j are unitary matrices.

In the simplest case without any other matrices sandwiched between U and U' in $V = UU'^{\dagger}$, there is however already some interesting structure present.

The initial project aimed at looking for mass matrices with a democratic structure, with the assumption that both the up- and down-sector mass matrices are derived from an initial structure of the type $M_0 = k\mathbf{N}$ and

$M'_0 = k'\mathbf{N}$ where

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

with the mass spectra $(0, 0, 3k)$, $(0, 0, 3k')$, and a mixing matrix equal to unity.

The initial flavour symmetry is subsequently broken, whereby the observed mass spectra with three non-zero masses are obtained, however keeping the hierarchical structure 'inherited' from the democratic origin.

Even though the matrix elements in M and M' are dominated by the hierarchical family structure, which does not look very ``democratic'', rewriting the matrices by extracting the dimensional coefficients ρ and μ unveils what can be perceived as traces of a democratic structure:

$$M \approx \rho \begin{pmatrix} A & Be^{-i\gamma} & -C \\ Be^{i\gamma} & H & -BCe^{i\gamma} \\ -C & -BCe^{-i\gamma} & C^2 \end{pmatrix}$$

and

$$M' \approx \mu \begin{pmatrix} A' & B'Ce^{i\gamma} & C \\ B'Ce^{-i\gamma} & H' & -B'e^{-i\gamma} \\ C & -B'e^{i\gamma} & C^2 \end{pmatrix},$$

with

$$\rho = (Y - X)s_\theta^2, A = (X\cot_\theta^2 + Y)/(Y - X), B = Z/(Y - X)s_\theta,$$
$$H = (Y + 2Z\cot 2\alpha)/(Y - X)s_\theta^2, C = \cot\theta$$
$$\mu = (X' - Y')s_\theta^2, A' = (X' + Y'\cot_\theta^2)/(X' - Y'), B' = Z'/(X' - Y')$$
$$H' = (Y' + 2Z'\cot 2\beta)/(X' - Y')s_\theta^2$$

Numerically (quark mass values taken at M_Z) this corresponds to

$\rho = 0.5269$ MeV,
$A = 3.3533, B = 23.608, H = 1743.71, C = \cot\theta \simeq 570.1$
$\mu = 0.00875$ MeV, $A' = 605.6, B' = 2.2514$, $H' = 5849.14$,

The up-sector mass matrix can be rewritten as as

$$M = \rho\left[\begin{pmatrix} 1 & & \\ & Be^{i\gamma} & \\ & & -C \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & Be^{-i\gamma} & \\ & & -C \end{pmatrix} + \Lambda\right] =$$

where

$$\Lambda = \begin{pmatrix} A-1 & & \\ & H-B^2 & \\ & & 0 \end{pmatrix}$$

Noticing that the matrix

$$\hat{M} = \begin{pmatrix} 1 & & \\ & Be^{i\gamma} & \\ & & -C \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & Be^{-i\gamma} & \\ & & -C \end{pmatrix} = D\mathbf{N}D^*$$

has only one non-zero eigenvalue, and that

$$\mathbf{N} = D^* \, \hat{M} \, D,$$

we can relate $\hat{M}$ to the democratic matrix $M_0 = k\mathbf{N}$, by equating the one non-zero eigenvalue of $\hat{M}$, $1 + B^2 + C^2 = trace(DD^*)$, to the one non-zero eigenvalue $3k$ of the democratic matrix M_0, which gives

$$3k = \rho(1 + B^2 + C^2),$$

If we in this way interpret the mass matrix

$$M = \rho[D\mathbf{N}D^* + \Lambda]$$

as starting out as a democratic matrix $M_0 = k\mathbf{N}$, the first flavour symmetry breaking is identified as

$$M_0 \Rightarrow \hat{M} = D\mathbf{N}D^*$$

where $\hat{M}$ has the same, one non-zero eigenvalue as M_0,
$3k = \rho(1 + B^2 + C^2)$, but the flavour symmetry of the fields $(\varphi_1, \varphi_2, \varphi_3)$ in the weak basis is broken.

By adding Λ, with the two non-zero eigenvalues Λ_1 and Λ_2, we finally get the full mass spectrum of M.

In this way we

- Identify the matrix $\hat{M}$ as having a kind of democratic texture.

- Determine the matrix Λ as the symmetry breaking term which finally gives the mass spectrum with the three observed non-zero masses.

The down-sector can be treated in a similar fashion, though here the traces of democracy are less transparent. Disregarding the signs, we can write

$$M' \sim \mu \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & B'e^{-i\gamma} & \\ & & C \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & B'e^{i\gamma} & \\ & & \end{pmatrix} \right.$$

where

$$\Lambda' = \begin{pmatrix} A' - C^2 & & \\ & H' - B'^2 & \\ & & C^2 - 1 \end{pmatrix}$$

Democratic mass matrices

In order to create explicitely democratic matrices, the unitary matrix

$$D = \frac{1}{6} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix},$$

that diagonalizes

$$\mathbf{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is sandwiched between the rotation matrices U and U', which gives $V = UDD^{\dagger}U'^{\dagger}$, and new rotation matrices $U_{new} = UD$ and $U'_{new} = U'D$.

With the mass values at M_Z, we get the numerical mass matrices

$$M_{new} = \begin{pmatrix} 54240.36 & 54477.93 & 58098.54e^{-i\eta} \\ 54477.93 & 54719.03 & 58355.76e^{-i\eta} \\ 58098.54e^{i\eta} & 58355.76e^{i\eta} & 63215.85 \end{pmatrix} MeV$$

and

$$M'_{new} = \begin{pmatrix} 970.6 & 955.88e^{i\varepsilon'} & 929.63e^{-i\varepsilon} \\ 955.88e^{-i\varepsilon'} & 951.75 & 936.26^{i\varepsilon} \\ 929.63e^{i\varepsilon} & 936.26^{-i\varepsilon} & 984.14 \end{pmatrix} MeV$$

where $\eta = 0.04861$ rad (2.785^o), $\varepsilon' = 0.00383$ rad (0.219^o), and $\varepsilon = 0.00392$ rad (0.225^o).

The up-matrix can be reparametrized as

$$M_{new} = \kappa \begin{pmatrix} 1 & B & Ce^{-i\eta} \\ B & B^2 & BCe^{-i\eta} \\ Ce^{i\eta} & BCe^{i\eta} & D \end{pmatrix} =$$

$$= \kappa \left[\begin{pmatrix} 1 & & \\ & B & \\ & & Ce^{i\eta} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & B & \\ & & Ce^{-i\eta} \end{pmatrix} + \Lambda \right] = \rho \Big[\hat{M}$$

where

$$\Lambda = \begin{pmatrix} 0 & & \\ & 0 & \\ & & A \end{pmatrix}$$

and $\kappa = 54240.36 MeV$, $B = 1.0044$, $C = 1.0711$, $A = 0.0183$.

For the down-sector we get a somewhat different texture.

Conclusion

Instead of "guessing" the form of the mass matrices, we obtain mass matrices by factorizing the mixing matrix in a specific way.

With the most "symmetric" factorization based on the standard parametrization of the Cabbibo-Kobayashi-Maskawa matrix, mass matrices are obtained, which in both charge sectors display certain internal relations, i.e. structures which are not identical but similar.

The mass matrices can moreover be parametrized in a way that indicates a democratic texture, and the breaking of the democracy is of the type

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \Bigg[\begin{pmatrix} 1 & & \\ & B & \\ & & Ce^{i\eta} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & B & \\ & & Ce^{-i\eta} \end{pmatrix} -$$

where Λ is a diagonal matrix.