Fermionization, Number of Families

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Fermionization, Number of Families

The works of major importance for the present talk are:

- Aratyn & Nielsen We made a theorem about the ratio of the number of bosons needed to represent a number of fermions from statistical mechanics in the free case, provided what we did not know that a bosonization existed.
- Kovner & Kurzeba They present an explicite bosonization of two complex fermion fields in 2+1 dimensions being equivalent to QED_3 meaning 2+1 dimensional quantum electrodynamics.
- Mankoc-Borstnik ... The Spin-Charge Unification model explains number of families from the number of fermion components needed in the formalism we use.

Use of Bosonization/Fermionization Justifying Number of Families

The governing philosophy and motivation for the present study is:

- Fermions do NOT exist fundamentally.
- Some boson degrees of freedom are rewritten by bosonization (better fermionization) to fermionic ones, which then make up the fermions in the world we see.
- We work here only with an at first free theory - for our presentation, it might be best if only bosonization worked for FREE theories in higher dimensions - i.e. free bosons can be rewritten as free fermions.
- We though suggest - hope- that exterior to both bosons and/or fermions, we can add a GRAVITATIONAL theory. So fundamentally: gravity with matter bosons. It gets rewritten to fermions in a gravitational field, just Normas theory.

Aratyn-Nielsen Theorem for massless free Bosonization

If there exist two free massless quantum field theories repectively with Boson and Fermion particles and they are equivalent w.r.t. to the number of states of given momenta and energies, **then** the two theories must have the same average energy densities for a given temperature T, or simply same average energies, if we take them with the same infrared cut off:

$$< U_{boson} > \; = \; < U_{fermion} > \text{ where} \tag{1}$$

$$< U_{boson} > \; = \; \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp\left(E(\vec{p})/T\right)} \tag{2}$$

$$< U_{fermion} > \; = \; \sum_{\vec{p}} \frac{E(\vec{p})}{1 + \exp\left(E(\vec{p})/T\right)}, \tag{3}$$

(4)

Aratyn-Nielsen Continued

Of course the single particle energy for a mass-less theory is

$$E(\vec{p}) = |\vec{p}|, \tag{5}$$

when c=1, and in $d_{spatial}$ dimensions and with an infrared cut off spatial volume V the sum gets replaced in the continuum limit by the integral

$$\sum_{\vec{p}} \cdots \rightarrow \int \sum_{components} \cdots \frac{V}{(2\pi)^{d_{spatial}}}, \tag{6}$$

where $\sum_{components} \ldots$ stands for the sum over the different polarization components of the particles in question. So effectively in the simplest case of all the particles having the same "spin"/the same set of components we have the replacement

$$\sum_{components} \ldots \rightarrow N_{families} * N_c \ldots \tag{7}$$

Some formulas for deriving Aratyn-Nielsen

$$< U_{boson} > = \sum_{\vec{p}} \frac{E(\vec{p})}{1 - \exp\left(E(\vec{p})/T\right)} \quad (8)$$

$$= \text{``}N_{families} * N_c'' * V/(2\pi)^{d_{spatial}} * \quad (9)$$

$$\int O(d_{spatial})|\vec{p}|^{d_{spatial}} E(\vec{p}) \sum_{n=0,1,\ldots} \exp\left(nE(\vec{p})/T\right) d|\vec{p}| \quad (10)$$

Simple Aratyn-Nielsen Relation

For a given temperature must the average energies of respectively the boson and the with it equivalent fermion theories

Our Realization Suggestion

- Fermions
  For the fermions we shall use the needed number of say Weyl fermions, i.e. we must adjust the number of families hoping that we get an integer number.
- Bosons
  For the bosons we let the number $2^{d_{spatial}} - 1$ suggest that we take a series of all Kalb-Ramond fields, one combination of fields for each value of the number p of indices on the "potential field" $A_{ab...k}$ (where then there are just p symbols in the chain $ab...k$). At first we take these symbols $a, b, ..., k$ to be only spatial coordinate numbers.

Free Kalb-Ramond

A Kalb-Ramond field with p indices on the “potential” and p+1 indices on the strength

$$F_{\mu\nu\rho\ldots\tau}(x) = \partial_{[\mu}A_{\nu\rho\ldots\tau]}(x), \tag{11}$$

where [...] means antisymmetrizing, and the “potential” $A_{\nu\rho\ldots\tau}$ is antisymmetric in its p indices $\nu\rho\ldots\tau$, is defined to have an action invariant under the gauge transformation:

$$A_{\nu\rho\ldots\tau}(x) \rightarrow A_{\nu\rho\ldots\tau}(x) + \partial_{[\nu}\lambda_{\rho\ldots\tau]}(x) \tag{12}$$

for any arbitrary antisymmetric gauge function $\lambda_{\rho\ldots\tau}(x)$ with $p-1$ indices.

Free Kalb-Ramond Action:

Note that the strength $F_{\mu\nu\rho...\tau} = \partial_{[\mu}A_{\nu\rho...\tau]}$ is gauge invariant, and that thus we could have a gauge invariant Lagrangian density as a square of this field strength

$$\mathcal{L}(x) = F_{\mu\nu...\tau}F_{\mu'\nu'...\tau'}g^{\mu\mu'} * g^{\nu\nu'} * ... * g^{\tau\tau'}. \quad (13)$$

Then the conjugate momentum of the potential becomes(formally):

$$\Pi_{\nu\rho...\tau} = \Pi_{A_{\nu\mu...\tau}} = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_{\nu\rho...\tau})} = F_{0\nu\rho...\tau}. \quad (14)$$

Problem with Components with the time index 0:

But full Kalb-Ramond fields require also components a 0 among the indices.(This is the main new thing to treat this problem of the components with one 0 among the indices.)
Remember about these components with a 0 index:

- Using a usual Minkowskian metric tensor $g^{\mu\nu}$ in constructing an inner product between Kalb-Ramond fields, say

$$g^{\mu\nu}g^{\rho\sigma}\cdots g^{\tau\kappa}A_{\mu\rho\ldots\tau}(\text{potentially an } \partial^0)A_{\nu\sigma\ldots\kappa}, \tag{15}$$

  we get the opposite signature (=sign of the square norm) depending on whether there is a 0 or not!
  This means that if particles produced by the components without the 0 index have normal positive norm square, then those produced by the ones with the 0 have negative normsquare!

Good Luck We Removed the Kalb-Ramond A with $p = 0$ Indices!

We could namely not have replaced one among zero indices by a 0. So we would not have known what to do for the fields with 0 indices.
We correspondingly also have to leave out the Kalb-Ramond-field with $p = d_{spatial} + 1$ indices, because for that there would be no components without an index 0.
For the unexceptional index numbers $p = 1, 2, ..., d_{spatial}$ there some components both with and without the 0.
For the two exceptions $p = 0$ and $d = d_{spatial} + 1$ we have chosen not to have a Kalb-Ramond-field in our scheme, using it to get the -1 in the from Aratyn-Nielsen required $2^{d_{spatial}} - 1$.

Simplest (Naive) Norm Square Assignment

Note that for each Kalb-Ramond-field we can choose an overall extra sign on the inner product, because we simply can define the overall inner product with an extra minus sign, if we so choose. But the simplest choice is to just let the particles corresponding to fields with only spatial indices (i.e. all p indices different from 0) to have positive norm square, while then those with one 0 have negative norm square.

This simple rule would lead to equally many components/particles with positive as with negative norm square, so that dreaming about imposing a constraint that removes equally many negative and positive norm square at a time would leave us with nothing.

Numbers of Components with and without 0.

An of course totally antisymmetric field $A_{\mu\nu\ldots\tau}$ with p indices has

$$\begin{aligned}
\#\ \text{components}_{KR\ p\text{indices}} &= \binom{d}{p} = \binom{d_{spatial}+1}{p}\\
\#\ \text{no 0 components}_{KR\ p\text{indices}} &= \binom{d_{spatial}}{p} = \binom{d-1}{p}\\
\#\ \text{cmps. with 0 \& p-1 non-0}_{KR\ p\text{indices}} &= \binom{d_{spatial}}{p-1} = \binom{d-1}{p-1}.
\end{aligned}$$

and so one must have as is easily checked

$$\binom{d}{p} = \binom{d-1}{p} + \binom{d-1}{p-1}$$

corresponding to

"All components" = "Without 0" + "With 0"

Using ONLY the Components WITHOUT 0 would fit $2^{d_{spatial}}$ Nicely !

Having decided to leave out the number of indices p values $p = 0$ and $p = d$ the number of components **without** the any component indices being 0 just makes up

$$\#\text{ without } 0 \text{ for all } p = 1, 2, ..., d-1 = \sum_{p=1,2,...,d-1} \binom{d-1}{p} = 2^{d-1} - 1$$

so these "only with spatial indices components" could elegantly correspond to $2^{d-1} = 2^{d_{spatial}}$ fermion components.
But problem: Kalb- Ramond fields need also the components with an index being 0.

Using ONLY the Components WITH 0 could also fit $2^{d_{spatial}}$ Nicely !

Having decided to leave out the number of indices p values $p = 0$ and $p = d$ the number of components **with** the 0 just makes up

$$\begin{aligned} \#\text{ with } 0 \text{ for all } p = 1, 2, ..., d-1 &= \sum_{p=1,2,...,d-1} \binom{d-1}{p-1} \\ &= 2^{d-1} - 1 \end{aligned}$$

also, so these "only with 0 index components" could elegantly correspond to $2^{d-1} = 2^{d_{spatial}}$ fermion components, also!
But problem: Kalb- Ramond fields need also the components without an index being 0, and these with 0 usually come with wrong norm square.

The Trick Suggested is to use for Some KR-fields Opposite Hilbert Norm Square

In other words we shall look along the chain of all the allowed p-values $p = 1, 2, ..., d - 1$; and for each of these p-values we can choose whether

- Normal: The states associated with the polarization components **without** the 0 among the indices shall be of **positive norm square**, as usual, and then from Lorentz invariance essentially the ones **with the** 0 **shall have negative norm square**, or
- Opposite The states **with** 0 **shall have positive** norm square, while the components **without** 0 **negative** norma square.

My proposal: Choose so that we get the largest number of positive norm square components.

How to get Maximal Number of Positive over Negative Norm Square Single Boson States

For each

Kovner and Kurzeba made 3+1

$$\psi_\alpha(x) = k\Lambda V_\alpha(x)\Phi(x)U_\alpha(x) \tag{16}$$

Does the Kovner Kurzeba Bosonization Match with the Aratyn-Nielsen Counting Rule?

First look at number of hermitean counted fields: Kovner and Kurzeba gets two complex meaning **4** real fermion fields $Re\psi_1(x)$, $Im\psi_1(x)$, $Re\psi_2(x)$, and $Im\psi_2(x)$ out of the for the construction relevant boson-fields $A_1(x)$, $A_2(x)$, $\partial_i E_i = \partial_1 E_1 + \partial_2 E_2$. This looks agreeing with the Aratyn Nielsen prediction that the ratio shall be

$$\frac{\#bosons}{\#fermions} = \frac{2^{d_s}-1}{2^{d_s}} = \frac{2^2-1}{2^2} \text{for the spatial dimension being} d_s = d-1 \quad (17)$$

Four real fermion fields bosonize to three real boson-fields! o.k.

What about the conjugate momenta to the fields?

While the fermion fields are normally each others conjugate variables(fields) in as far as they anticommute with each other having only no-zero anticommutators with themselves, the boson-fields typically are taken each to have associated an extra field - its conjugate - with which it does not commute, while of course any variable must commute with itself. But a field, that depends on an x-point or on a momentum, need NOT to commute with itself, though.

But then the question: Shall we for bosons somehow also count the conjugate momentum fields, when we shall compare the number of fermion and boson fields equivalent through bosonization ? For the fermions the conjugate fields are unavoidable already included into the set of fields describing the fermions, because the it is the field in question itself, but for bosons we could easily get the number of fields *doubled*, if we include for each field also its conjugate.

Conjugate Momentum Fields NOT to be Included in Counting.

Let us argue that it is enough in the counting to count the number of fields, from which you by Fourier resolution can extract the annihilation and creation operators needed to annihilate or create the particles, the species of which are to be counted:

- Normally we could extract the conjugate field by differentiating w.r.t. to time the field because usually you can replace the fields and their conjugate by the fields and their time derivatives.
- Using equations of motion these time derivatives can in turn be obtained by some way - also some sort of differentiation - from the field itself.
- Thus at the end the information on the conjugate is extractable from the field itself!

Further Support for NOT including also Conjugate Moementum Fields

We could very easily construct linear (or more complicated) combinations of boson fields and their conjugate fields. Such combinations would like the fermion fields typically not commute/anticommute with themselves.
So provided we can extra the particle creation and annihilation operators from the combined field we would have no rule to tell that we should include more. Thus we would need only the combined field, and with that rule have quite analogy to the fermion case.

Meaning of NOT Counting also the Conjugate Field

In QED_3 say $A_1(x)$ and $A_2(x)$ would be enough to represent both longitudinal and transversely polarized photons. It would NOT be needed also to have the essentially conjugate electric fields $E_1(x)$ and $E_2(x)$.
The field $\partial_i E_i$ is in fact the conjugate A_0 so that we - having the symmetry between a field and its conjugate, it being conjugate of its conjugate - can consider that timelike photons are described by this $\partial_i E_i$ field combination.

But in terms of Particles, How??

Usually one thinks of electrodynamics in 2+1 dimesnions as having only one particle polarisation, since there is only *one* transversely polarisation for a photon. So seemingly only one component of boson. This transversely polarized photon is even its own antiparticle, so even the anti-particle is not new.
On the contrary the fermions after the fermionization counts two complex fields meaning two different fermion components (ψ_1 and ψ_2) each with an a priori different antiparticle in as far as the fields ψ_1 and ψ_2 both are complex(non-Hermitean). That seems NOT to match!
Where have the two missing photon-polarizations gone?

Suggestion for How 3 photons.

To count independently both A_i (i=1,2.) as real fields, we need to consider it that we have not only the transverse photon, but also a **longitudinal photon** !
The third of the real fields $\partial_i E_i = div\vec{E}$ is actually the conjugate variable to the time component $A_0(x)$ of the fourcomponent photon field. So if we take it that conjugate or not does not matter it could correspond to the **timelike polarized photon**.
This would mean that we could hope for interpreting the three photon polarizations as being

- 1) The transverse photon.
- 2) The longitudinal photon.
- 3) The time-like photon.

But the time like photon has wrong signature ?!

Better Suggestion for the 3 particles ?

To avoid the problem with the ltime-like photon form Lorentz invariance having the signature with negative norm square states we can instead take a further scalar. If so we could have 3 bosons corresponding to the four (real) fermions.
In any case if we want a fermion system with positive definite Hilbert space we better have the bosons also give positive definite Hilbert space if they shall match in their Hilbert spaces.

How to count Hermitean Boson fields ?

To excercise we shall for the moment even begin with a 1+1 dimensional only right moving Hermitean field constructed as a superposition of momentum state creation $a^\dagger(p)$ and annihilation operators $a(p)$ for say a series discretized momentum values, which we for "elegance"(and later interest) shall take to be odd integers in some unit:

$$\phi(x) = \sum_{p \text{ odd}, p>0} \sqrt{p} a(p) \exp(ipx) + \sum_{p \text{ odd}, p<0} \sqrt{|p|} a^\dagger(|p|) \exp(ipx) \quad (18)$$

$$= \sum_{p \text{ odd}} \sqrt{|p|} a(p), \quad (19)$$

where we have put

$$a(p) = a^\dagger(-p) \text{ for all the odd p} \quad (20)$$

Properties of the Hermitean field

A Hermitean field of the form (in 1+1 dimension say)

$$\phi(x) = \sum_{p \text{ odd}, p>0} \sqrt{p} a(p) \exp(ipx) + \sum_{p \text{ odd}, p<0} \sqrt{|p|} a^\dagger(|p|) \exp((ipx) \quad (21)$$

$$= \sum_{p \text{ odd}} \sqrt{|p|} a(p) \quad (22)$$

obeys

$$\phi(x)^\dagger = \phi(x) \text{ (Hermiticity) } and \quad (23)$$

$$[\phi(x), \phi(y)] = \sum_{p \text{ odd}} \sum_{p' \text{ odd}} \sqrt{|p|}\sqrt{|p'|}[a(p), a(p')] \exp(ipx + ip'y) \quad (24)$$

$$= \sum_{p \text{ odd}} p \exp(ip(x-y)) = 2\pi \frac{d}{id(x-y)} \delta(x-y) \quad (25)$$

New, Reduce the Kovner Kurzeba model.

We claim, that in a way the Kovner and Kurzeba bosonization in 2 + 1 dimensions has included a kind of "funny extra bosonic degree of freedom" the charge density compared to our own plan of doing a completely free model.
Really we want to say: In a truly free electrodynamics "free QED_3" (in 2 +1 dimensions) the divergence of the electric field is zero:

$$\partial_i E_i \approx 0 \text{ (on physical states).} \tag{27}$$

When we use $\approx$ instead of $=$ it is because we may need the divergence $\partial_i E_i$ as an operator even though we may take it to be zero on the "physical states".

Reduction of Kovner Kurzeba model w.r.t. degrees of freedom

Inserting formally our claim of a constraint equation

$$\partial_i E_i \approx 0 \text{ (on physical states).} \tag{28}$$

into the expressions of Kovner and Kurzeba

$$V_1(x) = -i \exp\left(\frac{i}{2e}\int (\theta(x-y)-\pi)\partial_i E_i\right) \tag{29}$$

$$U_1(x) = \exp\left(-\frac{i}{2e}\theta(y-x)\partial_i E_i\right) \tag{30}$$

we get

$$V_1(x) \approx -i \tag{31}$$

$$U_1(x) \approx 1. \tag{32}$$

Using the constraint equation formally on Kovner and Kurzeba

In Kovner and Kurzeba one finds

$$\psi_\alpha(x) = k\Lambda V_\alpha(x)\Phi(x)U_\alpha(x) \quad (33)$$

$$\Phi(x) = \exp\left(ie\int e_i(y-x)A_i(y)d^2y\right); e_i(y-x) = \frac{y_i - x_i}{(y-x)^2} \quad (34)$$

$$V_1(x) = -i\exp\left(\frac{i}{2e}\int(\theta(x-y)-\pi)\partial_i E_i\right); V_2(x) = -iV_1^\dagger(x) \quad (35)$$

$$U_1(x) = \exp\left(-\frac{i}{2e}\theta(y-x)\partial_i E_i\right); U_2(x) = V_1^\dagger(x) \quad (36)$$

and thus with the constraint formally included

$$\psi_2(x) \approx i\psi_1(x) \quad (37)$$

Our Constraint would Spoil Rotation Symmetry

A constraint equation

$$\psi_2(x) \approx i\psi_1(x) \tag{38}$$

would *not* be consistent with the rotation symmetry and the transformation property for the fermion field suggested in Kovner and Kurzeba

$$\psi_1 \to \exp(i\phi/2)\psi_1; \psi_2 \to \exp(-i\phi/2)\psi_2. \tag{39}$$

So including the constraint would make the bosonization/fermionization become *non-rotational invariant*. But it is our philosophy not to take that as a so serious problem, because it is in any case *impossible* to get in a rotational invariant way spin 1/2 fermions from a purely bosonic theory with only integer spin!

Rotation symmetry broken in reduced model!