A trial to understand nature with the spin-charge-family theory
Why nature has not chosen Grassmann rather than Clifford space to describe the fermion degrees of freedom?

N.S. Mankoč Borštnik, H.B.Nielsen

21st workshop "What comes beyond the standard models?"

Bled 23 June - 1 July 2018

Some publications:

- Singapore talk 2018.
  Bled talk 2017, [arXiv:1802.05554v1v2] (arXiv:1806.01629 whole proceedings) with H.B.F. Nielsen.
  Bled discussion 2017, [arxiv:1805.06318] (arXiv:1806.01629 whole proceedings),With D. Lukman

More than century ago **quantum mechanics postulated** the **existence of fermions and bosons**, postulating for creation and annihilation operators the anticommutation relations

$$
\begin{aligned}
\{\hat{b}_i^{\alpha}(\vec{x}), \hat{b}_j^{\beta\dagger}(\vec{x'})\}_+ |\psi_{oc}> &= \delta_{ij}\delta(\vec{x}-\vec{x'})|\psi_{oc}>, \\
\{\hat{b}_i^{\alpha}(\vec{x}), \hat{b}_j^{\beta}(\vec{x'})\}_+ |\psi_{oc}> &= 0, \\
\{\hat{b}_i^{\alpha\dagger}(\vec{x}), \hat{b}_j^{\beta\dagger}(\vec{x'})\}_+ |\psi_{oc}> &= 0, \\
\hat{b}_i^{\alpha\dagger}(\vec{x})\,|\psi_{oc}> &= |\psi_i^{\alpha}>, \\
\hat{b}_i^{\alpha}(\vec{x})\,|\psi_{oc}> &= 0,
\end{aligned}
$$

$|\psi_{oc}>$ is the vacuum state.

The **standard model**, which offered an **elegant new step** in **understanding the origin of fermions and bosons**, **just kept this assumption**.

There are many phenomena, to which the *spin-charge-family* theory is offering the explanation:

- **The appearance of quarks and leptons and their charges,**
- **The appearance of the vector gauge fields and their charges,**
- **The appearance of the scalar gauge fields and their charges,**
- **The appearance of the dark matter**,
- **The appearance of matter-antimatter asymmetry**,

There are also the not yet understood phenomena to which the *spin-charge-family* theory might offer the right explanation:

- **The appearance of the dark energy**,
- **The number of (non) observed dimension of space time**,
- **many other phenomena, like why nature has made a choice of the Clifford rather than the Grassmann algebra, although both offer the anticommutation relations for spinors**.

Let me discuss this topic — the description of fermions in the second quantized theory in Grassmann in comparison with Clifford space.

It might help to understand better the appearance of fermions, beside the project fermionization of the Kalb-Ramond fields.

A simple action for **a spinor which carries in** $d = (13+1)$ **only two kinds of spins (no charges)** and for **gauge fields**:

$$
\begin{aligned}
\mathbf{S} &= \int d^d x \, E \, \mathcal{L}_f + \\
& \int d^d x \, E \, (\alpha \, R + \tilde{\alpha} \, \tilde{R})
\end{aligned}
$$

-

$$
\begin{aligned}
\mathcal{L}_f &= \frac{1}{2}(\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \{p_\alpha, E f^{\alpha}{}_a\}_- \\
\mathbf{p_{0\alpha}} &= \mathbf{p}_\alpha - \frac{\mathbf{1}}{\mathbf{2}} \mathbf{S^{ab}} \omega_{\mathbf{ab}\alpha} - \frac{\mathbf{1}}{\mathbf{2}} \tilde{\mathbf{S}}^{\mathbf{ab}} \tilde{\omega}_{\mathbf{ab}\alpha}
\end{aligned}
$$

- The Einstein action for a free gravitational field is assumed to be linear in the curvature

$$
\begin{aligned}
\mathcal{L}_{\mathbf{g}} &= \mathbf{E}\,(\alpha\,\mathbf{R} + \tilde{\alpha}\tilde{\mathbf{R}}), \\
\mathbf{R} &= \mathbf{f}^{\alpha[\mathbf{a}}\mathbf{f}^{\beta\mathbf{b}]}\,(\omega_{\mathbf{ab}\alpha,\beta} - \omega_{\mathbf{ca}\alpha}\omega^{\mathbf{c}}{}_{\mathbf{b}\beta}), \\
\tilde{\mathbf{R}} &= \mathbf{f}^{\alpha[\mathbf{a}}\mathbf{f}^{\beta\mathbf{b}]}\,(\tilde{\omega}_{\mathbf{ab}\alpha,\beta} - \tilde{\omega}_{\mathbf{ca}\alpha}\tilde{\omega}^{\mathbf{c}}{}_{\mathbf{b}\beta}),
\end{aligned}
$$

with $E = \det(e^{a}{}_{\alpha})$
and $f^{\alpha[a}f^{\beta b]} = f^{\alpha a}f^{\beta b} - f^{\alpha b}f^{\beta a}$.

- **Let us pay attention on the fermion — spinor part of the action.**

- **In the** *spin-charge-family* **theory only the Clifford algebra appears, but with the two kinds of the Clifford algebra operators:** γ^a **and** $\tilde{\gamma}^a$**,**

$$
\begin{aligned}
\{\gamma^{\mathbf{a}}, \gamma^{\mathbf{b}}\}_+ = \mathbf{2}\eta^{\mathbf{ab}} &= \{\tilde{\gamma}^{\mathbf{a}}, \tilde{\gamma}^{\mathbf{b}}\}_+, \\
\{\gamma^{\mathbf{a}}, \tilde{\gamma}^{\mathbf{b}}\}_+ &= 0, \\
(\tilde{\gamma}^{\mathbf{a}}\mathbf{B} : &= \mathbf{i}(-)^{\mathbf{n_B}}\mathbf{B}\gamma^{\mathbf{a}}\,)\,|\psi_0\,>, \\
(\mathbf{B} &= a_0 + a_a\gamma^a + a_{ab}\gamma^a\gamma^b + \cdots + a_{a_1\cdots a_d}\gamma^{a_1}\ldots\gamma^{a_d})|\psi_0
\end{aligned}
$$

$(-)^{n_B} = (+1, -1)$, when the object B has a Clifford even or odd character, respectively.
$|\psi_0>$ is a vacuum state on which the operators γ^a **apply**.

One could use $\tilde{\gamma}^a$ instead of γ^a to generate vectors

The two kinds of the Clifford algebra form two kind of the Lorentz generators:

$$\mathbf{S^{ab}} := (\mathbf{i}/\mathbf{4})(\gamma^{\mathbf{a}}\gamma^{\mathbf{b}} - \gamma^{\mathbf{b}}\gamma^{\mathbf{a}}),$$

$$\tilde{\mathbf{S}}^{\mathbf{ab}} := (\mathbf{i}/\mathbf{4})(\tilde{\gamma}^{\mathbf{a}}\tilde{\gamma}^{\mathbf{b}} - \tilde{\gamma}^{\mathbf{b}}\tilde{\gamma}^{\mathbf{a}}),$$

$$\{\mathbf{S^{ab}}, \tilde{\mathbf{S}}^{\mathbf{cd}}\}_- = \mathbf{0}.$$

- γ^a **are used to describe the spin and the charges of spinors,**
  $\tilde{\gamma}^a$ **must be used to describe families of spinors.**
- **States described by the Clifford algebra carry spin and correspondingly charges and family charges in fundamental representations.**

Phys. Lett. **B 292**, 25-29 (**1992**),
J. Math. Phys. **34**, 3731-3745 (**1993**),
Mod. Phys. Lett. **A 10**, 587-595 (**1995**)

Let us now have a look into Grassmann space.

- **A vector** $\{\theta^a\} = (\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \ldots, \theta^d)$ represents a point in d-dimensional Grassmann space of anticommuting coordinates θ^a, $\{\theta^a, \theta^b\}_+ = \theta^a\theta^b + \theta^b\theta^a = 0$, $(a = 0, 1, 2, 3, 5, \ldots, d)$,
- **A linear vector space has** 2^d **states,** $(\theta^{a_i})^2 = 0$**, for any** $a_i \in (0, 1, 2, 3, 5, \ldots, d)$**.**

$$\mathbf{B} \;=\; \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \theta^{a_1}\theta^{a_2}\ldots\theta^{a_k}|\phi_{og}>, \quad a_i \leq a_{i+1}\,,$$

$|\phi_{og}>$ is the vacuum state, here assumed to be $|\phi_{og}>=|1>$, $\frac{\partial}{\partial\theta^a}|\phi_{og}>=0$ for any θ^a.

- The **left derivative** $\frac{\partial}{\partial\theta_a}$ on $\mathbf{B}(\theta)$ is

$$\frac{\partial}{\partial\theta_a}\mathbf{B}(\theta) \;=\; \frac{\partial\mathbf{B}(\theta)}{\partial\theta_a}, \quad p^{\theta a} = i\frac{\partial}{\partial\theta_a}, \quad \{p^{\theta a}, \theta^b\}_+ = i\eta^{ab}$$

- **Let:**
  $\theta_a^\dagger = \frac{\partial}{\partial\theta_a}\eta^{aa} = -ip^{\theta a}\eta^{aa}\,,$
  $\frac{\partial}{\partial\theta_a}^\dagger = -i\theta^a\eta^{aa}\,,$
- $\mathbf{S}^{ab} = (\theta^a p^{\theta b} - \theta^b p^{\theta a})\,.$

- **One finds two superposition of θ^a and $p^{\theta a}$**

$$\begin{aligned}
\gamma^a &= (\theta^a - i\,p^{\theta a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a + i\,p^{\theta a})\,, \\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \\
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,
\end{aligned}$$

- $$\begin{aligned} \{d\theta^a, \theta^b\}_+ &= 0, \quad \int d\theta^a = 0\,, \quad \int d\theta^a \theta^a = 1\,, \\ \int d^d\theta\ \theta^0\theta^1 \ldots \theta^d &= 1\,, \\ d^d\theta &= d\theta^d \ldots d\theta^0\,, \quad \omega = \Pi_{k=0}^d (\frac{\partial}{\partial\theta_k} + \theta^k)\,. \end{aligned}$$

- **Norm in Grassmann space**

$$\begin{aligned} < \mathbf{B}|\mathbf{C} > &= \int d^d x d^d\theta^a\ \omega < \mathbf{B}|\theta > < \theta|\mathbf{C} > \\ &= \sum_{k=0}^{d} \int d^d x\ b^*_{b_1 \ldots b_k} c_{b_1 \ldots b_k}\,. \end{aligned}$$

$|\phi_{og} >= |1 >$.

- **Norm in Clifford space follows from the norm in Grassmann space**
- $$\begin{aligned} < \mathbf{F}|\mathbf{G} > &= \int d^d x d^d \theta^a \; \omega < \mathbf{F}|\gamma > < \gamma|\mathbf{G} > \\ &= \sum_{k=0}^{d} \int d^d x \; b^*_{b_1 \ldots b_k} c_{b_1 \ldots b_k} \; , \end{aligned}$$

  since γ^a can be expressed by θ^a.
- **We could as well forget on "Grassmann norm" and choose different $|\psi_{oc} >$ to define norm.**

In Clifford space the action for a free massless object, Lorentz invariant, is well known

o

$$\mathcal{A} \;=\; \int d^d x \, \frac{1}{2} \left(\psi^\dagger \gamma^0 \, \gamma^a p_a \psi \right) + h.c. \, ,$$

$p_a = i \, \frac{\partial}{\partial x_a}$, leading to the equations of motion

o

$$\begin{aligned} \gamma^a p_a |\psi^\alpha > \;&=\; 0 \, , \\ \gamma^a p_a \gamma^b p_b |\psi_i^\alpha > \;&=\; p^a p_a |\psi_i^\alpha >= 0 \, , \end{aligned}$$

fulfilling also the Klein-Gordon equation.

o γ^0 appears paying attention that

$S^{ab\dagger} \, \gamma^0 = \gamma^0 \, S^{ab}$, $S^\dagger \gamma^0 = \gamma^0 S^{-1}$, $S = e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})}$.

In Grassmann space we again require that the action for a free massless object is Lorentz invariant.

o

$$\mathcal{A} \;=\; \frac{1}{2}\{\int d^dx\, d^d\theta\, \omega\, (\phi^\dagger(1-2\theta^0\frac{\partial}{\partial\theta^0})\,\theta^a p_a\phi)\} + h.c.\,,$$

o We use the integral over θ^a coordinates, with the weight function ω.

o The operator $(1-2\theta^0\frac{\partial}{\partial\theta^0})$ takes care of the Lorentz invariance.

$\mathbf{S}^{ab\dagger}\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) = (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{ab}$,

$\mathbf{S}^\dagger\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) = (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{-1}$,

$\mathbf{S} = e^{-\frac{i}{2}\omega_{ab}(L^{ab}+\mathbf{S}^{ab})}$.

o The action leads to the equations of motion $\theta^a p_a|\phi_i^\theta >= 0$, $\frac{\partial}{\partial\theta^a}p_a|\phi_i^\theta >= 0$,

o $\{\theta^a p_a, \frac{\partial}{\partial\theta^b}\,p_b\}_+|\phi_i^\theta >= p^a p_a|\phi_i^\theta >= 0$.

- We can start with the creation operators as products of $\frac{d}{2}$ creation operators $\hat{b}^{\theta\dagger}_{a_i b_i} = \frac{1}{\sqrt{2}}(\theta^{a_i} \pm \epsilon \theta^{b_i})$.
- Then the corresponding annihilation operators have $\frac{d}{2}$ factors of $\hat{b}^{\theta}_{a_i b_i} = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \theta^{a_i}} \pm \epsilon^* \frac{\partial}{\partial \theta_{b_i}})$, $\epsilon = i$, if $\eta^{a_i a_i} = \eta^{b_i b_i}$ and $\epsilon = -1$, if $\eta^{a_i a_i} \neq \eta^{b_i b_i}$.
- In $d = 2(2n+1)$, n is a positive integer, we can start with the state

$$|\phi^{\theta 1}_1 >= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-1} + i\theta^d)|1> \ .$$

The state is the eigenstate of the Cartan subalgebra, with the integer eigenvalues.

- The rest of states, belonging to the same Lorentz representation, follows from the starting state by the application of the operators $\mathbf{S}^{cf}$.

In $d = 2(2n+1)$ we have

$$
\begin{aligned}
\hat{b}_j^{\theta 1\dagger} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}(\theta^0\theta^3 + i\theta^1\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,, \\
\hat{b}_j^{\theta 1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0} - i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1})\,. \\
\cdots
\end{aligned}
$$

It is not difficult to see that these creation and annihilation operators fulfill the anticommutation relations for fermions.

$$
\begin{aligned}
\{\hat{b}_i^{\theta}, \hat{b}_j^{\theta\dagger}\}_+ |\phi_{og}> &= \delta_{ij}\, |\phi_{og}>, \\
\{\hat{b}_i^{\theta}, \hat{b}_j^{\theta}\}_+ |\phi_{og}> &= 0\, |\phi_{og}>, \\
\{\hat{b}_i^{\theta\dagger}, \hat{b}_j^{\theta\dagger}\}_+ |\phi_{og}> &= 0\, |\phi_{og}>, \\
\hat{b}_j^{\theta\dagger} |\phi_{og}> &= |\phi_j> \\
\hat{b}_j^{\theta} |\phi_{og}> &= 0\, |\phi_{og}> .
\end{aligned}
$$

It is not difficult to see either that the number of all creation operators of an odd Grassmann character in $d = 2(2n+1)$-dimensional space is equal to $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$.

Formulation of creation and annihilation operators in Clifford space goes equivalently as in Grassmann space.
Making a choice of the Cartan subalgebra eigenstates of S^{ab}

$$\begin{aligned} \overset{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\ \overset{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \end{aligned}$$

where $k^2 = \eta^{aa}\eta^{bb}$, recognizing that the Hermitian conjugate values of $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$ are

$$\overset{ab}{(k)}^{\dagger} = \eta^{aa}\,\overset{ab}{(-k)}, \quad \overset{ab}{[k]}^{\dagger} = \overset{ab}{[k]}\ .$$

The corresponding eigenvalues of S^{ab} and $\tilde{S}^{ab}$, are

$$\begin{aligned} S^{ab}\,\overset{ab}{(k)} &= \frac{1}{2}k\,\overset{ab}{(k)}\,, \quad S^{ab}\,\overset{ab}{[k]} = \frac{1}{2}k\,\overset{ab}{[k]} \\ \tilde{S}^{ab}\,\overset{ab}{(k)} &= \frac{k}{2}\,\overset{ab}{(k)}\,, \quad \tilde{S}^{ab}\,\overset{ab}{[k]} = -\frac{k}{2}\,\overset{ab}{[k]}\ . \end{aligned}$$

We find in $d = 2(2n+1)$ that from the starting state with products of odd number of only nilpotents

$$|\psi_1^1> |_{2(2n+1)} \;=\; \overset{03}{(+i)}\overset{12}{(+)}\overset{35}{(+)}\cdots\overset{d-3\;d-2}{(+)}\overset{d-1\;d}{(+)}\;|\psi_{oc}>,$$

having correspondingly an odd Clifford character,
all the other states of the same Lorentz representation,
there are $2^{\frac{d}{2}-1}$ members,
follow by the application of S^{cd} on the starting state.

The operators $\tilde{S}^{cd}$ generate states with different eigenstates of the Cartan subalgebra ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$, $\cdots$, $\tilde{S}^{d-1\,d}$), we call the eigenvalues of their eigenstates the "family" quantum numbers.
There are $2^{\frac{d}{2}-1}$ families, each with $2^{\frac{d}{2}-1}$ familiy members.

The vacuum state

$$|\psi_{oc}> \;=\; \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\,d}{[-]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\,d}{[-]} + \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots$$
$$\text{for } d = 2(2n+1),$$

There is

$$2^{\frac{d}{2}-1}\,2^{\frac{d}{2}-1}$$

number of creation operators, defining the orthonormalized states when applying on the vacuum state and the same number of annihilation operators $\tilde{S}^{ab}$ connect members of different families, S^{ab} generates all the members of one family.

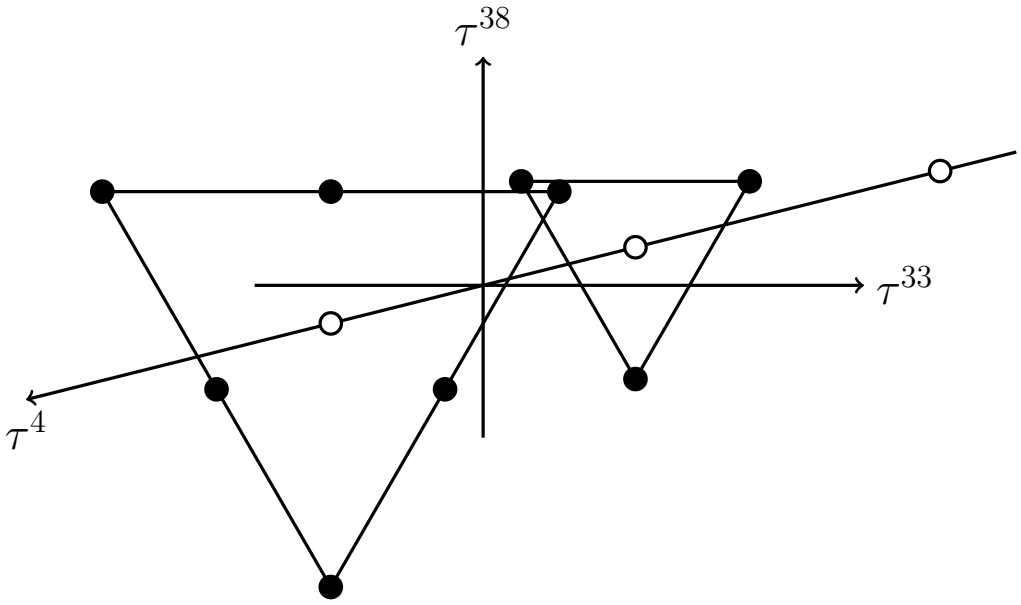
I		decuplet	$S^{9\,10}$	$S^{11\,12}$	$S^{13\,14}$	τ^{4}	τ^{33}	τ^{38}
	1	$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	1	1	1	-1	0	0
	2	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	1	0	0	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	3	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	1	-1	-1	$+\frac{1}{3}$	$+1$	$+\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	0	0	1	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	5	$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	-1	-1	-1	$+\frac{1}{3}$	0	$-\frac{2}{\sqrt{3}}$
	6	$(\theta^{11}+i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	0	1	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	7	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	-1	1	-1	$+\frac{1}{3}$	-1	$+\frac{1}{\sqrt{3}}$
	8	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	0	0	-1	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	9	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}-i\theta^{12})$	0	-1	0	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	10	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	-1	0	0	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$

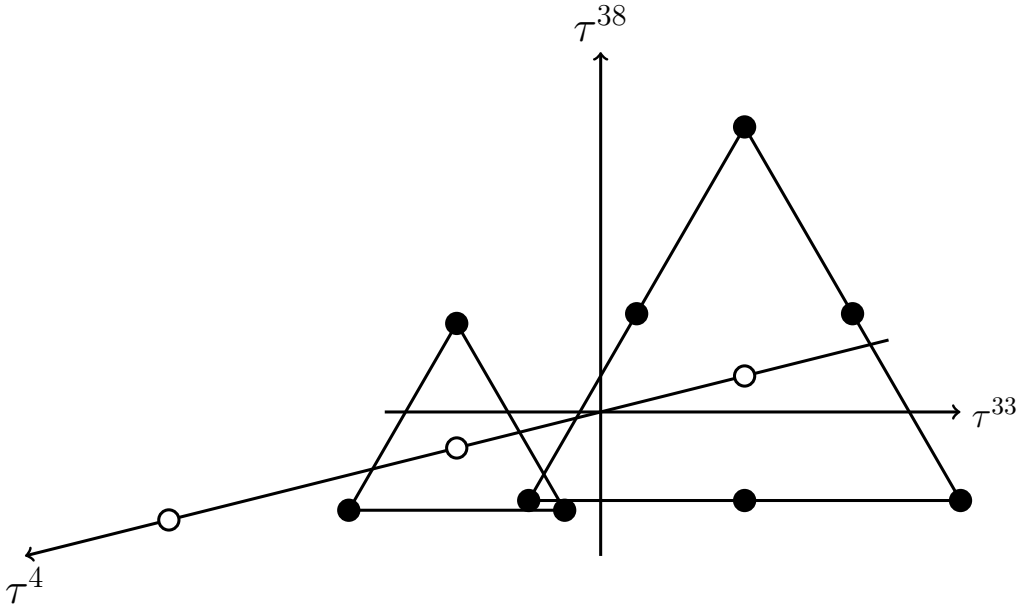
II		decuplet	$S^{9\,10}$	$S^{11\,12}$	$S^{13\,14}$	τ^{4}	τ^{33}	τ^{38}
	1	$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	-1	-1	-1	$+1$	0	0
	2	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	-1	0	0	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	3	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	-1	1	1	$-\frac{1}{3}$	-1	$-\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	0	0	-1	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	5	$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	1	1	-1	$-\frac{1}{3}$	0	$+\frac{2}{\sqrt{3}}$
	6	$(\theta^{11}-i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	0	-1	0	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	7	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	1	-1	1	$-\frac{1}{3}$	$+1$	$-\frac{1}{\sqrt{3}}$
	8	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	0	0	1	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	9	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}+i\theta^{12})$	0	1	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	10	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	1	0	0	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$

Table: The creation operators of the decuplet and the antidecouplet of the orthogonal group $SO(6)$ in Grassmann space are presented.

I			τ^4	τ^{33}	τ^{38}
singlet		$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	-1	0	0
triplet	1	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	2	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	3	$(\theta^{11}+i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
sextet	1	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	$\frac{1}{3}$	$+1$	$+\frac{1}{\sqrt{3}}$
	2	$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	$\frac{1}{3}$	0	$-\frac{2}{\sqrt{3}}$
	3	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	$\frac{1}{3}$	-1	$+\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	$\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	5	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}-i\theta^{12})$	$\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	6	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
II			τ^4	τ^{33}	τ^{38}
antisinglet		$(\theta^9-i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}-i\theta^{14})$	$+1$	0	0
antitriplet	1	$(\theta^9-i\theta^{10})(\theta^{11}\theta^{12}+\theta^{13}\theta^{14})$	$+\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
	2	$(\theta^9\theta^{10}+\theta^{11}\theta^{12})(\theta^{13}-i\theta^{14})$	$+\frac{1}{3}$	0	$+\frac{1}{\sqrt{3}}$
	3	$(\theta^{11}-i\theta^{12})(\theta^9\theta^{10}+\theta^{13}\theta^{14})$	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$
antisextet	1	$(\theta^9-i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}+i\theta^{14})$	$-\frac{1}{3}$	-1	$-\frac{1}{\sqrt{3}}$
	2	$(\theta^9+i\theta^{10})(\theta^{11}+i\theta^{12})(\theta^{13}-i\theta^{14})$	$-\frac{1}{3}$	0	$+\frac{2}{\sqrt{3}}$
	3	$(\theta^9+i\theta^{10})(\theta^{11}-i\theta^{12})(\theta^{13}+i\theta^{14})$	$-\frac{1}{3}$	$+1$	$-\frac{1}{\sqrt{3}}$
	4	$(\theta^9\theta^{10}-\theta^{11}\theta^{12})(\theta^{13}+i\theta^{14})$	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$
	5	$(\theta^9\theta^{10}-\theta^{13}\theta^{14})(\theta^{11}+i\theta^{12})$	$-\frac{1}{3}$	$-\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$
	6	$(\theta^9+i\theta^{10})(\theta^{11}\theta^{12}-\theta^{13}\theta^{14})$	$-\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{1}{2\sqrt{3}}$

Table: The creation operators in Grassmann space of the decuplet of Table 1 are arranged with respect to the $SU(3)$ and $U(1)$ subgroups of


τ^{38}
τ^{33}
τ^{4}


τ^{38}
τ^{33}
τ^{4}

- We have learned in the present study that one can use either Grassmann or Clifford space to express the internal degrees of freedom of fermions in any even dimensional space.
- In both spaces the creation operators and their Hermitian conjugated annihilation operators fulfill the anticommutation relation requirements, needed for fermions, provided that they are expressed as odd products of either Grassmann (θ^a, $(\theta^a)^\dagger = \frac{\partial}{\partial\theta_a}\eta^{aa}$.
- Or Clifford objects, either
  $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$.
  $\gamma^{a\dagger} = \gamma^a\eta^{aa}$,
  or
  $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$,
  and correspondingly
  $\tilde{\gamma}^{a\dagger} = \tilde{\gamma}^a\eta^{aa}$.

- But while in the Clifford case states appear in the fundamental representations of the Lorentz group, carrying half integer spins, the states in the Grassmann case are in adjoint representations of the Lorentz group with integer spin.
- The Clifford case, offering two kinds of the Clifford objects, γ^a and $\tilde{\gamma}^a$, enables to describe besides the spin degrees of freedom of fermion fields also their family degrees of freedom.
- The Grassmann case offers only one kind of objects.
- Assuming that "nature has both choices" for describing the internal degrees of freedom of fermion fields, the question arises why Grassmann choice is not chosen, or better, why the Clifford choice is chosen.

- In the case that spin degrees in $d \geq 5$ manifest as charges in $d = (3+1)$, fermions in the Grassmann case manifest charges in the adjoint representations — singlets, sextets, also color triplets.
- In the Clifford case — this is used in the *spin-charge-family* theory, which takes the Lorentz group $SO(13,1)$ — the spin and charges appear in the fundamental representations of the corresponding groups, offering also the family degrees of freedom.