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**Proceedings to the 22nd Workshop**

# What Comes Beyond the Standard Models

**Bled, July 6–14, 2019**

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# Contents

# Preface

The series of annual workshops on "What Comes Beyond the Standard Models?" started in 1998 with the idea of Norma and Holger for organizing a real workshop, in which participants would spend most of the time in discussions, confronting different approaches and ideas. Workshops take place in the picturesque town of Bled by the lake of the same name, surrounded by beautiful mountains and offering pleasant walks and mountaineering.

In our very open minded, friendly, cooperative, long, tough and demanding discussions several physicists and even some mathematicians have contributed. Most of topics presented and discussed in our Bled workshops concern the proposals how to explain physics beyond the so far accepted and experimentally confirmed both standard models — in elementary particle physics and cosmology — in order to understand the origin of assumptions of both standard models and be consequently able to make predictions for future experiments. Although most of participants are theoretical physicists, many of them with their own suggestions how to make the next step beyond the accepted models and theories, experts from experimental laboratories were and are very appreciated, helping a lot to understand what do measurements really tell and which kinds of predictions can best be tested.

The (long) presentations (with breaks and continuations over several days), followed by very detailed discussions, have been extremely useful, at least for the organizers. We hope and believe, however, that this is the case also for most of participants, including students. Many a time, namely, talks turned into very pedagogical presentations in order to clarify the assumptions and the detailed steps, analyzing the ideas, statements, proofs of statements and possible predictions, confronting participants' proposals with the proposals in the literature or with proposals of the other participants, so that all possible weak points of the proposals, those from the literature as well as our own, showed up very clearly. The ideas therefore seem to develop in these years considerably faster than they would without our workshops.

This year neither the cosmological nor the particle physics experiments offered much new, as also has not happened in the last two years, which would offer new insight into the elementary particles and fields and also into cosmological events, although a lot of work and effort have been put in, and although there are some indications for the existence of the fourth family to the observed three, due to the fact that the existence of the fourth family might explain the existing experimental data better, what is mentioned in this proceedings.

It looks like, that "nature" does not "like" to help us to better understand the assumptions, put into the standard models, as it is written in one of contributions to this proceedings.

There were talks accompanied by very lively discussions about the way which could lead to next step beyond both standard models, some of them appear in this proceedings, the others might contribute to the next year proceedings.
Understanding the universe through the cosmological theories and theories of the elementary fermion and boson fields, have, namely, so far never been so dependent on common knowledge and experiments in both fields. On both fields there appear proposals which should explain assumptions of these models. The competition, who will have right, is open and exciting.
We are keeping expecting that new cosmological experiments will help to resolve the origin of the dark matter. Since the results of the DAMA/LIBRA experiments, presented in this year proceedings, can hardly be explained in some other way than with the signal of the dark matter, it is expected that sooner or latter other laboratories will confirm the DAMA/LIBRA results. This has not yet happened and our discussions clarified the reasons for that.
Several contributions in this proceedings discuss proposals for the origin of the dark matter, suggesting that they might belong to the stable neutrons of the second group of four families, decoupled from the observed three, to the dark atoms made of dark baryons and ordinary baryons, and to the new scalar fields, new bosons, which manifest inside stars as a Bose-Einstein condensate. These contributions discuss also the possibilities that some of these kinds of the dark matter candidates were already observed by DAMA/LIBRA scattering events or if dark matter objects decay or annihilate too strongly they discuss reasons why experiments do not observe the corresponding gamma rays.
The experiments on the LHC and other laboratories around the world do not so far offer the accurately enough mixing matrices for quarks and leptons, so that it will become clear whether there is the fourth family to the observed three and whether there are several scalar fields, which determine the higgs and the Yukawa couplings, predicted by the spin-charge-family theory. The symmetry in all orders of corrections of the $4 \times 4$ mass matrices, determined by the scalars of this theory, studied in the previous proceedings, limits the number of free parameters of mass matrices, and would for accurately enough measured matrix elements of the $3 \times 3$ sub-matrices of the $4 \times 4$ mixing matrices predict properties of the fourth family of quarks and leptons. The fourth family with the masses close to 1 TeV for leptons and above 1 TeV for quarks is weaker coupled with the rest three families than it is the third u-quark coupled to the rest of quarks. Calculations show that the larger the masses of the fourth family – up to 6 TeV seems to be allowed by experiments – the smaller the unwanted mixing elements which could cause the flavour-changing neutral currents and the better is agreement with the experimental data, which require, that there should be the fourth family due to the nonunitarity of the $3 \times 3$ so far measured mixing matrix for quarks.
The new data might answer the question, whether laws of nature are elegant (as predicted by the spin-charge-family theory and also — up to the families — other Kaluza-Klein-like theories and the string theories) or "she is just using gauge groups when needed" (what many models assume, also some presented in this proceedings). Can the higgs scalars be guessed by smaller steps from the standard model case, appearing as pseudo Nambu Goldstone bosons and in many other

possibilities, or they originate in gravity in higher dimensions as also the gauge fields do?

Is there only gravity as the interacting field, which manifests in the low energy regime all the vector gauge fields as well as the scalar fields? Should correspondingly gravity be quantizable? Is masslessness of all the bosons and fermions essentail, while masses appear at low energy region due to interactions and break of symmetries? Do fermions charges manifest spins in higher dimensions? What is then the dimension of space-time? Infinite, or it emerges from zero?

Is the law of nature emerging from random mathematical structure, which then develope to differentiability, diffeomorphism symmetry, locality, Lorentz invariance, so that fermions spin in higher dimension manifests as charges at low energies? Why and how?

The evidences obviously tell that fermion fields have half integer spin and the charges in the fundamental representations of the so far observed groups. The Grassmann space offer on the other side the possibility that fermions would carry the integer spin and the charges in adjoint representations.

Shall the study of Grassmann space in confrontation with Clifford space for the description of the internal degrees of freedom for fermions, discussed in this proceedings, offering explanation for the second quantization of fermions, help to better understand the "elegance of the laws of nature"?

If "nature would make a choice" of the Grassmann instead of the Clifford algebra, all the atoms, molecules and correspondingly all the world would look completely different, but yet might be still possible. Why "she made a choice" of the Clifford algebra?

Is the working hypotheses that "all the mathematics is a part of nature" acceptable and must be taken seriously? We need and correspondingly use so many mathematical concepts in order to derive a consistent theory, but in most cases still several questions remain open.

Since, as every year also this year there has been not enough time to mature the very discerning and innovative discussions, for which we have spent a lot of time, into the written contributions, only two months, authors can not really polish their contributions. Organizers hope that this is well compensated with fresh contents.

Questions and answers as well as lectures enabled by M.Yu. Khlopov via Virtual Institute of Astroparticle Physics (viavca.in2p3.fr/site.html) of APC have in ample discussions helped to resolve many dilemmas. Google Analytics, showing more than 240 thousand visits to this site from 153 countries, indicates world wide interest to the problems of physics beyond the Standard models, discussed at Bled Workshop. At XXII Bled Workshop VIA streamning made possible to webcast practically all the talks.

The reader can find the records of all the talks delivered by cosmovia since Bled 2009 on viavca.in2p3.fr/site.html in Previous - Conferences. The three talks delivered by E. Kiritsis (Emergent gravity (from hidden sector)), Maxim Yu. Khlopov (Conspiracy of BSM Physics and BSM Cosmology) and Norma Mankoč Borštnik (Experimental consequences of spin-charge family theory) as well as students' scientific debuts talk by Valery Nikulin (Inflationary limits on the size of compact extra space) can be accessed directly at

http://viavca.in2p3.fr/what_comes_beyond_the_standard_models_XXII.html
Most of the talks can be found on the workshop homepage
http://bsm.fmf.uni-lj.si/.
Bled Workshops owe their success to participants who have at Bled in the heart of Slovene Julian Alps enabled friendly and active sharing of information and ideas, yet their success was boosted by vidoeconferences.
Let us conclude this preface by thanking cordially and warmly to all the participants, present personally or through the teleconferences at the Bled workshop, for their excellent presentations and in particular for really fruitful discussions and the good and friendly working atmosphere. We express our gratitude to MDPI journals "Symmetry" and "Particles" for travel support for young and Senior participants and our hope that this tradition will be continued and extended.

*Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,*
*(the Organizing comittee)*

*Norma Mankoč Borštnik, Holger Bech Nielsen, Dragan Lukman,*
*(the Editors)*

*Ljubljana, December 2019*

# 1 Predgovor (Preface in Slovenian Language)

Vsakoletne delavnice z naslovom „Kako preseči oba standardna modela, kozmološkega in elektrošibkega" ("What Comes Beyond the Standard Models?") sta postavila leta 1998 Norma in Holger z namenom, da bi udeleženci v izčrpnih diskusijah kritično soočali različne ideje in teorije. Delavnice domujejo v Plemljevi hiši na Bledu ob slikovitem jezeru, kjer prijetni sprehodi in pohodi na čudovite gore, ki kipijo nad mestom, ponujajo priložnosti in vzpodbudo za diskusije.
K našim zelo odprtim, prijateljskim, dolgim in zahtevnim diskusijam, polnim iskrivega sodelovanja, je prispevalo veliko fizikov in celo nekaj matematikov. Večina tem in vprašanj predstavljenih in diskutiranih na naših Blejskih delavnicah, zadeva predloge za razlago pojavov onkraj obeh standadnih modelov — v fiziki osnovnih delcev in kozmologiji — z namenom razumeti izvor predpostavk obeh standardnih modelov, kar bi omogočilo naovedi za nove poskuse. Čeprav je večina udeležencev teoretičnih fizikov, mnogi z lastnimi idejami kako narediti naslednji korak onkraj sprejetih modelov in teorij, so še posebej dobrodošli predstavniki eksperimentalnih laboratorijev, ki nam pomagajo v odprtih diskusijah razjasniti resnično sporočilo meritev in nam pomagajo razumeti kakšne napovedi so potrebne, da jih lahko s poskusi dovolj zanesljivo preverijo.
Organizatorji moramo priznati, da smo se na blejskih delavnicah v (dolgih) predstavitvah (z odmori in nadaljevanji preko več dni), ki so jim sledile zelo podrobne diskusije, naučili veliko, morda več kot večina udeležencev. Upamo in verjamemo, da so veliko odnesli tudi študentje in večina udeležencev. Velikokrat so se predavanja spremenila v zelo pedagoške predstavitve, ki so pojasnile predpostavke in podrobne korake, soočile predstavljene predloge s predlogi v literaturi ali s predlogi ostalih udeležencev ter jasno pokazale, kje utegnejo tičati šibke točke predlogov. Zdi se, da so se ideje v teh letih razvijale bistveno hitreje, zahvaljujoč prav tem delavnicam.
Tako kot v preteklih dveh letih tudi to leto niso eksperimenti v kozmologiji in fiziki osnovih fermionskih in bozonskih polj ponudili rezultatov, ki bi omogočili nov vpogled v fiziko osnovnih delcev in polj, čeprav je bilo vanje vloženega veliko truda in četudi razberemo iz eksperimentov, da četrta družina k že izmerjenim trem mora biti, saj lahko s štirimi družinami lažje pojasnimo izmerjene podatke, kar je omenjeno tudi v tem zborniku.
Zdi se, kot omenja eden od prispevkov v tem zborniku, da nam "narava no če pomagati", da bi bolje razumeli predpostavke v obeh standardnih modelih.
Nekatera predavanja so spremljale zelo živahne diskusije o predlogih, ki nam lahko pomagajo razumeti privzetke obeh standardnih modelov. Nekatere od teh razprav so v tem zborniku, druge so bodo mordna pojavile v zborniku prihodnje delavnice.
Kozmološka spoznanja in spoznanja v teoriji osnovnih fermionskih in bozonskih polj še nikoli doslej niso bila tako zelo povezana in soodvisna. Na obeh področjih

''rastejo'' novi predlogi, ki naj pojasnijo privzetke teh modelov. Tekma, kdo bo imel prav, je odprta in razburljiva.
Priǎkujemo, da bodo novi kozmološki poskusi razkrili izvor temne snovi. Ker rezultate poskusov DAMA/LIBRA, predstavljene v tem zborniku, težko pojasnimo drugače kot da gre za temno snov, je pričakovati, da bodo sčasoma tudi poskusi v drugih laboratorijih potrdili rezultate poskusa DAMA/LIBRA. To se še ni zgodilo, naše razprave so razjasnile razloge za to.
Kar nekaj je prispevkov v zborniku, ki obravnavajo izvor temne snovi: za delce temne snovi predlagajo stabilne nevtrone druge skupine štirih družin, ki niso sklopljene z že izmerjenimi tremi in pričakovano četrto, temne atome, ki jih sestavljajo temni in običajni barioni ali nova skalarna polja, nove bozone, ki se znotraj zvezd zgostijo v Bose-Einsteinov kondenzat. Avtorji v teh prispevkih obravnavajo tudi možnost, da so v poskusu DAMA/LIBRA nekatere od teh delcev že opazili. En prispevek obravnava možnost, da temna snov morda razpada ali se anihilira dovlj hitro, da bi morali opaziti pri tem nastale žarke gama, pa jih zaradi absorpcije ne opazimo.
Poskusom na pospeševalniku LHC in v drugih laboratorijih doslej ni uspelo izmeriti mešalnih matrik za leptone in kvarke dovolj natančno, da bi lahko ugotovili, ali poleg izmerjenih treh družin obstaja tudi četrta družina in ali obstaja tudi več skalarnih polj, ki določajo higgsov skalar in Yukawine sklopitve. Teorija spinov-nabojev-družin napoveduje obstoj četrte družine in obstoj več skalarnih polj. Simetrija masnih matrik $4 \times 4$ v vseh redih popravkov, obravavana v prispevkih v prejšnjih zbornikih, omeji število prostih parametrov masnih matrik. Za dovolj natančno izmerjene matrične elemente podmatrik $3 \times 3$ v mešalnih matrikah $4 \times 4$ bi ta teorija lahko napovedala lastnosti četrte družine kvarkov in leptonov.
Četrta družina, ki ima mase leptonov blizu 1 TeV, mase kvarkov pa nad 1 TeV, je šibkeje sklopljena s preostalimi tremi družinami, kot je tretji kvark u (top) sklopljen s preostalimi kvarki. Izračuni pokažejo, da se z večanjem mas četrte družine — poskusi dopuščajo mase do 6 TeV — zmanšujejo matrični elementi, ki povzročajo nevtralne tokove in spremembo družinskega kvantnega števila, Hkrati se izboljša ujemanje z eksperimentalnimi podatki, ki zahtevajo četrto družino zaradi neunitarnosti dosedaj izmerjene mešalne matrike $3 \times 3$ za kvarke.
Nove meritve bodo morda odgovorile na vprašanje, ali so zakoni narave elegantni (kot to napove teorija spinov-nabojev-družin in, z izjemo družin, ostale teorije Kaluza-Kleinovega tipa in teorije strun) ali pa samo "uporablja umeritvene grupe po potrebi" (kot to predpostavi veliko modelov, tudi nekateri v tem zborniku). Ali se da uganiti izvor higgsovega skalarja z metodo malih odmikov od standardnega modela, tako, denimo, da se pojavi kot psevdobozon Nambu-Goldstoneovega tipa (možnosti je še veliko več), ali pa izvirajo skalarji iz gravitacije v višjih dimenzijah, tako kot tudi umeritvena polja, in tudi naboji fermionov?
Je gravitacija edino polje, s katerimi fermioni interagirajo, pri nizkih energijah pa se manifestira kot običajna gravitacija in tudi kot poznana vektorska ter skalarno Higgsovo polje? Se gravitacijo da kvantizirati? Ali je brezmasnost vseh bozonov in fermionov osnovna lastnost, masam pa so pri nizkih energijah vzrok interakcije in zlomitve simetrij? Če so fermionskim nabojem vzrok spini fermionov v višjih

dimenzijah, kolikšna je tedaj dimenzija prostor-časa? Neskončna ali pa se pojavi iz nič?
Se zakon narave rodi iz naključnih matematičnih struktur, ki nato v svojem razvoju porodijo odvedlijivost, difeomorfno simetrijo, lokalnost, Lorentzovo invarianco, zaradi česar se spin fermionov iz višjih dimenzij kaže v nižjih kot naboji? Zakaj in kako?
Podatki kažejo, da imajo polja fermionov polštevilski spin in naboje v fundamentalni upodobitvi dosedaj opaženih grup. Grassmannov prostor pa ponuja možnost, da bi imeli fermioni celoštevilski spin in naboje v adjungirani upodobitvi grupo.
Lahko primerjava Grassmannovega in Cliffordovega prostora za opis notranjih prostostnih stopenj fermionov, obravnavana v tem zborniku, ponudi razlago za drugo kvantizacijo fermionov in pomaga bolje razumeti "eleganco zakonov narave"?
Če bi narava " izbrala" za opis notranjih prostostnih stopenj Grassmannovo algebro namesto Cliffordove, bi vsi atomi, molekule in posledično cel svet izgledali poplnoma drugače. Zakaj je"izbrala" Cliffordovo algebro?
Ali je domneva, da je "vsa matematika del narave" sprejemljiva in jo je potrebno vzeti resno? Za razvoj skladne teorije potrebujemo in zato uporabljamo veliko različnih matematičnih konceptov, vendar ostaja veliko vprašanj odprtih.
Ker je vsako leto le malo časa od delavnice do zaključka redakcije, manj kot dva meseca, avtorji ne morejo izpiliti prispevkov, vendar upamo, da to nadomesti svežina prispevkov.
Četudi so k uspehu „Blejskih delavnic" največ prispevali udeleženci, ki so na Bledu omogočili prijateljsko in aktivno izmenjavo mnenj v osrčju slovenskih Julijcev, so k uspehu prispevale tudi videokonference, ki so povezale delavnice z laboratoriji po svetu. Vprašanja in odgovori ter tudi predavanja, ki jih je v zadnjih letih omogočil M.Yu. Khlopov preko Virtual Institute of Astroparticle Physics (viavca.in2p3.fr/site.html, APC, Pariz), so v izčrpnih diskusijah pomagali razčistiti marsikatero vprašanje. Na letošnji delavnici je "pretočno predvajanje" omogočilo, da so vsa predavanja in diskusije tekle tudi preko spleta.
Bralec najde zapise vseh predavanj, objavljenih preko "cosmovia" od leta 2009, na viavca.in2p3.fr/site.html v povezavi Previous - Conferences. Letošnja predavanja na cosmoviji so prispevali:
E. Kiritsis (Emergent gravity (from hidden sector)),
Maxim Yu. Khlopov (Conspiracy of BSM Physics and BSM Cosmology),
Norma Mankoč Borštnik (Experimental consequences of spin-charge family theory).
Cosmovia predstavlja prvikrat študentsko predavanje, imel ga je Valerij Nikulin (Inflationary limits on the size of compact extra space).
http://viavca.in2p3.fr/what_comes_beyond_the_standard_models_XXII.html
Večino predavanj najde bralec na spletni strani delavnice na
http://bsm.fmf.uni-lj.si/.

Naj zaključimo ta predgovor s prisrčno in toplo zahvalo vsem udeležencem, prisotnim na Bledu osebno ali preko videokonferenc, za njihova predavanja in še posebno za zelo plodne diskusije in odlično vzdušje.

Zahvaljujemo se tudi revijam “Symmetry” in “Particles” založbe MDPI za podporo pri potovalnih stroških za mlade in starejše udeležence delavnice ter upamo, da se bo to sodelovanje lahko nadaljevalo in še razširilo.

*Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,*
*(Organizacijski odbor)*

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*(uredniki)*

*Ljubljana, grudna (decembra) 2019*

# Talk Section

All talk contributions are arranged alphabetically with respect to the authors' names.

*Proceedings to the 22nd Workshop*
***What Comes Beyond . . .*** *(p. 1)*
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# 1 Corollary Analyses After the Recent Model-Independent Results of DAMA/LIBRA–Phase2 ⋆

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**Abstract.** The first DAMA/LIBRA–phase2 model-independent results (exposure: 1.13 ton $\times$ yr, and software energy threshold at 1 keV) have recently been released. They further confirm — with high confidence level — the evidence already observed by DAMA/NaI and DAMA/LIBRA–phase1 on the basis of the exploited model-independent Dark Matter (DM) annual modulation signature. The total exposure above 2 keV of the three experiments is 2.46 ton $\times$ yr. Here several DM candidate particles and related scenarios are analyzed including the latest results. These analyses permit to constrain the parameters' space of the considered candidates in the given scenarios, restricting their values with respect to previous analyses thanks to the increase of the exposure and to the lower energy threshold.

**Povzetek.** Avtorji prispevka so pred nedavnim objavili zadnje analize svojih večletnih opazovanj temne snovi na eksperimentu DAMA/LIBRA, ki so jim dodali zadnjo fazo, Fazo 2. Ta zadnja faza ima skupno ekspozicijo 1.13 ton $\times$ let in energijski prag 1 keV. Avtorji zagotavljajo, da so rezultati poskusov, ki temeljijo na letni modulaciji signalov, neodvisni od modelov, ki poskušajo razložiti izvor temne snovi in njihovo detekcijo. Skupna ekspozicija obeh faz poskusov, Faza 1 je imela energijski prag 2 keV, je 2.46 ton $\times$ let. Zadnji rezultati potrjujejo z visoko zanesljivostjo rezultate prejšnje faze poskusa (DAMA/NaI and DAMA/LIBRA–Faza 1). Avtorji analizirajo ustreznost modelov glede na njihove meritve, ter omejijo prostor parametrov obravnavanih modelov.

Keywords: Scintillation detectors, elementary particle processes, Dark Matter

⋆ Talk presented by F. Cappella

## 1.1 Introduction

Recently the model-independent results of the first six full annual cycles measured by DAMA/LIBRA–phase2 with a software energy threshold of 1 keV [1] [1,2] have been released [3–7]. The model-independent evidence for the presence of DM particles in the galactic halo is further confirmed on the basis of the exploited DM annual modulation signature after the previous DAMA/LIBRA–phase1 [1,2,8–14] and the former DAMA/NaI [15,16] experiments. The cumulative Confidence Level (C.L.) is increased from the previous 9.3 $\sigma$ (data from 14 independent annual cycles for an exposure of 1.33 ton $\times$ yr) to 12.9 $\sigma$ (data from 20 independent annual cycles for an exposure of 2.46 ton $\times$ yr).

We recall that the expected DM particles differential counting rate depends on the Earth's velocity in the galactic frame: $v_E(t) = v_\odot + v_\oplus \cos\gamma \cos\omega(t - t_0)$, where the Sun velocity with respect to the galactic halo is $v_\odot \simeq v_0 + 12$ km/s with $v_0$ local velocity), and $v_\oplus \simeq 30$ km/s is the Earth's orbital velocity around the Sun on a plane with inclination $\gamma = 60^o$ with respect to the galactic one. Moreover, $\omega = 2\pi/T$ with $T = 1$ year and roughly $t_0 \simeq$ June $2^{nd}$ (when the Earth's speed in the galactic halo is at maximum). Thus, the expected counting rate averaged in a given energy interval can be conveniently worked out through a first order Taylor expansion:

$$\mathcal{S}(t) = \mathcal{S}_0 + \mathcal{S}_m \cos\omega(t - t_0), \tag{1.1}$$

with the contribution from the highest order terms being less than 0.1%. The $\mathcal{S}_m$ and $\mathcal{S}_0$ are the modulation amplitude and the un-modulated part of the expected differential counting rate, respectively.

In the DAMA experiments the experimental observable is the modulation amplitude, $\mathcal{S}_m$, as a function of the energy, and the identification of the constant part of the signal, $\mathcal{S}_0$, is not required to point out the presence of a signal in the exploited model-independent annual modulation approach. It has several advantages; in particular, the only background of interest is the one able to mimic the signature, i.e. able to account for the whole observed modulation amplitude and to simultaneously satisfy all its many specific peculiarities (see e.g. Ref. [5]). No background of this sort has been found, see Refs. [2–13].

The modulation amplitudes, $S_m$, for the whole data sets: DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2 (total exposure 2.46 ton$\times$yr) are plotted in Fig. 1.1; the data below 2 keV refer only to the DAMA/LIBRA–phase2 exposure (1.13 ton$\times$yr). It can be inferred that positive signal is present in the (1–6) keV energy interval, while $S_m$ values compatible with zero are present just above [5].

In the following the implications on some models, we already investigated with lower exposure and higher software energy threshold in the past, are updated by including the data of DAMA/LIBRA–phase2 [17].

[1] Throughout this paper: i) keV means keV electron equivalent, where not otherwise specified; ii) ton means metric ton (1000 kg).

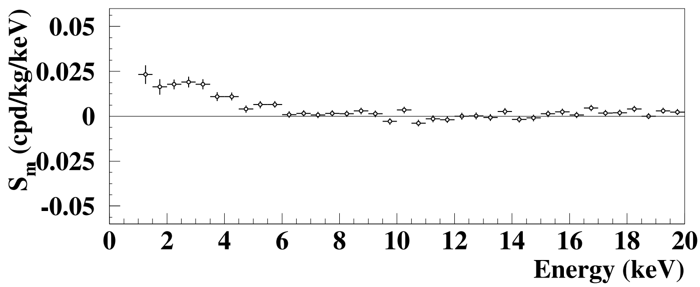


**Fig. 1.1.** Modulation amplitudes, $S_m$, for the whole data sets: DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2 (total exposure 2.46 ton×yr) above 2 keV; below 2 keV only the DAMA/LIBRA–phase2 exposure (1.13 ton × yr) is available and used. The energy bin $\Delta E$ is 0.5 keV. A clear modulation is present in the lowest energy region, while $S_m$ values compatible with zero are present just above.

## 1.2 Data analysis

The corollary analyses presented in the following are model-dependent; thus, it is important to point out at least the main topics which enter in the determination of the results and the related uncertainties. These arguments have been already addressed at various extents in previous corollary model-dependent analyses. The DM candidates considered here have been previously discussed in the Ref. [17] and references therein.

A specific phase-space distribution function (DF) in the galactic halo has to be adopted in order to derive the allowed regions of the parameter's space for the considered DM particles and scenarios. A large number of possibilities is available in literature; these models are continuously in evolution thanks to new simulations and astrophysical observations, as the recent GAIA ones (see e.g. Refs. [18,19] and references therein). Thus, large uncertainties in the predicted theoretical rate are present. Here, to account at some extent for the uncertainties in halo models and to allow direct comparisons, the same not-exhaustive set of halo models as in previous published analyses [15,16,20], is considered; they are illustrated in Table II of Ref. [20]. In particular, the considered classes of halo models correspond to: (1) spherically symmetric matter density with isotropic velocity dispersion (Class A); (2) spherically symmetric matter density with non-isotropic velocity dispersion (Class B); (3) axisymmetric models (Class C); (4) triaxial models (Class D); (5) moreover, in the case of axisymmetric models it is possible to include either an halo co-rotation or an halo counter-rotation.

We also consider the physical ranges of the local velocity $v_0$: from 170 km/s to 270 km/s, and of the local total DM density, $\rho_0$. For $\rho_0$, its minimal, $\rho_0^{min}$, and its maximal, $\rho_0^{max}$, values are estimated imposing essentially two astrophysical constraints: one on the amount of non-halo components and the other on the flatness of the rotational curve in the Galaxy. The values for $\rho_0^{min}$ and $\rho_0^{max}$ are related to the DF and the considered $v_0$; they are reported in Table III of Ref. [20]. The halo density $\rho_0$ ranges from 0.17 to 0.67 GeV/cm$^3$ for $v_0$ = 170 km/s, while $\rho_0$ ranges from 0.29 to 1.11 GeV/cm$^3$ for $v_0$ = 220 km/s, and $\rho_0$ ranges from 0.45 to

1.68 GeV/cm$^3$ for $v_0$ = 270 km/s, depending on the halo model. Moreover, to take into account that the considered DM candidate can be just one of the components of the dark halo, the $\xi$ parameter is introduced; it is defined as the fractional amount of local density in terms of the considered DM candidate ($\xi \leq 1$). Thus, the local density of the DM particles is $\rho_{DM} = \xi\rho_0$.

Finally, the DM escape velocity, $v_{esc}$, from the galactic gravitational potential is considered; actually, it is also affected by significant uncertainty. In the following $v_{esc}$ = 550 km/s is adopted as often considered in literature; however, no sizable differences are observed in the final results when $v_{esc}$ values ranging from 550 to 650 km/s are considered; in fact, for low-mass DM particles scattering off nuclei, the Na contribution is dominant and has a small dependence on the tail of the velocity distribution.

We note that the possible presence of non-virialized components, as streams in the dark halo coming from external sources with respect to our Galaxy [21–23] or other scenarios as e.g. that of Ref. [24–26], are not included in the present analyses.

In the interaction of DM particles in the NaI(Tl) detectors the detected energy, $E_{det}$, is a key quantity. It is connected with the energy released by the products of the interaction, $E_{rel}$; two possibilities exist: 1) the products of the interaction have electromagnetic nature (mainly electrons); 2) a nuclear recoil with $E_R$ kinetic energy is produced by the DM particle scattering either off sodium or off iodine nucleus. Since, the detectors are calibrated by using $\gamma$ sources, in the first case $E_{det} = E_{rel}$, while in the second case a quenching factor (q.f.) for each recoiling nucleus must be included: $E_{det} = q_{Na,I} \times E_{rel}$. In literature there are available a lot of measurements on the Na and I q.f.'s, that show a wide spread, since they are a property of the specific detector and not general properties of any NaI(Tl), particularly in the very low energy range. The same procedures previously adopted in Refs. [27–30] are considered here, i.e. the following three instances are accounted for:

- $(Q_I)$ Na and I q.f.'s "constants" with respect to the recoil energy $E_R$: the adopted values are $q_{Na} = 0.3$ and $q_I = 0.09$, measured with neutron source integrating the data over the 6.5 – 97 keV and the 22 – 330 keV recoil energy range, respectively [31];
- $(Q_{II})$ quenching factors depending on $E_R$, evaluated as in Ref. [32];
- $(Q_{III})$ quenching factors with the same behavior of Ref. [32], but normalized in order to have their mean values consistent with $Q_I$ in the energy range considered there.

Another important effect is the channeling of low energy ions along axes and planes of the NaI(Tl) DAMA crystals. This effect can lead to a further important deviation, in addition to the uncertainties discussed in section II of Ref. [27] and in Ref. [28]. In fact, the channeling effect in crystals implies that a fraction of nuclear recoils are channeled and experience much larger q.f.'s than those derived from neutron calibration (see Refs. [33,27] for a discussion of these aspects). Anyhow, the channeling effect in solid crystal detectors is not a well fixed issue and there could be several uncertainties in the modeling. Because of the difficulties of experimental measurements and of theoretical estimate of the channeling effect, in the following it will be either included using the procedure given in Ref. [33] or not in order to give idea on the related uncertainty.

Finally, three discrete cases are considered in the following to cautiously account for possible uncertainties on the quenching factors measured by DAMA in its detectors and on the parameters used in the SI and SD nuclear form factors [17]:

- Set A considers the mean values of the parameters of the used nuclear form factors [15] and of the quenching factors.
- Set B adopts the same procedure as in Refs. [34,35,16], by varying (i) the mean values of the $^{23}$Na and $^{127}$I quenching factors as measured in Ref. [31] up to $+2$ times the errors; (ii) the nuclear radius, $r_A$, and the nuclear surface thickness parameter, $s$, in the SI nuclear form factor from their central values down to $-20\%$; (iii) the $b$ parameter in the considered SD nuclear form factor from the given value down to $-20\%$.
- Set C where the iodine nucleus parameters are fixed at the values of set B, while for the sodium nucleus one considers [15]: (i) $^{23}$Na quenching factor at $q_{Na} = 0.25$; (ii) the nuclear radius, $r_A$, and the nuclear surface thickness parameter, $s$, in the SI nuclear form factor from their central values up to $+20\%$; (iii) the $b$ parameter in the considered SD nuclear form factor from the given value up to $+20\%$.

In conclusion, model-dependent analyses through a maximum likelihood procedure, which also takes into account the energy behavior of each detector, can be pursued. In particular, for each considered scenario, the allowed domains in the corresponding parameters' space will be obtained by marginalizing over the halo models, over halo parameters ($v_0$ and $\rho_0$) and over the sets A, B, C[2]. This procedure shows the impact of the uncertainties in the astrophysical, nuclear and particle physics on the model-dependent analyses. However, for simplicity the allowed regions in the parameters' space of each considered scenario can also be derived by comparing – for each $k$-$th$ energy bin of 1 keV – the measured DM annual modulation amplitude, $\mathcal{S}_{m,k}^{exp} \pm \sigma_k$ [3], with the theoretical expectation in each considered framework, $\mathcal{S}_{m,k}^{th}$. Of course, the $\mathcal{S}_{m,k}^{th}$ values depend on the free parameters of the model $\bar{\theta}$, such as the DM particle mass, the cross section, etc., on the uncertainties accounted for, on the proper accounting for the detector's features, and on priors.

In particular, as mentioned in previous works (as e.g. recently in Refs. [28,29]), a cautious prior on $\mathcal{S}_{0,k}$ – assuring safe and more realistic allowed regions/volumes – can be worked out from the measured counting rate in the cumulative energy spectrum; the latter is given by the sum of the un-modulated background contribution $b_k$ (whose existence is shown by the detailed analyses on residual radioactive contaminations in the detectors [8]) and of the constant part of the signal $\mathcal{S}_{0,k}$. By adopting a standard procedure, used in the past in several low background

[2] In particular, each allowed domain encloses all the allowed regions obtained for each chosen configuration of model and parameters.

[3] The distributions of the measured modulation amplitudes around their mean value show a perfect Gaussian behaviors, justifying the use of a symmetric uncertainty [9,2,11,3,5].

fields, one can derive lower limits on $b_k$ and, thus, upper limits on $\mathcal{S}_{0,k}$ ($\mathcal{S}_{0,k}^{max}$). In particular, in DAMA/LIBRA–phase2 is obtained: $\mathcal{S}_0 \lesssim 0.80$ cpd/kg/keV in the (1-2) keV energy interval; $\mathcal{S}_0 \lesssim 0.24$ cpd/kg/keV in (2-3) keV, and $\mathcal{S}_0 \lesssim 0.12$ cpd/kg/keV in (3-4) keV.

Thus, the following $\chi^2$ can be calculated for each considered model:

$$\chi^2(\bar{\theta}) = \sum_k \frac{\left(\mathcal{S}_{m,k}^{exp} - \mathcal{S}_{m,k}^{th}(\bar{\theta})\right)^2}{\sigma_k^2} + \sum_{k'} \frac{\left(\mathcal{S}_{0,k'}^{max} - \mathcal{S}_{0,k'}^{th}(\bar{\theta})\right)^2}{\sigma_{0,k'}^2} \Theta\left(\mathcal{S}_{0,k'}^{th}(\bar{\theta}) - \mathcal{S}_{0,k'}^{max}\right) \tag{1.2}$$

where the second term encodes the experimental bounds about the un-modulated part of the signal; $\sigma_{0,k'} \simeq 10^{-3}$ cpd/kg/keV, $\Theta$ is the Heaviside function, and $\mathcal{S}_{0,k'}^{th}$ is the average expected signal counting rate in the $k'$ energy bin. The sum in the first term in eq. 1.2 runs here from 1 keV to 20 keV.

The $\chi^2$ defined in eq. (1.2) can be calculated in each considered framework and is function of the model parameters $\bar{\theta}$. Thus, we can define:

$$\Delta\chi^2(\bar{\theta}) = \chi^2(\bar{\theta}) - \chi_0^2 \tag{1.3}$$

where $\chi_0^2$ is the $\chi^2$ for $\bar{\theta}$ values corresponding to absence of signal. The $\Delta\chi^2$ is used to determine the allowed intervals of the model parameters $\bar{\theta}$ at 10 $\sigma$ from the *null signal hypothesis*.

We have verified that the $Q_{III}$ option for the quenching factors provides results similar to the case of the $Q_I$ option; thus, to avoid the overloading of the figures in the following the $Q_{III}$ case is not considered.

## 1.3 Updated corollary model-dependent scenarios

### 1.3.1 DM particles elastically interacting with target nuclei

A lot of candidates have been proposed in theory extending the Standard Model of particles that includes candidates for DM elastically scattering off target nuclei.

In the DM particle-nucleus elastic scattering, the differential energy distribution of the recoil nuclei can be calculated by means of the differential cross section of the DM-nucleus elastic process [31,36,15,16,33]. The latter is given by the sum of two contributions: the SI and the SD one.

In the purely SI case, the nuclear parameters can be decoupled from the particle parameters and the nuclear cross sections, which are derived quantities, are usually scaled to a defined point-like SI DM particle-nucleon cross section, $\sigma_{SI}$. In principle, this procedure could allow – within a framework of several other assumptions (that in turn introduce uncertainties in final evaluations) – a model-dependent comparison among different target nuclei, otherwise impossible. In the following, the usually considered coherent scaling law for the nuclear cross sections is adopted:

$$\sigma_{SI}(A,Z) \propto m_{red}^2(A,DM)\,[f_p Z + f_n(A-Z)]^2, \tag{1.4}$$

where $\sigma_{SI}(A, Z)$ is the point-like cross section of DM particles scattering off nuclei of mass number A and atomic number Z, $m_{red}(A, DM)$ is the reduced mass of the system DM particle and nucleus, $f_p$ and $f_n$ are the effective DM particle couplings to protons and neutrons, respectively. The case of isospin violation $f_p \neq f_n$ will be discussed in Sect. 1.3.1; now we assume $f_p = f_n$ and, thus, we can write:

$$\sigma_{SI}(A, Z) = \frac{m^2_{red}(A, DM)}{m^2_{red}(1, DM)} A^2 \sigma_{SI}. \tag{1.5}$$

As for nuclear SI form factors, the Helm form factor [37,38] has been adopted[4] (for details on the used form factors see Ref. [15]). As described above, some uncertainties on the nuclear radius and on the nuclear surface thickness parameters in the Helm SI form factors have been included in the following analysis by considering three discrete cases, labeled as set A, B, and C in Sect. 1.2.

The purely SD case is even more uncertain since the nuclear and particle physics degrees of freedom cannot be decoupled and a dependence on the assumed nuclear potential exists. Also in the purely SD case all the nuclear cross sections are usually scaled to a defined point-like SD DM particle-nucleon cross section, $\sigma_{SD}$ [34,15]. The adopted scaling law for this case profits of the proportionality of the SD nuclear cross section to the nuclear spin factor $\Lambda^2 J(J+1)$ and to the squared reduced mass. To take into account the finiteness of the nucleus, a SD nuclear form factor is also used; for details of its parametrization used in the following see Ref. [15]. Moreover, a further parameter must be introduced; in fact, following the notations reported in Ref. [34]: $tan\theta = \frac{a_n}{a_p}$, where $a_{p,n}$ are the effective DM-nucleon coupling strengths for SD interactions. The mixing angle $\theta$ is defined in the $[0, \pi)$ interval; in particular, $\theta$ values in the second sector account for $a_p$ and $a_n$ with different signs. Therefore, further significant uncertainties in the evaluation of the SD interaction rate also arise from the adopted spin factor for the single target-nucleus. In fact, the available calculated values are well different in different models (and differently vary for each nucleus) and, in addition, at fixed model they depend on $\theta$ [34,15].

It is worth noting that for the SD part of the interaction not only the target nuclei should have spin different from zero (for example, this is not the case of Ar isotopes, and most of the Ca, Ge, Te, Xe, W isotopes) to be sensitive to DM particles with a SD component in the coupling, but also well different sensitivities can be expected among odd-nuclei having an unpaired proton (as e.g. $^{23}$Na and $^{127}$I, and $^{1}$H, $^{19}$F, $^{27}$Al, $^{133}$Cs) and odd-nuclei having an unpaired neutron (as e.g. the odd Xe and Te isotopes and $^{29}$Si, $^{43}$Ca, $^{73}$Ge, $^{183}$W).

**Spin-Independent interaction** For the purely SI scenario in the considered model frameworks the allowed region in the plane $m_{DM}$ and $\xi\sigma_{SI}$ have been calculated and are shown in Fig. 1.2. Of course, best fit values of cross section and DM mass span over a large range in the considered model frameworks.

[4] It should be noted that the Helm form factor is the least favorable one e.g. for iodine and requires larger SI cross-sections for a given signal rate; in case other form factor profiles,

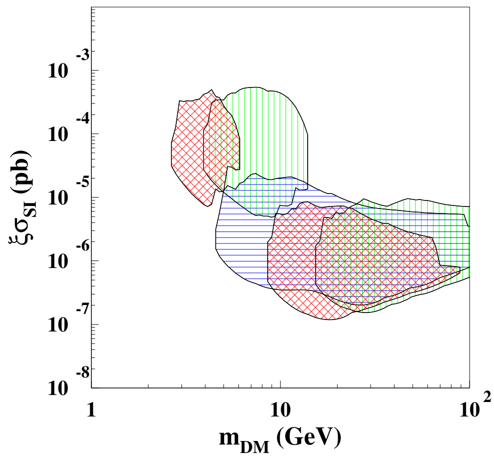


**Fig. 1.2.** Regions – allowed at 10 σ from absence of signal – in the nucleon cross-section vs DM particle mass plane allowed by DAMA experiments in the case of a DM candidate elastically scattering off target nuclei and SI interaction. Three different instances for the Na and I quenching factors have been considered: (i) $Q_I$ case [(green on-line) vertically-hatched region], (ii) with channeling effect [(blue on-line) horizontally-hatched region)] and (iii) $Q_{II}$ [(red on-line) cross-hatched region].

The allowed domains in Fig. 1.2 are obtained by marginalizing all the models for each considered scenario (see Sect. 1.2); they represent the domains where the likelihood-function values differ more than 10 σ from absence of signal. The three different instances described above for the Na and I quenching factors have been considered: (i) $Q_I$ case, (ii) with channeling effect, and (iii) $Q_{II}$.

When comparing with the previous results obtained with DAMA/NaI [15] and DAMA/LIBRA–phase1 [11] data, one can derive that: 1) the C.L. associated to the allowed regions is improved; 2) the allowed regions are restricted (i.e. several configurations are no more supported by the cumulative data at the given C.L.); 3) in the $Q_I$ and $Q_{II}$ cases the low and high mass regions, driven by the Na and I nuclei, respectively, are disconnected; 4) including the channeling effect the lower available mass is 4 GeV, instead of 2 GeV as in the previous analysis [27,2].

In conclusion, the purely SI scenario is still supported by the data both for low and high mass candidates; the inclusion of channeling effect also offers stringent agreement in many considered SI scenarios.

**Candidates with isospin violating SI coupling** To study the case of a DM candidate with SI isospin violating interaction, where $f_p \neq f_n$, a third parameter, namely the ratio $f_n/f_p$, must be considered together with $\xi\sigma_{SI}$ and $m_{DM}$. Obviously the previous case of isospin conserving is restored whenever the ratio $f_n/f_p = 1$.

considered in the literature, would be used, the allowed parameters' space would extend [15].

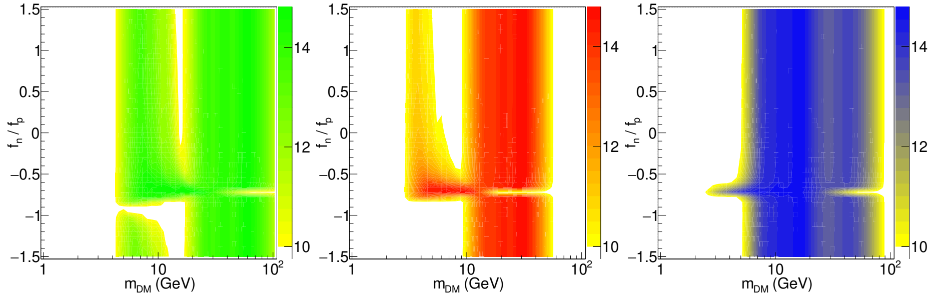


**Fig. 1.3.** Regions in the $f_n/f_p$ vs $m_{DM}$ plane allowed by DAMA experiments in the case of a DM candidate having isospin violating SI interaction. The Na and I quenching factors are: $Q_I$ [left (green on-line)], $Q_{II}$ [center (red on-line)], and with channeling effect [right (blue on-line)]. The considered halo is A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20]. The three possible sets of parameters A, B and C are considered (see Sect. 1.2). The color scales give the confidence level in units of $\sigma$ from the *null hypothesis*.

The results of the analysis for a single halo model hypothesis described later are reported in Fig. 1.3, where the allowed regions in the $f_n/f_p$ vs $m_{DM}$ plane are shown after marginalizing on $\xi\sigma_{SI}$. For simplicity the halo model A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20], and three choices of the Na and I quenching factors: $Q_I$, $Q_{II}$, and including the channeling effect are considered.

Typically, few considerations can be done:

- Two bands of $m_{DM}$ can be recognized, as expected: one at low mass and the other at higher mass.
- The low mass DM candidates have a good fit in correspondence of $f_n/f_p \simeq -53/74 = -0.72$, where the $^{127}$I contribution vanishes and the signal is mostly due to $^{23}$Na recoils.
- Similarly, at larger mass $f_n/f_p \simeq -0.72$ is instead disfavored.
- The case of isospin-conserving $f_n/f_p = 1$ is well supported at different extent both at lower and larger mass.
- When the channeling effect is included (panels on the right of Fig. 1.3), the case of $f_n/f_p = 1$ at low mass has even a stronger support, that is higher confidence level.
- Contrary to what was stated in Ref. [39–41] where the low mass DM candidates were disfavored for $f_n/f_p = 1$ by DAMA data, the inclusion of the uncertainties related to halo models, $v_0$ and $\rho_0$, quenching factors, channeling effect, nuclear form factors, etc., and correctly accounting for other aspects, can also support low mass DM candidates either including or not the channeling effect.

In conclusion, at present level of uncertainties the DAMA data, if interpreted in terms of DM particle inducing nuclear recoils through SI interaction, can account either for low and large DM particle mass and for a wide range of the ratio $f_n/f_p$, even including the "standard" case $f_n/f_p = 1$.

**Spin-Dependent interaction** The purely SD interaction, to which Na and I nuclei are fully sensitive, can also be considered.

The complete results would be described by a 3-dimensional volume: ($\xi\sigma_{SD}$, $m_{DM}$, $\theta$). Thus, a very large number of possible configurations are available; here for simplicity we show, as examples, the results obtained only for 4 particular couplings, which correspond to the following values of the mixing angle $\theta$: (i) $\theta = 0$ ($a_n = 0$ and $a_p \neq 0$ or $|a_p| \gg |a_n|$); (ii) $\theta = \pi/4$ ($a_p = a_n$); (iii) $\theta = \pi/2$ ($a_n \neq 0$ and $a_p = 0$ or $|a_n| \gg |a_p|$); (iv) $\theta = 2.435$ rad ($a_n/a_p = -0.85$, pure $Z_0$ coupling). The case $a_p = -a_n$ is nearly similar to the case (iv).

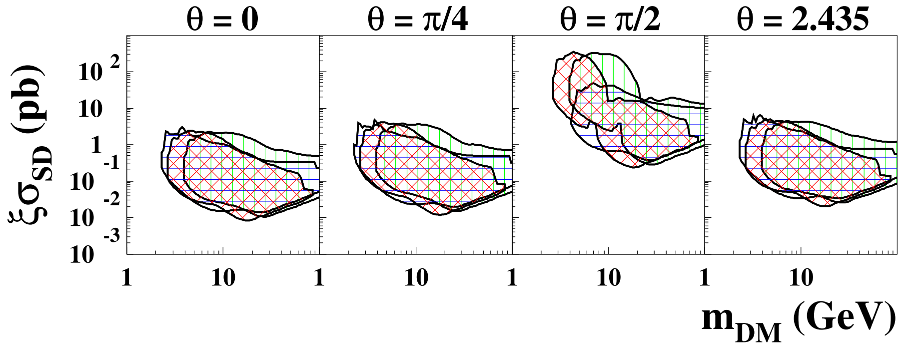


**Fig. 1.4.** Slices of the 3-dimensional volume ($\xi\sigma_{SD}$, $m_{DM}$, $\theta$) allowed at 10 $\sigma$ from absence of signal by the DAMA experiments in the case of a DM candidate elastically scattering off target nuclei and SD interaction. Three different instances for the Na and I quenching factors have been considered: (i) $Q_I$ case [(green on-line) vertically-hatched region], (ii) with channeling effect [(blue on-line) horizontally-hatched region)] and (iii) $Q_{II}$ [(red on-line) cross-hatched region].

In Fig. 1.4 slices of the 3-dimensional allowed volume ($\xi\sigma_{SD}$, $m_{DM}$, $\theta$) at 10 $\sigma$ from absence of signal are shown. For each configuration three regions are depicted accounting for the quenching factors uncertainties.

Finally, Fig. 1.5 shows the allowed regions in the $\tan\theta$ vs $m_{DM}$ plane after marginalizing on $\xi\sigma_{SD}$. For simplicity the halo model A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20], and three choices of the Na and I quenching factors: $Q_I$, $Q_{II}$, and including the channeling effect are considered.

In conclusion, the purely SD scenarios are in good agreement with the DAMA results and can explain the different capability of detection among targets with different unpaired nucleon. The large uncertainties e.g. in the spin factor also offer additional space for compatibility among different target nuclei.

**Mixed coupling framework** The most general case is when both SI and SD couplings are considered. Details of related calculations can be found in Ref. [34,15]. In this scenario, both the uncertainties on the SI and SD frameworks have to be accounted. The complete result is given by a 4-dimensional allowed volume: ($\xi\sigma_{SI}$, $\xi\sigma_{SD}$, $m_{DM}$, $\theta$). The isospin violating SI interaction is not included hereafter.

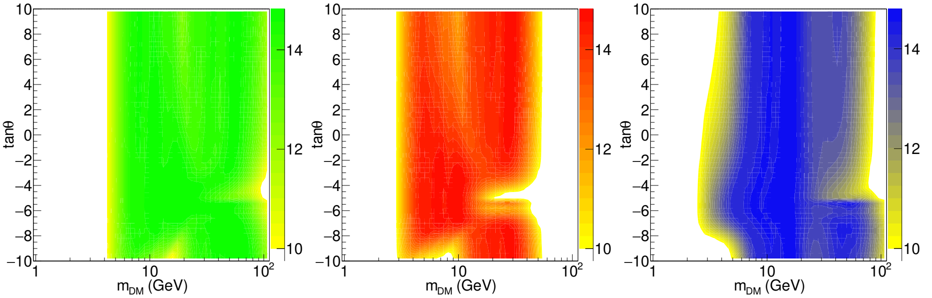


**Fig. 1.5.** Regions in the $\tan\theta$ vs $m_{DM}$ plane allowed by DAMA experiments in the case of a DM candidate with SD interaction. The Na and I quenching factors are: $Q_I$ [$\mathtt{left}$ (green on-line)], $Q_{II}$ [$\mathtt{center}$ (red on-line)], and with channeling effect [$\mathtt{right}$ (blue on-line)]. The considered halo is A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20]. The three possible sets of parameters A, B and C are considered (see Sect. 1.2). The color scales give the confidence level in units of $\sigma$ from the *null hypothesis*.

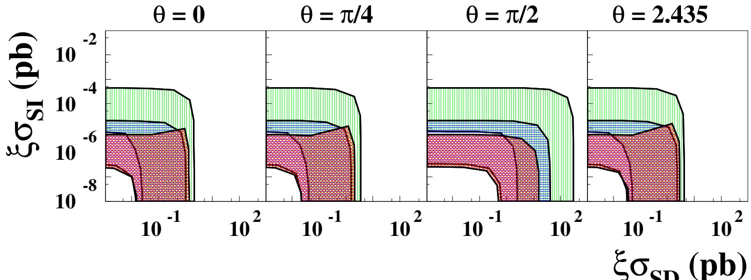


**Fig. 1.6.** Slices of the 4-dimensional volume ($\xi\sigma_{SI}$, $\xi\sigma_{SD}$, $m_{DM}$, $\theta$) allowed by all DAMA experiments in the case of a DM candidate with elastic scattering off target nuclei and mixed SI and SD interaction. Three different instances for the Na and I quenching factors have been considered: (i) $Q_I$ case [(green on-line) vertically-hatched region], (ii) with channeling effect [(blue on-line) horizontally-hatched region] and (iii) $Q_{II}$ [(red on-line) cross-hatched region].

Few examples of slices ($\xi\sigma_{SI}$, $\xi\sigma_{SD}$) at 10 $\sigma$ from the *null hypothesis* (absence of modulation) are shown in Fig. 1.6 for some values of $\theta$ and $m_{DM} = 10$ GeV.

Obviously, the proper accounting for the complete 4-dimensional allowed volume and the existing uncertainties and complementarity largely extend the results and any comparison.

Finally, let us now point out that configurations with $\xi\sigma_{SI}$ ($\xi\sigma_{SD}$) even much lower than those shown in Fig. 1.2 (Fig. 1.4) would be possible if a small SD (SI) contribution would be present in the interaction. This possibility is clearly pointed out in Fig. 1.7 where some examples of regions in the plane $\xi\sigma_{SI}$ vs $m_{DM}$ are reported. Similar plots can be obtained for the $\xi\sigma_{SD}$ vs $m_{DM}$ case (see Ref. [17]). As it can be seen, these arguments clearly show that even a relatively small SD (SI) contribution can drastically change the allowed region in the ($m_{DM}$, $\xi\sigma_{SI(SD)}$) plane; therefore, the typically shown model-dependent comparison plots between exclusion limits at a given C.L. and regions of allowed parameter space do not hold e.g. for mixed scenarios when comparing experiments with and without sensitivity to the SD component of the interaction. The same happens when comparing

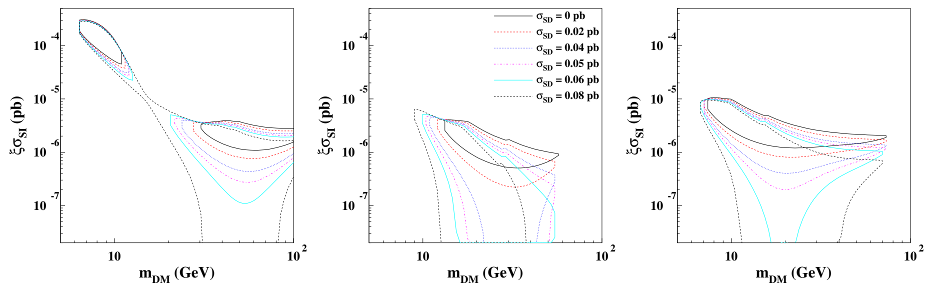


**Fig. 1.7.** An example of the effect induced by the inclusion of a SD component different from zero on allowed regions given in the plane $\xi\sigma_{SI}$ vs $m_{DM}$. In this example the B1 halo model with $v_0 = 170$ km/s and $\rho_0 = 0.42$ GeV/cm$^3$, the set of parameters A and the particular case of $\theta = 0$ for the SD interaction have been considered. The used quenching factors are $Q_I$ (left), $Q_{II}$ (center) and with channeling effect (right). From top to bottom the contours refer to different SD contributions: $\sigma_{SD} = 0$ pb (solid black line), 0.02 pb, 0.04 pb, 0.05 pb, 0.06 pb and 0.08 pb. Analogous situation is found for the other model frameworks.

regions allowed by experiments whose target-nuclei have unpaired proton with exclusion plots quoted by experiments using target-nuclei with unpaired neutron when the SD component of the interaction would correspond either to $\theta \simeq 0$ or $\theta \simeq \pi$.

### 1.3.2 DM particles with preferred electron interaction

Some extensions of the standard model provide DM candidate particles, which can have a dominant coupling with the lepton sector of the ordinary matter. Thus, such DM candidate particles can be directly detected only through their interaction with electrons in the detectors of a suitable experiment, while they cannot be studied in those experimental results where subtraction/rejection of the electromagnetic component of the experimental counting rate is applied[5]. These candidates can also offer a possible source of the 511 keV photons observed from the galactic bulge. This scenario was already investigated by DAMA with lower exposure [42]. The analyses updated by including the new data of the first six annual cycles of DAMA/LIBRA–phase2 with lower software energy threshold is reported in Ref. [17]. The lower energy threshold achieved by DAMA/LIBRA–phase2 at 1 keV prevents to find configurations for these DM candidates *distant* more than 10 $\sigma$ from the *null hypothesis*. However, allowed regions can be found when lowering the number of required $\sigma$ [17]. This is an example how to disentangle among some scenarios, improving the sensitivity of the set-up.

[5] If the electron is assumed at rest, considering the DM particle velocity, the released energy would be of order of few eV, well below the detectable energy in any considered detector in the field. However, the electron is bound in the atom and, even if the atom is at rest, the electron can have non-negligible momentum, as shown in Ref. [42].

### 1.3.3 Inelastic Dark Matter

Another scenario regards the inelastic DM: relic particles that cannot scatter elastically off nuclei. Following an inelastic scattering off a nucleus, the kinetic energy of the recoiling nucleus is quenched and is the detected quantity. As discussed in Refs. [43–45,35], the inelastic DM could arise from a massive complex scalar split into two approximately degenerate real scalars or from a Dirac fermion split into two approximately degenerate Majorana fermions, namely $\chi_+$ and $\chi_-$, with a $\delta$ mass splitting. In particular, a specific model featuring a real component of the sneutrino, in which the mass splitting naturally arises, has been given in Ref. [43].

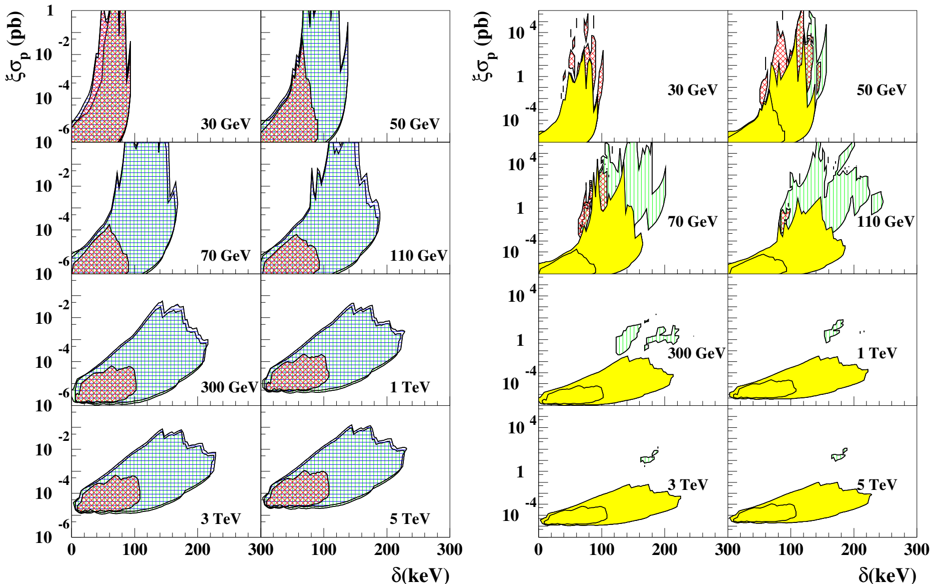


**Fig. 1.8.** Slices of the 3-dimensional volume ($\xi\sigma_p$, $\delta$, $m_{DM}$) allowed by DAMA experiments in the case of a DM candidate with preferred inelastic interaction. Three different instances for the target nuclei quenching factors have been considered: (i) $Q_I$ case [(green on-line) vertically-hatched region], (ii) with channeling effect [(blue on-line) horizontally-hatched region] and (iii) $Q_{II}$ [(red on-line) cross-hatched region]. In the right plots the inelastic scattering off thallium nuclei is also included; here the regions due to inelastic scattering only off Na and I nuclei, already shown on the left, are reported in (yellow on-line) light-filled.

The discussion of the theoretical arguments on such inelastic DM can be found e.g. in Ref. [43], where it was shown that for the $\chi_-$ inelastic scattering off target nuclei a kinematic constraint exists which favors heavy nuclei (such as $^{127}$I) with respect to lighter ones (such as e.g. $^{nat}$Ge) as target-detectors media. In fact, $\chi_-$ can only inelastically scatter by transitioning to $\chi_+$ (slightly heavier state than $\chi_-$) and this process can occur only if the $\chi_-$ velocity, $v$, is larger than:

$$v_{thr} = \sqrt{\frac{2\delta}{m_{red}(A,\chi)}}, \tag{1.6}$$

where $m_{red}(A,\chi)$ is the $\chi$−nucleus reduced mass. This kinematic constraint becomes increasingly severe as the nucleus mass, $m_N$, is decreased [43]. For example, if $\delta \gtrsim$100 keV, a signal rate measured e.g. in Iodine will be a factor about 10 or more higher than that measured in Ge [43]. Moreover, this model scenario implies some characteristic features when exploiting the DM annual modulation signature since it gives rise to an enhanced modulated component, $\mathcal{S}_m$, with respect to the un-modulated one, $\mathcal{S}_0$, and to largely different behaviors with energy for both $\mathcal{S}_0$ and $\mathcal{S}_m$ (both show a higher mean value) [43] with respect to elastic cases. Details of calculation procedures can be found in Ref. [35].

Accounting for the uncertainties mentioned above, in the inelastic DM scenario an allowed 3-dimensional volume in the space ($\xi\sigma_p$, $m_{DM}$, $\delta$) is obtained. Here, following the notation of Ref. [35], $\sigma_p$ is a generalized SI point-like $\chi$−nucleon cross section and $m_{DM}$ is the $\chi$ mass.

For simplicity, Fig. 1.8 left shows slices of such an allowed volume at 10 $\sigma$ from the *null hypothesis* for some values of $m_{DM}$; the different cases of quenching factors are considered as well. It can be noted that when $m_{DM} \gg m_N$, the expected differential energy spectrum is trivially dependent on $m_{DM}$ and, in particular, it is proportional to the ratio between $\xi\sigma_p$ and $m_{DM}$. Thus, allowed regions for other $m_{DM} \gg m_N$ can be obtained from the last panel of Fig. 1.8, straightforward.

Significant enlargement of such regions should be expected when including complete effects of model (and related experimental and theoretical parameters) uncertainties.

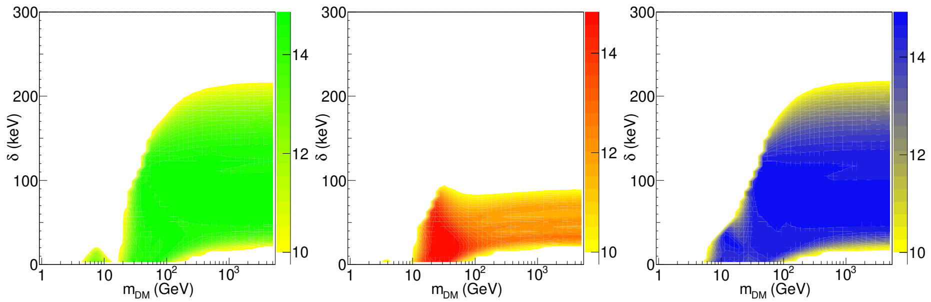


**Fig. 1.9.** Regions in the $\delta$ vs $m_{DM}$ plane allowed by DAMA experiments in the case of a DM candidate with preferred inelastic interaction. The Na and I quenching factors are: $Q_I$ [left (green on-line)], $Q_{II}$ [center (red on-line)], and with channeling effect [right (blue on-line)]. The considered halo is A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20]. The three possible sets of parameters A, B and C are considered (see Sect. 1.2). The color scales give the confidence level in units of $\sigma$ from the *null hypothesis*.

Fig. 1.9 shows the allowed regions in the $\delta$ vs $m_{DM}$ plane after marginalizing on $\xi\sigma_p$. For simplicity the halo model A0 (isothermal sphere) with the $v_0$ and $\rho_0$ in the range of Table III of Ref. [20], and three choices of the Na and I quenching factors: $Q_I$, $Q_{II}$, and including the channeling effect are considered.

It is worth noting that in the case of Inelastic DM the thallium dopant (stable isotopes with mass number 203 and 205, and natural abundances 29.5% and 70.5% respectively) can also play a role as it has been described in Ref. [46], where it has been shown how the DM interaction on thallium nuclei would give rise to a signal which cannot be detected with lower mass target-nuclei. This also can decouple theoretical and experimental aspects from different experiments. The slices of the 3-dimensional volume ($\xi\sigma_p$, $\delta$, $m_{DM}$), allowed by DAMA experiments when the inelastic scattering off thallium nuclei is also included, have been evaluated in Fig. 1.8 right marginalizing all the considered models (see Sect. 1.2). Two instances for the Tl quenching factor in NaI(Tl) are considered: (i) $Q_I$ case with $q_{Tl} = 0.075$, tentatively obtained by extrapolating the $q_{Na}$ and $q_I$ measured by DAMA with neutrons [31]; (ii) $Q_{II}$ quenching factors varying as a function of $E_R$ evaluated as in Ref. [32]. Moreover, the thallium is assumed to be homogeneously distributed in each crystal and among the crystals at level of 0.1% in mass (corresponding to $2.95 \times 10^{21}$ Tl atoms/kg). As shown in Fig. 1.8 right, new regions with $\xi\sigma_p \gtrsim 1$ pb and $\delta \gtrsim 100$ keV are allowed by DAMA after the inclusion of the inelastic scattering off thallium nuclei. Such regions are not fully accessible to detectors with target nuclei having mass lower than thallium.

In conclusion, we point out that here the analysis of the inelastic DM particle has been limited only to SI coupling. Recently analyses of the inelastic DM candidate with SD coupling have been reported in Refs. [47,48]. They show that also this scenario can be compatible with the DAMA result. This conclusion can be further confirmed considering e.g. the effects of uncertainties in the models that in those papers have not been included.

### 1.3.4 Investigation on light dark matter

Some extensions of the Standard Model provide DM candidate particles with sub-GeV mass; in the following these candidates will be indicated as Light Dark Matter (LDM). Several LDM candidates have been proposed in Warm DM scenarios, as keV-scale sterile neutrino, axino, gravitino, and MeV-scale particles (for details see Ref. [30]).

In this section the direct detection of LDM candidate particles is investigated considering the possible inelastic scattering channels either off the electrons or off the nuclei of the target. Firstly we note that – since the kinetic energy for LDM particles in the galactic halo does not exceed hundreds eV – the elastic scattering of such LDM particles both off electrons and off nuclei yields energy releases hardly detectable by the detectors used in the field; this might prevent the exploitation of the elastic scattering as detection approach for these candidates. Thus, the inelastic process could be the only possible viable one for the direct detection of LDM [30].

The following process is, therefore, considered for detection: the LDM candidate (hereafter named $\nu_H$ with mass $m_H$) interacts with the ordinary matter target, T, with mass $m_T$. The target T can be either an atomic nucleus or an atomic electron depending on the nature of the $\nu_H$ particle interaction. As result of the interaction a lighter particle is produced (hereafter $\nu_L$ with mass $m_L < m_H$) and the target recoils with an energy $E_R$, which can be detectable by suitable detectors. The lighter particle $\nu_L$ is neutral and it is required that it interacts very weakly with ordinary matter or not at all; thus, the $\nu_L$ particle escapes the detector. In particular, the $\nu_L$ particle can also be another DM halo component (dominant or sub-dominant with respect to the $\nu_H$ one), or it can simply be a Standard Model particle (e.g. $\nu_L$ can be identified with an active neutrino) [30].

Since the sub-GeV LDM wavelength ($\lambda = \frac{h}{k} > 10^3$ fm) is much larger than the nucleus size, the targets can be considered as point-like and the form factors of the targets can be approximated by one. The cross section of the processes, $\sigma_T$, is generally function of the LDM velocity, $v$, and can be written by adopting the approximation for the non-relativistic case [30]:

$$\sigma_T v \simeq a + b v^2 \ , \tag{1.7}$$

where $a$ and $b$ are constant depending on the peculiarity of the particle interaction with the target T. In the analysis, the cross sections $\sigma_0^T = \frac{a}{v_0}$ and $\sigma_m^T = b v_0$ are defined [30]; they are related to the $a$ and $b$ parameters rescaled with the DM local velocity, $v_0$. In particular, the $\sigma_m^T$ is responsible for the annual modulation of the expected counting rate for LDM interactions, and in the following it will be used as free parameter, together with $m_H$ and the mass splitting $\Delta = m_H - m_L$. Moreover, for the case of LDM interaction on nuclei, following the prescriptions given in Ref. [30], two different nuclear scaling laws are adopted: the coherent ($\sigma_m^{coh} \propto \sigma_m^{Na}/A_{Na}^2 \propto \sigma_m^I/A_I^2$) and the incoherent ($\sigma_m^{inc} \propto \sigma_m^A \propto \sigma_m^I$) ones.

**Interaction with atomic electrons** After the interaction of $\nu_H$ with an electron in the detector, the final state can have – beyond the $\nu_L$ particle – either a prompt electron and an ionized atom or an excited atom plus possible X-rays/Auger electrons. Therefore, the process produces X-rays and electrons of relatively low energy, which are mostly contained with efficiency $\simeq 1$ in a detector of a suitable size.

Comparing the expected modulated signal for this scenario with the experimental result it is possible to determine a 10 $\sigma$ C.L. allowed volume in the space ($m_H$, $\Delta$, $\xi\sigma_m^e$). The projection of such a region on the plane ($m_H$, $\Delta$) for the dark halo models and parameters described before is reported in Fig. 1.10. The allowed $m_H$ values and the splitting $\Delta$ are in the intervals 40 keV $\lesssim m_H \lesssim$O(GeV)[6] and 1.5 keV $\lesssim \Delta \lesssim$70 keV, respectively. It is worth noting that in such a case the decay through the detection channel: $\nu_H \to \nu_L e^+ e^-$, is energetically forbidden for the given $\Delta$ range. The configurations with $m_H \gtrsim 511$ keV (dark area in Fig. 1.10) are instead of interest for the possible annihilation processes: $\nu_H \bar{\nu}_H \to e^+ e^-$, $\nu_H \bar{\nu}_L \to e^+ e^-$, $\nu_L \bar{\nu}_H \to e^+ e^-$ and and $\nu_L \bar{\nu}_L \to e^+ e^-$ in the galactic center.

Some slices of the 3-dimensional allowed volume for various $m_H$ values (including the $m_H = \Delta$ case, that is a massless or a very light $\nu_L$ particle) in the ($\xi\sigma_m^e$ vs $\Delta$) plane are reported in Ref. [17].

In conclusion, it is worthwhile to summarize that electron interacting LDM candidates in the few-tens-keV/sub-MeV range are allowed by DAMA experiments (see Fig. 1.10). This can be of interest, for example, in the models of Warm DM particles, such as e.g. weakly sterile neutrino. Moreover, configurations with

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[6] For values of $m_H$ greater than O(GeV), the definition of LDM is no longer appropriate. Moreover, the kinetic energy of the particle would be enough for the detection in DAMA experiments also through the elastic scattering process, as demonstrated in Ref. [42].

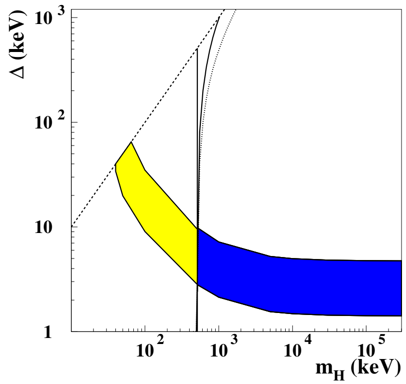


**Fig. 1.10.** Projection of the allowed 3-dimensional volume on the plane $(m_H, \Delta)$ for electron interacting LDM. The dashed line $(m_H = \Delta)$ marks the case where $\nu_L$ is a massless particle. The decay through the detection channel, $\nu_H \to \nu_L e^+ e^-$, is energetically not allowed for the selected configurations. The configurations with $m_H \gtrsim m_e$ (dark area) are interesting for the possible annihilation processes: $\nu_H \bar{\nu}_H \to e^+ e^-$, $\nu_H \bar{\nu}_L \to e^+ e^-$, $\nu_L \bar{\nu}_H \to e^+ e^-$ and $\nu_L \bar{\nu}_L \to e^+ e^-$ in the galactic center. The three nearly vertical curves are the thresholds of these latter processes as mentioned in Ref. [30].

$m_H$ in the MeV/sub-GeV range are also allowed; similar LDM candidates can also be of interest for the production mechanism of the 511 keV gammas from the galactic bulge.

**Interaction with nuclei** With regard to the interaction of LDM with target nuclei, an allowed volume can be obtained in the space $(m_H, \Delta, \xi\sigma_m^{nucleus})$. The projections of such a region on the plane $(m_H, \Delta)$ are reported in Fig. 1.11 for the two above-mentioned illustrative cases of coherent and incoherent nuclear scaling laws. They have been obtained by marginalizing all the models for each considered scenario (see Sect. 1.2) and they represent the domain where the likelihood-function values differ more than 10 $\sigma$ from the *null hypothesis* (absence of modulation). The allowed $m_H$ values and the splitting $\Delta$ are in the intervals 8 MeV $\lesssim m_H \lesssim$O(GeV) and 29 keV $\lesssim \Delta \lesssim$150 MeV, respectively (see Fig. 1.11). It is worth to note that in such a case the decays through the diagram involved in the detection channel (e.g. in nucleon anti-nucleon pairs or in meson(s), as $\nu_H \to \nu_L \pi^0$) are obviously energetically forbidden. Moreover, there are allowed configurations that could contribute – in principle, if suitable couplings exist – to the positron generation in the galactic center; in fact, the decay $\nu_H \to \nu_L e^+ e^-$ is energetically allowed for $\Delta > 2m_e$ (dark area in Fig. 1.11), while the annihilation processes into $e^+e^-$ pairs are energetically allowed for almost all the allowed configurations.

It is worth noting that for nuclear interacting LDM the 3-dimensional allowed configurations are contained in two disconnected volumes, as seen e.g. in their projections in Fig. 1.11. The one at larger $\Delta$ at $m_H$ fixed is mostly due to interaction on Iodine target, while the other one is mostly due to interaction on Sodium target.

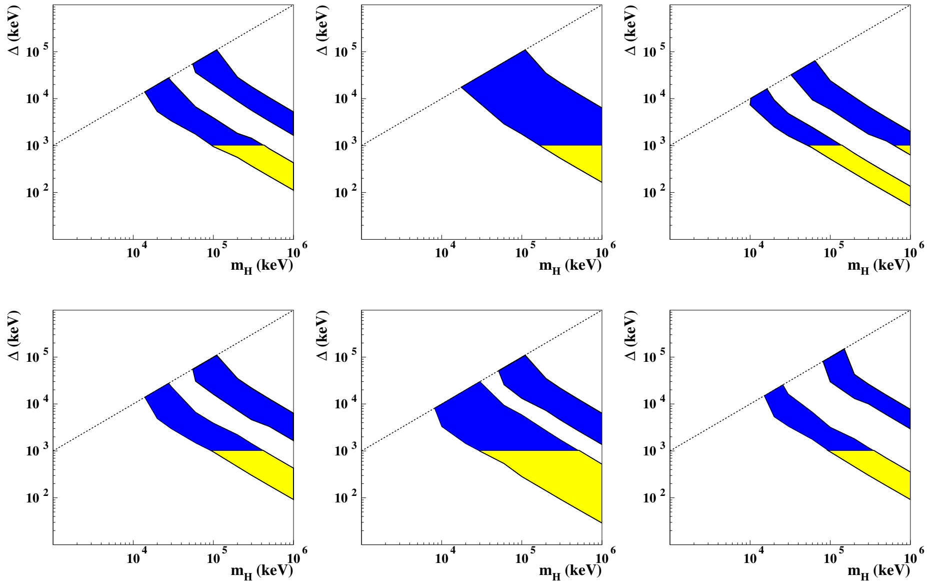


**Fig. 1.11.** Case of nucleus interacting LDM. Projections of allowed 3-dimensional volumes on the plane ($m_H$, $\Delta$) for coherent (top) and incoherent (bottom) nuclear scaling law, considering for the quenching factors: (i) $Q_I$ case (left), (ii) with channeling effect (center), and (iii) $Q_{II}$ (right). The dashed lines ($m_H = \Delta$) mark the case where $\nu_L$ is a massless particle. The decays through the diagram involved in the detection channel are energetically forbidden.

Some slices of the 3-dimensional allowed volumes for various $m_H$ values (including the $m_H = \Delta$ case, that is a massless or a very light $\nu_L$ particle) in the ($\xi\sigma_m^{coh,inc}$ vs $\Delta$) plane are reported in Ref. [17].

Finally, it is worthwhile to summarize that LDM candidates in the MeV/sub-GeV range are allowed by DAMA experiments (see Fig. 1.11). Also these candidates, such as e.g. axino, sterile neutrino, can be of interest for the positron production in the galactic bulge.

### 1.3.5 Mirror Matter

Well-motivated DM candidates are represented by the so called Mirror particles. The Mirror scenario can be introduced by considering a parallel gauge sector with particle physics exactly identical to that of ordinary particles, coined as mirror world. In this theory the Mirror particles belong to the hidden or shadow gauge sector and can constitute the DM particles of the Universe. A comprehensive discussion about Mirror Matter as DM component can be found in Refs. [28,29]. In these two papers the annual modulation effect measured by DAMA experiments with lower exposure has been analyzed in the framework of Asymmetric and Symmetric Mirror Matter scenarios. The analyses updated by including the new data of the first six annual cycles of DAMA/LIBRA–phase2 with lower software energy threshold is reported in Ref. [17]. This new analysis restricts a significant part of the parameters' space of the Mirror DM scenarios.

## 1.4 Conclusions

A high C.L. model-independent evidence for the presence of DM particles in the galactic halo has been achieved by DAMA/NaI, DAMA/LIBRA–phase1 and by the first six full annual cycles of DAMA/LIBRA–phase2 on the basis of the exploited signature.

The corollary investigation on the nature of the DM particles is an open problem; it always requires a large number of assumptions. In this paper several possible scenarios [17] for DM candidates are analyzed on the basis of the long-standing DAMA results exploiting the DM annual modulation signature.

In particular, the DAMA/LIBRA–phase2 data, collected over the first six full annual cycles (1.13 ton $\times$ yr) with a software energy threshold down to 1 keV, are analyzed with the DAMA/NaI and DAMA/LIBRA–phase1 data for several scenarios, improving the confidence levels and restricting the allowed parameters' space of the considered DM candidate particles with respect to previous analyses.

Several scenarios are compatible with the observed signal; other possibilities are open as well. For example other scenarios as e.g. Refs. [49,50] are planned to be analysed as well. It is also worth noting that even a suitable particle not yet foreseen by theories may be the- or one-of-the- solutions for DM particles.

The improved results presented in this paper show how important is to improve the capability of the experiment to effectively disentangle among the many possible different scenarios. For such a purpose an increase of exposure in the new lowest energy bins and the lowering of the software energy threshold below 1 keV are important. Thus, DAMA/LIBRA–phase2 has continued its data taking. Moreover, related R&D's towards the so-called phase3 have been funded and are in progress.

In conclusion, the new data have allowed significantly improving the confidence levels and restricting the allowed parameters' space for the various considered scenarios with respect to previous DAMA analyses; efforts towards further improvements are in progress.

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# 2 Conspiracy of BSM Physics and Cosmology

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**Abstract.** The lack of experimental evidence at the LHC for physics beyond the Standard model (BSM) of elementary particles together with necessity of its existence to provide solutions of internal problems of the Standard model (SM) as well as of physical nature of the basic elements of the modern cosmology demonstrates the *conspiracy* of BSM physics. Simultaneously the data of precision cosmology only tighten the constraints on the deviations from the now standard ΛCDM model and thus exhibit *conspiracy* of the nonstandard cosmological scenarios. We show that studying new physics in combination of its physical, astrophysical and cosmological probes, can not only unveil the conspiracy of BSM physics but will also inevitably reveal nonstandard features in the cosmological scenario.

**Povzetek.** Poskusi na pospeševalniku LHC niso ponudili doslej ničesar, kar bi prispevalo k razumevanju izvora predpostavk standardnega modela osnovnih delcev. Tudi vedno bolj natančne kozmološke meritve ne kažejo odstopanja od standardnega modela. Zdi se, meni avtor, kot da se je narava ”zarotila” in vstraja na standardnem modelu in “nam noče pokazati poti” k razumevanju privzetkov standardnega modela. Avtor v prispevku demonstrira, kako lahko teoretične raziskave skupaj s fizikalnimi, astrofizikalnimi in kozmološkimi meritvami pokažejo nestandardne značilnosti v kozmološkem scenariju.

Keywords: cosmology, particle physics, cosmoparticle physics, inflation, baryosynthesis, dark matter, primordial black holes, antimatter, dark atoms, composite dark matter, stable double charged particles

## 2.1 Introduction

The now standard description of the structure and evolution of the Universe is based on inflationary models with baryosynthesis and dark matter/ energy. The interpretation of the data of precision cosmology ascribes about 95% of the modern cosmological energy density to the impact of physics beyond the Standard model (BSM) of elementary particles. BSM physics is involved in virtually all the

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mechanisms of inflation and baryosynthesis, explaining the initial conditions of the cosmological evolution. It makes the observed homogeneous and isotropic expanding Universe, origin and structure of its inhomogeneities with their observed baryon asymmetry an evident reflection of the BSM physics.

The problem of experimental studies of BSM physics is generally related with necessity to address effects of a high energy scale $F$ [1]. At the energy release $E \geq F$ it leads to appearance of new heavy particles with the mass $M \sim F$ or new interactions that manifest their full strength at these energies. If the energy is much less, than $F$, only virtual effects of new physical scale are possible, which are suppressed by some power of $E/F$. Therefore we can either turn to rare low energy processes, in which new high energy physics phenomena can appear, like proton decay, or probe at the currently available energies $E$ the extensions of the Standard model (SM), which involve new physics at scales $F \leq E$. Probes for supersymmetric (SUSY) models at the LHC corresponded to the latter case, but the lack of positive evidence for existence of SUSY particles at the energy of hundreds GeV probably moves the SUSY scale to higher energies, at which direct search of SUSY particle production at the LHC is not possible.

The only experimentally proven evidence for new physics is the effect of neutrino oscillations, but the physical nature of neutrino mass is still unknown.

Following [1] we characterize here the current situation as the *conspiracy* of the BSM physics: there is no doubt in its existence, but all its features are hidden, since the experimental data put only more and more stringent constraints on the new physics effects. We discuss the physical motivation for extension of SM model and their possible physical, astrophysical and cosmological signatures in Section 2.2. We draw attention in the Section 2.3 that BSM physics involved in the description of the now standard cosmological model (which we consider in Section 2.2 as the motivation for the SM extension) should inevitably add nonstandard model dependent features like a plethora of non-WIMP forms of dark matter, primordial black holes or antimatter domains in the baryon asymmetrical Universe. We express the hope in the Conclusion (Section 2.4) that revealing of specific model dependent signatures of BSM physics can not only unveil its conspiracy, but can also enrich the theory of structure and evolution of the Universe by nonstandard cosmological scenarios.

## 2.2 Motivations for the SM extension

### 2.2.1 Physics of neutrino mass

The discovery of neutrino oscillations proves the existence of the nonzero mass of neutrino. It may be considered as a manifestation of BSM physics, since neutrinos are strictly massless in the Standard model. However, the very existence of neutrino mass doesn't shed light on its physical nature and the corresponding new physics.

Neutrino mass term relates ordinary left-handed neutrino state to some right handed state. The latter can be ordinary right-handed antineutrino. It corresponds

[1] Henceforth,if it is not otherwise specified, we use the units $\hbar = c = k = 1$

to Majorana mass term, in which lepton number L conservation is violated and L changes as $\Delta L = 2$. In the SM lepton number is conserved at the tree level and Majorana mass term is the example of BSM physics.

Smallness of ordinary neutrino Majorana mass $m_\nu$ relative to the Dirac mass $m_D$ of the corresponding charged lepton is explained by "see-saw" mechanism, involving right handed neutrino with large Majorana mass $M$, so that ordinary neutrino mass is given by

$$m_\nu = \frac{m_D^2}{M} = \frac{m_D}{M} m_D \ll m_D. \tag{2.1}$$

Majorana mass term of electron neutrino leads to neutrinoless double beta decay. In the nonrelativistic limit interaction of Majorana neutrino with nuclei is proportional to spin operator acting on nuclear wave function. It leads to spin dependent interaction of nonrelativistic Majorana neutrino with nuclei.

Another possibility is a Dirac neutrino mass term. It corresponds to transition to a new state of sterile right handed neutrino. Such neutrino doesn't participate in the ordinary weak interactions, being another possible example of BSM physics, related to the mechanism of neutrino mass generation.

In the nonrelativistic limit Dirac neutrino interaction with nuclei is spin independent and leads to coherent scattering of low energy neutrinos in the matter. V.Shwatsman has noted in his diploma work in late 1960s that neutrino with mass $m$ and velocity $v$ can scatter coherently on the piece of matter with size $l \sim \hbar/(mv)$ and cause its acceleration. This idea, published in [2,3] was probably the first step towards direct detection of cosmological dark matter.

It is the stable prediction of the Big Bang theory that primordial thermal neutrino background should exist with number density

$$n_{\nu\bar{\nu}} = \frac{3}{11} n_\gamma, \tag{2.2}$$

where $n_\gamma \approx 400 cm^{-3}$ is the the number density of CMB photons. Multiplied by neutrino mass it gives the predicted contribution of relic massive neutrinos to the cosmological density. Experimental constraints on the mass of electron neutrinos (see [4] for the latest results) together with the data on the neutrino oscillations exclude explanation of the measured dark matter density by this contribution. However, while ordinary massive neutrinos cannot play dominant dynamical role in the Universe, BSM physics of neutrino mass can lead to important cosmological effects, like sterile neutrino dark matter [5].

### 2.2.2 Supersymmetry and the SM problems

SUSY models provide natural solution for the internal SM problems, if the SUSY scale is in the range of several hundred GeV.

Then contribution of SUSY partners in loop diagrams of radiative effects in the Higgs boson mass cancel the quadratic divergent contribution of the corresponding SM particles. Renormalization group analysis of evolution of scalar field potential from superhigh energy scale leads to the Higgs form of this potential at lower energy, explaining the nature of the electroweak symmetry breaking.

R-parity or some continuous symmetry provides stability of the lightest SUSY particle. Such particle with mass of several hundred GeV has interaction cross section at the level of weak interaction and can play the role of Weakly Interacting Massive Particle (WIMP) candidate for dark matter.

The lack of experimental signatures for SUSY particles at the LHC as well as of positive result of underground WIMP searches [2] implies nontrivial ways of search for SUSY (see [11] for the latest review).

In the extreme case SUSY scale may be close to the scale of Grand Unfication (GUT). This case implies non-SUSY solution for the problem of divergence of the Higgs mass and origin of the electroweak symmetry (see the next subsection), but has the advantage to unify all the four fundamental natural forces, including gravity, in the framework of Supergravity. Starobinsky supergravity can provide simultaneous BSM solution for dark matter in the form of superheavy gravitino [12–14] and Starobinsky inflation [15]. This solution can be hardly probed by any direct experimental mean and makes cosmological consequences the unique way for its indirect test.

### 2.2.3 Nonsupersymmetric solutions. Composite Higgs. Multiple charged particles

Nonsupersymmetric solution for the problem of Higgs mass divergence may be related to the composite nature of Higgs boson [16–21]. Then this divergence is cut at the scale, at which Higgs constituents are bound. In parallel such constituents can form bound states with exotic charges. Such situation can take place in the model of composite Higgs based on Walking Technicolor (WTC) [22–27].

The minimal walking technicolor model (WTC) involves two techniquarks,$U$ and $D$. They transform under the adjoint representation of a $SU(2)$ technicolor gauge group. Neutral techniquark-antitechniquark state is associated with the Higgs boson. Six bosons $UU$, $UD$, $DD$, and their corresponding antiparticles carry a technibaryon number. If the technibaryon number is conserved, the lightest technibaryon should be stable.

Electric charges of $UU$, $UD$ and $DD$ are given in general by $q + 1$, $q$, and $q - 1$, respectively, where $q$ is an arbitrary real number [28–30]. To compensate the anomalies the model includes in addition technileptons $\nu'$ and $\zeta$ that are technicolor singlets. Their electric charges are in terms of $q$, respectively, $(1-3q)/2$ and $(-1 - 3q)/2$.

Fractional value of $q$ would correspond to stable fractionally charged techniparticles. Their creation in the early Unvierse would lead to their presence in the terrestrial matter that is severely constrained by the experimental data. On the same reason, stable techniparticles should not have odd charge $2n + 1$. Positively

[2] Though interpretation of positive result of DAMA/NaI and DAMA/LIBRA experiments in the terms of WIMPs is not excluded [6,7], theoretical analysis [8], proving such a possibility indicates its contradiction with the results of XENON1T [9] and PICO [10] experiments.

charged $+(2n+1)$ stable particles are bound with electrons in anomalous isotopes of elements with $Z = 2n+1$. Negatively charged particles with charge $-(2n+1)$, created in the early Unvierse, bind with $n+1$ nuclei of primordial helium, produced in the Big Bang Nucleosynthesis, and form a +1 charged ion that binds with electrons in atoms of anomalous hydrogen. The experimental data put severe constraints on such anomalous isotopes.

The case of stable multiple charged particles with even value of negative charge $-2n$ avoids these troubles, since it forms with $n$ nuclei of primordial helium neutral dark atom. Their bound states with primordial helium can play the role of dark matter and can even solve the puzzles of dark matter searches (see [1,31–33] for the latest review).

### 2.2.4 Axion and axion-like models

The popular solution for the problem of strong CP violation in QCD involves the additional $U(1)_{PQ}$ symmetry which provides automatic suppression of the CP-violating $\theta$-term [34]. Breaking of this Peccei-Quinn symmetry spontaneously at the scale $f$, followed by its manifest breaking at the scale $\Lambda \ll f$ results in appearance of a pseudo-Nambu-Goldstone (PNG) particle, axion, $a$. In the axion models the second step of breaking is generated by instanton transitions.

The mass of axion is given by [35]

$$m_a = C m_\pi f_\pi / f, \tag{2.3}$$

where $m_\pi$ and $f_\pi \approx m_\pi$ are the pion mass and constant, respectively. The constant $C \sim 1$ depends on the choice of the axion model. The relationship (2.3) of axion to neutral pion makes possible to estimate the cross section of axion interactions from the corresponding cross section of pion processes multiplied by the factor $(f_\pi/f)^2$.

The existence of $a\gamma\gamma$ vertex leads to a two-photon decay of axion, as well as to effects of $a\gamma$ conversion [38] like axion-photon conversion in magnetic field (see e.g. [39] for review and references). The principles of experimental search for axion by "light shining through walls" effects are based on such a conversion [40].

Axion couplings to nondiagonal quark and lepton transition can lead to rare processes like $K \to \pi a$ or $\mu \to e a$. In the gauge model of family symmetry breaking [41] the PNG particle called *archion* shares properties of axion with the ones of singlet Majoron and familon, being related to the mechanism of neutrino mass generation.

In the axion-like models the condition of Eq. (2.3) is absent and the mass of the PNG particle may be very small.

In cosmology, in spite of a very small mass (2.3) primordial axions appear in the ground state of Bose-Einstein condensate and, being created initially nonrelativistic, represents a specific form of Cold Dark Matter.

### 2.2.5 BSM physics of the standard cosmology

The now Standard cosmological model involves inflation to explain the homogeneity and isotropy of the Universe as well as initial impulse for Big Bang expansion.

Observed absence of antimatter objects is explained by baryosynthesis, in which baryon asymmetry was generated in the intially baryon symmetric Universe. Formation and evolution of Large Scale Structure is described in the framework of the standard $\Lambda$CDM model, assuming dominance in the modern total cosmological density of dark energy with vacuum-like equation of state (cosmological constant $\Lambda$ in the simplest case) and dark matter dominating in the matter content of the Universe. All these elements of the Standard Cosmological model imply BSM physics, making the observational confirmation of this model an evidence for existence of BSM physics.

On the other hand, the data of precision cosmology (planck15,planck18) analysed in the terms of parameters of this standard cosmological model continuously tighten the constraints on deviations of the measured parameters from the model predictions. These measured parameters involve dark matter density $\Omega_{DM}h^2 = 0.120 \pm 0.001$, baryon density $\Omega_b h^2 = 0.0224 \pm 0.0001$ (where the dimensionless constant $h$ is the modern Hubble constant $H_0$ in the units of 100 km/s/Mpc), scalar spectral index $n_s = 0.965 \pm 0.004$, and optical depth $\tau = 0.054 \pm 0.007$ [43]. These results are only weakly dependent on the cosmological model and remain stable, with somewhat increased errors, in many commonly considered extensions. Assuming the $\Lambda$CDM cosmology, the inferred late-Universe parameters were determined: the Hubble constant $H_0 = (67.4 \pm 0.5)$ km/s/Mpc; matter density parameter $\Omega_m = 0.315 \pm 0.007$; and matter fluctuation amplitude $\sigma_8 = 0.811 \pm 0.006$. Combining with the results of studies of baryon acoustic oscillations (BAO) by measurement of large scale distribution of galaxies [44] [3] Planck collaboration has constrained the effective extra relativistic degrees of freedom to be $N_{eff} = 2.99 \pm 0.17$, and the sum of neutrino mass was tightly constrained to $\sum m_\nu < 0.12$. These results prove the basic ideas of inflationary model with baryosynthesis and dark matter/energy, but cannot provide definite choice for the corresponding BSM physics.

PLANCK collaboration has found no compelling evidence for extensions to the $\Lambda$CDM model, but has mentioned the $3\sigma$ difference with the results of local determination of $H_0$ [46]. Such a discrepancy may be a hint to a necessity to extend the standard cosmological model.

Indeed, the *conspiracy* of Beyond the Standard model (BSM) Cosmology [1] is puzzling taking into account the plethora of nontrivial cosmological consequences of BSM particle models. Some of these nonstandard features which have probably found their experimental evidence are discussed in the next Section 2.3.

## 2.3 Features of BSM cosmology

### 2.3.1 Plethora of dark matter candidates

Well motivated BSM models offer a plethora of dark matter candidates. In the essence such candidates follow from the extension of the SM symmetry. If the additional symmetry acting on new sets of particles is strict or nearly strict, the

[3] Such oscillations were first discussed by A.D. Sakharov [45] and are also called *Sakharov oscillations*

lightest particles that possess this symmetry are stable or sufficiently long living to play the role of dark matter. In addition to massive sterile neutrinos, superheavy gravitino or invisible axion that follow respectively from solutions of the origin of neutrino mass, Starobinsky supergravity or solution of the problem of strong CP violation in QCD there are mirror or shadow particles, whose existence is related to restoration of equivalence of left- and right-handed coordinate systems. Grand Unification, string phenomenology or phenomenology of extra dimensions extend this list by many other nontrivial candidates accompanied by a very extensive hidden sector of new particles and fields. Such extensions naturally lead to multi-component dark matter that can include unstable or decaying components, like it takes place in the model of broken family gauge symmetry [41] (see e.g. [28,37,35] for review and references).

In this large list of possibilities the model of dark atoms, in which stable $-2n$ charged particles are bound with $n$ nuclei of primordial helium, is of special interest not only owing to the minimal set of the involved new physics parameters (their number is reduced to the mass of a hypothetical negatively charged stable particles only), but also since it may provide a solution for controversial results of direct dark matter searches.

The idea of this solution is that nuclear interacting dark atoms are slowed down in the terrestrial matter and thus cannot cause significant nuclear recoil in the underground detector. However, in the matter of these detectors dark atoms can bind with intermediate mass nuclei with the binding energy of few keV (see [1,31,33] for recent review and references). Since the concentration of dark atoms in the matter of underground detectors is adjusted to their incoming cosmic flux, energy release in such binding should experience annual modulations. It explains positive results of DAMA/NaI and DAMA/LIBRA experiments. In a simple rectangular wall and well approximation it was shown in [47] that a level of about 3 keV can exist in binding of dark atoms with intermediate mass nuclei and doesn't exist for heavy nuclei, like xenon, explaining absence of positive results in the corresponding detectors. If such level exists, transition to it is determined by isospin violating electric dipole operator and its rate is proportional to the temperature, being suppressed in cryogenic detectors [1,31,33].

The open problem of this explanation is a selfconsistent treatment of Coulomb and nuclear interactions of dark atoms. Such treatment needs special study in the lack of all the usual approximations of atomic physics: there are no small parameters like small ratio of sizes of nucleus and atom and the electroweak interaction of electronic shell. Dark atoms has strongly interacting nuclear shell with the radius of the order or equal to the nuclear radius.

Dark atom cosmology contains such notrivial features as Warmer than Cold Dark Matter scenario and can explain the observed excess of radiation in positron annihilation line from the center of Galaxy as indirect effect of dark atoms (see [1,31–33] for recent review and references). This explanation assuming electron-positron pair production in de-excitation of dark atoms excited in collisions in the center of Galaxy is possible only for a narrow range around 1.25 TeV of the mass of dark atom, which is determined by its constituent with multiple negative charge [1,35]. It challenges search for multiple charged stable particles at the LHC that provides complete test of such an explanation [48].

In a two-component dark atom model, a possibility to explain the observed excess of high energy positrons by decays of $+2$ charged dark atom constituents was proposed in [49]. However, any source of positrons is simultaneously the source of gamma radiation and to avoid contradiction with the observed gamma background the mass of the decaying $+2$ consituent of dark atom should be less, than 1 TeV. Moreover, in view of the difference of propagation in the Galaxy by gamma radiation and positrons the condition not to exceed the observed gamma background may cause troubles for any explanation for the high energy positron excess, involving indirect effects of dark matter [50]. In any case, the results of searches for stable double charged particles in the ATLAS experiment at the LHC put lower limit on the mass of such particles [51], excluding explanation of high energy positron anomaly by decaying $+2$ charged constituents of dark atoms [1,35].

### 2.3.2 Primordial Black holes

Strong primordial inhomogeneities are a prominent tracer of BSM physics of very early Universe and Primordial Black Holes (PBH) are the most popular example of this kind (see e.g. [12,52] for review and references). To form a black hole in the homogeneously expanding Universe the expansion should stop in some region and it corresponds to a very strong inhomogeneity [53–55]. In the universe with equation of state

$$p = \gamma\epsilon \tag{2.4}$$

with numerical factor $\gamma$ being in the range

$$0 \leq \gamma \leq 1 \tag{2.5}$$

the probability of forming a black hole from fluctuations within the cosmological horizon is given by [56]

$$W_{PBH} \propto \exp\left(-\frac{\gamma^2}{2\left\langle\delta^2\right\rangle}\right), \tag{2.6}$$

where $\left\langle\delta^2\right\rangle \ll 1$ is the amplitude of density fluctuations. For relativistic equation of state ($\gamma = 1/3$) the probability (2.6) is exponentially small. It can increase, if the amplitude of density fluctuations in the early Universe was much larger, than in the period of galaxy formation, or the equation of state was much softer, corresponding to matter dominated stage with $\gamma = 0$.

Therefore PBH origin may be related with early matter dominated stages, phase transitions in the early Universe or nonflat features in the spectrum of primordial density fluctuations. All these phenomena are not only originated from BSM physics, but also represent strong deviation from the Standard cosmological scenario.

PBHs with mass $M \leq 10^{15}$ g evaporate by the mechanism of Hawking [57,58]. This process is the universal process of production of any type of particles with mass

$$m \leq T_{evap} \approx 10^{13}\,\mathrm{GeV}\frac{1\,\mathrm{g}}{M}.$$

It can be the source of superweakly interacting particles, like gravitino [59] as well as of fluxes of particles with energy much larger, than the thermal energy of particles in the surrounding medium. It causes non equilibrium processes in the hot Big Bang Universe, nonequilibrium cosmological nucleosynthesis [60], in particular.

PBHs with mass $M \geq 10^{15}$ g should survive to the present time and represent a specific form of dark matter. It was noticed in [61] that taking into account PBH formation in clusters the constraints on PBH contribution into the total density [62] can be relaxed and even the possibility of PBH dominant dark matter is not excluded. It would make primordial nonhomogeneities in the form of PBHs the dominant matter content of the modern nonhomogeneities.

Mechanism of PBH cluster formation can be illustrated with the use of the axion-like model, discussed in subsection 2.2.4, in which the first step of symmetry breaking at scale $f$ takes place on the inflationary stage [35,52]. Then at the second stage of the symmetry breaking at $T \sim \Lambda$ closed massive walls are formed so that the larger wall is accompanied by a set of smaller walls. Their collapse form a PBH cluster, in which the range of PBH masses $M$ is determined by the model parameters $f$ and $\Lambda$ [35,36]

$$f(\frac{m_{pl}}{\Lambda})^2 \leq M \leq \frac{m_{pl}}{f} m_{pl}(\frac{m_{pl}}{\Lambda})^2 \tag{2.7}$$

Here the minimal mass is determined by the condition that the width of wall doesn't exceed its gravitational radius, while the upper limit comes from the condition that the wall enters horizon, before it starts to dominate within it [36]. At $\Lambda < 100\,\mathrm{MeV}(m_{pl}/f)^{1/2}$ the maximal mass exceeds $100 M_{odot}$. Collapse of massive walls to such black holes takes place at

$$t > \frac{m_{pl}}{f}\frac{m_{pl}}{\Lambda^2}. \tag{2.8}$$

At $\Lambda < 1\,\mathrm{GeV}$ and $f = 10^{14}\,\mathrm{GeV}$ it happens at $t > 0.1$ s, what can lead to interesting observable consequences.

Closed wall collapse leads to primordial gravitational wave (GW) spectrum, estimated as peaked at [35]

$$\nu_0 = 3 \times 10^{11} (\Lambda/f)\mathrm{Hz}. \tag{2.9}$$

Their estimated contribution to the total density can reach

$$\Omega_{GW} \approx 10^{-4}(f/m_{pl}), \tag{2.10}$$

being at $f \sim 10^{14}$ GeV $\Omega_{GW} \approx 10^{-9}$. For $1 < \Lambda < 10^8\,\mathrm{GeV}$ the maximum of the spectrum corresponds to

$$3 \times 10^{-3} < \nu_0 < 3 \times 10^5\,\mathrm{Hz}, \tag{2.11}$$

being in the range from tens to thousands of Hz a challenge for LIGO/VIRGO grvitational wave searches.

Predictions for Gravitational wave signals from PBH coalescence in cluster involve study of cluster evolution, which is now under way [61].

Being in cluster, PBHs with the masses of tens $M_{\odot}$ form binaries much easier, than in the case of their random distribution, as well as formation of such PBHs in collapse of first stars is rather problematic. In this aspect detection of signals from binary BH coalescence in the gravitational wave experiments [63–67] may be considered as a positive evidence for this scenario [35]. Repeatedly detected signals localized in the same place would provide successive support in its favor or exclusion [35,61,68]. The existing statistics is evidently not sufficient to make any definite conclusion on this possibility. However, repeating detection of four GW signals in the August of 2017 noted in GWTC catalog [69] may be an interesting hint to such a possibility [1,35].

Primordial black holes reflect strong inhomogeneity of the very early Universe. Their production in a significant amount is not a necessary consequence of all the models of very early Universe. However, it is just this model dependent character provides a very sensitive probe of BSM physics and the confirmation of PBH existence can severely tighten the class of possible realistic BSM models.

The same is true for the existence of antimatter objects in baryon asymmetric Universe, which can reflect strong nonhomogeneity of the baryosynthesis.

### 2.3.3 Antimatter and Baryon Asymmetry

The baryon asymmetry of the Universe is related with the evident dominance of matter over antimatter in the visible part of the Universe. The set of astrophysical data puts only constraints on the possible amount of macroscopic antimatter. However severe, these constraints still don't exclude completely the existence of antimatter objects, which can be formed in antimatter domains in baryon asymmetric Universe originated from the strongly nonhomogeneous baryosynthesis [70–76] (see [36,12,75] for review and references).

If created, antimatter domains should survive in the surrounding matter to the present time. It puts a lower limit on its size being in terms of its mass about $10^3 M_{odot}$ [72–74] that corresponds to a minimal mass of globular clusters. If antimatter globular cluster is formed in our Galaxy, it may be the source of cosmic ray antinuclei.

However exotic, the hypothesis on antimatter globular cluster in our Galaxy [72] doesn't contradict observations, if the mass of the cluster doesn't exceed the limit

$$M \leq 10^5 M_{odot}. \tag{2.12}$$

Indeed, globular clusters are an old population of the Galaxy being dominantly in halo, where matter gas density is low. Their gravitational potentials are not sufficient to hold matter, lost by stars by stellar winds or supernova explasions. In the case of antimatter cluster, it means that there is no antimatter gas within it and matter gas that enters the cluster annihilate only on antimatter stellar surfaces. Taking into account low density of matter gas in halo and relatively small surface on which it can annihilate, one can conclude that antimatter globular cluster is expected to be a rather faint gamma ray source. The upper limit (2.12) follows from the condition that the antimatter lost by antimatter stars and polluting the Galaxy doesn't cause overproduction of gamma ray background from annihilation with the matter gas [72–74].

It was noted in [72–74] that cosmic antihelium flux may be a profound signature for an antimatter globular cluster in our Galaxy. Symmetry in physics of matter and antimatter would make antihelium-4 the second by abundance element of antimatter. In addition to antihelium lost by antimatter stars its cosmic fluxes can increase due to destruction of heavier antinuclei in their annihilation with matter. Rough estimation of the expected antihelium flux as simply proportional to the ratio of the mass of antimatter cluster to the total mass of the Galaxy predicts that it should be within the reach by AMS02 experiment to 2024.

This prospect makes necessary to specify the predictions for the cosmic antihelium flux in more details and such analysis can be based on our knowledge of properties of globular clusters. However, one should take into account that antimatter stars may have properties much different from ordinary stars with correspondingly different observational signatures [77].

Possible detection of cosmic antihelium-3 nuclei by AMS02 experiment together with some detected events that may correspond to antihelium-4 cannot find natural astrophysical explanation [78] and may be strong evidence for existence of macroscopic forms of antimatter in our Galaxy.

## 2.4 Conclusions

Plethora of BSM physics involves, pending on the particular model, various combinations of its physical, astophysical and cosmological signatures. Such model dependent predictions lead to nontrivial cosmological features that can provide potentially observable deviations from the predictions of the standard cosmological model.

We have drawn special attention to some, at first glance exotic, predictions, like nuclear interacting dark atoms, massive PBH clusters or antimatter stars in our Galaxy, since they can explain the corresponding experimental anomalies, such as the puzzles of direct dark matter searches, origin of coalescensing massive black holes or experimental evidence for cosmic antihelium. If these explanations are confirmed, they strongly tighten the class of viable BSM models and add the corresponding nonstandard features to the cosmological scenario. Reminding Ya.B.Zeldovich, we can repeat after him that "though the probability for existence of these phenomena seems low, the expectation value of their discovery can be hardly overestimated".

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# 3 New Way of Second Quantized Theory of Fermions With Either Clifford or Grassmann Coordinates and *Spin-Charge-Family* Theory ⋆

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**Abstract.** Fermions with the internal degrees of freedom described in Clifford space carry in any dimension a half integer spin. There are two kinds of spins in Clifford space. The spin-charge-family theory ([1–7,9,8], and the references therein), assuming even $d = (13+1)$, uses one kind of spins to describe in $d = (3 + 1)$ spins and charges of quarks and leptons and antiquarks and antileptons, while the other kind is used to describe families.
In this work the new way of second quantization, suggested by the spin-charge-family theory, is presented. It is shown that the creation and annihilation operators of 1-fermion states, written as products of nilpotents and projectors of an odd Clifford character, fulfill the anticommutation relations as required in the second quantization procedure for fermions, what means that 1-fermion states are in Clifford space already second quantized, and that the creation operators for $n$-fermion second quantized vectors are products of one fermion creation operators, operating on the empty vacuum state. There is no need in this theory for the negative energy states fulfilled with fermions.
It is demonstrated that also in Grassmann space there exist the creation and annihilation operators of an odd Grassmann character, generating "fermions", which fulfill as well the anticommutation relations for fermions, representing correspondingly the second quantized 1-"fermion" states. However, while the internal spins determined by the generators of the Lorentz group of the Clifford objects of both kinds are half integer, the internal spins determined by the Grassmann objects are integer. Grassmann space offers no families.
We discuss the new second quantization procedure of the fields in both spaces. For the Grassmann case we present the action, the basic states, the solution of the "Weyl" equation for free massless "fermions" and the discrete symmetry operators. A short overview of the achievements of the spin-charge-family theory is done, and open problems of this theory still waiting to be solved are presented. We compare the Grassmann and the Clifford case in order to better understand to how many open questions in physics of elementary fermion and boson fields and in cosmology the spin-charge-family theory is able to answer.

**Povzetek.** Cliffordova algebra ponudi v vseh dimenzijah dva neodvisna vektorska prostora za opis fermionov. Teorija spinov-nabojev-družin ([1–7,9,8], in reference v teh člankih), ki predpostavi da ima prostor-čas $d \geq (13 + 1)$ dimenzij, uporabi eno vrsto spina za opis spina in nabojev kvarkov in leptonov in antikvarkov in antileptonov, drugo vrsto spina pa

⋆ Talk presented by N.S. Mankoč Borštnik

za opis družin.
Avtorica teorije spinov-nabojev-družin je dokazala, da vektorji, ki so lastni vektorji Cartanove podalgebre Lorentzove algebre in so produkt lihega stevila Cliffordovih operatorjev, izpolnjujejo vse lastnosti fermionov v drugi kvantizaciji. To pomeni, da opis fermionov v Cliffordovi algebri razloži Diracove postulate za drugo kvantizacijo fermionov. Kreacijski in anihilacijski operatorji, ki določajo v tej drugi kvantizaciji 1-fermionska stanja, zadostijo antikomutacijskim relacijam za drugo kvantizacijo fermionov, če jih zapišemo kot produkt niloptentov in projektorjev lihega števila Cliffordovih operatorjev. Kreacijski operatorji za $n$ fermionska stanja so v tej drugi kvantizaciji produkti enofermionskih kreacijskih operatorjev, ki delujejo na praznem vakuumskem stanju. V tej teoriji ni potrebe po negativnih energijskih stanjih zapolnjenih s fermioni.
Avtorja postavita zahtevo, ki ohrani le enega od obeh vektorskih prostorov, druga vrsta operatorjev pa poveže neodvisne nerazcepne upodobitve Lorentzove algebre v tem prostoru in jim " podeli" kvantno število "družin ". Tako omogoči Cliffordova algebra opis ne le spinov in nabojev kvarkov in leptonov in antikavarkov in antileptonov, ampak tudi njihovih družin.

Članek pokaže, da tudi Grassmannova algebra ponudi kreacijske in anihilacijske operatorje, ki zadoščajo antikomutacijskim relacijam za 1 fermionska stanja. Vendar so spini teh vektorjev celoštevilski. Grassmannov prostor ne ponudi družin.
Akcija in enečbe gibanja, ko so v Cliffordovi algebri poznani, za Grassmannov algebro pa članek predlaga akcijo in diskretne operatorje. Za obe algebri ponudi rešitve ustrezne "Weylove" enačbe za proste "fermione" brez mase in jih komentira. Avtorja ponudita tudi kratek pregled dosežkov teorije spinov-nabojev-družin in njenih odprtih problemov. Primerjava Grassmannovega in Cliffordovega primera osvetli mnoga odprta vprašanja fizike osnovnih fermionov in bozonov ter kozmologije.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmannspace, Kaluza-Klein-like theories, Discrete symmetries, Higher dimensional spaces, Beyond the standard model

## 3.1 Introduction

More than 50 years ago the *standard model* offered an elegant new step in understanding elementary fermion and boson fields by postulating:
**i.** Massless family members of coloured quarks and colouress leptons, the left handed members as the weak charged doublets and the weak chargeless right hand members, the left handed quarks distinguishing in the hyper charge from the left handed leptons, each right handed member having a different hyper charge. All fermion charges are in the fundamental representation of the corresponding groups. Antifermions carry the corresponding anticharges and opposite handedness. The existence of massless families to each family member is as well postulated. There is no right handed neutrino, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino.
**ii.** The existence of the massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the corresponding adjoint representations.

**iii.** The existence of a massive scalar Higgs, gaining at some step of the expanding universe the nonzero vacuum expectation value, causing masses of fermions and heavy bosons and the Yukawa couplings. The Higgs carry a half integer weak and hyper charge.
**iv.** Fermion and boson fields can be (second) quantized.

The *standard model* assumptions have in the literature several explanations, mostly with many new not explained assumptions. The most successful seem to be the grand unifying theories [12–28], if postulating in addition the family group and the corresponding gauge scalar fields.

The *spin-charge-family* theory, the project of N.S.M.B. [1–7,9,8,10], is offering the explanation for all the assumptions of the *standard model*, unifying not only charges, but also charges and spins and families, explaining the appearance of families, of the vector gauge fields, of the scalar field and the Yukawa couplings, offering the explanation for the matter-antimatter asymmetry, making several predictions. This theory also offers the explanation for the appearance of creation and annihilation operators, fulfilling the anticommutation relations for fermions,which in the Dirac theory [67] is just assumed.

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [29–36,8] due to the assumption that in $\mathrm{d} \geq 5$ (in the *spin-charge-family* theory $\mathrm{d} \geq (13+1)$) fermions interact with the gravity only. Correspondingly this theory shares with the Kaluza-Klein like theories their weak points, at least: **a.** Not yet solved the quantization problem of the gravitational field. **b.** Breaking spontaneously the starting symmetry, which would at low energies manifest the observed almost massless fermions [30]. Concerning this second point we proved on the toy model of $\mathrm{d} = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [68–70]. It remains to study how does appear the spontaneous breaking of the starting symmetry in $\mathrm{d} = (13+1)$, first with the appearance of the condensate of two right handed neutrinos, Table 3.3, Ref. [4], and then when scalar fields with space index $(7,8)$ obtain nonzero vacuum expectation values. (This second point is common to all the unifying theories.)

Since in $\mathrm{d} = (3+1)$-dimensional space — at low energies — the gauge gravitational fields manifest as the observed vector gauge fields [5], which can be quantized in the usual way, quantization procedure of gravity can wait to be made. The author is in mean time trying to find out (together with the collaborators) how far can the *spin-charge-family* theory — starting in $\mathrm{d} = (13+1)$-dimensional space with a simple and "elegant" action, Eq. (3.1) — reproduce in $\mathrm{d} = (3+1)$ the observed properties of quarks and leptons [3–7,9,8,10], the observed gauge fields, the assumed scalar field, the appearance of the dark matter and of the matter-antimatter asymmetry, as well as the other open questions, connecting elementary fermion and boson fields and cosmology. The work done so far seems promising.

Let us in what follows and in Subsect. 3.1.1 overview shortly the starting assumptions and so far achievements of the *spin-charge-family* theory, and discuss as well open problems.

The recognition that there are in Grassmann space two kinds of the Clifford algebra objects [2] ($\gamma^a$ and $\tilde{\gamma}^a$) enables that the *spin-charge-family* theory is explaining the origin of families [47–49,1,2], Table 3.1.

The assumption made in the *spin-charge-family* theory that the dimension of space is $\geq (13+1)$ enables the explanation for by the *standard model* assumed spins and charges of quarks and leptons [71,72], explaining as well the miraculous cancellation of triangle anomalies [8,9,4] in the *standard model*, however, without relating handedness and charges "by hand" as needed in $SO(10)$ [37–39].

Since there are in $SO(13+1)$ additional quantum numbers to those assumed by the *standard model*, the theory predicts that right handed neutrinos and left handed antineutrinos, carrying nonzero additional quantum numbers — $\tau^{23}$ and $\tau^4$ instead of $Y$ in the *standard model* ($Y = (\tau^{23} + \tau^4)$ in the *spin-charge-family* theory as presented in Table 3.6 and in Eqs. (3.111, 3.112, 3.113, 3.114)) — are regular members of families of quarks and leptons [71,72,3,9]. This prediction is common also to $SO(10)$ [37–39].

In the *spin-charge-family* theory spins and charges are described by the superposition of $S^{ab}$ $(= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, Eq. (3.2)), with $\gamma^a$ belonging to the first kind of the Clifford algebra objects and with $S^{mn}, (m,n) = (0,1,2,3)$, describing spins and handedness of quarks and leptons (Eq. (3.111)), and $S^{st}, (s,t) = (5,6,\cdots,14)$, describing their charges, Table 3.6, Eqs. (3.112, 3.113) and Refs. [2,47,49,72].

Family quantum numbers are determined by the second kind of the Clifford algebra objects, by the superposition of $\tilde{S}^{ab}$ $(= \frac{i}{4}\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a))$, Eq. (3.2), Table 3.1 [2,48].

The vector gauge fields, assumed in the *standard model* as the gauge fields of the corresponding fermion charges, are in the *spin-charge-family* theory explainable as the superposition of the gauge fields of the generators of the Lorentz transformations $S^{st}$ ($S^{st}\,\omega_{stm}, (s,t) = (5,6,\cdots,14)$, Eqs. (3.1, 3.9, 3.111)), with the vector index $m = (0,1,2,3)$ , Eq. (3.10), Ref. [5].

In the *standard model* the scalar fields appear as the Higgs scalar and the Yukawa couplings by the assumption. In the *spin-charge-family* theory both kinds of the gauge fields, $\sum_{s',t'} c^{s't'}\,\omega_{s't's}$, which are the gauge fields of $S^{st}$ with $(s',t') = (5,6,7,8)$, and $\sum_{a,b} \tilde{c}^{ab}\,\tilde{\omega}_{abs}$, which are the gauge fields of $\tilde{S}^{ab}$, with $(a,b) = (0,1,\cdots,8)$, both with the scalar index $s = (7,8)$, manifesting properties of the Higss scalar (by carrying weak and hyper charges in the "fundamental representation"), define masses of quarks and leptons and of heavy bosons, Eq. (3.10), Refs. [72,9,3].
These scalar fields determine in the *spin-charge-family* theory masses of the two groups of four families [51,53–56,3,9]. The lower group predicts the existence of the fourth family of quarks and leptons, coupled to the observed three families [51,53,56,54,70]. From the symmetry of the mass matrices predicted the $4\times 4$ mixing matrix of quarks [56] appear to be in better agreement with the experiments than if only three families are assumed [40].

The lowest family of the upper four families offers the explanation for the existence of the dark matter [54,61].

There are additional scalar fields in the *spin-charge-family* theory [4], having the scalar space index $t \in (9,10,\ldots,14)$. They carry colour charges in the "fundamental" representations, cause transitions of antileptons and antiquarks into quarks and back, enabling the decay of baryons. These scalar fields are offering in the presence of the right handed neutrino condensate, Table 3.3, Ref. [4], which breaks the $\mathcal{CP}$ symmetry, the answer to the question about the matter-antimatter asymmetry in the universe [4].

Authors of this paper proved on the toy model of $d = (5+1)$ that breaking the symmetry in Kaluza-Klein theories can lead to massless fermions [68–70]. The authors determine as well the discrete symmetries operators in observable dimensions $d = (3+1)$ for any d, Eqs. (3.94), Ref. [65].

The breaking of the starting symmetry $SO(13+1)$ is in the *spin-charge-family* theory triggered by the appearance of the condensate (Table 3.3) of the right handed neutrinos [4] and, like in the *standard model*, by the nonzero vacuum expectation values of the scalar fields with the space index $s = (7, 8)$.

In this paper it is demonstrated that the odd products of nilpotents and projectors, which are the "egienfunctions" of the Cartan subalgebra of the Lorentz algebra in Clifford space, and which solve the Weyl equations for free massless fermions, fulfill together with the corresponding Hermitian conjugated annihilation operators the anti-commutation relations as needed in the second quantized fermion fields [50]. No assumption of the Dirac kind about the creation and annihilation operators is needed.

The *spin-charge-family* theory has many common points with other unifying theories ([12–17,29–36] and other references), and because of that and because of the fact that by starting with the very simple action, Eq. (3.1), the theory is able to offer explanations for so many observed phenomena, built into assumptions of the *standard model*(s) of the elementary boson and fermion fields and also of cosmology, and also in other unifying theories, it might be that it is the right next step beyond the standard models.

The achievements of the *spin-charge-family* theory are discussed in more details in Subsect. 3.1.1. There also problems waiting to be solved are presented.

Let us present a very simple starting action of the *spin-charge-family* theory of N.S.M.B., in which massless fermions in $d = (13+1)$-dimensional space interact with massless bosons, that is only with gravity — the vielbeins $f^{\alpha}{}_{a}$ (the gauge fields of moments $p_a$) and the two kinds of the spin connections ($\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the Clifford algebra objects $\gamma^a$ and $\tilde{\gamma}^a$, respectively).

$$\begin{aligned}
\mathcal{A} = & \int d^d x\, E\, \frac{1}{2}\, (\bar{\psi}\, \gamma^a p_{0a} \psi) + \text{h.c.} + \\
& \int d^d x\, E\, (\alpha\, R + \tilde{\alpha}\, \tilde{R})\,,
\end{aligned} \tag{3.1}$$

with $p_{0a} = f^{\alpha}{}_{a} p_{0\alpha} + \frac{1}{2E}\{p_\alpha, E f^{\alpha}{}_{a}\}_-$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$ and $R = \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\ (\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^{c}{}_{b\beta})\} + \text{h.c.}$, $\tilde{R} = \frac{1}{2}\{f^{\alpha[a}f^{\beta b]}\ (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\,\tilde{\omega}^{c}{}_{b\beta})\} + \text{h.c.}$. Here [1] $f^{\alpha[a}f^{\beta b]} = f^{\alpha a}f^{\beta b} - f^{\alpha b}f^{\beta a}$.

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[1] $f^{\alpha}{}_{a}$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}{}_{b},\ e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while

The two kinds of the Clifford algebra objects, $\gamma^a$ and $\tilde{\gamma}^a$, Eq. (3.2), anticommute and determine the infinitesimal generators of the Lorentz transformations in the internal space of fermions — $S^{ab}$ for $SO(13,1)$, arranging states into representations (Table 3.6), and $\tilde{S}^{ab}$ for $\widetilde{SO}(13,1)$, arranging states into families (Table 3.1). Eq. (3.69) relates these two internal dgrees of freedom, keeping the relations of Eq. (3.2) unchanged.

$$
\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,, \\
&S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) \,, \\
&\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a) \,.
\end{aligned} \tag{3.2}
$$

The generators $S^{ab}$ are used in the *spin-charge-family* theory to determine spins and charges of spinors of any family, Table 3.6, another kind, $\tilde{S}^{ab}$, determines the family quantum numbers, Table 3.1. These two degrees of freedom are connected by the requirement, presented in Eq. (3.69).

The scalar curvatures $R$ and $\tilde{R}$ determine dynamics of the gauge fields — the spin connections and the vielbeins — manifesting in $d = (3+1)$ as all the known vector gauge fields, as well as the scalar fields [5], which offer the explanation for the appearance of the Higgs and the Yukawa couplings, of the ordinary matter-antimatter asymmetry [4] and the dark matter [54], provided that the symmetry breaks from the starting $SO(13,1)$ to $SO(3,1) \times SU(3) \times U(1)$.

In this paper we start to study the possibility that fermions are described in Grassmann space, in order to better understand how far can the simple starting action, Eq. (3.1), of the *spin-charge-family* theory agree with the at low energies observed properties of fermions and bosons.

We demonstrate in this paper that besides Clifford space also Grassmann space offers the description of the internal degrees of freedom of fermions in the second quantized procedure. In both cases there exist the creation and annihilation operators, which fulfill the anticommutation relations required for fermions, Eqs. (3.54, 3.81). But while the internal spins determined by the generators of the Lorentz group of the Clifford objects $S^{ab}$ and $\tilde{S}^{ab}$ — we repeat here that in the *spin-charge-family* theory $S^{ab}$ determine the spin degrees of freedom and $\tilde{S}^{ab}$ the family degrees of freedom — are half integer, the internal spin determined by $\mathbf{S}^{ab}$ (expressible with $S^{ab} + \tilde{S}^{ab}$) is integer.

Correspondingly Clifford space offers according to the *spin-charge family* theory the description of spins, charges and families, all in the fundamental representations of the subgroups of the Lorentz group $SO(d-1,1)$, while Grassmann space offers spins and charges in the adjoint representations of the subgroups

Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$ ), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

of the Lorentz group $SO(d-1,1)$ and no family degrees of freedom. Fermions with integer spins would lead to completely different nucleons, nuclei, atoms, molecules, matter than the so far observed ones.

Let us make a short introduction into the Grassmann space as well.

In Grassmann space the infinitesimal generators of the Lorentz transformations $\mathbf{S}^{ab}$ are expressible with anticommuting coordinates $\theta^a$ and their conjugate momenta $p^{\theta a} = i\frac{\partial}{\partial\theta_a}$ [2],

$$\begin{aligned}\{\theta^a,\theta^b\}_+ &= 0\,, \quad \{p^{\theta a},p^{\theta b}\}_+ = 0\,, \quad \{p^{\theta a},\theta^b\}_+ = i\eta^{ab}\,,\\ \mathbf{S}^{ab} &= \theta^a p^{\theta b} - \theta^b p^{\theta a}\,.\end{aligned} \tag{3.3}$$

Taking into account that $\gamma^a$ and $\tilde{\gamma}^a$, expressible in terms of $\theta^a$ and their conjugate momenta $p^{\theta a}$, anticommute [2],

$$\gamma^a = (\theta^a - i\,p^{\theta a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a + i\,p^{\theta a})\,, \tag{3.4}$$

one recognizes

$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,, \tag{3.5}$$

from where one concludes, if taking into account Eq. (3.1), that in the Grassmann case the covariant momenta $p_{0\alpha}$ are

$$p_{0\alpha} = p_\alpha - \frac{1}{2}\mathbf{S}^{ab}\Omega_{ab\alpha}\,, \tag{3.6}$$

with $\Omega_{ab\alpha}$ as the only kind of the connection fields (instead of the two kinds in the Clifford case — $\omega_{ab\alpha}$, which is the gauge fields of $S^{ab}$ and $\tilde{\omega}_{ab\alpha}$, which is the gauge fields of $\tilde{S}^{ab}$).

Let us point out that Eq. (3.69) relates the two anticommuting degrees of freedom, $\{\gamma^a,\tilde{\gamma}^b\}_+ = 0$, making a choice of $\gamma^a$ to detremine the internal degrees of freedom in Clifford space, while keeping all the relation of Eq. (3.2) unchanged.

It follows for $\mathbf{S}^{ab}$

$$\begin{aligned}\{\mathbf{S}^{ab},\mathbf{S}^{cd}\}_- &= i\{\mathbf{S}^{ad}\eta^{bc} + \mathbf{S}^{bc}\eta^{ad} - \mathbf{S}^{ac}\eta^{bd} - \mathbf{S}^{bd}\eta^{ac}\}\,,\\ \mathbf{S}^{ab\dagger} &= \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,.\end{aligned} \tag{3.7}$$

The same relations are true also if $\mathbf{S}^{ab}$ is replaced with either $S^{ab}$ or $\tilde{S}^{ab}$. These infinitesimal generators of the Lorentz group — the two kinds of the Clifford operators and the Grassmann operators — operate as follows

$$\begin{aligned}\{S^{ab},\,\gamma^e\}_- &= -i\,(\eta^{ae}\,\gamma^b - \eta^{be}\,\gamma^a)\,,\\ \{\tilde{S}^{ab},\,\tilde{\gamma}^e\}_- &= -i\,(\eta^{ae}\,\tilde{\gamma}^b - \eta^{be}\,\tilde{\gamma}^a)\,,\\ \{S^{ab},\,\tilde{S}^{cd}\}_- &= 0\,,\\ \{\mathbf{S}^{ab},\,\theta^e\}_- &= -i\,(\eta^{ae}\,\theta^b - \eta^{be}\,\theta^a)\,,\\ \{\mathbf{S}^{ab},\,p^{\theta e}\}_- &= -i\,(\eta^{ae}\,p^{\theta b} - \eta^{be}\,p^{\theta a})\,,\\ \{\mathbf{M}^{ab},\,A^{d\ldots e\ldots g}\}_- &= -i\,(\eta^{ae}\,A^{d\ldots b\ldots g} - \eta^{be}\,A^{d\ldots a\ldots g})\,,\end{aligned} \tag{3.8}$$

where $\mathbf{M}^{ab}$ are defined in the Clifford case by the sum of $L^{ab}$ plus either $S^{ab}$ (if $\gamma^a$'s are chosen to describe the basis, otherwise $\tilde{S}^{ab}$ replace $S^{ab}$), while in the Grassmann case $\mathbf{M}^{ab}$ is $L^{ab} + \mathbf{S}^{ab}$ (which is, Eq. (3.5), $\mathbf{M}^{ab} = L^{ab} + S^{ab} + \tilde{S}^{ab}$).

In Sect. 3.2 the actions and norms for free massless fermions, with the internal degrees of freedom described in Clifford and in Grassmann space in d-dimensional spaces are presented. The discrete symmetry operators in d-dimensional space — Clifford and Grassmann — and their manifestation in $d = (3 + 1)$-dimensional space are presented in Subsect. 3.3.3 of Sect. 3.3. While the action and the discrete symmetry operators in Clifford space are known from before [9,65], the action in Grassmann space as well as the discrete symmetry operators are here assumed by N.S.M.B..

The new way of second quantization of fermion fields in both spaces is discussed in Sect. 3.3. We treat in both spaces only massless free particles. Sect. 3.4 presents what we learn from this work.

This work is a part of the project of both authors, which includes the *fermionization* procedure of boson fields (or the *bosonization* procedure of fermion fields), discussed in Refs. [42,43,45] for any dimension d (by the authors of this contribution, while one of them, H.B.F.N. [44], has succeeded with another author to do the *fermionization* for $d = (1 + 1)$), and which would hopefully also help to understand a little better the content and dynamics of our universe.

### 3.1.1 Comments on the achievements of the *spin-charge-family* theory so far and the open questions to be solved

Let us illustrate the achievements of the *spin-charge-family* theory, presented in the introduction, by adding some comments.

**I.** In the action, Eq. (3.1), fermions carry in $d = (13 + 1)$ two kinds of spins — no charges and interact with gravity only — with the vielbeins $f^{\alpha}{}_a$ and the two kinds of the spin connection fields, the gauge fields of $S^{ab}$ — $\omega_{ab\alpha}$ — and the gauge fields of $\tilde{S}^{ab}$ — $\tilde{\omega}_{ab\alpha}$.

One can formally rewrite the fermion part of the action so that it manifests in $d = (3 + 1)$ the free massless fermion part (first line in Eq. (3.9)), the interaction of fermions with the vector gauge fields (the second line in Eq. (3.9)), the interaction of fermions with the scalar fields (the third line in Eq. (3.9)), and the rest.

$$
\begin{aligned}
\mathcal{L}_f = & \sum_m \bar{\psi}\gamma^m p_m \psi \\
& - \sum_{A,i} \bar{\psi}\,\gamma^m \tau^{Ai} A^{Ai}_m\,\psi + \\
& + \sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\;\psi \\
& + \sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\;\psi\,,
\end{aligned}
\tag{3.9}
$$

with $\tau^{Ai} = \sum_{st} c_{st}{}^{Ai} S^{st}$, $(s,t) = (5, 6, \cdots, 13, 14)$, which are generators of the subgroups of $SO(13,1)$, determining charges of fermions, Eq. (3.112, 3.113, 3.114), with $A^{Ai}_{m}$, which are the corresponding superposition of $\omega_{stm}$ ([4,9] and the references therein), $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$ and $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, while $m \in (0,1,2,3)$, $s \in (7,8)$, $(s', s") \in (5,6,7,8)$, $(a,b)$ (appearing in $\tilde{S}^{ab}$) run within $(0,1,2,3)$ and $(5,6,7,8)$, $t \in (5,6,9,\ldots,13,14)$, $(t', t") \in (5,6,7,8)$ and $\in (9,10,\ldots,14)$.

**I.i** The spinor function $\psi$ represents all the family members, $2^{\frac{d}{2}-1} = 64$ for $d = 13+1$, of all the $2^{\frac{7+1}{2}-1} = 8$ families, including fermions and antifermions. Tables 3.6 and 3.1 represent the creation operators for the states of one family and the creation operators for the eight families, respectively. The rest of families are assumed to have very large masses as discussed and proved for a toy model in Ref. [68–70,73]. The creation operators operate on a vacuum state, Eq. (3.79).

**I. A.** The Clifford object $\gamma^a$ are in the *spin-charge-family* theory used to determine from the point of view of $d = (3+1)$ spins and all the charges of fermions.

**I. A.i.** $d = (13+1)$-dimensional space offers $2^{\frac{d}{2}-1} = 64$ members of $SO(13,1)$. In Table 3.6 the properties of quarks and leptons and antiquarks and antileptons, forming 64 members, are presented from the point of view of subgroups of $SO(13,1)$ breaking first into $SO(7,1) \times SU(3) \times U(1)$, keeping connection between handedness and the two $SU(2)_{I,II}$ charges, and further to — $SU(2)_R \times SU(2)_L \times SU(2)_I \times SU(2)_{II} \times SU(3) \times U(1)$ — representing in $d = (3+1)$ the spin and handedness, the weak charge $\tau^{13}$ of $SU(2)_I$, the second $\tau^{23}$ of $SU(2)_{II}$, the colour charge $\tau^{33}$ and $\tau^{38}$ of $SU(3)$ and $\tau^4$ of $U(1)$ *for quarks and leptons and for antiquarks and antileptons*.
Cartan subalgebra has $\frac{d}{2} = 7$ members, the *standard model* assumes one commuting operator less.

**I. A.ii.** Due to the additional commuting operator (the member of the Cartan subalgebra of $S^{ab}$) in the *spin-charge-family* theory, the neutrinos become a regular members of quarks and leptons, with masses determined by the interaction with the scalar fields as all the rest of family members [51,53–56,3,9] (in Eq. (3.9) the interaction of fermions with the scalar fields is contained in the third line). This is the case also in $SO(10)$ theories [12–15]. The difference in the *spin-charge-family* theory is, that spin and handedness are correlated with charges, while in $SO(10)$ this is not the case (and must be correlated by "hand"). This fact is discussed in details in Ref. [8].

Let us point out that colour chargeless leptons and quarks of any of the three colours have completely the same $SO(7,1)$ part. Quarks and leptons distinguish only in the $SU(3) \times U(1)$ part.

**I. B.** The second Clifford object $\tilde{\gamma}^a$ offers the explanation for the existence of families.

**I. B.i.** There are twice four families of quarks and leptons in the *spin-charge-family* theory ([3] and the references therein) after the appearance of the condensate of the two right handed neutrinos, presented in Table 3.3, Ref. [4]. Since we have not really shown yet how this dynamically happens (we did this so far only for the toy model [68–70]), this remains as an open problem. All eight families obtain masses when the scalar gauge fields with the space index (7,8) — third line in Eq. (3.9) — gain nonzero vacuum expectation values at the electroweak phase transition. Table 3.1 represents in the left column eight families of creation operators of $\hat{u}_R^{c1\dagger}$ — the first member in Table 3.6 — and of chargeless $\hat{\nu}_R^{\dagger}$ — the $25^{th}$ member in Table 3.6. ($S^{11\,12}$, for example, transforms $\hat{u}_R^{ci\dagger}$ into $\hat{\nu}_R^{\dagger}$ and opposite).

**I. B.ii.** The eight-plets separate into two groups of four families: One group contains doublets with respect to $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$, these families are singlets with respect to $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$. Another group of families contains doublets with respect to $\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^1$, these families are singlets with respect to $\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^2$. Mass matrices of both groups manifest correspondingly, when the scalar fields — the gauge fields of ($\vec{\tilde{N}}_R$, $\vec{\tilde{\tau}}^2$, $U(1)$) and ($\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^1$, $U(1)$) — obtain nonzero vacuum expectation values. Correspondingly both groups manifest $SU(2) \times SU(2) \times U(1)$ symmetry, with the same $U(1)$ and two different $SU(2)_{(L,R)} \times SU(2)_{(I,II)}$ symmetries, Ref. [57].

To the lower four families the observed three families of quarks and leptons contribute [51–53,55,56,58]. By the *spin-charge-family* theory predicted $SU(2) \times SU(2) \times U(1)$ symmetry of mass matrices, which limits the number of free parameters of mass matrices, the properties of the fourth family could be predicted by fitting free parameters to the experimental data. However, the accuracy of the so far measured $3 \times 3$ mixing (sub)matrices are even for quarks far from the required precision, which would enable prediction of masses of the fourth family members [55,56]. We predict for the assumed masses of the fourth family of quarks the corresponding matrix elements. Calculations show [56] that the larger the masses of the fourth family — up to 6 TeV seems to be allowed by experiments [40] — the smaller the unwanted mixing elements which could cause the flavour-changing neutral currents and the better is agreement with the experimental data, which require, due to the observations in Refs. [40,41], that there should be the fourth family due to the nonunitarity of the $3 \times 3$ so far measured mixing matrix for quarks and that the $4 \times 4$ mixing matrix elements should have the properties: $V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$. Here $u_i, d_i, i = 1, 2, 3, 4$ represent $u, c, t, u_4$ and $d, s, b, d_4$ quarks.

The lowest of the upper four families is, as evaluated in Refs. [54,61], the candidate, which can explain (or at least can contribute to) the appearance of the dark matter in the universe. Comparing the results from following the fifth family members in the expanding universe with the astrophysical observations of dark matter and the direct measurements of the dark matter, the predicted masses of the fifth family quarks would be $10^2$ TeV $< m_{q_5} c^2 < 4 \cdot 10^2$ TeV, and the scattering cross section $\sigma$ for the fifth family neutron at least $10^{-6}\times$ smaller than the cross section for the first family neutron. These values change if the fifth family neutron is not the only source of the dark matter.

The fifth family would correspondingly manifest completely different "nuclear force" than the members of the lower four families [54], leading to different atoms and molecules, if they would success to form a matter in the expanding universe.

**II.** The gauge fields — the vielbeins, $f^{\alpha}{}_{a}$, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, of Eq. (3.1), appearing in the $2^{nd}, 3^{rd}$ and $4^{th}$ lines in Eq. (3.9) — manifest in $d = (3 + 1)$ as the vector gauge fields of $\vec{\tau}^3$,

| | | | | | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{N}^3_L$ | $\tilde{N}^3_R$ | $\tilde{\tau}^4$ |
|---|---|---|---|---|---|---|---|---|---|
| I | $\hat{u}^{c1\dagger}_{R\,1}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\vert\overset{56}{[+]}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,1}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\vert\overset{56}{[+]}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $\hat{u}^{c1\dagger}_{R\,2}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\vert\overset{56}{[+]}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,2}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\vert\overset{56}{[+]}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $\hat{u}^{c1\dagger}_{R\,3}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\vert\overset{56}{(+)}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,3}$ | $\overset{03}{(+i)}\,\overset{12}{[+]}\vert\overset{56}{(+)}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| I | $\hat{u}^{c1\dagger}_{R\,4}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\vert\overset{56}{(+)}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,4}$ | $\overset{03}{[+i]}\,\overset{12}{(+)}\vert\overset{56}{(+)}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ |
| II | $\hat{u}^{c1\dagger}_{R\,5}$ | $\overset{03}{[+i]}\,\overset{12}{[+]}\vert\overset{56}{[+]}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,5}$ | $\overset{03}{[+i]}\,\overset{12}{[+]}\vert\overset{56}{[+]}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $0$ | $-\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $\hat{u}^{c1\dagger}_{R\,6}$ | $\overset{03}{(+i)}\,\overset{12}{(+)}\vert\overset{56}{[+]}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,6}$ | $\overset{03}{(+i)}\,\overset{12}{(+)}\vert\overset{56}{[+]}\,\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $0$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $\hat{u}^{c1\dagger}_{R\,7}$ | $\overset{03}{[+i]}\,\overset{12}{[+]}\vert\overset{56}{(+)}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,7}$ | $\overset{03}{[+i]}\,\overset{12}{[+]}\vert\overset{56}{(+)}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| II | $\hat{u}^{c1\dagger}_{R\,8}$ | $\overset{03}{(+i)}\,\overset{12}{(+)}\vert\overset{56}{(+)}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{(-)}\,\overset{13\,14}{(-)}$ | $\hat{\nu}^{\dagger}_{R\,8}$ | $\overset{03}{(+i)}\,\overset{12}{(+)}\vert\overset{56}{(+)}\,\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\,\overset{11\,12}{[+]}\,\overset{13\,14}{[+]}$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $-\frac{1}{3}$ |

**Table 3.1.** Eight families of creation operators of $\hat{u}^{c1\dagger}_R$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33}=1/2$, $\tau^{38}=1/(2\sqrt{3})$), appearing in the first line of Table 3.6 — and of the colourless right handed neutrino $\hat{\nu}^{\dagger}_R$ — of spin $\frac{1}{2}$, appearing in the $25^{th}$ line of Table 3.6 — are presented in the left and in the right column, respectively. Table is taken from [9]. Families belong to two groups of four families, one (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eq. (3.111). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (3.129). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$) (Eq. (3.129)) transform $\hat{u}^{\dagger}_{1R}$ to all the members of one family of the same colour. The same generators transform equivalently the right handed neutrino $\hat{\nu}^{\dagger}_{1R}$ to all the colourless members of the same family.

Eq .(3.113), $\tau^4$, Eq. (3.113), $\vec{\tau}^1$, Eq. (3.112), and $\vec{\tau}^2$, Eq. (3.112), if the space index is $m = (0,1,2,3)$ ($2^{nd}$ line in Eq. (3.9)), as well as the scalar gauge fields, if the space index is $s \geq 5$ ($3^{rd}$ and $4^{th}$ line in Eq. (3.9)), of the same operators as in the vector gauge fields case, Ref. [5].

Only if there are no fermion present, then both, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, are uniquely expressed by vielbeins, Ref. ([9], Eq. (C9)).

$$\begin{aligned}
\omega_{ab\alpha} = \tilde{\omega}_{ab\alpha} = &-\frac{1}{2E}\Big\{e_{e\alpha}e_{b\gamma}\,\partial_\beta(Ef^{\gamma[e}f^{\beta}{}_{a]}) + e_{e\alpha}e_{a\gamma}\,\partial_\beta(Ef^{\gamma}{}_{[b}f^{\beta e]}) \\
&\qquad - e_{e\alpha}e^{e}{}_{\gamma}\,\partial_\beta\left(Ef^{\gamma}{}_{[a}f^{\beta}{}_{b]}\right)\Big\} \\
&-\frac{1}{d-2}\Big\{e_{a\alpha}\frac{1}{E}e^{d}{}_{\gamma}\partial_\beta\,\left(Ef^{\gamma}{}_{[d}f^{\beta}{}_{b]}\right) \\
&\qquad - e_{b\alpha}\frac{1}{E}e^{d}{}_{\gamma}\partial_\beta\,\left(Ef^{\gamma}{}_{[d}f^{\beta}{}_{a]}\right)\Big\}\,. 
\end{aligned} \tag{3.10}$$

**II. A.** It is proven in Ref. [5] that the vector (as well as the scalar gauge fields) can indeed be expressed with the spin connections (rather than with the vielbeins),

$$A^{Ai}_{m} = \sum_{s,t} c^{Ai}{}_{st}\,\omega^{st}{}_{m}, \tag{3.11}$$

demonstrating the symmetry of space with $(s,t) \geq 5$, making the *spin-charge-family* theory transparent and correspondingly "elegant", so that it is easier to recognize that the origin of charges of the observed fermions, vector gauge fields, Higgs's scalar and Yukawa couplings might really be in $(d-4)$ space.

In the presence of the condensate, Table 3.3, of the right handed neutrinos, all the vector gauge fields and the scalar gauge fields, which interact with the condensate, gain masses. Only the weak ($SU(2)_I$), the colour ($SU(3)$) and the hyper ($U(1)$, $Y = \tau^4 + \tau^{23}$) gauge fields, which do not interact with the condensate, remain massless.

**II. A.i.** The weak vector gauge fields $\vec{A}^1_m$ , the gauge field of $SU(2)_I$, and $\vec{A}^2_m$, the gauge fields of $SU(2)_{II}$, are the superposition of gauge fields $\omega_{s't's}$ (Ref. [9], Eqs. (8,9,10)),

$$\begin{aligned}
\vec{A}^1_m &= (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m})\,, \\
\vec{A}^2_m &= (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m})\,.
\end{aligned} \tag{3.12}$$

Taking into account Eq. (3.113) one easily finds the colour vector gauge field expressed with $\omega_{stm}$. $\vec{A}^2_m$ get masses by interaction with the condensate.

In Ref. [5], Eqs. (24-25), the reader can find Lagrange density for the $R^{(d-4)}$ part of the gravity field $R$, Eq.(3.1), expressed by the vector gauge fields $\vec{A}^A_m$.

**II. B.** The scalar gauge fields are the superposition of either $\omega_{s't's}$, with $(s',t',s) = (5,6,\cdots,14)$, Ref. [5], or $\tilde{\omega}_{abs}$, with $(a,b) = (0,1,\cdots,8)$ and $(s) = (5,6,7,8)$, Refs. [4,7,9], the fourth line in Eq. (3.9).

Both kinds of scalar fields with $s = (7,8)$ contribute to the masses of the two groups of four families. Scalar fields $\omega_{s's's}$, with $(s',t') = (5,6,\cdots,14)$, $s = (9,10,\cdots,14)$ contribute to matter-antimatter asymmetry and to proton decay [4].

**II. B.i.** In the *spin-charge-family* theory the scalar fields with the space index $s = (7,8)$ carry with respect to this space index the weak charge and the hyper charge $(\mp\frac{1}{2},\pm\frac{1}{2})$, respectively, independent of whether they are superposition of $\omega_{s't's}$ or of $\tilde{\omega}_{abs}$, $s=(7,8)$, Refs. [9,3,4].

There are twice two triplets, the superposition of $\tilde{\omega}_{abs}$, Eqs. (3.111, 3.112) with $S^{ab}$ replaced by $\tilde{S}^{ab}$, the gauge scalar fields of either the group $\widetilde{SU}(2)_{\widetilde{SO}(3,1)_L} \times \widetilde{SU}(2)_I$ or of the group $\widetilde{SU}(2)_{\widetilde{SO}(3,1)_R} \times \widetilde{SU}(2)_{II}$, the first two triplets interacting with one group of four families, the second two triplets interacting with another group of four families, both groups presented in Table 3.1. There are also three singlets, the gauge scalar fields of $= (Q,Q',Y')$, Eq. (3.114), which are the superposition of $\omega_{s't's}$ and interact with members of all the eight families of Table 3.1 [7,9,3,4].

Let us use a common notation $A^{Ai}_s$ for all the scalar fields, independently of whether they originate in $\tilde{\omega}_{abs}$ or $\omega_{abs}$, $s=(7,8)$,

$$
\begin{aligned}
&A^{Ai}_s \in (\,A^{Q}_s\,, A^{Q'}_s\,, A^{Y'}_s\,, \vec{\tilde{A}}^{\tilde{1}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s\,, \vec{\tilde{A}}^{\tilde{2}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s\,)\,,\\
&\tau^{Ai} \supset (Q,\, Q',\, Y',\, \vec{\tilde{\tau}}^{1},\, \vec{\tilde{N}}_L,\, \vec{\tilde{\tau}}^{2},\, \vec{\tilde{N}}_R)\,.
\end{aligned}
\tag{3.13}
$$

Here $\tau^{Ai}$ represent the operators of the groups the gauge scalar fields of which are $A^{Ai}_s$.

Let us rewrite the third line in Eq. (3.9) as follows, Ref. ([9], Eqs. (18-19)).

$$
\begin{aligned}
&\sum_{s=(7,8),A,i} \bar{\psi}\,\gamma^{s}\,(-\tau^{Ai}\,A^{Ai}_s\,)\,\psi =\\
&\sum_{A,i} -\bar{\psi}\{\stackrel{78}{(+)}\;\tau^{Ai}\,(A^{Ai}_7 - i\,A^{Ai}_8) + \stackrel{78}{(-)}\;(\tau^{Ai}\,(A^{Ai}_7 + i\,A^{Ai}_8)\}\psi\,,\\
&\stackrel{78}{(\pm)} = \frac{1}{2}\,(\gamma^7 \pm i\gamma^8\,)\,, \quad A^{Ai}_{\stackrel{78}{(\pm)}} := (A^{Ai}_7 \mp i\,A^{Ai}_8)\,,
\end{aligned}
\tag{3.14}
$$

with the summation over $A, i$ performed, since $A^{Ai}_s$ represent the scalar fields ($A^{Q}_s$, $A^{Q'}_s$, $A^{Y'}_s$, $\tilde{A}^{\tilde{4}}_s$, $\vec{\tilde{A}}^{\tilde{1}}_s$, $\vec{\tilde{A}}^{\tilde{2}}_s$, $\vec{\tilde{A}}^{\tilde{N}_R}_s$ and $\vec{\tilde{A}}^{\tilde{N}_L}_s$). In the low energy regime the momentum $p_s$, $s=(7,8)$ can be neglected.

Taking into account that $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, $Y = (\tau^{23} + \tau^{4})$, $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, while $\tau^{4} = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, and $S^{ab}A_c = i(A^a\delta^b_c - A^b\delta^a_c)$, one finds

$$
\begin{aligned}
\tau^{13}\,(A^{Ai}_7 \mp i\,A^{Ai}_8) &= \pm\,\frac{1}{2}\,(A^{Ai}_7 \mp i\,A^{Ai}_8)\,,\\
Y\,(A^{Ai}_7 \mp i\,A^{Ai}_8) &= \mp\,\frac{1}{2}\,(A^{Ai}_7 \mp i\,A^{Ai}_8)\,,\\
Q\,(A^{Ai}_7 \mp i\,A^{Ai}_8) &= 0\,.
\end{aligned}
\tag{3.15}
$$

This are quantum numbers of the by the *standard model* assumed Higgs. These scalar gauge fields with the space index $(7,8)$, gaining nonzero vacuum expectation values (by assumption as in the *standard model* so far), cause the electroweak break, breaking the weak and the hyper charge, explaining the appearance of in the *standard model* assumed Higgs and the Yukawa couplings, predicting the existence of several scalars — two triplets and three singlets, which couple to the lower four families, making them massive and giving masses to weak bosons.

These scalar fields manifest the $SU(2)\times SU(2)\times U(1)$ symmetry, which reduces the number of free parameters in mass matrices of quarks and leptons, enabling predictions of properties of the four families [55–57].

**II. B.ii.** The scalar fields with the space index $s = (9, 10, \cdots, 14)$, presented in Table 3.2, carry triplet or antitriplet colour charges and the "spinor" charge equal to twice the quark or antiquark "spinor" charge, and the fractional hyper and electromagnetic charge.

They carry in addition the quantum numbers of the adjoint representations originating in $S^{ab}$ or in $\tilde{S}^{ab}$. (Although carrying the colour charge of one of the triplet or antitriplet quantum numbers, these fields can not be interpreted as superpartners of the quarks, since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.)

Let us have a look what do the scalar fields, appearing in the fourth line of Eq. (3.9) and in the seventh line of Table 3.2, do when applying on the left handed members of the Weyl representation presented on Table 3.6, containing quarks and leptons and antiquarks and antileptons [71,72,65].

Fig. 3.1 presents the creation of proton due to the interaction of quarks and leptons with these scalar fields. One can read on this Fig. 3.1 all the quantum numbers of a positron ($57^{th}$ line of Table 3.6), an antiquark ($43^{rd}$ line of Table 3.6), and a quark ($9^{th}$ line of Table 3.6), as well as of the scalar field $A^{2\boxminus}_{\substack{9\,10\\(+)}}$, seventh line of Table 3.2, involved in the proton birth. The opposite transition at low energies would make the proton decay.

After the appearance of the condensate of the two right handed neutrinos, Table 3.3, the discrete symmetry $\mathbb{C}_N\mathcal{P}_N$ is obviously broken. In the expanding universe, fulfilling the Sakharov request for appropriate non-thermal equilibrium, the triplet scalars from Table 3.2 have a chance to explain the matter-antimatter asymmetry in the universe [4].

**III.** The *spin-charge-family* theory suggests two kinds of phase transitions — two kinds of breaking symmetries: The appearance of the condensate and the nonzero vacuum expectation values of the scalar fields with the space index $s = (7, 8)$.

**III. A.** Table 3.3 represents the properties of the condensate of the two right handed neutrinos $\hat{\nu}^{\dagger}_{R8}$ — Table 3.1 — of spin up and spin down, breaking the discrete $\mathbb{C}_N\mathcal{P}_N$ symmetry Subsect. 3.3.3, [4,65].

Due to the interaction with the condensate of Table 3.3 the gauge vector fields of $\vec{\tau}^2$ and $\tau^4$ become massive. The colour vector gauge fields of $\vec{\tau}^3$, the weak vector gauge fields of $\vec{\tau}^1$ and the hyper vector gauge field of $Y$ do not interact with the condensate (the corresponding quantum numbers of the condensate are zero) and correspondingly remain massless, the gravity in $d = (3 + 1)$, which is the gauge field of $S^{mn}$ and $p_m$, remains massless as well.

| field | prop. | $\tau^4$ | $\tau^{13}$ | $\tau^{23}$ | $(\tau^{33}, \tau^{38})$ | Y | Q | $\tilde{\tau}^4$ | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{N}^3_L$ | $\tilde{N}^3_R$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A^{1\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | $\boxed{\pm}\ 1$ | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3} + \boxed{\mp}\ 1$ | 0 | 0 | 0 | 0 | 0 |
| $A^{13}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $A^{1\boxed{\pm}}_{11\,12}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | $\boxed{\mp}\ 1$ | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3} + \boxed{\mp}\ 1$ | 0 | 0 | 0 | 0 | 0 |
| $A^{13}_{11\,12}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $A^{1\boxed{\pm}}_{13\,14}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | $\boxed{\mp}\ 1$ | 0 | $(0, \oplus\ \frac{1}{\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3} + \boxed{\mp}\ 1$ | 0 | 0 | 0 | 0 | 0 |
| $A^{13}_{13\,14}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(0, \oplus\ \frac{1}{\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $A^{2\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | $\boxed{\pm}\ 1$ | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3} + \boxed{\mp}\ 1$ | $\oplus\ \frac{1}{3} + \boxed{\mp}\ 1$ | 0 | 0 | 0 | 0 | 0 |
| $A^{23}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $\tilde{A}^{1\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | $\boxed{\pm}\ 1$ | 0 | 0 | 0 |
| $\tilde{A}^{13}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $\tilde{A}^{2\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | $\boxed{\pm}\ 1$ | 0 | 0 |
| $\tilde{A}^{23}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $\tilde{A}^{N_L\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | $\boxed{\pm}\ 1$ | 0 |
| $\tilde{A}^{N_L 3}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $\tilde{A}^{N_R\boxed{\pm}}_{9\,10}(\oplus)$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | $\boxed{\pm}\ 1$ |
| $\tilde{A}^{N_R 3}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $A^{3i}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\boxed{\pm}\ 1 + \oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |
| $A^{4}_{9\,10}(\oplus)$ $\cdots$ | scalar | $\oplus\ \frac{1}{3}$ | 0 | 0 | $(\oplus\ \frac{1}{2}, \oplus\ \frac{1}{2\sqrt{3}})$ | $\oplus\ \frac{1}{3}$ | $\oplus\ \frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 |

**Table 3.2.** Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in the fourth line of Eq. (3.9), are presented. To the colour charge of all these scalar fields the space degrees of freedom — the space index — contribute one of the triplets or antitriplet values. These scalars are with respect to the two $SU(2)$ charges, ($\vec{\tau}^1$ and $\vec{\tau}^2$), and the two $\widetilde{SU}(2)$ charges, ($\vec{\tilde{\tau}}^1$ and $\vec{\tilde{\tau}}^2$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "spinor" number ($\tau^4$) of the quarks or antiquarks. The quantum numbers of the two vector gauge fields, the colour and the $U(1)_{II}$ ones, are added. These Table is taken from Ref. [4], Table I. We invite the reader to visit Ref. [4] for more details.

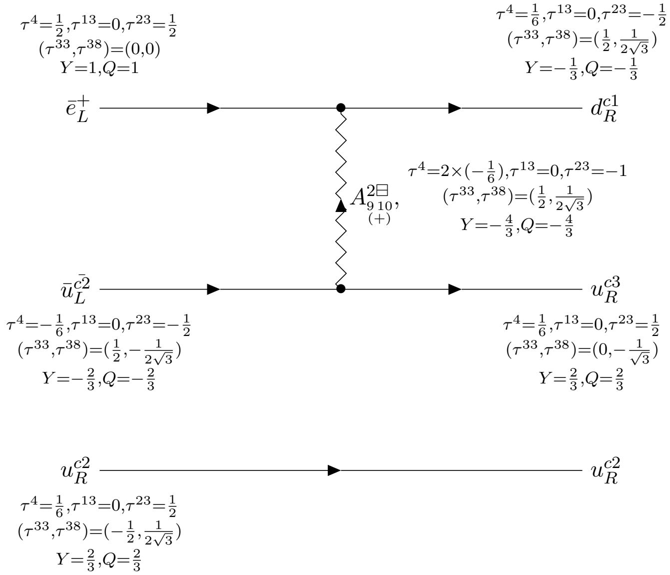


**Fig. 3.1.** The birth of a "right handed proton" out of a positron $\bar{e}_L^+$, antiquark $\bar{u}_L^{\bar{c2}}$ and quark (spectator) $u_R^{c2}$. The family quantum number can be any.

| state | $S^{03}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^4$ | $Y$ | $Q$ | $\tilde{\tau}^{13}$ | $\tilde{\tau}^{23}$ | $\tilde{\tau}^4$ | $\tilde{Y}$ | $\tilde{Q}$ | $\tilde{N}_L^3$ | $\tilde{N}_R^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(\lvert\nu^{\mathbf{VIII}}_{\mathbf{1R}}>_{\mathbf{1}}\ \lvert\nu^{\mathbf{VIII}}_{\mathbf{2R}}>_{\mathbf{2}})$ | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |
| $(\lvert\nu^{VIII}_{1R}>_1\ \lvert e^{VIII}_{2R}>_2)$ | 0 | 0 | 0 | 0 | −1 | −1 | −1 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |
| $(\lvert e^{VIII}_{1R}>_1\ \lvert e^{VIII}_{2R}>_2)$ | 0 | 0 | 0 | −1 | −1 | −2 | −2 | 0 | 1 | −1 | 0 | 0 | 0 | 1 |

**Table 3.3.** The condensate of the two right handed neutrinos $\nu_R$, with the quantum numbers of the $\mathrm{VIII}^{th}$ family, coupled to spin zero and belonging to a triplet with respect to the generators $\tau^{2i}$, is presented, together with its two partners. The condensate carries $\vec{\tau}^1 = 0$, $\tau^{23} = 1$, $\tau^4 = -1$ and $Q = 0 = Y$. The triplet carries $\tilde{\tau}^4 = -1$, $\tilde{\tau}^{23} = 1$ and $\tilde{N}_R^3 = 1$, $\tilde{N}_L^3 = 0$, $\tilde{Y} = 0$, $\tilde{Q} = 0$. The family quantum numbers of quarks and leptons are presented in Table 3.1.

Due to nonzero family quantum numbers of the condensate the corresponding scalar gauge fields become massive. The condensate gives masses to all the scalars from Table 3.2, either because they couple to the condensate due to $\tau^4$ or $\tilde{\tau}^4$ or $\tau^{23}$ or $\tilde{\tau}^{23}$ quantum numbers. It gives masses also to all the scalar fields with $s \in (5,6,7,8)$, since they couple to the condensate due to the nonzero $\tau^{23}$. The scalar fields with the quantum numbers of the upper four families couple in addition through their family quantum numbers.

**III. B.** The electroweak phase transition is caused by the nonzero vacuum expectation values of twice two triplets and three singlet scalars, giving masses to the lower fourth families — two of twice two triplets and three singlets — and to the upper four families — another two triplets and the same three singlets.

**IV.** Predictions of the *spin-charge-family* theory so far.

**IV. A.** The *spin-charge-family* theory predicts the fourth family to the observed three to be observed at the LHC [53]. By predicting symmetry of mass matrices (in all orders of loop corrections [57]) the theory enables for accurate enough measured mixing matrices of the $3 \times 3$ submatrices (the sensitivity of the fitting procedure on masses of the so far measured quarks and leptons is much smaller [55,56]), and due to other measured properties of quarks and leptons [40], to predict the properties of the $4 \times 4$ mixing matrices and to explain correspondingly the origin of Higgs and Yukawa couplings. The $4 \times 4$ mixing matrix elements for quarks are predicted to have the properties: $V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$, here $u_i, d_i, i = 1, 2, 3$ represent $u, c, t, u_4$ and $d, s, b, d_4$ quarks.

The theory explains [58] why the fourth family has not yet been observed, which is the main argument against the existence of four families [59,60] among experts in high energy physics.

**IV. B.** The theory predicts the existence of several scalar fields — there are two triplets and three singlets which determine masses of the lower four families [9,7,3,6] — some of which will be observed in the near future measurements.

**IV. C.** The theory predicts the second group of four families, the stable one of these four families contributing to the dark matter [54]. The nuclear force among these baryons differs a lot from the so far observed nuclear force [54,61].

**IV. D.** The masses of quarks and leptons are, according to these two groups of four families, spread from $10^{-3}$ eV to few TeV — at least 12 orders of magnitude for the first four families — and from 100 TeV to $10^{13}$ TeV — at least 11 orders of magnitude for the second four families, offering the explanation for the hierarchy problem. (The mass matrices of the two groups of mass matrices are very closed to the democratic ones [55,56]).

**IV. E.** The *spin-charge-family* theory predicts the masses of the dark matter baryons [54].

**IV. F.** The *spin-charge-family* theory predicts the scalar fields which contribute to the matter-antimatter asymmetry in the universe [4] and correspondingly also to the proton decay.

**V.** The *spin-charge-family* theory has (so far) several open problems, although it is also true that the more work is done, the more solutions of the open problems follow.

**V. A.** In the *spin-charge-family* the vector and scalar gauge fields originate in gravity as the two kinds of the spin connection fields and the vielbeins. In the low energy region these vector and scalar gauge fields can be quantized in the usual way [5]. Yet the quantization of gravity remains as an open problem when the energies rise up to $10^{16}$ GeV and above.

**V. B.** The dimension of space time — $13 + 1$ — remains as an open problem: Why $d = (13 + 1)$, why not $\infty$? (Only 0 and $\infty$ need no explanation.) How has the universe come to $d = (13 + 1)$ [77]?

**V. C.** Breaking the symmetry with the appearance of the condensate [4], which lead to observable properties of fermion and boson fields, explaining all the assumptions of the *standard model*s, needs to be studied as a dynamical appearance of the condensate in the expanding universe.

**V. D.** It should be demonstrated dynamically how do the scalar fields gain nonzero vacuum expectation values, leading to the effective fields as assumed for the Higgs. The demonstrations, made in Refs. [68–70] for the toy model in $d = (5 + 1)$ must be done also for $d = (13 + 1)$.

**V. F.** The coupling constants of the gauge and scalar fields in the low energy regime should be evaluated when starting with the simple action of Eq. (3.1) in $d = (13 + 1)$, with only one (or already with two) coupling constants.

**V. G.** There are additional open problems which we already see and either solve, like the one treated in this paper about the internal degrees of freedom of fermions in Clifford and Grassmann space and the new way of second quantization procedure, which explains the usual way of second quantization, or they wait to be solved, like the lepton number non conservation in the *spin-charge-family* theory. And there are open problems which we do not see yet or which we could better understand if learning more from all the trials to understand the evolution of the universe and the creation of hadrons of all kinds in the literature.

## 3.2 Fermions in Grassmann and in Clifford space

In the literature the Clifford algebra is frequently discussed as an useful tool to describe internal degrees of freedom of fermions [62–64]. In the *spin-charge-family* theory Clifford space is used to describe all the internal degrees of fermions — quarks and leptons with their families included [1,2,9].

In this paper we demonstrate that the Clifford algebra offers an elegant and transparent way to better understand fermions properties: In even dimensional spaces — we make a choice of $d = 2(2n + 1)$, $n = 3$ — the creation operators of an odd Clifford character can be defined (they are superposition of odd numbers of the Clifford algebra objects ($\gamma^a$'s or $\tilde{\gamma}^a$'s, Eq. (3.2)), each of them is a product of $\frac{d}{2}$ nilpotents and projectors, Eq. (3.27, 3.70) [47,48], so that they are the eigenvectors of twice all the $\frac{d}{2}$ members of the two kinds of the Cartan subalgebras of the Lorentz algebra — $S^{ab}$ and $\tilde{S}^{ab}$ — with the half integer eigenvalues, Eq. (3.72). These creation operators, Eq. (3.76), and their Hermitian conjugated partners — the annihilation operators, Eq. (3.77) — fulfill on the vacuum state, Eq. (3.79), the anti commutation relations required for fermions, Eq. 3.81.
The superposition of these creation operators solve for a particular momentum $p^a$ the equation of motions for free massless fermions, Eq. (3.36), determining in $d = (3 + 1)$ spins, handedness, charges and family quantum numbers. Again they fulfill on the vacuum state, Eq. (3.79), together with their Hermitian conjugated annihilation operators, the anti commutation relations required for fermions, Eq. (3.83). Correspondingly the *creation and annihilation operators are indeed defined with the first quantized fermion fields already*.

We demonstrate in this paper that there exist also in Grassmann space of anticommuting coordinates, Eq. (3.3), the eigenvectors of the Cartan commuting subalgebra of the Lorentz algebra $\mathbf{S}^{ab}$, Eq. (3.3, 3.21), the $\frac{d}{2}$ products of which form creation operators, Eq. (3.51), and which fulfill together with their Hermitian

conjugated partners the annihilation operators, Eq. (3.18), as well the anticommutation relations required for fermions, Eq. (3.54). However, the *eigenvalues of the Cartan subalgebra are in this case integer*.
Also in the Grassmann case the superposition of these creation operators solve for a particular momentum $p^a$ the equation of motions for free massless fermions, presented in Eq. (3.43), determining in $d = (3+1)$ spins, handedness and charges. There are no families in this case.

For both cases, Clifford and Grassmann, we present the proofs for the above statements and illustrate the properties of fermions of both kinds on a few examples.

### 3.2.1 Actions and equation of motion in Clifford and in Grassmann space

We define in $d = ((d-1)+1)$-dimensional space states with integer spin — in Grassmann space — and states with half integer spin — in Clifford space — proving that norms in both spaces can be determined by the integral in Grassmann space, Eqs. (3.32, 3.33), since the Clifford algebra objects are expressible with the Grassmann algebra objects, Eq. (3.4) [2]. When reformulating the vacuum in the Clifford case, Eq. (3.79), half integer spinors presentation in Clifford space become more elegant, that is easier to recognize properties of fermions.

We present as well actions in both cases, Grassmann, Eq. (3.41), and Clifford, Eq. (3.36), leading to the equations of motion (in the Clifford case the Weyl equation is known for a long time, in the Grassmann case it is present for the first time by N.S.M.B.). We compare Euler-Lagrange equations in both cases to compare properties of Grassmann "fermions" with the Clifford fermions.

**a.** *Fields with the integer spin in Grassmann space*

A point in d-dimensional Grassmann space of anticommuting coordinates $\theta^a$, $(a = 0, 1, 2, 3, 5, \ldots, d)$, is determined by a vector $\{\theta^a\} = (\theta^0, \theta^1, \theta^2, \theta^3, \theta^5, \ldots, \theta^d)$. A linear vector space over the coordinate Grassmann space has correspondingly the dimension $2^d$, due to the fact that $(\theta^{a_i})^2 = 0$ for any $a_i \in (0, 1, 2, 3, 5, \ldots, d)$.

Correspondingly are fields in Grassmann space expressible in terms of the Grassmann algebra objects

$$\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\, \theta^{a_1} \theta^{a_2} \ldots \theta^{a_k} |\phi_{og}>, \quad a_i \le a_{i+1}\,, \tag{3.16}$$

where $|\phi_{og}>$ is the vacuum state, here assumed to be $|\phi_{og}> = |1>$, so that $\frac{\partial}{\partial \theta^a}|\phi_{og}> = 0$ for any $\theta^a$. The *Kalb-Ramond* boson fields $a_{a_1 a_2 \ldots a_k}$ are antisymmetric with respect to the permutation of indexes, since the Grassmann coordinates anticommute $\{\theta^a, \theta^b\}_+ = 0$, Eq. (3.3).

[2] Observations in this paper might help also when fermionizing boson fields or bosonizing fermion fields [42].

The left derivative $\frac{\partial}{\partial \theta_a}$ on vectors of the space of monomials $\mathbf{B}(\theta)$ is defined as follows

$$\frac{\partial}{\partial \theta_a}\mathbf{B}(\theta) = \frac{\partial \mathbf{B}(\theta)}{\partial \theta_a}\,,$$
$$\left\{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\right\}_+ \mathbf{B} = 0\,, \text{ for all } \mathbf{B}\,. \tag{3.17}$$

The commutation relations are for $p^{\theta a} = i\frac{\partial}{\partial \theta_a}$ defined in Eq. (3.3), where the metric tensor $\eta^{ab}$ $(= diag(1,-1,-1,\dots,-1))$ lowers the indexes of a vector $\{\theta^a\}$: $\theta_a = \eta_{ab}\,\theta^b$ (the same metric tensor lowers the indexes of the ordinary vector $x^a$ of commuting coordinates).

Defining [3]

$$(\theta^a)^\dagger = \frac{\partial}{\partial \theta_a}\eta^{aa} = -i\,p^{\theta a}\eta^{aa}\,, \tag{3.18}$$

it follows

$$(\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa}\,\theta^a\,, \quad (p^{\theta a})^\dagger = -i\eta^{aa}\theta^a\,. \tag{3.19}$$

Making a choice for the complex properties of $\theta^a$, and correspondingly of $\frac{\partial}{\partial \theta_a}$, as follows

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,,$$
$$\{\frac{\partial}{\partial \theta_a}\}^* = (\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial \theta_1}, -\frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}, -\frac{\partial}{\partial \theta_5}, \frac{\partial}{\partial \theta_6}, ..., -\frac{\partial}{\partial \theta_{d-1}}, \frac{\partial}{\partial \theta_d})\,, \tag{3.20}$$

it follows for the two Clifford algebra objects $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, and $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$, Eq. (3.4), that $\gamma^a$ is real if $\theta^a$ is real, and $\gamma^a$ is imaginary if $\theta^a$ is imaginary, while $\tilde{\gamma}^a$ is imaginary when $\theta^a$ is real and $\tilde{\gamma}^a$ is real if $\theta^a$ is imaginary, just as it is required in Eq. (3.26).

Applying the operator $\mathbf{S}^{ab}$ of Eq. (3.3) on the "states" $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, $a \neq b$, and $\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b)$, $a \neq b$, it follows

$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,$$
$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) = 0\,, \tag{3.21}$$

$k^2 = \eta^{aa}\eta^{bb}$.

We define here the commuting objects $\gamma^a_G$, which will be helpful when looking for the appropriate action for Grassmann fermions, Eq. (3.41). These operators will be needed also when looking for the definition of appropriate discrete symmetry operators in the Grassmann case. Following the definition of the discrete symmetry

[3] In Ref. [2] the definition of $\theta^{a\dagger}$ was differently chosen. Correspondingly also the scalar product needed a (slightly) different weight function in Eq. (3.32).

operators in the Clifford algebra case [65] in $((d-1)+1)$ space-time and in $(3+1)$ space-time, the discrete symmetry operators ($\mathcal{C}_{\mathcal{G}}$, $\mathcal{T}_{\mathcal{G}}$, $\mathcal{P}_{\mathcal{G}}$) in $((d-1)+1)$ and ($\mathcal{C}_{\mathcal{NG}}$, $\mathcal{T}_{\mathcal{NG}}$, $\mathcal{P}_{\mathcal{NG}}$) in $(3+1)$ will be defined in Subsct. 3.3.3, respectively.

$$\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a}) = -i\eta^{aa}\gamma^a \tilde{\gamma}^a \,, \quad \{\gamma^a_G, \gamma^b_G\}_- = 0 \,. \tag{3.22}$$

Index $^a$ is not the Lorentz index in the usual sense. $\gamma^a_G$ are commuting operators for all $(a,b)$. They are real and Hermitian.

$$\gamma^{a\dagger}_G = \gamma^a_G \,, \quad (\gamma^a_G)^* = \gamma^a_G \,. \tag{3.23}$$

Correspondingly it follows: $\gamma^{a\dagger}_G \gamma^a_G = I$, $\gamma^a_G \gamma^a_G = I$. I represents the unit operator.

By introducing [2] the generators of the infinitesimal Lorentz transformations in Grassmann space, as presented in Eq. (3.3), and making use of the Cartan subalgebra of commuting operators, Eq. (3.110), the basic states in Grassmann space can be arranged into representations of the eigenstates of the Cartan subalgebra operators, Eq. (3.21), Ref. [2,46]. All these states have integer spins ($k$ is $\pm i$ or $\pm 1$). The starting state in d-dimensional space, for example, with the eigenvalues of the Cartan subalgebra equal to either $i$ or 1 is: $(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, with $|\phi_{og}> = |1\;\;>$, Eq. (3.21). All the states of the representation, which starts with this state, follow by the application of those $\mathbf{S}^{ab}$, which do not belong to the Cartan subalgebra of the Lorentz algebra. $\mathbf{S}^{01}$, for example, transforms this starting state into $(\theta^0\theta^3 + i\theta^1 i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$, while $(\mathbf{S}^{01} - i\mathbf{S}^{02})$ transforms this state into $(\theta^0 + \theta^3)(\theta^1 - i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)|\phi_{og}>$.

**b.** *Fields with the half integer spin in Clifford space*

Let us present as well the properties of the fermion fields with the half integer spin, expressed by the Clifford algebra objects $\gamma^a$'s ([1,2,9,3,5,4,47] and the references therein)

$$\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k} \gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}> \,, \quad a_i \le a_{i+1} \,, \tag{3.24}$$

where $|\psi_{oc}>$ is the vacuum state. The *Kalb-Ramond* fields $a_{a_1 a_2 \ldots a_k}$ are again in general boson fields, which are antisymmetric with respect to the permutation of indexes, since the Clifford objects have the anticommutation relations, Eq. (3.2), $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$. The linear vector space over the Clifford coordinate space has, as in the Grassmann case, the dimension $2^d$, due to the fact that $(\gamma^{a_i})^2 = \eta^{a_i a_i}$ for any $a_i \in (0,1,2,3,5,\ldots,d)$.

As written in Eq. (3.4), $\gamma^a$ are expressible in terms of the Grassmann coordinates and their conjugate momenta, as $\gamma^a = (\theta^a - i\,p^{\theta a})$, and $\tilde{\gamma}^a = i\,(\theta^a + i\,p^{\theta a})$, with the anticommutation relation of Eq. (3.2), $\{\gamma, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$. Taking into account Eqs. (3.18, 3.19, 3.4) one finds

$$\begin{aligned}
&(\gamma^a)^\dagger = \gamma^a \eta^{aa} \,, \quad (\tilde{\gamma}^a)^\dagger = \tilde{\gamma}^a \eta^{aa} \,, \\
&\gamma^a \gamma^a = \eta^{aa} \,, \quad \gamma^a (\gamma^a)^\dagger = I \,, \quad \tilde{\gamma}^a \tilde{\gamma}^a = \eta^{aa} \,, \quad \tilde{\gamma}^a (\tilde{\gamma}^a)^\dagger = I \,,
\end{aligned} \tag{3.25}$$

where I represents the unit operator. Making a choice for the $\theta^a$ properties as presented in Eq. (3.20), it follows for the Clifford objects

$$\begin{aligned}
\{\gamma^a\}^* &= (\gamma^0, \gamma^1, -\gamma^2, \gamma^3, -\gamma^5, \gamma^6, ..., -\gamma^{d-1}, \gamma^d)\,, \\
\{\tilde{\gamma}^a\}^* &= (-\tilde{\gamma}^0, -\tilde{\gamma}^1, \tilde{\gamma}^2, -\tilde{\gamma}^3, \tilde{\gamma}^5, -\tilde{\gamma}^6, ..., \tilde{\gamma}^{d-1}, -\tilde{\gamma}^d)\,,
\end{aligned} \tag{3.26}$$

Applying the operators $S^{ab}$ and $\tilde{S}^{ab}$, Eq. (3.2), on $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$ and on $\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, and taking into account the relation of Eq. (3.69), one obtains

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) \\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= -\frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,.
\end{aligned} \tag{3.27}$$

One could make a choice of $\tilde{\gamma}^a$ instead of $\gamma^a$ and change correspondingly the relations in Eqs. (3.69, 3.27).

All the three choices for the linear vector space — spanned over either the Grassmann $\theta^a$'s, or over the vector space of $\gamma^a$'s, or over the vector space of $\tilde{\gamma}^a$'s — have the dimension $2^d$. More about the meaning of these degrees of freedom in any of theses cases can be found in Ref. [11].

Let us point out here that $\theta^a$'s and $\frac{\partial}{\partial\theta_a}$'s (each of them has $2^d$ degrees of freedom) are expressible with $\gamma^a$'s and $\tilde{\gamma}^a$'s (with $2^d$ degrees of freedom each) and opposite. Since $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $\gamma^a$'s and $\tilde{\gamma}^a$'s form independent degrees of freedom. We should therefore allow also $\tilde{\gamma}^a$'s to form the vector space.

We can express Grassmann coordinates $\theta^a$ and momenta $p^{\theta a} = i\frac{\partial}{\partial\theta_a}$ in terms of $\gamma^a$ and $\tilde{\gamma}^a$ as well [4]

$$\begin{aligned}
\theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a)\,, \\
\frac{\partial}{\partial\theta_a} &= \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{3.28}$$

with $\frac{\partial}{\partial\theta_b}\theta^a|1> = \eta^{ab}|1>$.

Requiring that the application of $\tilde{\gamma}^a$'s on $\gamma^a$'s are determined by Eq. (3.69), the $\tilde{\gamma}^a$'s part is sacrificed [11]. The two possibilities are no longer acceptable: $\gamma^a$'s are chosen to span the basis, while $\tilde{\gamma}^a$'s become operators which determine the family quantum numbers. From Eqs. (3.28, 3.69) follows that $\frac{\partial}{\partial\theta_b}\theta^a = 0$ and $\theta^a = \gamma$. All the relations of Eq. (3.2) remain valid, while the space of $\tilde{\gamma}^a$'s is sacrificed and the Grassmann space has lost $\frac{\partial}{\partial\theta_b}$, the Hermitian conjugated partner of $\theta^a$.

(Of course, we can still replace $\gamma^a$ by $\tilde{\gamma}^a$, if we change correspondingly the vacuum state $|\psi_{oc}>$ and relation in Eq. (3.69)).

[4] In Ref. [76] the author suggested in Eq. (47) a choice of superposition of $\gamma^a$ and $\tilde{\gamma}^a$, which resembles the choice of one of the authors (N.S.M.B.) in Ref. [2] and both authors in Ref. [47,48] and in present article.

The vacuum state $|\phi_{og}>=|1>$ must after Eq. (3.69) be transformed into $|\psi_{oc}>$ with the property [2,7,9]

$$
\begin{aligned}
&<\psi_{oc}|\gamma^a|\psi_{oc}>=0\,, \quad \tilde{\gamma}^a|\psi_{oc}>= i\gamma^a|\psi_{oc}>\,, \quad \tilde{\gamma}^a\gamma^b|\psi_{oc}>= -i\gamma^b\gamma^a|\psi_{oc}>\,,\\
&\tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a\neq b} = -\gamma^a\gamma^b|\psi_{oc}>\,, \quad \tilde{\gamma}^a\tilde{\gamma}^b|\psi_{oc}>|_{a=b} = \eta^{ab}|\psi_{oc}>\,. \qquad (3.29)
\end{aligned}
$$

This is in agreement with the requirement

$$
\begin{aligned}
\gamma^a\,\mathbf{F}(\gamma)\,|\psi_{oc}>:=&\,(\,a_0\,\gamma^a\,+a_{a_1}\,\gamma^a\,\gamma^{a_1}+a_{a_1 a_2}\,\gamma^a\,\gamma^{a_1}\gamma^{a_2}+\cdots+\\
&\,a_{a_1\cdots a_d}\,\gamma^a\,\gamma^{a_1}\cdots\gamma^{a_d}\,)\,|\psi_{oc}\,>\,,\\
\tilde{\gamma}^a\,\mathbf{F}(\gamma)\,|\psi_{oc}>:=&\,(\,i\,a_0\gamma^a - i\,a_{a_1}\gamma^{a_1}\,\gamma^a + i\,a_{a_1 a_2}\gamma^{a_1}\gamma^{a_2}\,\gamma^a+\cdots+\\
&\,i\,(-1)^d\,a_{a_1\cdots a_d}\gamma^{a_1}\cdots\gamma^{a_d}\,\gamma^a\,)\,|\psi_{oc}\,>\,. \qquad (3.30)
\end{aligned}
$$

The basic states in Clifford space can be arranged in representations, in which any state is the eigenstate of the Cartan subalgebra operators of Eq. (3.110). The state, for example, in d-dimensional space with the eigenvalues of $S^{03}, S^{12}, S^{56}, \ldots, S^{d-1\,d}$ and of $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \ldots, \tilde{S}^{d-1\,d}$ equal to $\frac{1}{2}(i,1,1,\ldots,1)$ is $(\gamma^0-\gamma^3)(\gamma^1+i\gamma^2)(\gamma^5+i\gamma^6)\cdots(\gamma^{d-1}+i\gamma^d)$. The states of one representation follow from the starting state by the application of $S^{ab}$, which do not belong to the Cartan subalgebra operators, while $\tilde{S}^{ab}$, which operate on family quantum numbers, cause jumps from the starting family to the new one.

**Norms of vectors in Grassmann and Clifford space** Let us look for the norm of vectors in Grassmann space, $\mathbf{B} = \sum_{k=0}^{d} a_{a_1 a_2 \ldots a_k}\,\theta^{a_1}\theta^{a_2}\ldots\theta^{a_k}|\phi_{og}>$, and in Clifford space, $\mathbf{F} = \sum_{k=0}^{d} a_{a_1 a_2}\ldots a_k\,\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}|\psi_{oc}>$, where $|\phi_{og}>$ and $|\psi_{oc}>$ are the vacuum states in the Grassmann and Clifford case, respectively. In what follows we refer to Ref. [2].

**a.** *Norms of Grassmann vectors*

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta>=<\theta|\mathbf{B}>^\dagger$, by requiring

$$
\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,, \quad \int d^d\theta\;\theta^0\theta^1\cdots\theta^d = 1\,,\\
&d^d\theta = d\theta^d\ldots d\theta^0\,, \quad \omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial\theta_k}+\theta^k)\,, \qquad (3.31)
\end{aligned}
$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function $\omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial\theta_k}+\theta^k)$ to define the scalar product $<\mathbf{B}|\mathbf{C}>$

$$
<\mathbf{B}|\mathbf{C}> = \int d^{d-1}x d^d\theta^a\;\omega <\mathbf{B}|\theta><\theta|\mathbf{C}>= \sum_{k=0}^{d}\int d^{d-1}x\, b^*_{b_1\ldots b_k} c_{b_1\ldots b_k}\,, \qquad (3.32)
$$

where, according to Eq. (3.18), it follows:

$$
<\mathbf{B}|\theta>= \sum_{p=0}^{d}(-i)^p\; a^*_{a_1\ldots a_p}\,p^{\theta a_p}\,\eta^{a_p a_p}\cdots p^{\theta a_1}\,\eta^{a_1 a_1}.
$$

The vacuum state is chosen to be $|\phi_{og}>=|1>$, as assumed in Eq. (3.16).

The norm $<\mathbf{B}|\mathbf{B}>$ is correspondingly always nonnegative. Let us notice that the choice of the Hermitian conjugated value of $\theta^a$ is $\frac{\partial}{\partial\theta^a}$ ($(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}$, Eq. (3.18)) makes that we easily evaluate in any $d$ the scalar product

$$<\phi_{og}|(\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}\frac{\partial}{\partial\theta^{d-2}}\cdots\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0})(\theta^0\theta^1\cdots\theta^{d-2}\theta^{d-1}\theta^d)|\phi_{og}>=1$$

for $|\phi_{og}>=|1>$ (without integration over coordinate space of $\theta^a$'s).

**b.** *Norms of Clifford vectors*

To evaluate norms in the Clifford space for vectors of Eq. (3.24) we can use as well Eqs. (3.31, 3.32), if expressing $\gamma^a$ in terms of $\theta^a$ and $p^{\theta a}$: $<(\theta^a - ip^{\theta a})|\mathbf{F}>$. In this case $|\psi_{oc}>=|\phi_{og}>=|1>$. It follows

$$<\mathbf{F}|\mathbf{G}> = \int d^{d-1}x d^d\theta^a\ \omega <\mathbf{F}|\gamma><\gamma|\mathbf{G}>=\sum_{k=0}^{d}\int d^{d-1}x\, a^*_{a_1\ldots a_k} b_{b_1\ldots b_k}\,. \tag{3.33}$$

To simplify the evaluation we use instead [3,47] in the Clifford case the vacuum state $|\psi_{oc}>$, Eq. (3.79), which is the product of projectors, Eq. (3.70). It takes care of the orthogonality of states (like if we would evaluate the integration in Grassmann space).

Correspondingly we can write

$$\int d^d\theta^a\ \omega(a_{a_1a_2\ldots a_k}\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k})^\dagger(a_{a_1a_2\ldots a_k}\gamma^{a_1}\gamma^{a_2}\ldots\gamma^{a_k}) = a^*_{a_1a_2\ldots a_k}\,a_{a_1a_2\ldots a_k}\,. \tag{3.34}$$

The norm of each scalar term in the sum of $\mathbf{F}$ is nonnegative.

**Actions in Grassmann and Clifford space** We construct an action for free massless fermion in which the internal degrees of freedom is described: **i.** in Grassmann space, **ii.** in Clifford space. In the first case the internal degrees of freedom manifest integer spins, in the second case the half integer spin.

While the action in Clifford space is well known since long [67], the action in Grassmann space will be defined here (by N.S.M.B.). In both cases we present an action for free massless fermions in $((d-1)+1)$ space [5]. States in Grassmann space as well as states in Clifford space will be arranged to be the eigenstates of

[5] In $d=(3+1)$ space masses of fermions are in the *spin-charge-family* theory in the Clifford case caused by the interaction of fermions with scalar gauge fields with the space index $(7,8)$, that is the vielbeins and the spin connections of two kinds — the gauge scalar fields of $S^{ab}$ and of $\tilde{S}^{ab}$. We expect that masses of "fermions" appear also in the Grassmann case due to the interaction of fermions with scalar gauge fields with the space index $(7,8)$, but in this case due to the vielbeins and the spin connection of one kind only — the gauge field of $\mathbf{S}^{ab}$

the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space and with respect to $S^{ab}$ and $\tilde{S}^{ab}$ in Clifford space, Eq. (3.110), and orthogonal and normalized with respect to Eq. (3.31) [6].

In both spaces the requirement that states are obtained by the application of creation operators on the vacuum state — in the Grassmann case $\hat{b}^{\theta^k \dagger}_i$ on $|1>$, Eq. (3.58), obeying together with the $\hat{b}^{\theta^k}_i$ the anti commutation relations of Eq. (3.54) on the vacuum state $|\phi_{og}>=|1>$, and in the Clifford case $\hat{b}^{\alpha\dagger}_i$, Eq.(3.76), obeying together with the $\hat{b}^{\beta}_i$ the equivalent anticommutation relations of Eq. (3.81) on the vacuum states $|\psi_{oc}>$, Eq. (3.79) — reduces the number of states, in Clifford space more than in Grassmann space. But while in Clifford space all physically applicable states are reachable by either $S^{ab}$ (defining family members quantum numbers) or by $\tilde{S}^{ab}$ (defining family quantum numbers), the states in Grassmann space, belonging to different representations with respect to the Lorentz generators, seem not to be connected.

**a.** *Action in Clifford space*

In Clifford space the action for a free massless fermion must be Lorentz invariant

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^\dagger \gamma^0 \, \gamma^a p_a \psi) + h.c. \,, \tag{3.35}$$

$p_a = i\,\frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a |\psi> = 0 \,, \tag{3.36}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi> = 0 \,, \tag{3.37}$$

for each of the basic states $\hat{b}^{\alpha\dagger}_i |\psi_{0c}> = |\psi^\alpha_i>$. $\gamma^0$ appears in the action since we pay attention that

$$\begin{aligned} S^{ab\dagger}\gamma^0 &= \gamma^0 \, S^{ab} \,, \quad S^\dagger \gamma^0 = \gamma^0 S^{-1} \,, \\ S &= e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})} \,. \end{aligned} \tag{3.38}$$

The Lagrange density, Eq. (3.35),

$$\mathcal{L}_C = \frac{1}{2}\{\psi^\dagger \, \gamma^0 \, \gamma^a \, \hat{p}_a \psi - \hat{p}_a \psi^\dagger \, \gamma^0 \, \gamma^a \, \psi \} \,, \tag{3.39}$$

leads to

$$\begin{aligned} \frac{\partial \mathcal{L}_C}{\partial \psi^\dagger} - \hat{p}_a \, \frac{\partial \mathcal{L}_C}{\partial \hat{p}_a \psi^\dagger} &= 0 = \gamma^0 \gamma^a \, \hat{p}_a \, \psi \,, \\ \frac{\partial \mathcal{L}_C}{\partial \psi} - \hat{p}_a \, \frac{\partial \mathcal{L}_C}{\partial (\hat{p}_a \psi)} &= 0 = -\hat{p}_a \, \psi^\dagger \gamma^0 \, \gamma^a \,. \end{aligned} \tag{3.40}$$

[6] In the Clifford case the states can be orthogonalized also with respect to Eq. (3.79), while taking into account Eq. (3.71).

All the states, belonging to different values of the Cartan subalgebra — they differ at least in one value of either the set of $S^{ab}$ or the set of $\tilde{S}^{ab}$, Eq. (3.110) — are orthogonal according to the scalar product, defined as the integral over the Grassmann coordinates, Eq. (3.31), for a chosen vacuum state $|1\;>$. Correspondingly the states generated by the creation operators, Eq. (3.76), on the vacuum state, Eq. (3.79), are orthogonal as well.

**b.** *Action in Grassmann space*

We define here the action in Grassmann space, for which we require — similarly as in the Clifford case — that the action for a free massless fermion is Lorentz invariant

$$\mathcal{A}_G = \int d^d x \; d^d\theta \; \omega \{\phi^\dagger (1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) \, \frac{1}{2}\, \theta^a p_a \phi\} + h.c.. \tag{3.41}$$

We use the integral over $\theta^a$ coordinates with the weight function $\omega$ from Eq. (3.31, 3.32). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator $\gamma^0_G$ ($\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial\theta^a})$), which takes care of the Lorentz invariance. Namely

$$\begin{aligned}
\mathbf{S}^{ab\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial\theta^0})\,\mathbf{S}^{ab}\,,\\
\mathbf{S}^{\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial\theta^0})\,\mathbf{S}^{-1}\,,\\
\mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab}+\mathbf{S}^{ab})}\,,
\end{aligned} \tag{3.42}$$

while $\theta^a$, $\frac{\partial}{\partial\theta_a}$ and $p^a$ transform as Lorentz vectors. The equations of motion follow from the action, Eq. (3.41),

$$\begin{aligned}
\frac{1}{2}\,\gamma^0_G\,(\theta^a - \frac{\partial}{\partial\theta^a})\,p_a\,|\phi> &= 0\,,\\
\gamma^0_G = (1 - 2\theta^0 \frac{\partial}{\partial\theta^0}) \quad &,
\end{aligned} \tag{3.43}$$

as well as the Klein-Gordon equation, $\gamma^0_G\,(\theta^a - \frac{\partial}{\partial\theta^a})\,p_a\,\gamma^0_G\,(\theta^b - \frac{\partial}{\partial\theta^a})\,p_b\,|\phi>= 0$, leading to

$$\{\theta^a p_a, \frac{\partial}{\partial\theta_b}\,p_b\}_+ = p^a p_a = 0\,. \tag{3.44}$$

From the Lagrange density, presented in Eq. (3.41), using Eqs. (3.18, 3.19, 3.28) ($\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$, $(\theta^a - \frac{\partial}{\partial\theta_a}) = -i\tilde{\gamma}^a$) it follows, up to the surface term,

$$\begin{aligned}
\mathcal{L}_G &= -i\frac{1}{2}\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,(\hat{p}_a\phi)\\
&= -i\frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\phi\,-\hat{p}_a\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\phi\}.
\end{aligned} \tag{3.45}$$

One correspondingly finds

$$\begin{aligned}
\frac{\partial\mathcal{L}_G}{\partial\phi^\dagger} - \hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial\hat{p}_a\phi^\dagger} &= 0 = \frac{-i}{2}\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\,\phi\,,\\
\frac{\partial\mathcal{L}_G}{\partial\phi} - \hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial(\hat{p}_a\phi)} &= 0 = \frac{i}{2}\hat{p}_a\,\phi^\dagger\gamma^0_G\,\tilde{\gamma}^a\,,
\end{aligned} \tag{3.46}$$

The solutions of these equations are presented in Eq. (3.98).

We shall see that, if one identifies the creation operators in both spaces with the products of odd numbers of either $\theta^a$ — in the Grassmann case — or $\gamma^a$ — in the Clifford case — and the annihilation operators as the Hermitian conjugated operators of the creation operators, the creation and annihilation operators fulfill the anticommutation relations, required for fermions. The internal parts of states are then defined by the application of the creation operators on the vacuum state.

But while the Clifford subalgebra defines states with the half integer "eigenvalues" of the Cartan subalgebra operators of the corresponding Lorentz algebra, the Grassmann algebra defines states with the integer "eigenvalues" of the Cartan subalgebra operators of the corresponding Lorentz algebra.

## 3.3 Second quantization of Grassmann and Clifford vectors

It is proven in this section that solutions of the Weyl equations — following from the Hermitian and Lorentz invariant actions for free massless fermions, using to describe their internal degrees of freedom either Clifford space, Eqs. (3.35, 3.36), or Grassmann space, Eq. (3.41, 3.43), — can be represented as creation operators, operating on the appropriate vacuum state. The corresponding Hermitian conjugated operators, taken as their annihilation partners, fulfill together with the creation operators, if both are of an odd either Clifford or Grassmann character, the anticommutation relations required for fermions.

Correspondingly there is no need to assume the anticommutation relations as done in the Dirac theory [67,74,75], since the creation and annihilation operators of an odd either Clifford or Grasmmann character by themselves fulfill the anticommutation relations for fermions without postulating them.

Creation operators in both spaces determine the Hilbert space of $n$ fermions for any integer $n$ and have all the properties of the corresponding Slater determinants, if we recognize that a product of two creation operators of two different moments in the ordinary space $(p_k, p_l)$ — $\hat{b}^{\alpha\dagger}_{i\,p_k} \cdot \hat{b}^{\beta\dagger}_{j\,p_l}$, applying on the vacuum state $|\psi_{oc}>$, are zero if and only if $i = j$, $\alpha = \beta$ and $p_k = p_l$. In the Grassmann case $\hat{b}^{\alpha\dagger}_{i\,p_k} \cdot \hat{b}^{\beta\dagger}_{j\,p_l}$ is replaced by $\hat{b}^{\theta\dagger}_{i\,p_k} \cdot \hat{b}^{\theta\dagger}_{j\,p_l}$ and the vacuum $|\psi_{oc}>$ by $|\psi_{og}>$.

Let us point out that fermions with the internal degrees of freedom described in Clifford space manifests half integer spins, while "fermions" with the internal degrees of freedom described in Grassmann space demonstrate integer spins.

We pay attention in this paper on $d = 2(2n+1)$-dimensional spaces, arranging all the vectors to be "eigenvectors" of the Cartan subalgebra operators of $S^{ab}$ and $\tilde{S}^{ab}$ in the Clifford case and of $\mathbf{S}^{ab}$ in the Grassmann case, Eqs. (3.110, 3.2, 3.3).

In $d$-dimensional spaces the linear vector space, spanned over either the Clifford coordinates $\gamma^a$'s or the Grassmann coordinates $\theta^a$'s, has the dimension $2^d$. One can in both cases represent the vector space as $2^d$ operators, which — when applied on the vacuum state — create $2^d$ vectors. Half of these operators have an odd and half an even either Clifford (with respect to odd or even products of $\gamma^a$'s) or Grassmann (with respect to odd or even products of $\theta^s$'s) character.

In the Clifford case there are in the group of an odd Clifford character two groups of operators: each member of one group has its Hermitian conjugated partner in another group. One of the two groups can be therefore chosen to represent the creation operators, the other to represent the corresponding annihilation operators. Each of the two groups has $2^{d-2}$ members.

Each of the two Clifford odd groups, one with $2^{d-2}$ creation the other with $2^{d-2}$ annihilation operators, further divides into $2^{\frac{d}{2}-1}$ subgroups with $2^{\frac{d}{2}-1}$ members. All the $2^{\frac{d}{2}-1}$ members of one particular subgroup are related by the operators $S^{ab}$, while $\tilde{S}^{ab}$ transform each member of this subgroup of particular family into the same member of one of $2^{\frac{d}{2}-1}$ families.

In the group of the Clifford even operators there are again two groups, each with $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ operators related by either $S^{ab}$ or by $\tilde{S}^{ab}$. Within each of the group there are $2^{\frac{d}{2}-1}$ subgroups with $2^{\frac{d}{2}-1}$ members, related by the application of $S^{ab}$, while $\tilde{S}^{ab}$ transform each member of a particular subgroup into the same member — with respect to the operators $S^{ab}$ — of another subgroup with again $2^{\frac{d}{2}-1}$ members.

These two groups are not related by the Hermitian conjugation as in the case of odd Clifford objects. In each of the two groups of an even Clifford character there are $2^{\frac{d}{2}-1}$ self adjoint operators. The rest of $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ Clifford even operators have the Hermitian conjugated partners within the same group.

$\tilde{\gamma}^a \gamma^a$ transform $2^{\frac{d}{2}-1}$ self adjoint operators of one Clifford even group into $2^{\frac{d}{2}-1}$ self adjoint operators of another Clifford even group, while $\tilde{\gamma}^a \gamma^a$ transform the rest of this group — that is $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ operators, having the Hermitian conjugated partners within the same subgroup — into $2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1)$ operators of another Clifford even group, having again the Hermitian conjugated partners within the same subgroup.

Any odd Clifford member of the assumed (chosen to be) creation operators gives, when applied on one (only one) of the even self adjoint operators of only one of the two groups with $(2^{\frac{d}{2}-1})^2$ members, a nonzero contribution, which is the same creation operator back. It gives nonzero contribution also on one (only one) of the rest $2^{\frac{d}{2}-1} \cdot (2^{\frac{d}{2}-1} - 1)$ operators of the same group to which also the self adjoint operator belong, transforming it to one of creation operators, belonging to another family of the creation operators. On all the others Clifford even objects this creation operator gives zero.

The annihilation operators manifest, when applied on the Clifford even objects, equivalent properties as creation operators.

Let $\hat{b}^{\alpha\dagger}_i$ be the creation operator of an odd Clifford character, $\alpha$ denoting the subgroup with a particular value of the Cartan subalgebra of $\tilde{S}^{ab}$ (family) and with $i$ denoting a particular member of a family $\alpha$. To all the members of particular $\alpha$ one and only one of the selfadjoint operators of an even Clifford character corresponds, which, when any of these members applies on it, gives the same creation operator back.

$(\hat{b}^{\alpha\dagger}_i)^\dagger = \hat{b}^{\alpha}_i$, denoting the corresponding annihilation operator of an odd Clifford character, gives zero when applied on the selfadjoint operators on which $\hat{b}^{\alpha\dagger}_i$ gives nonzero contribution.

We choose the superposition of these selfadjoint operators to determine the vacuum state in the Clifford case, Eq. (3.79).

All the members of the odd Clifford character, half of them creation operators and half of them annihilation operators, fulfill the anticommutation relations, required for fermions. Correspondingly there are only $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ creation operators, determining $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ family members each, which when applied on the superposition of selfadjoint operators of one group of Clifford even operators, create fermion states. These creation operators determine $n$ fermions Hilbert space.

In the Grassmann case there are two kinds of operators, $\theta^a$ and $\frac{\partial}{\partial\theta^a}$, Hermitian conjugated to each other, Eqs. (3.18, 3.19). If $\theta^a$ represent the creation operators, then $\frac{\partial}{\partial\theta^a}$ are the corresponding annihilation operators. Not having the Hermitian conjugated partner with the property that when applying on $|\psi_{og}>=|!>$ gives zero, the identity (I) can not belong either to creation or to annihilation operators.

In $d=2(2n+1)$-dimensional Grassmann spaces there are correspondingly $2^d-1$ creation operators. The largest two representations have together $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators and the same number of annihilation operators of an odd Grassmann character, Eq. (3.59), chosen to be eigenstates of the Cartan subalgebra, Eq. (3.110), of $\mathbf{S}^{ab}$. All the irreducible representations of the Grassmann case are decoupled. The application of the creation operators, which are products of $\frac{d}{2}$ $\theta^a$'s, on the identity (I) gives them back, while the annihilation operators applied on I give zero.

The $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators split into two by the generators of the Lorentz transformations $\mathbf{S}^{ab}$ unconnected groups, each with $\frac{1}{2}\ \frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members.

We introduce common notation for the Clifford and Grassmann case to simplify the discussion: Let $\hat{b}^{\alpha\dagger}_i$ be the creation operator of an odd Grassmann character with $\alpha=(1,2)$ denoting one of the two (by $\mathbf{S}^{ab}$ unconnected) the largest subgroups and let $i$ denotes one of the $\frac{1}{2}\ \frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members related among themselves by $\mathbf{S}^{ab}$. We make a choice of the vacuum state in the Grassmann case to be $|\psi_{o\alpha}>=|\ 1>$.

All members of two groups of $\frac{1}{2}\ \frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ number of creation operators of an odd Grassmann character, and their Hermitian conjugated partners, fulfill the anticommutation relations, required for fermions.

The number of vectors in the Hilbert space of n-fermions depends for a chosen momentum $p^a_k$ on the number of the creation operators, creating a particular fermion in the Clifford case or a particular "fermion" in the Grassmann case.

There are for each $p^a_k$ in the odd Clifford case $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ and in the odd Grassmann case (for the two the largest representations) $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators $\hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}$ of an odd character — either Clifford odd character, with $\alpha=(1,\cdots,2^{\frac{d}{2}-1}), i=(1,\cdots,2^{\frac{d}{2}-1})$, or Grassmann odd character, with $\alpha=(1,2), i=(1,\cdots,\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$), creating the corresponding single particle states, when applied on the vacuum states $|\psi_o>$ — in the Clifford case is the vacuum state $|\psi_{oc}>$, the superposition of all selfadjoint operators, on which an odd $\hat{\mathbf{b}}^{\alpha i\dagger}_{p_k}$ gives a nonzero contribution, and in the Grassmann case the vacuum state is $|\psi_{og}>=|\ 1>$.

Let the zero fermion state for any $p^a_k$ in either Clifford or Grassmann space, be written as $|\psi_o>:=|0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, 0^{\alpha=1}_{i=3\,p_1},\ldots, 0^{\alpha=1}_{i_{max}\,p_1},\ldots, 0^{\alpha=\alpha_{max}}_{i_{max}\,p_1},\ldots, 0^{\alpha=1}_{i=1\,p_2},$

$0^{\alpha=1}_{i=2\,p_2}, 0^{\alpha 1}_{i=3\,p_2}, \ldots, 0^{\alpha=1}_{i_{\max}\,p_2}, \ldots, 0^{\alpha=\alpha_{\max}}_{i_{\max}\,p_k}, \cdots, \cdots |\psi_o>$, with $|\psi_o>= (|\psi_{oc}>, |\ 1>$ ), in the Clifford and the Grassmann case, respectively, and $\alpha_{max} = (2^{\frac{d}{2}-1}, 2)$ and $i_{max} = (2^{\frac{d}{2}-1}, \frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$, again in the Clifford and the Grassmann case, respectively. Then the vector space with $n$ fermions in the Clifford case or $n$ "fermions" in the Grassmann case, for any $n$ looks like

$$
\begin{aligned}
\hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |\psi_o> = & |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, 0^{\alpha=\alpha_{\max}}_{i_{\max}\,p_1}, \ldots, 1^{\alpha}_{i\,p_k} \cdots, |\psi_o> \\
& \ldots \\
& \text{there are } \alpha_{\max} \cdot i_{\max} \text{ such } 1-\text{fermion states for each } p_k\,, \\
& \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}\, \hat{\mathbf{b}}^{\alpha\dagger}_{j\,p_k} |\psi_o>, \\
& \ldots \\
& \ldots \\
& \Pi_{\alpha=1,\alpha_{\max}} \Pi_{i=1,i_{\max}} \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |\psi_o>,\,, \\
& \ldots \\
& \Pi_{\alpha=1,\alpha_{\max}} \Pi_{i=1,i_{\max}} \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_l} |\psi_o>,\,, \\
& \ldots \\
& \text{there are } 2^{\alpha_{\max}\cdot i_{\max}} \text{ Slater determinants of fermions for each } p_k\,, \\
& \ldots
\end{aligned} \tag{3.47}
$$

$\alpha_{\max} = (2^{\frac{d}{2}-1}, 2)$ and $i_{\max} = (2^{\frac{d}{2}-1}, \frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$ in the Clifford and Grassmann case, respectively.

One sees that

$$
\begin{aligned}
& \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}\, \hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l} |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, \\
& \qquad 1^{\alpha'}_{i'p_{k'}}, \ldots, 0^{\alpha'''}_{i'''=1\,p_{k'''}}, \ldots, 1^{\alpha^{iv}}_{i^{iv}p_{k^{iv}}}, \ldots, 1^{\beta'}_{j'\,p_{l'}}, \ldots, |\psi_o>= \\
& -\hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l}\, \hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k} |0^{\alpha=1}_{i=1\,p_1}, 0^{\alpha=1}_{i=2\,p_1}, \ldots, 1^{\alpha'}_{i'p_{k'}}, \ldots, 0^{\alpha'''}_{i'''=1\,p_{k'''}}, \ldots 1^{\alpha^{iv}}_{i^{iv}p_{k^{iv}}}, \ldots, \\
& \qquad\qquad 1^{\beta'}_{j'\,p_{l'}}, \ldots, |\psi_o>,
\end{aligned} \tag{3.48}
$$

and is zero only if any of the occupied states is the same as one (or both) of the two states determined by $\hat{\mathbf{b}}^{\alpha\dagger}_{i\,p_k}$ or $\hat{\mathbf{b}}^{\beta\dagger}_{j\,p_l}$ applied on $|\psi_o>$ [7].

[7] Each single particle state caries its own internal space, described by a creation operator with a superposition of an odd number of $\gamma^a_i$'s, and its own coordinate space, described by $x^a_i$'s (or $p^a_i$). The creation operators of any two pairs of particles therefore anti-commute. Correspondingly the two states of two particles must distinguish in either internal space or in the coordinate space, as it follows from Eq. (3.86). The property of the creation operators $\hat{b}^{\alpha\dagger}_{spi}\hat{b}^{\alpha'\dagger}_{s'p'j}$ applying on the n-particle state $|1^{\alpha}_{sp1}, 1^{\alpha'}_{s'p'2}, 1^{\alpha"}_{s"p"3}, \ldots, 0^{\alpha"'}_{s"'p_i\,i}, \ldots, 0^{\alpha^{iv}}_{s^{iv}p_j\,j}, \ldots, >$,presented in Eq. (3.86), can be as well described by (superposition of) Slater determinants of single particle states. Let us add that the vacuum state, having the sum of the spins of both kinds of operators, $S^{ab}$ and $\tilde{S}^{ab}$, equal to zero and therefore neutral, remains neutral also when filled with fermions of all the spins, $S^{ab}$ and $\tilde{S}^{ab}$.

One fermion states are either in Clifford or in Grassmann space already second quantized, since in both cases they fulfill the anticommutation relations required for fermions, Eqs. (3.66, 3.87).

All together there are $2^{2^{d-2}}$ Slater determinants for a chosen $p_k$ in the Clifford case and $2^{\frac{d!}{\frac{d}{2}!\frac{d}{2}!}}$ Slater determinants for a chosen $p_k$ in the Grassmann case (if only the two largest group of odd irreducible representations are taken into account, if we take all odd representations into account we have $2^{2^{d-1}}$ Slater determinants), $p_k$ has a continuously changing value, $p^0 = (0, \infty)$, $-\infty \leq p^l \leq \infty$, $l = (1, 2, 3, 5, \cdot, d)$.

It can be concluded that there are only second quantized states, since the anticommuting creation and annihilation operators, creating a Clifford fermion or Grassmann "fermion" states, determine all the properties of the n-particle Hilbert space for any n.

We shall as well recognize that no Dirac sea is needed either in the Clifford or in the Grassmann case, since the same Lorentz representation includes in both cases fermions and antifermions.

We discuss in the subsections the second quantization procedure in both spaces, Clifford and Grassmann, when dimension of the space-time is larger than four. We demonstrate that if the dynamics manifests only in $d = (3+1)$, that is when momentum is different from zero only in $d = (3+1)$, $p^a = (p^0, p^1, p^2, p^3, 0, 0, \cdots, 0)$ — what happens at low energies after the break of Lorentz symmetries in $d \geq 5$ — spins in $d \geq 5$ manifest as charges in $d = (3+1)$.

While the Clifford case offers the explanation for all the properties of observable fermions (after sacrificing the space of $\tilde{\gamma}^a$'s), the Grassmann case, having difficulties in describing energy within the usual second quantized procedure, as long as the Lorentz invariance in internal space is unbroken, leads to unobserved "fermions" with integer spins.

Let us point out that states in Grassmann space as well as states in Clifford space are organized to be — within each of the two spaces — orthogonal and normalized with respect to Eq. (3.31, 3.32, 3.33). All the states in each of spaces are chosen to be eigenstates of the Cartan subalgebra — with respect to $\mathbf{S}^{ab}$ in Grassmann space, Eqs. (3.3, 3.5, 3.110), and with respect to $S^{ab}$ and $\tilde{S}^{ab}$, Eq. (3.2), in Clifford space, Eq. (3.110).

We pay attention in this paper almost only to spaces with $d = 2(2n+1)$ [8].

### 3.3.1 Second quantization in Grassmann space

There are $2^d$ states in Grassmann space, orthogonal to each other with respect to Eqs. (3.31, 3.32). To any coordinate there exists the conjugate momentum. We pay attentionin what follows mostly to spaces with $d = 2(2n+1)$. The states, which

[8] The main reason that we treat here mostly $d = 2(2n+1)$ spaces is that one Weyl representation, expressed by the product of the Clifford algebra objects, manifests in $d = (1+3)$ all the observed properties of quarks and leptons, if $d \geq 2(2n+1), n = 3$, and that the breaks of the starting symmetry down to $d = (3+1)$ can lead to massless fermions [68,69].

contribute in the second quantization procedure and manifest anticommutation relations required for fermions, are Grassmann odd products of eigenstates of the Cartan subalgebra, Eq. (3.110), of the Lorentz algebra. In $d = 2(2n+1)$ spaces there are two Grassmann odd irreducible representations of the Lorentz algebra with the largest number of members, divided into two separated groups of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, Eq. (3.59). (All states of one group are reachable from a starting state by the application of $\mathbf{S}^{ab}$.) Any Grassmann odd state can be written as a creation operator, operating on the vacuum state, while the Hermitian conjugated creation operator is the corresponding annihilation operator. Creation and annihilation operators of an odd Grassmann character fulfill the anticommutation relations of Eq. (3.50, 3.54). Let us see how it works.

If $\hat{b}^{\theta\dagger}_i$ is a creation operator, which creates a state in the Grassmann space when operating on a vacuum state $|\psi_{og}>$ and $\hat{b}^{\theta}_i = (\hat{b}^{\theta\dagger}_i)^\dagger$ is the corresponding annihilation operator, then for a set of creation operators $\hat{b}^{\theta\dagger}_i$ and the corresponding annihilation operators $\hat{b}^{\theta}_i$ it must be

$$\begin{aligned}
\hat{b}^{\theta}_i|\phi_{og}> &= 0\,,\\
\hat{b}^{\theta\dagger}_i|\phi_{og}> &\neq 0\,.
\end{aligned} \tag{3.49}$$

We first pay attention on only the internal degrees of freedom of the Grassmann "fermions": the spin in any dimension $d = 2(2n+1)$, $n$ is a positive integer.

Choosing $\hat{b}^{\theta\dagger}_a = \theta^a$, then it follows that $(\hat{b}^{\theta\dagger}_a)^\dagger = \frac{\partial}{\partial\theta^a}$, Eqs. (3.18, 3.19). One correspondingly finds

$$\begin{aligned}
\hat{b}^{\theta\dagger}_a = \theta^a\,, \quad \hat{b}^{\theta}_a &= \frac{\partial}{\partial\theta^a}\,,\\
\{\hat{b}^{\theta}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= \delta_{ab}|\phi_{og}>\,,\\
\{\hat{b}^{\theta}_a, \hat{b}^{\theta}_b\}_+|\phi_{og}> &= 0\,,\\
\{\hat{b}^{\theta\dagger}_a, \hat{b}^{\theta\dagger}_b\}_+|\phi_{og}> &= 0\,,\\
\hat{b}^{\dagger\theta}_a\,|\phi_{og}> &= \theta^a|\phi_{og}>\,,\\
\hat{b}^{\theta}_a\,|\phi_{og}> &= 0\,.
\end{aligned} \tag{3.50}$$

The vacuum state $|\phi_{og}>$ will in this case be chosen as $|\phi_{og}>=|\,1\,>$.

The number operator $\hat{N}^{\theta}_a = \hat{b}^{\theta\dagger}_a\hat{b}^{\theta}_a$ has the property, due to thefirst line in Eq. (3.49) and the second line in Eq. (3.50), that $(\hat{N}^{\theta}_a)^2 = \hat{N}^{\theta}_a$, with the eigenvalue $0$ or $1$.

The identity $I$ ($I^\dagger = I$) can not be taken as a creation operator, since its annihilation partner does not fulfill Eq. (3.49). The identity is obviously selfadjoint operator determining the vacuum state $|\phi_{og}>=|\,1\,>$.

We can use the superposition of products of $\theta^a$'s as creation operators and the corresponding superposition of products of $\frac{\partial}{\partial\theta_a}$'s as the corresponding annihilation operators, provided that they fulfill the requirements for the creation and annihilation operators, Eq. (3.54), with the vacuum state $|\phi_{og}>=|1\,>$. In general they would not. Only an odd number of $\theta^a$ in any superposition would have the required anticommutation properties.

To construct creation operators it is convenient to take products of such superposition of vectors $\theta^a$ and $\theta^b$ that each factor is the "eigenstate" of one of the Cartan subalgebra members of the Lorentz algebra (3.110). Let us start in $d = 2(2n+1)$ with the creation operator, which is a product of $\frac{d}{2}$ "eigenstates" of an odd Grassmann character of the Cartan subalgebra $S^{ab}$ $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, Eq. (3.21). Then the corresponding annihilation is a product of $\frac{d}{2}$ of the corresponding factors $\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta^a} + \frac{\eta^{aa}}{-ik}\frac{\partial}{\partial\theta_b})$, In both cases $(a,b)$ belong to $(0,3), (1,2), (5,6), \cdots, (d-1,d)$.

Let us in $d = 2(2n+1)$, $n$ is a positive integer, start with the state

$$
\begin{aligned}
|\phi^1_1> &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-1}+i\theta^d)\hat{b}^{\theta 1}_1\,|1>,\\
&= \hat{b}^{\theta 1\dagger}_1\,|1>, \quad \text{with}\\
\hat{b}^{\theta 1\dagger}_1 &:= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-1}+i\theta^d)\,. \qquad (3.51)
\end{aligned}
$$

One finds for the eigenvalues of the Cartan subalgebra operators, Eq. (3.110), the values $(+i, +1, +1, \cdots +1)$.

The rest of states, belonging to the same Lorentz irreducible representation, follow from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators.

One can find creation and annihilation operators for $d = 4n$ in App. 3.5.

**i.** We proposed in Eq. (3.51) the starting creation operator $\hat{b}^{\theta 1\dagger}_1$, the upper index indicates one of the two groups, the lower index indicates the starting member. By taking into account Eqs. (3.18, 3.19) the starting creation operator and its annihilation partner are for $d = 2(2n+1)$ equal to

$$
\begin{aligned}
\hat{b}^{\theta 1\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-1}+i\theta^d)\,,\\
\hat{b}^{\theta 1}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\,,\\
&\text{for } d = 2(2n+1)\,. \qquad (3.52)
\end{aligned}
$$

The rest of creation operators belonging to this group (group 1) in $d = 2(2n+1)$ follow by the application of operators $\mathbf{S}^{ef}$. The corresponding annihilation operators are the Hermitian conjugated partners of the corresponding of creation operators. For $d = 2(2n+1)$ one finds by the application of $\mathbf{S}^{01}$ another creation operator and the corresponding annihilation operator as follows

$$
\begin{aligned}
\hat{b}^{\theta 1\dagger}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-1}+i\theta^d)\,,\\
\hat{b}^{\theta 1}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0} - i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1})\,,\\
&\text{in general}:\\
\hat{b}^{\theta 1\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta 1\dagger}_1\,,\\
\hat{b}^{\theta 1}_i &= (\hat{b}^{\theta 1\dagger}_i)^\dagger\,. \qquad (3.53)
\end{aligned}
$$

It was taken into account in the above equation that any $\mathbf{S}^{ac}$ $(a \neq c)$, which does not belong to the Cartan subalgebra, Eq.(3.110), transforms $(\frac{1}{\sqrt{2}})^2(\theta^a + i\theta^b)(\theta^c + i\theta^d)$ $(a \neq c$ and $a \neq d$, $b \neq c$ and $b \neq d$, $\eta^{aa} = \eta^{bb})$ into $\frac{1}{\sqrt{2}}(\theta^a\theta^b + \theta^c\theta^d)$. The states are normalized and the simplest phases are assumed. One evaluates that either $\mathbf{S}^{ab}$ or $\mathbf{S}^{cd}$, applied on $(\theta^a\theta^b \pm \theta^c\theta^d)$, gives zero. The vacuum state is in all these cases $|\,1\,>$.

All the creation operators of an odd Grassmann character — the Grassmann even $S^{ac}$ does not change the oddness of the creation operators and neither do the Hermitian conjugation — fulfill the anticommutation relations

$$\begin{aligned}
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l\dagger}_j\}_+|\phi_{og}> &= \delta_{ij}\,\delta_{kl}\,|\phi_{og}>\,,\\
\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l}_j\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\
\{\hat{b}^{\theta k\dagger}_i, \hat{b}^{\theta l\dagger}_j\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\
\hat{b}^{\theta k\dagger}_i|\phi_{og}> &= |\phi^k_i>\,,\\
\hat{b}^{\theta k}_j|\phi_{og}> &= 0\,|\phi_{og}>\,,\\
(k,l) &= (1,2)\,.
\end{aligned} \tag{3.54}$$

Since there is another group of states, presented in Eq. (3.56), not reachable from the starting state by $\mathbf{S}^{ab}$, we denote, to generalize the notation, creation operator with $\hat{b}^{\theta k\dagger}_i$ and the annihilationoperator with $\hat{b}^{\theta k}_i$.

It is not difficult to see that states included into one representation, which started with $\hat{b}^{\theta 1\dagger}_i\,|1\,>$ as presented in Eq. (3.52) for $d = (2n+1)2$ *have the properties, required by* Eq. (3.54) for $k = 1$:

**i.a.** In any d-dimensional space the product $\frac{\partial}{\partial\theta^{a_1}}\cdots\frac{\partial}{\partial\theta^{a_k}}$, with all different $a_i$ (if all or some of them are equal, then this is trivially true since $(\frac{\partial}{\partial\theta^a})^2 = 0$), if applied on the vacuum $|1\,>$, is equal to zero. Correspondingly the second equation and the fifth equation of Eq. (3.54) are fulfilled.

**i.b.** In any d-dimensional space the product of different $\theta^a$s — $\theta^{a_1}\theta^{a_2}\cdots\theta^{a_k}$ with all different $\theta^a$'s ($a_i \neq a_j$ for all $a_i$ and $a_j$) — applied on the vacuum $|\,1\,>$, is different from zero. Since all the $\theta$'s, appearing in Eqs. (3.52, 3.53), are different, forming orthogonal and normalized states, the fourth equation of Eq. (3.54) is fulfilled.

**i.c.** The third equation of Eq. (3.54) is fulfilled provided that there is an odd number of $\theta^s$ in the expression for a creation operator. Then, when in the anticommutation relation different $\theta^a$'s appear (like in the case of $d = 6$ $\{\theta^0\theta^3\theta^5, \theta^1\theta^2\theta^6\}_+$), such a contribution gives zero. When two or several equal $\theta$'s appear in the anticommutation relation, the contribution is zero (since $(\theta^a)^2 = 0$).

**i.d.** Also for the first equation in Eq. (3.54) it is not difficult to show that it is fulfilled only for a particular creation operator and its Hermitian conjugated partner: Let us show this for $d = (3+1)$ and the creation operator $\frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\,\theta^1\theta^2$ and its Hermitian conjugate (annihilation) operator: $\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}\,(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3}), \frac{1}{\sqrt{2}}(\theta^0 - \theta^3)\,\theta^1\theta^2\}_+$. Applying $(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})$ on $(\theta^0 - \theta^3)$ gives two, while $\frac{\partial}{\partial\theta^2}\,\frac{\partial}{\partial\theta^1}$ applied on $\theta^1\theta^2$ gives one.

**i.e.** If we define the number operator $N^{\theta k}_i$ as follows

$$\hat{N}^{\theta k}_i = \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i \,, \tag{3.55}$$

it follows, taking into account the third equation of Eq. (3.54), that $(\hat{N}^{\theta k}_i)^2 = \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i \hat{b}^{\theta k\dagger}_i \hat{b}^{\theta k}_i = \hat{N}^{\theta k}_i$, requiring that the eigenvalue of this operator $\hat{N}^{\theta k}_i$ on the state $\hat{b}^{\theta k\dagger}_i |\phi^{k'}_{i'}>$ is 0 or 1.

**ii.** There is one additional irreducible representation of creation and annihilation operators in $d = 2(2n+1)$, which follows from the starting state

$$\begin{aligned}
&|\phi^2_1> = \hat{b}^{\theta 2\dagger}_{01} |1>\,,\\
&\hat{b}^{\theta 2\dagger}_{01} := (\frac{1}{\sqrt{2}})^{\frac{1}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})(\theta^{d-1} + i\theta^d)\,,\\
&\qquad \text{for } d = 2(2n+1)\,.
\end{aligned} \tag{3.56}$$

This state can not be obtained from the previous group of states, presented in Eqs. (3.52, 3.53) by the application of $\mathbf{S}^{ef}$, since each $\mathbf{S}^{ef}$ changes an even number of factors, never an odd one. All the other states of this new group of states follow from the starting one by the application of $\mathbf{S}^{ef}$. The corresponding creation and annihilation operators are

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
\hat{b}^{\theta 2}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0} + \frac{\partial}{\partial\theta^3})\,,\\
&\text{for } d = 2(2n+1)\,.
\end{aligned} \tag{3.57}$$

The corresponding annihilation operators follow by the Hermitian conjugation of the creation operators.

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta 2\dagger}_1\,,\\
\hat{b}^{\theta 2}_i &= (\hat{b}^{\theta 2\dagger}_i)^\dagger\,.
\end{aligned} \tag{3.58}$$

Also all these creation and annihilation operators fulfill the requirements for the creation and annihilation operators, presented in Eq. (3.54), due to the same reasons as in the first case.

It is true also in this case, as stated below Eq. (3.55), that $\hat{N}^{\theta k}_i$ applied on the state $|\phi^k_i>$ gives 0 or 1, due to the fact that $(\hat{N}^{\theta k}_i)^2 = \hat{N}^{\theta k}_i$. Thus the basic states, determined by the application of creation operators of Eqs. (3.53, 3.58) on the vacuum state $|1>$ have the properties required for fermions.

Let us now count the number of states in each of the two groups presented in Eqs. (3.53, 3.58).

There are in $(d = 2)$ two creation $((\theta^0 \mp \theta^1$, for $\eta^{ab} = diag(1,-1))$ and correspondingly two annihilation operators $(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^1})$, each belonging to its own group with respect to the Lorentz transformation operators, both fulfilling Eq. (3.54).

It is not difficult to see that the number of all creation operators of an odd Grassmann character in $d = 2(2n+1)$-dimensional space, with all $\gamma^a$'s included is equal to $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$.

We namely ask: In how many ways can one put on $\frac{d}{2}$ places $d$ different $\theta^a$'s. And the answer is — the central binomial coefficient for $x^{\frac{d}{2}}\, 1^{\frac{d}{2}}$ — with all $x$ different. This is just $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$. But we have counted all the states with an odd Grassmann character, while we know that these states belong to two different groups of representations with respect to the Lorentz group.

Correspondingly one concludes: *There are two groups of states in* $d = 2(2n+1)$ *with an odd Grassmann character with all* $\theta^a$*'s included, each of these two groups has*

$$\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!} \tag{3.59}$$

*members*.

In $d = 2$ we have two groups with one state, which have an odd Grassmann character, in $d = 6$ we have two groups of 10 states, in $d = 10$ we have two groups of 126 states with an odd Grassmann characters. And so on. All together there are $2^{d-1}$ the states of an odd Grassmann character.

Correspondingly we have in $d = 2(2n+1)$-dimensional spaces two groups of creation operators of the kind presented in Eqs. (3.53, 3.58), each kind with $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, creating states with an odd Grassmann character and the same number of annihilation operators. Creation and annihilation operators fulfill anticommutation relations presented in Eq. (3.54).

The rest of creation operators [and the corresponding annihilation operators] with the opposite Grassmann character than the ones studied so far — like $\theta^0\theta^1$ $[\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^0}]$ in $d = (1+1)$ $(\theta^0 \mp \theta^3)(\theta^1 \pm i\theta^2)$ $[(\frac{\partial}{\partial\theta^1} \mp i\frac{\partial}{\partial\theta^2})(\frac{\partial}{\partial\theta^0} \mp \frac{\partial}{\partial\theta^3}]$, $\theta^0\theta^3\theta^1\theta^2$ $[\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}]$ in $d = (3+1)$, do not fulfill the anticommutation relations required for fermions in Eq. (3.54), with $\hat{b}^{\theta 1}_i$ and $\hat{b}^{\theta 1\dagger}_i$ replaced by $\hat{b}^{\theta k}_i$ and $\hat{b}^{\theta k\dagger}_i$, $k = (1,2)$ and correspondingly with $\{\hat{b}^{\theta k}_i, \hat{b}^{\theta l\dagger}_j\}|\phi_{og}>= \delta_{kl}\,\delta_{ij}|\phi_{og}>$, $(k,l) = (1,2), (i,j)$ running from $(1,\ldots,\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!})$.

All the states $|\phi^k_i>$, $k = (1,2)$, generated by the creation operators, Eqs. (3.54, 3.58), on the vacuum state $|\phi_{og}>$ $(= |1>)$ are the eigenstates of the Cartan subalgebra operators and are orthogonal and normalized with respect to the norm of Eq. (3.31)

$$\begin{aligned} &< \phi^k_i|\phi^{k'}_j> = \delta_{ij}\,\delta^{kk'}\,,\\ &(k,k') = (1,2)\,,\ (i,j) = (1,2,\ldots,\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!})\,. \end{aligned} \tag{3.60}$$

All these basic states describing the internal degrees of freedom can be used to solve Eq. (3.43) for free massless "fermions", with the part in ordinary space proportional to $e^{-ip^a x_a}$. The eigenstates of Eq. (3.43) are superposition of the basic states $|\phi^k_i>$ with coefficients depending on momentum $p^a$, $a = (0,1,2,3,5,\dots,d)$

$$\begin{aligned}\hat{b}^{\theta k\dagger}_{sp} &= \sum_i c^k{}_{spi}\,\hat{b}^{\theta k\dagger}_i\,,\\ |\phi^k_{sp}> &= \hat{b}^{\theta k\dagger}_{sp}\,|\phi_{og}>\,,\\ |\phi^k_{sp}> &= \sum_i c^k{}_{spi}\,|\phi^k_i>\,,\end{aligned} \tag{3.61}$$

$s$ represents different solutions of the equations of motion, and, since they are orthogonalized, they fulfill the relation $<\phi^k_{sp}|\phi^{k'}_{s'p'}>=\delta_{kk'}\,\delta_{ss'}\,\delta^{pp'}$, where we assumed the discretization of momenta.

The corresponding creation operators, creating the basic states describing free massless "fermions" — $\hat{b}^{\theta k\dagger}_{sp}$ — are superposition of creation operators $\hat{b}^{\theta k\dagger}_i$, $\hat{b}^{\theta k\dagger}_{sp} = \sum_i c^k{}_{spi}\hat{b}^{\theta k\dagger}_i$ and fulfill together with the corresponding annihilation operators $\hat{b}^{\theta k}_{sp} = (\hat{b}^{\theta k\dagger}_{sp})^\dagger$ the relations

$$\begin{aligned}\{\hat{b}^{\theta k}_{sp}, \hat{b}^{\theta k'\dagger}_{s'p'}\}_+|\phi_{og}> &= \delta^k_{k'}\,\delta_{ss'}\delta_{pp'}\,|\phi_{og}>\,,\\ \{\hat{b}^{\theta k}_{sp}, \hat{b}^{\theta k'}_{s'p'}\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\ \{\hat{b}^{\theta k\dagger}_{sp}, \hat{b}^{\theta k'\dagger}_{s'p'}\}_+|\phi_{og}> &= 0\,|\phi_{og}>\,,\\ \hat{b}^{\theta k}_{sp}|\phi_{og}> &= 0\,|\phi_{og}>\,,\\ \hat{b}^{\theta k\dagger}_{sp}|\phi_{og}> &= |\phi^k_{sp}>\,,\\ |\phi_{og}> &= |1>\ .\end{aligned} \tag{3.62}$$

Again index $k = (1,2)$ in $(\hat{b}^{\theta 1}_{sp}, \hat{b}^{\theta 1\dagger}_{sp})$ $(\hat{b}^{\theta 2}_i, \hat{b}^{\theta\dagger 2}_i)$ denotes creation and annihilation operators of one of the two groups of states describing the internal space of "fermions", reachable by $\mathbf{S}^{ab}$, and $\hat{b}^{\theta k\dagger}_{sp}$ creates the state for a particular momentum in ordinary space $p^a$, solving Eq. (3.43).

The number operator for a "fermion" state $|\phi^k_{sp}>$ is now

$$\begin{aligned}\hat{N}^{\theta k}_{sp} &= \hat{b}^{\theta k\dagger}_{sp}\hat{b}^{\theta k}_{sp}\,,\\ (\hat{N}^{\theta k}_{sp})^2 &= \hat{N}^{\theta k}_{sp}\,,\end{aligned} \tag{3.63}$$

with the eigenvalues 0 or 1, since the states of a chosen discretized $p^a$ are orthogonal. Correspondingly each state can be occupied or empty. If $|1^{\theta k}_{s_1p_1}, 1^{\theta k}_{s_2p_2}, 1^{\theta k}_{s_3p_3},\dots, 0^{\theta k}_{s_kp_k},\dots,0^{\theta k}_{s_lp_l},\dots,>$ is a $n$ particle state of "fermions" (and "antifermions"), where 1 denotes the occupied state and 0 the unoccupied state, then it follows, for example, due to the third line in Eq. (3.62), that

$$\begin{aligned}&\hat{b}^{\theta k\dagger}_{s_ip_i}\,\hat{b}^{\theta k\dagger}_{s_jp_j}\,|1^{\theta k}_{s_1p_1}, 1^{\theta k}_{s_2p_2}, 1^{\theta k}_{s_3p_3},\dots,0^{\theta k}_{s_ip_i},\dots,0^{\theta k}_{s_jp_j},\dots,>=\\ &-\hat{b}^{\theta k\dagger}_{s_jp_j}\,\hat{b}^{\theta k\dagger}_{s_ip_i}|1^{\theta k}_{s_1p_1}, 1^{\theta k}_{s_2p_2}, 1^{\theta k}_{s_3p_3},\dots,0^{\theta k}_{s_ip_i},\dots,0^{\theta k}_{s_jp_j},\dots,>\ .\end{aligned} \tag{3.64}$$

Any $n$ "fermion" state is therefore a product of $n$ creation operators $\hat{b}^{\theta k\dagger}_{i\,p_k}$ as presented in Eq. (3.47).

The number operator for "fermions" in the n-particle state of Eq. (3.64) is correspondingly

$$\hat{N}^{\theta} = \sum_{k,s_ip_i} \hat{N}^{\theta k}_{s_ip_i} \tag{3.65}$$

When coefficients ${c^k}_{spi}$ depend also on coordinates $x^a$ (for free "fermions" ${\mathbf{c}^k}_{spi}(x) = {c^k}_{spi} \cdot e^{-ip_a x^a}$), it follows for $p^a p_a = 0$

$$\hat{\mathbf{b}}^{\theta k\dagger}_{s}(x^0, \vec{x}) = \sum_i \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\, {\mathbf{c}^k}_{spi}(x)\, \hat{b}^{\theta k\dagger}_{i}\,.$$

$$\begin{aligned}
&\{\hat{\mathbf{b}}^{\theta k}_{s}(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_{+}|\psi_{oc}> = \delta^{kk'}\,\delta^{ss'}\delta^{d-1}(\vec{x}-\vec{y})\,|\psi_{oc}>\,,\\
&\{\hat{\mathbf{b}}^{\theta k\dagger}_{s}(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_{+}|\psi_{oc}> = 0\,, \quad \{\hat{\mathbf{b}}^{\theta k}_{s}(x^0,\vec{x}),\, \hat{\mathbf{b}}^{\theta k'\dagger}_{s'}(x^0,\vec{y})\}_{+}|\psi_{oc}>= 0\,.
\end{aligned} \tag{3.66}$$

It is discussed in the subsection 3.3.3 how do discrete symmetry operators in the Grassmann case take care of "fermion" and "antifermion" states.

Let us now take into account Eq. (3.45) with

$$\mathcal{L}_G = \frac{1}{4}\{\hat{\phi}^{\dagger}\gamma^0_G\tilde{\gamma}^a(\hat{p}_a\phi) - (\hat{p}_a\phi^{\dagger})\gamma^0_G\tilde{\gamma}^a\phi\}.$$

The Euler-Lagrange equations lead to $-i\frac{1}{2}\gamma^0_G\tilde{\gamma}^a\hat{p}_a\phi = 0$ and $i\frac{1}{2}\hat{p}_a\phi^{\dagger}\gamma^0_G\tilde{\gamma}^a = 0$.

Let us find the Hamilton function for a second quantized field: $\hat{\phi}(x^0,\vec{y})$, generated by one of the creation operators $\hat{\mathbf{b}}^{\theta\dagger}_{s}$ on the vacuum state $|\phi_{og}>$,

$$\begin{aligned}
\Pi_{\hat{\phi}} &= \frac{\partial\mathcal{L}_G}{\partial(\hat{p}_0\hat{\phi})} = \frac{1}{4}\,\hat{\phi}^{\dagger}\gamma^0_G\tilde{\gamma}^0\,, \quad \Pi_{\hat{\phi}^{\dagger}} = \frac{\partial\mathcal{L}_G}{\partial(\hat{p}_0\hat{\phi}^{\dagger})} = -\frac{1}{4}\,\gamma^0_G\tilde{\gamma}^0\hat{\phi}\,,\\
\mathcal{H}_G &= \Pi_{\phi}\,(\hat{p}_0\hat{\phi}) + (\hat{p}_0\hat{\phi}^{\dagger})\Pi_{\hat{\phi}^{\dagger}} - \mathcal{L}_G\,,\\
&= \frac{i}{4}\,[\hat{\phi}^{\dagger}\gamma^0_G\tilde{\gamma}^i(\hat{p}_i\hat{\phi}) - (\hat{p}_i\hat{\psi}^{\dagger})\gamma^0_G\tilde{\gamma}^i\hat{\phi}\,]\,,\\
H_G &= \int d^{d-1}x\,\mathcal{H}_G\,.
\end{aligned} \tag{3.67}$$

A vector $\hat{\phi}$ depends on $k = (I, II)$ and on spins (what in $d = (3+1)$ manifests as spins and charges).

Hamilton function is obviously an odd Grassmann object and *does not define the energy of the system*. However, if assuming the relation: $\frac{i}{2}\gamma^0 p_0\,\hat{\phi}^k(x^0,\vec{x}) = \{\hat{\phi}^k(x^0,\vec{x}),\, H_G\}_{-}$, one still ends up with the equations of motion, Eq. (3.45). One namely obtains

$$\gamma^0\hat{p_0}\hat{\phi}^k(t,\vec{x}) = \left\{\,\hat{\phi}^k(t,\vec{x})\,,\, H_G\right\}_{-} = -\gamma^0_G\tilde{\gamma}^i\hat{p}_i\hat{\phi}^k(t,\vec{x})\,, \tag{3.68}$$

what might help to find the procedure to define the energy for the interacting "Grassmann fermions". One must at this point either give up with the Grassmann "fermions" with the integer spins or find a consistent unconventional way to define the energy, like the one suggested in Eq. (3.68).

### 3.3.2 Second quantization in Clifford space

In Grassmann space the requirement that products of "eigenstates" of the Cartan subalgebra operators form the creation and annihilation operators, obeying the relations of Eq. (3.54), reduces the number of creation operators and correspondingly the number of states from $2^d$ (allowed for "eigenstates" of the Cartan subalgebra operators) to two isolated groups of $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators. (There are no generators of the Lorentz transformations $\mathbf{S}^{ab}$ that would connect both groups of states and correspondingly there are no families.)

Let us study what happens, when, let say, $\gamma^a$'s are used to create the basis and correspondingly also to create the creation and annihilation operators. Here we briefly follow Ref. [50].

Let us point out that $\gamma^a$ is expressible with $\theta^a$ and its derivative ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$), Eq. (3.4), and that we again require that creation (annihilation) operators create (annihilate) states, which are "eigenstates" (Eq. (3.72)) of the Cartan subalgebra operators, Eq. (3.110). Then the application of $\tilde{\gamma}^a$ on any Clifford algebra object $A(\gamma^a)$, (determined by $\gamma^a$'s), can be evaluated as follows, Eq. (3.29, 3.30),

$$(\tilde{\gamma}^a A = i(-)^{(A)} A\gamma^a)|\psi_{oc}>\,, \tag{3.69}$$

where $(-)^{(A)} = -1$, if $A$ is an odd Clifford algebra object and $(-)^{(A)} = 1$, if $A$ is an even Clifford algebra object, while $|\psi_{oc}>$ is the vacuum state, replacing the vacuum state in the Grassmann case $|\psi_{og}>= |1\ >$ with the one of Eq. (3.79), in accordance with the relation of Eqs. (3.4, 3.32, 3.31), Refs. [50,10]. We could as well make a choice of $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$ instead of $\gamma^a$'s to create the basic states, exchanging correspondingly the role of $\gamma^a$ and $\tilde{\gamma}^a$ [9]).

Making a choice of the Cartan subalgebra "eigenstates" of $S^{ab}$, Eq. (3.27), one defines nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$

$$\begin{aligned}
\stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{(k)}{}^2 = 0\,,\\
\stackrel{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad \stackrel{ab}{[k]}{}^2 = \stackrel{ab}{[k]}\,,
\end{aligned} \tag{3.70}$$

where $k^2 = \eta^{aa}\eta^{bb}$. Recognizing that the Hermitian conjugate values of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ are

$$\stackrel{ab}{(k)}{}^\dagger = \eta^{aa}\stackrel{ab}{(-k)}, \quad \stackrel{ab}{[k]}{}^\dagger = \stackrel{ab}{[k]}\,, \tag{3.71}$$

while the corresponding "eigenvalues" of $S^{ab}$ and $\tilde{S}^{ab}$ on nilpotents and projectors, Eq. (3.27), are

$$\begin{aligned}
S^{ab}\stackrel{ab}{(k)} &= \frac{k}{2}\stackrel{ab}{(k)}\,, \quad S^{ab}\stackrel{ab}{[k]} = \frac{k}{2}\stackrel{ab}{[k]}\,,\\
\tilde{S}^{ab}\stackrel{ab}{(k)} &= \frac{k}{2}\stackrel{ab}{(k)}\,, \quad \tilde{S}^{ab}\stackrel{ab}{[k]} = -\frac{k}{2}\stackrel{ab}{[k]}\,,
\end{aligned} \tag{3.72}$$

[9] In the case that we would choose $\tilde{\gamma}^a$'s instead of $\gamma^a$'s, Eq.(3.4), the role of $\tilde{\gamma}^a$ and $\gamma^a$ should be then correspondingly exchanged in Eq. (3.69).

we find for $d = 2(2n+1)$ that from the starting state made as a product of an odd number of only nilpotents

$$
\begin{aligned}
|\psi^1_1> &= \hat{b}^{1\dagger}_1|\psi_{oc}>\,,\\
\hat{b}^{1\dagger}_1 &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{35}{(+)}\cdots \overset{d-3\; d-2}{(+)} \;\overset{d-1\; d}{(+)}\,,\\
\hat{b}^{1}_1 &= (\hat{b}^{1\dagger}_1)^\dagger = \overset{d-1\; d}{(-)}\; \overset{d-3\; d-2}{(-)} \cdots \overset{35}{(-)}\overset{12}{(-)}\overset{01}{(-i)}\,,
\end{aligned} \tag{3.73}
$$

having correspondingly an odd Clifford character, all other states of the same Lorentz representation, there are $2^{\frac{d}{2}-1}$ members, follow by the application of $S^{cd}$ (which do not belong to the Cartan subalgebra) on the starting state [10], Eq. (3.110), ($S^{cd}\,|\psi^1_1>=|\psi^1_i>$).

$$
\begin{aligned}
\hat{b}^{1\dagger}_i &\propto S^{ab}\ldots S^{ef}\hat{b}^{1\dagger}_1\,, \quad |\psi^1_i>= S^{ab}..S^{ef}|\psi^1_1>\,,\\
\hat{b}^{1}_i &\propto \hat{b}^{1}_1 S^{ef}\ldots S^{ab}\,,
\end{aligned} \tag{3.74}
$$

with $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$. We make a choice of the proportionality factors so that the corresponding states $|\psi^1_i>= \hat{b}^{1\dagger}_i|\psi_{oc}>$ are normalized [50,10].

The operators $\tilde{S}^{cd}$, which belong to the Cartan subalgebra of $\tilde{S}^{ab}$, Eq. (3.110), generate "eigenstates" of the Cartan subalgebra operators ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}$), with the eigenvalues which determine the "family" quantum numbers. There are $2^{\frac{d}{2}-1}$ families. From the starting new member with a different "family" quantum number the whole Lorentz representation of family members with this "family" quantum number follows by the application of $S^{ef}$: $S^{ab}\cdots S^{ef}\;\tilde{S}^{cd}|\psi^1_1>=|\psi^\alpha_i>$. All states of one Lorentz representation of any particular "family" quantum number have an odd Clifford character, since neither $S^{cd}$ nor $\tilde{S}^{cd}$ — both of an even Clifford character — can change the odd character of the starting state.

Any vector $|\psi^\alpha_i>$ follows from the starting vector, Eqs. (3.73), by the application of either $\tilde{S}^{ef}$, which change the family quantum number, or $S^{gh}$, which change the family member quantum number of a particular family or with the corresponding product of $S^{ef}$ and $\tilde{S}^{ef}$

$$
|\psi^\alpha_i> \propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}|\psi^1_i>\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}|\psi^1_1>\,. \tag{3.75}
$$

Again, $\alpha$ denotes "family" quantum numbers, $i$ denotes family member quantum number. Correspondingly we define $\hat{b}^{\alpha\dagger}_i$ (up to a constant) to be

$$
\begin{aligned}
\hat{b}^{\alpha\dagger}_i &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1\\
&\propto S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_1 S^{ab}\cdots S^{ef}\,.
\end{aligned} \tag{3.76}
$$

This last expression follows due to the property of the Clifford object $\tilde{\gamma}^a$ and correspondingly of $\tilde{S}^{ab}$, presented in Eqs. (3.69, 3.120).

We accordingly have for an annihilation operator $\hat{b}^\alpha_i (= (\hat{b}^{\alpha\dagger}_i)^\dagger)$

$$
\hat{b}^\alpha_i = (\hat{b}^{\alpha\dagger}_i)^\dagger \propto S^{ef}\cdots S^{ab}\hat{b}^1_1 S^{pr}\cdots S^{mn}\,. \tag{3.77}
$$

[10] The smallest number of all the generators $S^{ac}$, which do not belong to the Cartan subalgebra, Eq. (3.110), needed to create from the starting state all the other members, is $2^{\frac{d}{2}-1}-1$. This is true for both even dimensional spaces – $2(2n+1)$ and $4n$.

The proportionality factor ought to be chosen so that the corresponding states $|\psi^{\alpha}_{i}>=\hat{b}^{\alpha\dagger}_{i}|\psi_{oc}>$ are normalized when the vacuum state $|\psi_{oc}>$ is normalized, $<\psi_{oc}|\psi_{oc}>=1$, while all the states belonging to the physically acceptable states, like $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\stackrel{78}{[-]}\cdots\stackrel{d-3\,d-2}{(+)}\;\stackrel{d-1\,d}{(+)}|\psi_{oc}>$, must not give zero for either $d=2(2n+1)$ or for $d=4n$. We also want that states, obtained by the application of ether $S^{cd}$ or $\tilde{S}^{cd}$ or both, are orthogonal. To make a choice of the vacuum it is needed to know the relations of Eq. (3.116). It must be

$$
\begin{aligned}
&<\psi_{oc}|\cdots\stackrel{ab}{(k)}^{\dagger}\cdots|\cdots\stackrel{ab}{(k')}\cdots|\psi_{oc}>=\delta_{kk'}\,,\\
&<\psi_{oc}|\cdots\stackrel{ab}{[k]}^{\dagger}\cdots|\cdots\stackrel{ab}{[k']}\cdots|\psi_{oc}>=\delta_{kk'}\,,\\
&<\psi_{oc}|\cdots\stackrel{ab}{[k]}^{\dagger}\cdots|\cdots\stackrel{ab}{(k')}\cdots|\psi_{oc}>=0\,.
\end{aligned}
\tag{3.78}
$$

We must choose the vacuum state in a way that fulfills the above requirements as well as the requirements $\hat{b}^{\beta\dagger}_{i}|\psi_{oc}>\neq 0$ and $\hat{b}^{\beta}_{i}|\psi_{oc}>=0$ for all members $i$ of any family $\beta$. Since any $\tilde{S}^{eg}$ changes $\stackrel{ef}{(+)}\stackrel{gh}{(+)}$ into $\stackrel{ef}{[+]}\stackrel{gh}{[+]}$ and $\stackrel{ab}{[+]}^{\dagger}=\stackrel{ab}{[+]}$, while $\stackrel{ab}{(+)}^{\dagger}\stackrel{ab}{(+)}=\stackrel{ab}{[-]}$, the vacuum state $|\psi_{oc}>$ must be

$$
\begin{aligned}
&|\psi_{oc}>=\\
&\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}+\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}+\stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]}+\cdots|0>\,,\\
&\text{for } d=2(2n+1)\,,
\end{aligned}
\tag{3.79}
$$

$n$ is a positive integer. There are $2^{\frac{d}{2}-1}$ summands, since we can start with the vacuum state $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}|1>$, which fulfills the requirement for $\hat{b}^{1\dagger}_{1}|\psi_{oc}>\neq 0$ and $\hat{b}^{1}_{1}|\psi_{oc}>=0$, and then we must step by step replace all possible pairs of $\stackrel{ab}{[-]}\cdots\stackrel{ef}{[-]}$ in the starting part $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-1\,d}{[-]}$ into $\stackrel{ab}{[+]}\cdots\stackrel{ef}{[+]}$ and include new terms into the vacuum state so that the last $(2n+1)$ summands have for $d=2(2n+1)$ case, $n$ is a positive integer, only one factor $[-]$ and all the rest $[+]$, each $[-]$ at different position [11].

This vacuum has all the spins, either with respect to $S^{ab}$ or with respect to $\tilde{S}^{ab}$, equal to zero.

The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$,
while there is

$$
2^{\frac{d}{2}-1}\,2^{\frac{d}{2}-1}
\tag{3.80}
$$

[11] The choice of Eq. (3.79) for the vacuum state is not unique. If one would multiply any of summands by a number $\beta_{\alpha}$, where $\alpha$ represents the $\alpha$-th family, and then multiply each of $2^{\frac{d}{2}-1}$ members of creation operators belonging to this family $\hat{b}^{\alpha\dagger}_{i}$ by $\sqrt{\beta_{\alpha}}$ and the corresponding annihilation operator $\hat{b}^{\alpha}_{i}$ by $\sqrt{\beta^{*}_{\alpha}}$, $\beta^{*}_{\alpha}$ is the complex conjugated value of $\beta_{\alpha}$, it would still be true that $\hat{b}^{\alpha}_{i}\,\hat{b}^{\beta\dagger}_{j}=\delta^{\alpha\beta}\delta_{ij}$ times the corresponding summand of the vacuum back.

number of creation operators, defining the orthonormalized states when applying on the vacuum state of Eqs. (3.79) and the same number of annihilation operators, which are Hermitian conjugated to creation operators. Again, operators $\tilde{S}^{ab}$ connect members of different families, operators $S^{ab}$ generate all the members of one family.

Paying attention on only internal degrees of freedom, that is on the spin, the creation and annihilation operators must fulfill the relations

$$
\begin{aligned}
\{\hat{b}^{\alpha}_{i}, \hat{b}^{\alpha'\dagger}_{j}\}_{+}|\psi_{oc}> &= \delta^{\alpha\alpha'}\,\delta_{ij}|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha}_{i}, \hat{b}^{\alpha'}_{j}\}_{+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\dagger}_{i}, \hat{b}^{\alpha'\dagger}_{j}\}_{+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{\alpha}_{i}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{\alpha\dagger}_{i}|\psi_{oc}> &= |\psi^{\alpha}_{i}>\,,
\end{aligned}
\tag{3.81}
$$

with $(i, j)$ determining family members quantum numbers and $(\alpha, \alpha')$ denoting "family" quantum numbers.

Only Clifford odd objects fulfill the relations of Eq. (3.81), since the odd Clifford objects anti-commute (like: $\{(\gamma^0 - \gamma^3), (\gamma^1 + i\gamma^2)\}_+ = 0$), while the Clifford even objects commute (like: $\{(1-\gamma^0\gamma^3), (1-i\gamma^1\gamma^2)\}_+ = 2\,(1-\gamma^0\gamma^3)(1-i\gamma^1\gamma^2)$).

The reader can find the detailed proofs for the above statements, for either $d = 2(2n+1)$ or $d = 4n$, in Refs. [50,10].

Let us again, like in the Grassmann case, Eq. (3.62), look for the creation (and their annihilation operators) which, when applied on the vacuum state, Eq. (3.79), solve the equation of motion, Eq. (3.36). The solution for each momentum $p^a_k$, $a = (1, \dots, d)$, for discretized values of momenta, is a superposition of $\hat{b}^{\alpha\dagger}_{i}$,

$$
\hat{b}^{\alpha\dagger}_{s\,p_k} = \sum_i c^{\alpha}{}_{si}(p_k)\,\hat{b}^{\alpha\dagger}_{i}\,,
\tag{3.82}
$$

applied on the vacuum state, Eq. (3.79). Since $\hat{b}^{\alpha\dagger}_{i}$ and $\hat{b}^{\alpha}_{j}$ fulfill the relations of Eq. (3.81) and, if the states for two different momenta are orthogonalized, it follows

$$
\begin{aligned}
\{\hat{b}^{\alpha}_{s\,p_k}, \hat{b}^{\alpha'\dagger}_{s'\,p_l}\}_{+}|\psi_{oc}> &= \delta^{\alpha\alpha'}\,\delta_{ss'}\,\delta_{p_k\,p_l}\,|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha}_{s\,p_k}, \hat{b}^{\alpha'}_{s'\,p_l}\}_{+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\{\hat{b}^{\alpha\dagger}_{s\,p_k}, \hat{b}^{\alpha'\dagger}_{s'\,p_l}\}_{+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{\alpha}_{s\,p_k}|\psi_{oc}> &= 0\,|\psi_{oc}>\,,\\
\hat{b}^{\beta\dagger}_{s\,p_k}|\psi_{oc}> &= |\psi^{\alpha}_{s\,p_k}>\,,
\end{aligned}
\tag{3.83}
$$

with the vacuum state $|\psi_{oc}>$ defined in Eq. (3.79), with $s$ denoting the corresponding solution of equations of motion and for a discretized momentum space.

The number operator of a particular solution $s$, a particular momentum $p_k$ and a particular "family" $\alpha$,

$$
\hat{N}^{\alpha}_{sp_k} = \hat{b}^{\alpha\dagger}_{sp_k}\,\hat{b}^{\alpha}_{sp_k}\,, \quad (\hat{N}^{\alpha}_{sp_k})^2 = \hat{N}^{\alpha}_{sp_k}\,,
\tag{3.84}
$$

has the eigenvalues 1 or 0.

The number of fermions in the $n$-particle state of Eq. (3.86) is correspondingly

$$\hat{N} = \sum_{\alpha, s, p_k} \hat{N}^{\alpha}_{s p_k} \,. \tag{3.85}$$

For a $n$ fermion and antifermion state, Eqs. (3.47, 3.48) in the introduction to Sect. 3.3, $|1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv}p_j}, \dots, >$ it follows, for example, due to the third line in Eq. (3.83), that

$$\begin{aligned} &\hat{b}^{\alpha'\dagger}_{s'p_i}\, \hat{b}^{\alpha''\dagger}_{s''p_j} |1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv}p_j}, \dots, >= \\ -\;&\hat{b}^{\alpha''\dagger}_{s''p_j}\, \hat{b}^{\alpha'\dagger}_{s'p_i} |1^{\alpha=1}_{s=1\,p_1}, 1^{\alpha=1}_{s=2\,p_1}, 1^{\alpha=1}_{s=3\,p_1}, \dots, 0^{\alpha}_{s\,p_i}, \dots, 0^{\alpha}_{s^{iv}p_j}, \dots, >, \end{aligned} \tag{3.86}$$

where 1 denotes the occupied state and 0 the unoccupied state, and $|1^{\alpha=1}_{s=1\,p_1} >= \hat{b}^{\alpha=1\dagger}_{s=1 p_1} |\psi oc >$.

Eq. (3.86, 3.47) demonstrates properties of Slater determinants. One fermion state is obviously second quantized by construction.

Two states with $n_1$ and $n_2$ fermions each, defined by $\hat{A}^{a\dagger}$ as $n_1$ products of $\hat{b}^{\alpha\dagger}_{sp_i}$ (which distinguish among themselves in at least one of the properties $(\alpha, s, p_i)$) and by $\hat{A}^{b\dagger}$ as $n_2$ products of $\hat{b}^{\beta\dagger}_{s'p_j}$ (which distinguish among themselves in at least one of the properties $(\alpha', s', p_j)$), applying on $|\psi_{oc}>$, must distinguish in either internal space or in the coordinate space, as it follows from Eq. (3.86), that the product of $\hat{A}^{a\dagger}$ and $\hat{A}^{b\dagger}$ applying on $|\psi_{oc}>$ would give a state with $(n_1 + n_2)$ fermions.

Let us add that the vacuum state, having the sum of the spins of both kinds of operators, $S^{ab}$ and $\tilde{S}^{ab}$, equal to zero and therefore neutral, remains neutral also when filled with fermions of all the spins, $S^{ab}$ and $\tilde{S}^{ab}$.

When coefficients $c^{\alpha}{}_{si}(p_k)$ depend also on coordinates $x^a$ (for free fermions $\mathbf{c}^{\alpha}{}_{si}(p_k, x) = c^{\alpha}{}_{si}(p_k) \cdot e^{-ip_a x^a}$), it follows for $p^a p_a = 0$,

$$\hat{\mathbf{b}}^{\alpha\dagger}_{s}(x^0, \vec{x}) = \sum_i \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\, \mathbf{c}^{\alpha}{}_{si}(p_k, x)\, \hat{b}^{\alpha\dagger}_{i} \,.$$

$$\begin{aligned} &\{\hat{\mathbf{b}}^{\alpha}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha'\dagger}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}> = \delta^{\alpha\alpha'}\, \delta_{ss'}\, \delta^{d-1}(\vec{x} - \vec{y})\, |\psi_{oc}>, \\ &\{\hat{\mathbf{b}}^{\alpha\dagger}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha'\dagger}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}> = 0, \quad \{\hat{\mathbf{b}}^{\alpha}_{s}(x^0, \vec{x}),\, \hat{\mathbf{b}}^{\alpha'}_{s'}(x^0, \vec{y})\}_+ |\psi_{oc}>= 0 \,. \end{aligned} \tag{3.87}$$

Let us now take into account Eq. (3.35) with

$$\mathcal{L}_C = \frac{1}{2}\{\hat{\psi}^{\dagger}\gamma^0\gamma^a(\hat{p}_a\psi) - (\hat{p}_a\psi^{\dagger})\gamma^0\gamma^a\psi\}.$$

The Euler-Lagrange equations lead to $\gamma^0\gamma^a\hat{p}_a\psi = 0$ and $-\hat{p}_a\phi^{\dagger}\gamma^0\gamma^a = 0$.

Let us look for the Hamilton function for fermions determined by one of the creation operators, like $\hat{\psi}^{\alpha}_{s}(x^0, \vec{x}) = \hat{\mathbf{b}}^{\alpha\dagger}_{s}(x^0, \vec{x})]|\psi_{oc}>$, which is already the second quantized state.

For a vector $\hat{\psi}$ and $\hat{\psi}^\dagger$ it therefore follows

$$\begin{aligned}
\Pi_{\hat{\psi}} &= \frac{\partial \mathcal{L}_C}{\partial(\hat{p}_0\hat{\psi})} = \frac{1}{2}\,\hat{\psi}^\dagger\,, \quad \Pi_{\hat{\psi}^\dagger} = \frac{\partial \mathcal{L}_C}{\partial(\hat{p}_0\hat{\psi}^\dagger)} = \frac{1}{2}\,\hat{\psi}\,,\\
\mathcal{H}_C &= \Pi_{\psi}\,(\hat{p}_0\hat{\psi}) + (\hat{p}_0\hat{\psi}^\dagger)\Pi_{\hat{\psi}^\dagger} - \mathcal{L}_C\,,\\
&= -\frac{1}{2}\left[\hat{\psi}^\dagger\,\gamma^0\,\gamma^i\,(\hat{p}_i\hat{\psi}) - (\hat{p}_i\hat{\psi}^\dagger)\gamma^0\gamma^i\hat{\psi}\,\right],\\
H_C &= \int d^{d-1}x\,\mathcal{H}_C\,,
\end{aligned} \tag{3.88}$$

Correspondingly one finds for a component $\hat{\psi}^\alpha_s(x^0,\vec{x})$ [74], $\vec{x}$ is a vector in $(d-1)$-dimensional coordinate space,

$$\begin{aligned}
\hat{p_0}\hat{\psi}^\alpha_s\,(x^0,\vec{x}) &= \left\{\,\hat{\psi}^\alpha_s\,(x^0,\vec{x})\,,\,H_C\right\}_-\\
&= \left\{\,\hat{\psi}^\alpha_s(x^0,\vec{x})\,,\int d^{d-1}x'\sum_{\alpha',s'}\,\hat{\psi}^{\alpha'\dagger}_{s'}\,(x^0,\vec{x'})\,\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{s'}(x^0,\vec{x'}))\,\right\}_-\\
&= \int d^{d-1}x'\sum_{\alpha's'}\left\{\,\hat{\psi}^\alpha_s(x^0,\vec{x})\,,\hat{\psi}^{\alpha'\dagger}_{s'}(x^0,\vec{x'})\,\right\}_+\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{\mathbf{s}'}\,(x^0,\vec{x'}))\\
&= -\gamma^0\gamma^i\,(\hat{p}_i\hat{\psi}^\alpha_{\mathbf{s}}(x^0,\vec{x}))\,.
\end{aligned} \tag{3.89}$$

(We took into account that $\gamma^0\gamma^i$ transforms $\hat{\psi}^{\alpha'}_{s'}\,(x^0,\vec{x'})$ into $\sum_{s''} c^{\alpha'}{}_{s's''}\,\hat{\psi}^{\alpha'}_{s''}(x^0,\vec{x'})$, which anticommute with $\hat{\psi}^\alpha_s(x^0,\vec{x})$ (Eq. (3.87)), we also assumed that states, obtained when operators operate on a vacuum state, do not contribute to the surface term. Integrating per partes and dropping the surface term simplifies $H_C$ into $-\int\sum d^{d-1}x'\,\hat{\psi}^{\alpha'\dagger}_{s'}(x^0,\vec{x'})\,\gamma^0\gamma^i\,(\hat{p'}_i\hat{\psi}^{\alpha'}_{s'}(x^0,\vec{x'}))$.) The obtained equations of motion agree with the ones from Eqs. (3.39, 3.40). Correspondingly the energy of the n-fermion state of free massless fermions created by $\hat{\mathbf{b}}^{\alpha\dagger}_s$ on the vacuum state $|\psi_{oc}>$ all with zero momentum $p_o$ (solving the Weyl equation Eqs. (3.36,3.40)) is equal to $E=\sum_{\alpha s}\hat{N}^\alpha_s p_0$. The current is correspondingly $\hat{\mathbf{j}}^a=\hat{\psi}^{\alpha\dagger}_{\mathbf{s}}\gamma^0\gamma^a\hat{\psi}^\alpha_{\mathbf{s}}$.

The observed fermions — quarks and leptons — manifest their properties obviously in $d=(3+1)$. The internal space in $d=(3+1)$ can therefore be used to describe the spin and handedness of massless fermions, in the *spin-charge-family* theory also families, while the internal space in $d\geq 5$ can be used to describe charges of fermions, contributing in the *spin-charge-family* theory as well to families.

One family representation contains in $d=2(2n+1), n=3$, $2^{\frac{d}{2}-1}=64$ members, described by the creation and annihilation operators fulfilling the anticommutation relations of Eq. (3.81), explaining from the point of view of $d=(3+1)$ spins, handedness and charges of the observed quarks and leptons and antiquarks and antileptons. Correspondingly there is no need for the negative energy "Dirac sea".

We discuss below discrete symmetry operators for both cases, the Clifford one and the Grassmann one, in d and in observable dimension $d=(3+1)$. In Subsect. 3.3.4 we present a few examples.

### 3.3.3 Discrete symmetries in Grassmann space and in Clifford space in $d$ and in $d = (3+1)$ part of the space

We have treated so far free massless fermions in Grassmann and in Clifford space. The fermion "nature" of states are in both spaces demonstrated by the fact that the corresponding creation and annihilation operators fulfill the anticommutation relations of Eq. (3.62) in Grassmann case and of Eq. (3.83) in Clifford space. Fermions — in both spaces — are in superposition of eigenstates of the Cartan subalgebra operators of $\mathbf{S}^{ab}$ in the Grassmann case, in the Clifford case they are in superposition of the Cartan subalgebra operators of $S^{ab}$ as well as of $\tilde{S}^{ab}$.

We distinguish in d-dimensional space two kinds of discrete symmetry $\mathcal{C}, \mathcal{P}$ and $\mathcal{T}$ operators with respect to the internal space in which the fermion properties are described.

In the Clifford case we have [65]

$$
\begin{aligned}
\mathcal{C}_{\mathcal{H}} &= \prod_{\gamma^a \in \Im} \gamma^a \, K\,, \\
\mathcal{T}_{\mathcal{H}} &= \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a \, K\, I_{x^0}\,, \\
\mathcal{P}^{(d-1)}_{\mathcal{H}} &= \gamma^0 \, I_{\vec{x}}\,, \\
I_x x^a &= -x^a\,, \quad I_{x^0} x^a = (-x^0, \vec{x})\,, \quad I_{\vec{x}} \vec{x} = -\vec{x}\,, \\
I_{\vec{x}_3} x^a &= (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)\,.
\end{aligned}
\tag{3.90}
$$

The product $\prod \gamma^a$ is meant in the ascending order in $\gamma^a$, $K$ stands for complex conjugation.

In the Grassmann case we correspondingly define

$$
\begin{aligned}
\mathcal{C}_G &= \prod_{\gamma^a_G \in \Im \gamma^a} \gamma^a_G \, K\,, \\
\mathcal{T}_G &= \gamma^0_G \prod_{\gamma^a_G \in \Re \gamma^a} \gamma^a_G \, K\, I_{x^0}\,, \\
\mathcal{P}^{(d-1)}_G &= \gamma^0_G \, I_{\vec{x}}\,,
\end{aligned}
\tag{3.91}
$$

$\gamma^a_G$ is defined in Eq. (3.22) as $\gamma^a_G = (1 - 2\theta^a \eta^{aa} \frac{\partial}{\partial \theta_a})$, while $I_x x^a = -x^a$, $I_{x^0} x^a = (-x^0, \vec{x})$, $I_{\vec{x}} \vec{x} = -\vec{x}$, $I_{\vec{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)$, like in the Clifford case. Let be noticed, that since $\gamma^a_G$ $(= -i\eta^{aa} \gamma^a \tilde{\gamma}^a)$ is always real as we see in Eq. (3.28) [12]. Since $\gamma^a$ is either real or imaginary, Eq. (3.22), we use in Eq. (3.91) $\gamma^a$ to make a choice of appropriate $\gamma^a_G$. In what follows we shall use the notation as in Eq. (3.91).

Let us define in the Clifford case and in the Grassmann case the operator "emptying" [13]. The operation "emptying$_{NH}$" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ in

[12] If we choose a real $\theta^a$, then $\gamma^a$ is real and $\tilde{\gamma}^a$ imaginary, if $\theta^a$ is imaginary, then $\gamma^a$ is imaginary and $\tilde{\gamma}^a$ real, as is demonstrated in Eq. (3.28).

[13] The operator "emptying" empties the "Dirac sea" of negative energies [65], although in the *spin-charge-family* theory is no need for the "Dirac sea" of negative energies, as

the Clifford case [65,7,9] (arxiv:1312.1541) and "emptying$_{NG}$" after the charge conjugation $\mathcal{C}_G$ in the Grassmann case, namely transforms the positive energy fermions into positive energy antifermions in both cases, solving Eq. (3.36) in the Clifford case, and Eq. (3.43) in the Grassmann case.

$$\begin{aligned}
\text{"emptying}_{NH}\text{"} &= \prod_{\Re\gamma^a} \gamma^a\, K \quad \text{in Clifford space}\,, \\
\text{"emptying}_{NG}\text{"} &= \prod_{\Re\gamma^a} \gamma^a_G\, K \quad \text{in Grassmann space}\,.
\end{aligned} \tag{3.92}$$

Then the anti-particle state creation operator to the corresponding particle state creation operator can be obtained by the application of

$$\mathbb{C}_{\mathcal{H},\mathcal{G}} = \text{"emptying}_{NH,NG}\text{"} \cdot \mathcal{C}_{\mathcal{H},\mathcal{G}} \tag{3.93}$$

$\mathbb{C}_{\mathcal{H}}$ and $\mathbb{C}_G$, with indexes $_{\mathcal{H}}$ and $_{NH}$ denoting the Clifford case and with $_{\mathcal{G}}$ and $_{NG}$ denoting the Grassman case, on the creation operator for a particle state, or opposite. Let us remind the reader that in the *spin-charge-family* theory, using the Clifford algebra, the family members of each family include fermions and antifermions — quarks and leptons and antiquarks and antileptons. This is the case also for Grassmann fermions and antifermions, but in this casethere are instead of families two by $\mathbf{S}^{ab}$ unconnected representations.

Ref. [65] proposes in the Clifford case the following discrete symmetry operators, manifesting dynamics in $d = (3+1)$

$$\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \prod_{\Im\gamma^m, m=0}^{3} \gamma^m \;\; \Gamma^{(3+1)}\, K\, I_{x^6,x^8,\ldots,x^d}\,, \\
\mathcal{T}_{\mathcal{N}} &= \prod_{\Re\gamma^m, m=1}^{3} \gamma^m \;\; \Gamma^{(3+1)}\, K\, I_{x^0}\, I_{x^5,x^7,\ldots,x^{d-1}}\,, \\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0\, \Gamma^{(3+1)}\, \Gamma^{(d)}\, I_{\vec{x}_3}\,, \\
\mathbb{C}_{\mathcal{N}} &= \gamma^0\gamma^5\gamma^7 \cdots \gamma^{d-1} I_{\vec{x}_3}\, I_{x^6,x^8,\ldots,x^d} \\
\mathcal{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0\,\gamma^2\, I_{\vec{x}_3} K\, I_{x^6,x^8,\ldots,x^d}\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0\,\gamma^5\, \cdots \gamma^{d-1}\, I_{\vec{x}_3}\, I_{x^6,x^8,\ldots,x^d}\,.
\end{aligned} \tag{3.94}$$

In the Grassmann case we use the Grassmann even, Hermitian and real operators $\gamma^a_G$, Eq. (3.22), to determine discrete symmetries in $((d-1)+1)$ space (as presented in Eq. (3.91)) and in $d = (3+1)$ space. In $(3+1)$ space we proceed —

we discussed already in the introduction of Sect. 3.3, for either Clifford or Grassmann fermions. The operation of "emptying$_{NH}$" after the charge conjugation $\mathcal{C}_{\mathcal{H}}$ in the Clifford case, which transforms the state put on the top of the Clifford "Dirac sea" into the corresponding negative energy state, namely creates the anti-particle state to the starting particle state, each anti-particle state, put on the top of the "Dirac sea", solving the Weyl equation in the Clifford case, Eq. (3.36).

in analogy with the operators in the Clifford case [65] — as follows

$$
\begin{aligned}
\mathcal{C}_{NG} &= \prod_{\gamma^m_G \in \Re\gamma^m} \gamma^m_G \, K \, I_{x^6 x^8 \ldots x^d} \,, \\
\mathcal{T}_{NG} &= \gamma^0_G \prod_{\gamma^m_G \in \Im\gamma^m} \gamma^m_G \, K \, I_{x^0} I_{x^5 x^7 \ldots x^{d-1}} \,, \\
\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{s=5}^{d} \gamma^s_G I_{\vec{x}} \,, \\
\mathbb{C}_{NG} &= \prod_{\gamma^s_G \in \Re\gamma^s} \gamma^s_G \,, I_{x^6 x^8 \ldots x^d} \,, \\
\mathcal{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \, \gamma^2_G \, K \, I_{\vec{x}_3} \, I_{x^6 x^8 \ldots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \gamma^0_G \prod_{\gamma^s_G \in \Im\gamma^s, s=5}^{d} \gamma^s_G \, I_{\vec{x}_3} \, I_{x^6 x^8 \ldots x^d} \,, \\
\mathbb{C}_{NG}\mathcal{T}_{NG}\mathcal{P}^{(d-1)}_{NG} &= \prod_{\gamma^s_G \in \Im\gamma^a} \gamma^a_G \, I_x K \,.
\end{aligned}
\tag{3.95}
$$

### 3.3.4 Examples of massless fermion and antifermion states in Clifford and in Grassmann space

Let us illustrate solutions for free fermion states, represented by the creation operators applied on the vacuum states for the Clifford and the Grassmann case in $((d-1)+1)$-dimensional space, representing indeed the contribution of a one fermion second quantized state in the Fock space of any number of fermions. We analyze states in both cases from the point of view $d = (3+1)$-dimensional space, with the momentum in ordinary space $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, so that the charges "seen" in $d = (3+1)$ are determined by the generators of the Lorentz transformations in the internal space — $S^{st}, (s,t) = (5,6,7,\cdots,d)$ in the Clifford case and $\mathbf{S}^{st}, (s,t) = (5,6,7,\cdots,d)$ in the Grassmann case. In the Clifford case we discuss one family in details (let be reminded that the generators $S^{ab}$ connect all the members belonging to one family, while $\tilde{S}^{ab}$ transform a particular member of one family into the same member of another family), commenting also on the appearance of families (all the families are reachable by $\tilde{S}^{ab}$) and present them briefly. In the Grassmann case different representations can not be reached by the generators of the Lorentz representations $\mathbf{S}^{ab}$. The discrete symmetry operators are in the Clifford case presented in Eq. (3.94), and in the Grassmann case in Eq. (3.95).

We start with examples in $d = (5+1)$-dimensional space, with charges determined by $S^{st}, (s,t) = (5,6)$ in the Clifford case and $\mathbf{S}^{st}, (s,t) = (5,6)$ in the Grassmann case.

The dimension $(13+1)$, used in the *spin-charge-family* theory to describe quarks and lepton as well the gauge fields and scalar fields, offers to free fermions at low energies additional charges, what explains observable properties of quarks and leptons. We present the creation operators creating all the states of one family members in Clifford space. The family members creation operators are reachable by $S^{ab}$. All the families are reachable from the starting family by $\tilde{S}^{ab}$ in the case of Clifford odd representations. In the case of the Clifford even representations there are $\tilde{S}^{ab}$ and $\gamma^a \tilde{\gamma}^a$, which take care of all irreducible representations.

In Ref. [50,68–70] $(d = 5+1)$-dimensional space is studied as a toy model to manifest that the break of symmetry from the higher dimensional space to the $(3+1)$-dimensional space *can* lead to massless fermions. Fermions were described in Clifford space. Here we briefly follow these references, and Refs. [65,66], adding new observations.

The first study of Grassmann case can be found in Ref. [46].

**Clifford fermions and antifermions** Let us start with the examples in the Clifford case. To make discussions transparent let us first treat the $d = (5+1)$ case. The $d = (13+1)$ case is not so easy to present in particular when also families are treated.

**Clifford case in $d = (5+1)$:**

In Table 3.4 the basic creation operators $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$ and their annihilation partners $\hat{b}^{\alpha}_{i=(ch,s)}$ in $d = (5+1)$ are presented for all four $(2^{\frac{d}{2}-1})$ families $\alpha = (I, II, III, IV)$. Index $i$ is devided into $s$, determining spin and into $ch$ to point out that $S^{56}$ represents the charge from the point of view of $d = (3+1)$, having two values, $+\frac{1}{2}$ and $-\frac{1}{2}$. The vacuum state, Eq. (3.79), is the sum of selfajoint operators $(\stackrel{03}{[-i]}\stackrel{12}{[-]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[+]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[-]}|\stackrel{56}{[+]}$, and $\stackrel{03}{[-i]}\stackrel{12}{[+]}|\stackrel{56}{[+]})$, needed that the first, second, third and fourth family creation operators, respectively, applying on the vacuum state, give nonzero value.

There are superposition of the basic creation operators — $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$ — which solve, applied on the vacuum state, the Weyl equation Eq. (3.36). Let us make the choice of $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$ to see how the spin in $d = (5,6)$ manifest charges in $d = (3+1)$.

$$p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)\,,$$
$$\hat{b}^{\alpha\dagger}_{(ch,sol)}(p)|\psi_{oc}> = \sum_{s} c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)\, \hat{b}^{\alpha\dagger}_{i=(ch,s)} e^{-ip_a x^a} |\psi_{oc}>\,, \tag{3.96}$$

where index $_{(ch,sol)}$, represents charges and different solutions, respectively, of the Weyl equation for massless free fermions.

We present in Eq. (3.97) the creation operators, the superposition of the first family members, presented in Table 3.4, which solve the Weyl equation, Eq. (3.36), for $p^a = (p^0, p^1, p^2, p^3, 0, 0)$. The corresponding annihilation operators follow by the Hermitian conjugation of the creation operators.

There are two fermion solutions with the charge $\frac{1}{2}$ and two antifermion solutions with the charge $-\frac{1}{2}$, both having the positive energy. The first two creation operators are related by the time reversal operator ($\mathcal{T}_{\mathcal{N}} = \gamma^1 \gamma^3 K I_{x^0} I_{x^5, x^7, \cdots, x^{d-1}}$), while the second two follow from the first two by the application of $\mathbb{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}}$ $= \gamma^0 \gamma^5 \cdots \gamma^{d-1} I_{\vec{x}_3} I_{x^6, x^8, \ldots, x^d}$, both are presented in Eq. (3.94).

| family $\alpha$ | $i=(ch,s)$ | $\hat{b}^{\alpha\dagger}_{ch,s}$ | $\hat{b}^{\alpha}_{ch,s}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{3+1}$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| I | $(\frac{1}{2},\frac{1}{2})$ | $\overset{03}{(+i)}\overset{12}{(+)}\vert\overset{56}{(+)}$ | $(-)\overset{56}{(-)}\vert(-)\overset{12}{(-)}\overset{03}{(-i)}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| I | $(\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{[-i]}\overset{12}{[-]}\vert\overset{56}{(+)}$ | $(-)\overset{56}{(-)}\vert\overset{12}{[-]}\overset{03}{[-i]}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| I | $(-\frac{1}{2},\frac{1}{2})$ | $\overset{03}{[-i]}\overset{12}{(+)}\vert\overset{56}{[-]}$ | $\overset{56}{[-]}\vert(-)\overset{12}{(-)}\overset{03}{[-i]}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| I | $(-\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{(+i)}\overset{12}{[-]}\vert\overset{56}{[-]}$ | $\overset{56}{[-]}\vert\overset{12}{[-]}\overset{03}{(-i)}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| II | $(\frac{1}{2},\frac{1}{2})$ | $\overset{03}{[+i]}\overset{12}{[+]}\vert\overset{56}{(+)}$ | $(-)\overset{56}{(-)}\vert\overset{12}{[+]}\overset{03}{[+i]}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| II | $(\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{(-i)}\overset{12}{(-)}\vert\overset{56}{(+)}$ | $(-)\overset{56}{(-)}\vert(-)\overset{12}{(+)}\overset{03}{(+i)}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| II | $(-\frac{1}{2},\frac{1}{2})$ | $\overset{03}{(-i)}\overset{12}{[+]}\vert\overset{56}{[-]}$ | $\overset{56}{[-]}\vert\overset{12}{[+]}\overset{03}{(+i)}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| II | $(-\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{[+i]}\overset{12}{(-)}\vert\overset{56}{[-]}$ | $\overset{56}{[-]}\vert(-)\overset{12}{(+)}\overset{03}{[+i]}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| III | $(\frac{1}{2},\frac{1}{2})$ | $\overset{03}{[+i]}\overset{12}{(+)}\vert\overset{56}{[+]}$ | $\overset{56}{[+]}\vert(-)\overset{12}{(-)}\overset{03}{[+i]}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| III | $(\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{(-i)}\overset{12}{[-]}\vert\overset{56}{[+]}$ | $\overset{56}{[+]}\vert\overset{12}{[-]}\overset{03}{(+i)}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| III | $(-\frac{1}{2},\frac{1}{2})$ | $\overset{03}{(-i)}\overset{12}{(+)}\vert\overset{56}{(-)}$ | $(-)\overset{56}{(+)}\vert(-)\overset{12}{(-)}\overset{03}{(+i)}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| III | $(-\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{[+i]}\overset{12}{[-]}\vert\overset{56}{(-)}$ | $(-)\overset{56}{(+)}\vert\overset{12}{[-]}\overset{03}{[+i]}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| IV | $(\frac{1}{2},\frac{1}{2})$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}$ | $\overset{56}{[+]}\vert\overset{12}{[+]}\overset{03}{(-i)}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| IV | $(\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}$ | $\overset{56}{[+]}\vert(-)\overset{12}{(+)}\overset{03}{[-]}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| IV | $(-\frac{1}{2},\frac{1}{2})$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}$ | $(-)\overset{56}{(+)}\vert\overset{12}{[+]}\overset{03}{[-i]}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| IV | $(-\frac{1}{2},-\frac{1}{2})$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}$ | $(-)\overset{56}{(+)}\vert(-)\overset{12}{(+)}\overset{03}{(-i)}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

**Table 3.4.** The basic creation operators — $\hat{b}^{\alpha\dagger}_{i=(ch,s)}$, $ch$ (charge) and $s$ (spin) explain the index $i$ — and their annihilation partners — $\hat{b}^{\alpha}_{i=(ch,s)}$ — are presented for the $d=(5+1)$-dimensional case. The basic creation operators are the products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra generators, ($S^{03}$, $S^{12}$, $S^{56}$), ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), presented in Eq. (3.110). Operators $\hat{b}^{\dagger}_{ch,s}$ and $\hat{b}_{ch,s}$ fulfill the commutaion relations of Eq. (3.81).

$$
\begin{aligned}
&\text{Creation operators for the fermion states in Clifford space for } d=(5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{I_1\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p}) &= \beta\left(\overset{03}{(+i)}\overset{12}{(+)}\,|\,\overset{56}{(+)}+\frac{p^1+ip^2}{|p^0|+|p^3|}\overset{03}{[-i]}\overset{12}{[-]}\,|\,\overset{56}{(+)}\right)e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{I_2\dagger}_{\frac{1}{2},-\frac{1}{2}}(\vec{p}) &= \beta^*\left(\overset{03}{[-i]}\overset{12}{[-]}\,|\,\overset{56}{(+)}-\frac{p^1-ip^2}{|p^0|+|p^3|}\overset{03}{(+i)}\overset{12}{(+)}\,|\,\overset{56}{(+)}\right)e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
&\text{Creation operators for the antifermion states in Clifford space for } d=(5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{I_3\dagger}_{-\frac{1}{2},\frac{1}{2}}(\vec{p}) &= -\beta\left(\overset{03}{[-i]}\overset{12}{(+)}\,|\,\overset{56}{[-]}+\frac{p^1+ip^2}{|p^0|+|p^3|}\overset{03}{(+i)}\overset{12}{[-]}\,|\,\overset{56}{[-]}\right)e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{I_4\dagger}_{-\frac{1}{2},-\frac{1}{2}}(\vec{p}) &= -\beta^*\left(\overset{03}{(+i)}\overset{12}{[-]}\,|\,\overset{56}{[-]}-\frac{p^1-ip^2}{|p^0|+|p^3|}\overset{03}{[-i]}\overset{12}{(+)}\,|\,\overset{56}{[-]}\right)e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,
\end{aligned}
\tag{3.97}
$$

Index ${}_{i=(1,2,3,4)}$ counts the solutions, while $\beta^*\beta=\frac{|p^0|+|p^3|}{2|p^0|}$ takes care that the corresponding states are normalized. All the states are correspondingly orthogonalized. The coefficients $c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)$ can be read from the solutions. The solutions have the definite handedness and orientation of the spin with respect to the momentum: $\hat{b}^{I_1\dagger}_{\frac{1}{2},\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=1$ and the spin and momentum both up, $\hat{b}^{I_2\dagger}_{\frac{1}{2},-\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=1$ and with spin and momentum both down, $\hat{b}^{I_3\dagger}_{-\frac{1}{2},\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)}=-1$ and the spin up and the momentum down, $\hat{b}^{I_4\dagger}_{-\frac{1}{2},-\frac{1}{2}}$ defines the state with $\Gamma^{(3+1)} = -1$, the spin down and the momentum up.

The same indexes — $c^{\alpha\, i=(ch,s)}{}_{(ch,sol)}(p)$ — define the solution of the Weyl equation also for for the rest three families presented in Table 3.4.

The phases of creation operators are in agreement with the application of discrete symmetry operators $\mathbb{C}_{\mathcal{N}} \cdot \mathcal{P}_{\mathcal{N}}$, and $\mathcal{T}_{\mathcal{N}}$.

Let us point out that the scalar fields, interacting with fermions (in the *spin-charge-family* theory [[4,3] and the references cited therein] the scalar fields origin in the spin connection fields — $\omega_{abc}$, the gauge fields of $S^{ab}$, and $\tilde{\omega}_{abc}$, the gauge fields of $\tilde{S}^{ab}$, appearing in Eq. (3.1) — with the space indexes $c \geq 5$) can make massless fermions massive [68,69,73,66]. In this case the creation operators (and correspondingly the annihilation operators) start to be superposition of basic operators of different charges $ch$ as well :

$$(\hat{b}^{\alpha\dagger}_{sol'}(p) = \sum_{ch,sol} c^{\alpha,ch,sol}{}_{ch,sol'}(p)\, \hat{b}^{\alpha\dagger}_{ch,sol} e^{-ip_a x^a})\, |\psi_{oc}> .$$

In this case the solutions of the corresponding equations of motion, presented in Eq. (3.97) for massless states, become superposition of different charges and different families.

For $p^m = (0,0,0), m = (1,2,3)$ and one massive family only [66] the creation operators for the basic states (usually used in text books [74,75] for massive states) are presented at Table 3.5. The creation operators, presented in Table 3.5, define

| family $\alpha$ | $\hat{b}^{\alpha\dagger}_{s,m}$ | $\tilde{S}^{12}$ |
|---|---|---|
| 1 | $\frac{1}{\sqrt{2}}\,(\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)} + \frac{m}{m_+}\,\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]})$ | $\frac{1}{2}$ |
| 2 | $\frac{1}{\sqrt{2}}\,(\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)} + \frac{m}{m_+}\,\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]})$ | $-\frac{1}{2}$ |

**Table 3.5.** The basic creation operators — $\hat{b}^{\alpha\dagger}_{s,m}$ — for massive states, the first with spin up and the second with spin down, are presented. $\hat{b}^{\alpha\dagger}_{s,m}\, e^{-imx^0}$, $s = \pm\frac{1}{2}$, solve the equations of motion $\{p_0 + \gamma^0(\stackrel{56}{(+)}\; m_+ + \stackrel{56}{(-)}\; m_-)\}\, \hat{b}^{\alpha\dagger}_{s,m}\, e^{-imx^0} = 0$, for the two positive energy states, (1,2), (one with spin up and the other with spin down). $m^2 = m_+ m_-$, $m_+ = -m_-$, $(p_0)^2 = m^2$, $p_{0a} = -\frac{1}{2} S^{cd}\omega_{cda}$ is assumed to be real [66].

orthonormal states when applied on the vacuum state and fulfill, together with the annihilation operators, the anticommutation relations presented in Eq. (3.83).

**Clifford case in $d = (13+1)$:**

There are $2^{\frac{14}{2}-1} = 64$ creation operators for family members of one family, all reachable from the starting one by $S^{ab}$. They are presented in Table 3.6, analyzed so that the internal degrees of freedom manifest in $d = (3+1)$ quantum numbers of the observed quarks and leptons. Applied on the vacuum state $|\psi_{oc}>$ they form in the *spin-charge-family* theory 64 basic states *for quarks and leptons and anti-quarks and anti-leptons* for each family. In the *spin-charge-family* theory there are two times four families — $2^{\frac{8}{2}-1}$ — getting masses after the two triplet scalar fields, the superposition of $\tilde{\omega}_{abc}$, $(a,b) = (0,1,\cdots,8)$ and three singlet scalar fields, the superposition of $\omega_{abc}$, $(a,b) = (5,6)$ or $(7,8)$ or $(9,\cdots 14)$, while $c = (5,6,7,8)$ for all these scalar fields, get nonzero vacuum expectation values at low energies [9,3,4,6,7].

Table 3.1 represents the creation operators creating 8 families of $\hat{u}^{c1\dagger}_R$ and of $\hat{\nu}^\dagger_R$. All the family members of each of these families follow by the application of $S^{ab}$.

All the rest of families not included in these eight families get in the spin-charge-family theory masses by the interaction with the condensate [9,3,4,6,7].

To the lower four families the three so far observed families of quarks and leptons belong.

| i | | $^a\hat{b}^\dagger_i$ | $\Gamma^{(3+1)}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^{33}$ | $\tau^{38}$ | $\tau^4$ | Y | Q |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | (Anti)octet, $\Gamma^{(7+1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons | | | | | | | | | |
| 1 | $\hat{u}^{c1\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 2 | $\hat{u}^{c1\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 3 | $\hat{d}^{c1\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 4 | $\hat{d}^{c1\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 5 | $\hat{d}^{c1\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 6 | $\hat{d}^{c1\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 7 | $\hat{u}^{c1\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 8 | $\hat{u}^{c1\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 9 | $\hat{u}^{c2\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 10 | $\hat{u}^{c2\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 11 | $\hat{d}^{c2\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 12 | $\hat{d}^{c2\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 13 | $\hat{d}^{c2\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 14 | $\hat{d}^{c2\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 15 | $\hat{u}^{c2\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 16 | $\hat{u}^{c2\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 17 | $\hat{u}^{c3\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 18 | $\hat{u}^{c3\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{2}{3}$ |
| 19 | $\hat{d}^{c3\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 20 | $\hat{d}^{c3\dagger}_R$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| 21 | $\hat{d}^{c3\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 22 | $\hat{d}^{c3\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ |
| 23 | $\hat{u}^{c3\dagger}_L$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 24 | $\hat{u}^{c3\dagger}_L$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{[-]}\,\vert\vert\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{\sqrt{3}}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
| 25 | $\hat{\nu}^{\dagger}_R$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}\stackrel{78}{(+)}\,\vert\vert\,\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | 0 | 0 |

*Continued on next page*

| i | | $^a\hat{b}^\dagger_i$ | $\Gamma^{(3+1)}$ | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^{33}$ | $\tau^{38}$ | $\tau^4$ | Y | Q |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | (Anti)octet, $\Gamma^{(7+1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons | | | | | | | | | |
| 26 | $\hat{\nu}^\dagger_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | 0 | 0 |
| 27 | $\hat{e}^\dagger_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-1$ | $-1$ |
| 28 | $\hat{e}^\dagger_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{2}$ | $-1$ | $-1$ |
| 29 | $\hat{e}^\dagger_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ |
| 30 | $\hat{e}^\dagger_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ |
| 31 | $\hat{\nu}^\dagger_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| 32 | $\hat{\nu}^\dagger_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 |
| 33 | $\hat{d}^{\bar{c}1\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 34 | $\hat{d}^{\bar{c}1\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 35 | $\hat{u}^{\bar{c}1\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 36 | $\hat{u}^{\bar{c}1\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 37 | $\hat{d}^{\bar{c}1\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 38 | $\hat{d}^{\bar{c}1\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 39 | $\hat{u}^{\bar{c}1\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 40 | $\hat{u}^{\bar{c}1\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 41 | $\hat{d}^{\bar{c}2\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 42 | $\hat{d}^{\bar{c}2\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 43 | $\hat{u}^{\bar{c}2\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 44 | $\hat{u}^{\bar{c}2\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 45 | $\hat{d}^{\bar{c}2\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 46 | $\hat{d}^{\bar{c}2\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 47 | $\hat{u}^{\bar{c}2\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 48 | $\hat{u}^{\bar{c}2\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{(+)}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 49 | $\hat{d}^{\bar{c}3\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 50 | $\hat{d}^{\bar{c}3\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 51 | $\hat{u}^{\bar{c}3\dagger}_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 52 | $\hat{u}^{\bar{c}3\dagger}_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| 53 | $\hat{d}^{\bar{c}3\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 54 | $\hat{d}^{\bar{c}3\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $\frac{1}{3}$ |
| 55 | $\hat{u}^{\bar{c}3\dagger}_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 56 | $\hat{u}^{\bar{c}3\dagger}_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{\sqrt{3}}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ | $-\frac{2}{3}$ |
| 57 | $\hat{e}^\dagger_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | 1 |
| 58 | $\hat{e}^\dagger_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | 1 |
| 59 | $\hat{\nu}^\dagger_L$ | $\overset{03}{[-i]}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | -1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| 60 | $\hat{\nu}^\dagger_L$ | $\overset{03}{(+i)}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | -1 | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 0 | 0 |
| 61 | $\hat{\nu}^\dagger_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| 62 | $\hat{\nu}^\dagger_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{(-)}\overset{78}{(+)}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| 63 | $\hat{e}^\dagger_R$ | $\overset{03}{(+i)}\overset{12}{[+]}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |
| 64 | $\hat{e}^\dagger_R$ | $\overset{03}{[-i]}\overset{12}{(-)}\vert\overset{56}{[+]}\overset{78}{[-]}\Vert\overset{9\,10}{[-]}\overset{11\,12}{[-]}\overset{13\,14}{[-]}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |

**Table 3.6.** The left handed ($\Gamma^{(13,1)} = -1$), multiplet of creation operators of spinors — the members of the fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and anti-quarks and the colourless leptons and anti-leptons — is presented in the massless basis using the technique presented in App. 3.7. It represent the left handed ($\Gamma^{(3+1)} = -1$, App. 3.7) weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, ($\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})$) and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, $\vec{\tau}^2 = \frac{1}{2}(S^{58} +$

$S^{67}, S^{57} - S^{68}, S^{56} + S^{78}$)) quarks and leptons and the right handed ($\Gamma^{(3+1)} = 1$), weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin $S^{12}$ up and down ($\pm\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ( $= (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, ($\vec{\tau}^3 = \frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}$)), carrying the "fermion charge" ($\tau^4 = \frac{1}{6}, = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$. The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). The same multiplet of creation operators represents also the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged anti-quarks and anti-leptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless anti-quarks and anti-leptons. Anti-quarks distinguish from anti-leptons again only in the $SU(3) \times U(1)$ part: Anti-quarks are anti-triplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anti-colourless anti-leptons carry the "fermion charge" ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The creation operators of opposite charges (anti-particle creation operators) are reachable from the particle ones besides by $S^{ab}$ also by the application of the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}_{\mathcal{N}}$, presented in Refs. [65,66]. The reader can find this Weyl representation also in Refs. [4,71,72,9] and in the references therein.

Table 3.6 represents in the *spin-charge-family* theory the basic creation operators for observed *quarks and leptons and anti-quarks and anti-leptons* for a particular family. Hermitian conjugation of the creation operators of Table 3.6 generates the corresponding annihilation operators, fulfilling together with the creation operators anticommutation relations for fermions of Eq. (3.81).

In observable dimension $d = (3+1)$ the $d = (13+1)$ case differs from $d = (5+1)$ case, Table 3.5, in a much reacher offer of charges. The kinematics of the fermion states in $d = (13+1)$, Table 3.6, in $d = (3+1)$ is, however, very similar to the one of Table 3.97.

The coefficients of the superposition of the basic creation operators — $\hat{b}_i^{\alpha\dagger}$ — which solve, applied on the vacuum state, the Weyl equation, Eq. (3.36), for the choice of $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, can be taken from Eq. (3.97). For the positive energy solution of spin $\frac{1}{2}$ one only has to replace $\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}$ by $\hat{u}^{c1\dagger}_{R,1/2}$ with spin $\frac{1}{2}$ and $\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}$ by $\hat{u}^{c1\dagger}_{R,-1/2}$ with spin $-\frac{1}{2}$. The coefficients, $\beta$ and $\frac{p^1+ip^2}{|p^0|+|p^3|}$, remain the one of the case with $d = (5+1)$.

The operator $\mathcal{T}_{\mathcal{N}} = \gamma^1\gamma^3 K I_{x^0} I_{x^5, x^7, \cdots, x^{d-1}}$ transforms this superposition of creation operators, $\beta\,(\hat{u}^{c1\dagger}_{R,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\hat{u}^{c1\dagger}_{R,-1/2}) \cdot e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})}$, into $\beta^*\,(\hat{u}^{c1\dagger}_{R,-1/2} - \frac{p^1-ip^2}{|p^0|+|p^3|}\hat{u}^{c1\dagger}_{R,1/2}) \cdot e^{-i(p^0x^0 + \vec{p}\cdot\vec{x})}$.

The operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^0\gamma^5\gamma^7\cdots\gamma^{d-1} I_{\vec{x}_3}\cdots I_{x^6, x^8, \cdots, x^d}$ transforms the positive energy solution creation operator for u quark, $\beta\,(\hat{u}^{c1\dagger}_{R,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\hat{u}^{c1\dagger}_{R,-1/2}) \cdot e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})}$, into the positive energy solution of anti-u quark, $-\beta\,(\hat{\bar{u}}^{\bar{c}1\dagger}_{L,1/2} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\hat{\bar{u}}^{\bar{c}1\dagger}_{L,-1/2}) \cdot e^{-i(p^0x^0 + \vec{p}\cdot\vec{x})}$.

One can proceed in the same way also for the $\hat{u}^{c1\dagger}_L$, $\hat{d}^{c1\dagger}_R$, and all the other quarks $^{ci}$, as well as for leptons.

Spins in higher dimensional space manifest charges in $d = (3+1)$, Table 3.6, provided that the angular momentum in ordinary space at higher dimensions do not contribute, which is supposed to be the case at low energies. All the creation operators of any family and any family member, or the orthogonal superposition of them, together with their Hermitian conjugate annihilation operators fulfill the anticommutation relations of Eqs. (3.81, 3.82, 3.83).

The commuting part of the operators of $S^{ab}$, Eq. (3.110), determine in $d = (3+1)$ the handedness ($\Gamma^{(3+1)}) = -4i\cdot S^{03}S^{12}$)), the spin ($S^{12}$), the third component of the weak $SU(2)$ charge ($\tau^{13}$), the third component of the second $SU(2)$ charge

($\tau^{23}$), the two components of the $SU(3)$ colour charge ($\tau^{33}, \tau^{38}$) and the "fermion charge" ($\tau^4$, originating in $U(1)$ from $SO(6)$, which includes $SU(3) \times U(1)$). The hypercharge $Y$, which is in the *standard model* "guessed" from the experimental data, is in the *spin-charge-family* theory equal to ($\tau^4 + \tau^{23}$), while electromagnetic charge $Q$ is, like in the *standard model*, equal to ($Y + \tau^{13}$).

One representation of creation operators with $2^{\frac{d}{2}-1}$ members includes all the left and the right handed coloured quarks and colourless leptons and left and right handed (anti coloured) antiquarks and (anti colourless) antileptons. The right handed neutrinos and the left handed antineutrinos, like all the other members of one Lorentz representation, carry the additional hypercharge (the additional superposition of $\tau^4$ and $\tau^{23}$) and are correspondingly not chargeless like in the *standard model*.

The sum of the charges, the sum of the spins and the sum of the handedness —properties defined with respect to $d = (3+1)$ — over all the members of one representation are equal to zero in any $d$, as it is the case of $d = (5+1)$. However, in the $d = (13+1)$ case this is true even within quarks and leptons separately and within antiquarks and antileptons separately. Let be repeated that this is so since the right handed neutrinos and the left handed antineutrinos are the regular members of one representation, as it is true for quarks and charged leptons. This can be checked in Table 3.6. Exclusion of the right handed neutrinos and left handed antineutrinos makes nonzero the sum of $(\Gamma^{(3+1)})$, $\tau^{23}$ and $\tau^4$ over the spinor part separately and correspondingly also over the antispinor part. The whole representation has even in this case sums over all the quantum numbers of spins and charges equal to zero.

**Grassmann "fermions" and "antifermions"** Let us represent creation and annihilation operators in Grassmann space, like we did in the Clifford case.

In the Grassmann case the representations in $d = (13+1)$ space start to be very large and correspondingly almost uncontrollable, Eq. (3.59). We learn in the Clifford case that at the low energy regime, when we treat the equations of motion for free massless fermions with nonzero momentum only in $d = (3+1)$, the higher dimensional space contributes charges, which are reacher the larger is space, but kinematics in $d = (3+1)$ are in all such cases the same. We treat therefore only the $d = (5+1)$ case.

In Table 3.7 the basic creation operators for $d = (5+1)$ case, with Grassmann space used to describe internal degrees of freedom of "fermions" and "antifermions", are presented. "Fermions" carry in Grassmann space integer spins and charges in the adjoint representations.

There are two independent decuplets (unconnected by $\mathbf{S}^{ab}$).

Both decuplets [46] of creation operators are of an odd Grassmann character, representing the second quantized $n = 1$ "fermion" states, Eq. (3.54), which belong in general to $n$ (any $n$) "fermion" states. There are, from the point of view of $d = (3+1)$ space, two triplets, one doublet and two singlets in each of the two decouplets.

In Subsect. 3.3.3 the discrete symmetry operators in Grassmann space are discussed, with the discrete symmetry operators for the case that "fermions"

| I | $i$ | decuplet of creation operators $\hat{b}_i^{\theta k\dagger}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|---|---|---|---|---|---|
| | 1 | $(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$ | $i$ | 1 | 1 |
| | 2 | $(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$ | 0 | 0 | 1 |
| | 3 | $(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$ | $-i$ | $-1$ | 1 |
| | 4 | $(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$ | $i$ | $-1$ | $-1$ |
| | 5 | $(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$ | 0 | 0 | $-1$ |
| | 6 | $(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$ | $-i$ | 1 | $-1$ |
| | 7 | $(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$ | $i$ | 0 | 0 |
| | 8 | $(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$ | $-i$ | 0 | 0 |
| | 9 | $(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$ | 0 | 1 | 0 |
| | 10 | $(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$ | 0 | $-1$ | 0 |

| II | $i$ | decuplet of creation operators $\hat{b}_i^{\theta k\dagger}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|---|---|---|---|---|---|
| | 1 | $(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$ | $-i$ | 1 | 1 |
| | 2 | $(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$ | 0 | 0 | 1 |
| | 3 | $(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$ | $i$ | $-1$ | 1 |
| | 4 | $(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$ | $-i$ | $-1$ | $-1$ |
| | 5 | $(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$ | 0 | 0 | $-1$ |
| | 6 | $(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$ | $i$ | 1 | $-1$ |
| | 7 | $(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$ | $-i$ | 0 | 0 |
| | 8 | $(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$ | $i$ | 0 | 0 |
| | 9 | $(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$ | 0 | 1 | 0 |
| | 10 | $(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$ | 0 | $-1$ | 0 |

**Table 3.7.** Two decuplets of the basic creation operators $\hat{b}_i^{\theta k\dagger}$, $k=(I,II)$, $i=(1,\ldots,10)$, of the orthogonal group $SO(5,1)$ in Grassmann space are presented. The creation operators form "eigenstates" of the Cartan subalgebra, Eq. (3.110), ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{56}$ for $SO(5,1)$) with integer spins and charges, defining "fermions" and "antifermions". The creation operators within each decuplet are reachable from any member by (a product of) $\mathbf{S}^{ab}$'s (which do not belong to the Cartan subalgebra). Creation operators $\hat{b}_i^{\theta k\dagger}$ and their Hermitian conjugated annihilation operators $\hat{b}_i^{\theta k}$ fulfill the anticommutation relations for fermions, Eq. (3.62). The product of the discrete symmetry operators $\mathbb{C}_{NG}$ and $\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), ($\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$ $=\gamma_G^0\gamma_G^5\, I_{\vec{x}_3} I_{x^6}$ in $d=(5+1)$) transforms, for example, $\hat{b}_1^{\theta I\dagger}$ into $\hat{b}_6^{\theta I\dagger}$, $\hat{b}_2^{\theta I\dagger}$ $\hat{b}_5^{\theta I\dagger}$ and $\hat{b}_3^{\theta I\dagger}$ into $\hat{b}_4^{\theta I\dagger}$, transforming "fermions" with the charge 1 into "antifermions" with the charge $-1$.

manifest kinematics only in $d=(3+1)$-dimensional space, while the higher dimensions contribute charges, included.

Let us notice that the Grassmann even operator $\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), transforms the creation operator creating the positive energy particle state ($p^a = (|p^0|,0,0,|p^3|,0,0)$) with the charge 1, $\hat{b}_1^{\theta I\dagger}$, into the creation operator of the anti-particle state, $\hat{b}_6^{\theta I\dagger}$, with the positive energy $|p^0|$ and with $-|p^3|$ and with the charge $-1$ , for example. Correspondingly $\mathbb{C}_{NG}\mathcal{P}_{NG}^{(d-1)}$, Eq. (3.95), transforms the particle state $\hat{b}_3^{\theta I\dagger}$ with the positive energy into the anti-particle state $\hat{b}_4^{\theta I\dagger}$ with the positive energy. All these states belong to the same representation, the same decuplet.

In Eq. (3.98) the superposition of the creation operators of the two triplets of the first decuplet of creation operators — ($\hat{b}_1^{\theta I\dagger}$, $\hat{b}_2^{\theta I\dagger}$, $\hat{b}_3^{\theta I\dagger}$) — which solve Eq. (3.43) for free massless "fermions" in Grassmann space, with the space function $e^{-ip_a x^a}$, $p^a=(p^0,p^1,p^2,p^3,0,0)$, Eq. (3.66), is presented. Two indexes — $(ch, s)$ — replace the index $i$, $ch$ represents the charge, defined by $\mathbf{S}^{56}$, and $s$ represents the spin, $\mathbf{S}^{12}$.

$$
\begin{aligned}
&\text{Creation operators for "fermion" states in Grassmann space for } d = (5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{\theta 1\dagger}_{1,1}(\vec{p}) &= \beta\,\{(\frac{1}{\sqrt{2}})^3\,(\theta^0-\theta^3)(\theta^1+i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1-ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)\\
&\quad -(\frac{p^1+ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3\,(\theta^0+\theta^3)(\theta^1-i\theta^2)\}\,(\theta^5+i\theta^6)\,e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{\theta 2\dagger}_{1,-1}(\vec{p}) &= \beta^*\,\{(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1+ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)\\
&\quad -(\frac{p^1-ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)\}\,(\theta^5+i\theta^6)\,e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
&\text{Creation operators for "anti}-\text{fermion" states in Grassmann space for } d = (5+1)\\
p^0 &= |p^0|\,,\\
\hat{b}^{\theta 3\dagger}_{-1,1}(\vec{p}) &= \beta\,\{(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1-ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)\\
&\quad -(\frac{p^1+ip^2}{|p^0|+|p^3|})^2\,(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)\}\,(\theta^5-i\theta^6)\,e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
\hat{b}^{\theta 4\dagger}_{-1,-1}(\vec{p}) &= \beta^*\,\{(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1+ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)\\
&\quad -(\frac{p^1-ip^2}{|p^0|+|p^3|})^2(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)\}\,(\theta^5-i\theta^6)\,e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,
\end{aligned}
\tag{3.98}
$$

Here $\beta^*\beta = \frac{(|p^0|+|p^3|)^2}{2(3(p^0)^2-(p^3)^2)}$. All the corresponding states are orthonormal.

The corresponding annihilation operators follow from the creation ones by taking into account Eq. (3.18). Let us write down, as an example, the annihilation operator partner to the creation operator $\hat{b}^{\theta 1\dagger}_{1,1}(\vec{p})$ from Eq. (3.98). Taking into account Eq. (3.18) (saying that $\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial\theta_a} = \frac{\partial}{\partial\theta^a}$), it follows $\hat{b}^{\theta 1}_{1,1}(\vec{p}) = (\frac{1}{\sqrt{2}})^3\beta^*(\partial_{\theta^5}-i\partial_{\theta^6})\Big\{(\partial_{\theta^1}-i\partial_{\theta^2})(\partial_{\theta^0}-\partial_{\theta^3}) - \frac{2(|p|^0-|p^3|)}{p^1+ip^2}\sqrt{2}(\partial_{\theta^3}\partial_{\theta^0}-i\partial_{\theta^2}\partial_{\theta^1}) - (\frac{p^1-ip^2}{|p^0|+|p^3|})^2\,(\partial_{\theta^1}+i\partial_{\theta^2})(\partial_{\theta^0}+\partial_{\theta^3})\Big\}e^{i(|p^0|x^0-\vec{p}\cdot\vec{x})}$.

The creation and annihilation operators fulfill the anti-commutation relations of Eq. (3.62).

Creation operators $\hat{b}^{\theta k\dagger}_{ch,\,s}(\vec{p})\,e^{-i(p_m x^m)}$, $m = (0,\cdots,3)$, while $p^5 = 0 = p^6$, generate states, which solve the equation of motion $(\theta^a - \frac{\partial}{\partial\theta_a})\,p_a\,\phi^{\theta k}_{ch,\,s}(x^0,\vec{x}) = 0$, Eq. (3.43), [14].

Let be noticed that the second creation operator $\hat{b}^{\theta 2\dagger}_{1,-1}$ follows from the first one — $\hat{b}^{\theta 1\dagger}_{1,1}$ — by the application of the operator $\mathcal{T}_{\mathcal{NG}} = \gamma^1_G\,\gamma^3_G\,K\,I_{x^0}\,I_{x^5,x^7,\cdots,x^{d-1}}$, Eq. (3.95).

When applying on the first two creation operators of positive charge ($\hat{b}^{\theta 1\dagger}_{1,1}$, $\hat{b}^{\theta 2\dagger}_{1,-1}$), defining the "fermion" states of positive energy, the operator $\mathbb{C}_{\mathcal{NG}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{NG}}(= \gamma^0_G\gamma^5_G\gamma^7_G\cdots\gamma^{d-1}_G\,I_{\vec{x}_3}\,I_{x^6,x^8,\cdots,x^d})$, the third and the fourth creation operators follow, defining the "antifermion" states of negative charge and positive energy ($\hat{b}^{\theta\dagger}_{-1,1}$, $\hat{b}^{\theta 4\dagger}_{-1,-1}$).

Solutions of the equation of motion of the second decouplet, and correspondingly the creation and annihilation operators, can be obtained in equivalent way.

[14] The equation $(\theta^a - \frac{\partial}{\partial\theta_a})\,p_a\,\phi(\theta,x) = 0$ can be rewritten into $-i\tilde{\gamma}^a\,p_a\,\phi = 0$, from where the equation $\{\tilde{\Gamma}^{(3+1)}\,\hat{p}^0 = 2(\tilde{S}^{23}\,\hat{p}^1 + \tilde{S}^{31}\,\hat{p}^2 + \tilde{S}^{12}\,\hat{p}^3)\}\,\phi(\theta,x)$ follows, leading to the same solutions as presented in Eq. (3.98). Similar relation appears also in the Clifford case.

*We learned that states transform under the application of the discrete symmetry operators* (defined in the Clifford case in Eq. (3.90) and Eq. (17) in Ref. [65], or Eq. (10) in Ref. [66], and in the Grassmann case in Eqs. (3.91, 3.95)), *equivalently in the Clifford and in the Grassmann case.*

### 3.3.5 What do we learn from the second quantization procedure in Grassmann and in Clifford space?

We proved that in both spaces, in Clifford space and in Grassmann space, the corresponding creation operators and their Hermitian conjugated annihilation operators of an odd (either Clifford or Grassmann) character fulfill the anticommutation relations as required for fermions, Eqs (3.83, 3.62), if operating on an appropriate vacuum state, representing in both spaces a $n = 1$ fermion space out of $n$, any $n$, fermion Hilbert space.

*No postulated creation operators are needed as in ordinary second quantization procedure.*

In Clifford space the creation operators are (after the requirement of Eq. (3.69)) products of odd numbers of $\gamma^a$'s, arranged into nilpotents and projectors, Eq. (3.70), which are the "eigenstates" of the Cartan subalgebras of $S^{ab}$, Eq. (3.72), generating spins and charges, and of $\tilde{S}^{ab}$, generating families, Eqs. (3.2, 3.4). In Grassmann space they are products of $\theta^a$, arranged in "eigenstates" of the Cartan subalgebra of $\mathbf{S}^{ab}$, Eq. (3.5, 3.52)).

While in the Grassmann case the vacuum state is simple, $|\phi_{og}>=|1>$, in the Clifford case the vacuum state is a sum of $2^{\frac{d}{2}-1}$ products of projectors, Eq. (3.79).

In $2(2n+1)$-dimensional spaces there are in the Clifford case in one representation $2^{\frac{d}{2}-1}$ creation operators. The whole representation is reachable from the (any) starting operator by products of $S^{ab}$, while products of $\tilde{S}^{ab}$ transform each of these creation operators into the creation operator of the same family member, but belonging to another family, Eq. (3.76). There are correspondingly $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation operators, and correspondingly the same number of states, reachable by products of $S^{ab}$'s or $\tilde{S}^{ab}$'s or of both, $S^{ab}$'s and $\tilde{S}^{ab}$'s. Each state follows by the corresponding creation operator on the vacuum state and it is annihilated by its Hermitian conjugated operator, Eq.(3.71).

In $2(2n+1)$-dimensional spaces there are in the Grassmann case (before the requirement of Eq. (3.69)) two decoupled representations with all the $\theta^a$'s included into the representations, each with $\frac{1}{2}\,\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ creation operators, and correspondingly with the same number of states. Each state can be obtained by the corresponding creation operator operating on the vacuum state and any state is annihilated by the corresponding Hermitian conjugated creation operator. While all of $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ states in Clifford space of an odd character are reachable from any of Clifford odd states by either products of $S^{ab}$'s or by products of $\tilde{S}^{ab}$'s or by products of both, and states of an even Clifford character by either products of $S^{ab}$'s or by products of $\tilde{S}^{ab}$'s or $\tilde{\gamma}^a\gamma^a$ or all of them, in Grassmann space all the irreducible representations are decoupled — no products of $\mathbf{S}^{ab}$'s transform states of one group into states of another groups.

The creation (annihilation) operators — which are superposition of the creation (annihilation) operators defining the eigenstates of the Cartan subalgebra in the internal space, fulfilling the relations of Eqs. (3.62, 3.83), respectively — form the eigenstates of the equations of motion for free massless "fermions" with integer spins and no families in the Grassmann case, Eqs. (3.43, 3.61), and for free massless fermions with half integer spins and families in the Clifford case, Eqs. (3.36, 3.82).

The number operators for the odd part of either Clifford or Grassmann case have the eigenvalues 0 or 1, Eqs. (3.55, 3.84).

One can as well define in both cases the Hamilton functions, which lead to the equations of motion in the Grassmann case, Eqs. (3.67, 3.68), and in the Clifford case, Eqs. (3.88, 3.89). While in the Clifford case the procedure to find the Hamilton function is the usual one, that is the known one, in the Grassmann case is not. It remains to understand better the Hamilton function in the Grassmann case.

Comparing solutions for free massless states in a toy model with $d = (5+1)$ from the point of view of $d = (3+1)$ (assuming that $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$) for the Clifford case and for the Grassmann case, one observes several similarities. The main differences are: **i.** that spins and charges are in the Clifford case half integer while in the Grassmann case are integer, **ii.** that Clifford space offers, after the assumption of Eq. (3.69), the existence of families, while Grassmann space, before the assumption of Eq. (3.69), does not, and **iii.** that the requirement that the action is Lorentz invariant leads in Clifford space to well defined Hamilton function, while in the Grassmann case this point needs further study.

We can conclude: **a.** The — odd part of the — Clifford algebra presentation of the internal degrees of freedom of fermions offers the $n = 1$ second quantized fermion part of the $n$ second quantized Hilbert space, offering the fermion creation and annihilation operators, fulfilling the required relations, explaining therefore the assumption of Dirac about introducing creation and annihilation operators in the second quantized fields.

**b.** The *spin-charge-family* theory of N.S.M.B., assuming $d \geq (13+1)$-dimensional space and the Clifford algebra to explain internal degrees of freedom of fermions, enables to justify the assumption of the usual second quantized procedure. The group theory alone, without connecting the internal degrees of freedom with the *Clifford objects for explaining spins, charges, and families*, can not do that.

**c.** Table 3.6 demonstrates that any family contains all the fermions and antifermions, what in the *spin-charge-family* theory means all the quarks and the antiquarks and leptons and anileptons, left and right handed. No Dirac sea of negative energy states is needed to explain the existence of antifermions. Correspondingly the vacuum state is simple, of an even Clifford character, with the sum of all the quantum numbers over the family members equal to zero.

**d.** The sum of all the quantum numbers within one family representation, but also separately within fermions and separately within antifermions within the same representation, is zero. Also the sum over family quantum numbers is zero.

**e.** In the Clifford case the operator $\mathbb{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}}$, Eq. (3.94), transforms the fermion state into the anti-fermion state.

In the Grassmann case it is the operator $\mathbb{C}_{NG} \mathcal{P}^{(d-1)}_{NG}$, which transforms the Grassmann "fermion" into the "antifermion".

## 3.4 Conclusions

We have learned in the present study that both Clifford and Grassmann space offer 1-fermion second quantized part of vector space, with creation and annihilation operators — defined as an odd products of either Clifford or Grassmann eigenstates of the corresponding Cartan subalgebra operators in even dimensional space, Eq. (3.110) — fulfilling the desired anticommutation relations for fermions, Eqs. (3.62, 3.83). The corresponding number operators have the eigenvalues 0 or 1 in both cases. The fact that states, solving equations of motions, fulfill the desired anticommutation relations for second quantized fermions explains the second quantization postulates of Dirac.

Grassmann coordinates and Clifford coordinates offer the same degrees of freedom: Two times $2^d$ each. $\theta^a$'s and their Hermitian conjugated partners $\frac{\partial}{\partial\theta_a}$ are expressible with the two kinds of Clifford coordinates, $\gamma^a$'s and $\tilde{\gamma}^a$'s — defining two independent spaces — and opposite. The vacuum states ought to be changed from $|1>$in the Grassmann case to the one presented in Eq. (3.79) for either $\gamma^a$'s or $\tilde{\gamma}^a$'s. The Grassmann states carry integer spins, while Clifford states carry in both spaces half integer spins.

The requirement of Eq. (3.69) breaks the equivalence of both kinds of the Clifford coordinates and opens the possibility for the appearance of families. Clifford space, defined by the two kinds of objects, narrow now to only one of the two, determined by $\gamma^a$'s, while $\tilde{\gamma}^a$'s take care of families. Correspondingly also in Grassmann space there remain only $\theta^a$'s, becoming $\gamma^a$'s, while their Hermitian conjugated partners $\frac{\partial}{\partial\theta_a}$ no longer exist. Consequently, after the requirement of Eq. (3.69), the possibility of having integer spins "fermions" no longer exists.

The 1-fermion second quantized vector space has for a chosen momentum $p^a_k$ in the Clifford case (after the requirement of Eq. (3.69)) $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ members (that is $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members), and in the Grassmann case (before the requirement of Eq. (3.69)), when all $\theta^a$'s contribute in forming a state, $\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members in two decoupled representations.

In both spaces the members of one representation include fermions and antifermions and correspondingly there is no need for the Dirac sea of negative energies filled by fermions.

In both cases the creation and annihilation operators of different momentum $p^a$ and the same internal part represent different creation operators.

The $n$ (any $n$) second quantized vector space of fermions (or "fermions" in the Grassmann case) follows in both cases as products of $n$ creation operators defining each one fermion states when applying on the corresponding vacuum state (in the Clifford case on $|\psi_{oc}>$, Eq. (3.79), in the Grassmann case $|\psi_{og}>=|\ 1>$), if the creation operators distinguish at least either in one of the quantum numbers of the corresponding Cartan subalgebra or in momentum $p^a_k$.

But while in the Clifford case states carry spin and charges from the point of view of $d=(3+1)$ in the fundamental representations of the Lorentz group carrying therefore half integer spins, states in the Grassmann case are in adjoint representations of the Lorentz group, carrying therefore integer spins.

We present in this paper as well the action (Eq. (3.41, 3.42)), describing free massless "fermions" with the internal degrees of freedom describable in Grassmann space. The action leads to the equations of motion (Eq. (3.43)), analogous to the Weyl equation in Clifford space (Eq. (3.36)), fulfilling as well the Klein-Gordon equation (Eq. (3.44)). We also present the discrete symmetry operators in the Grassmann case.

Since the Clifford objects $\gamma^a$ and $\tilde{\gamma}^a$ are expressible with the Grassmann coordinates $\theta^a$ and their conjugate moments $\frac{\partial}{\partial \theta^a}$ — $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$, Eq. (3.4) — either basic states in Grassmann space, Eq. (3.16), or basic states in Clifford space, Eq. (3.73), can be normalized with the same integral, Eq. (3.31, 3.32, 3.33).

To understand better the difference in the description of the fermion internal degrees of freedom either with Clifford algebra (after the requirement of Eq. (3.69)) or with Grassmann algebra (before the requirement of Eq. (3.69)), let us replace in the starting action of the *spin-charge-family* theory, Eq. (3.1), using the Clifford algebra (after the requirement of Eq. (3.69)) to describe fermion degrees of freedom, the covariant momentum $p_{0a} = f^{\alpha}{}_a\, p_{0\alpha}$, $p_{0\alpha} = p_\alpha - \frac{1}{2} S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$, with $p_{0\alpha} = p_\alpha - \frac{1}{2}\, \mathbf{S}^{ab}\Omega_{ab\alpha}$, where $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (3.5), and $\Omega_{ab\alpha}$ are the spin connection gauge fields of $\mathbf{S}^{ab}$ (which are the generators of the Lorentz transformations in Grassmann space), while $f^{\alpha}{}_a\, p_{0\alpha}$ replaces the ordinary momentum when massless objects start to interact with the gravitational field through the vielbeins and the spin connections. Let us add that it follows, if varying the action with respect to either $\omega_{ab\alpha}$ or $\tilde{\omega}_{ab\alpha}$ when no fermions are present, that both spin connections are uniquely determined by the vielbeins ([9,3,5] and references therein) and correspondingly in this particular case $\Omega_{ab\alpha} = \omega_{ab\alpha} = \tilde{\omega}_{ab\alpha}$.

The present study was stimulated by one of the author in order to better understand whether and to which extend the *spin-charge-family* theory offers the next step to both *standard models* — the one of the fermion and boson fields and the cosmological one. Correspondingly we present in Subsect. 3.1.1 of the introductory Sect. 3.1, the achievements so far of the *spin-charge-family* theory as well as the open problems of this theory, both suggested by the referees.

In shortly, the *spin-charge-family* theory (using Clifford objects to describe the internal space of fermions) offers, while starting with the simple action in $d \geq (13+1)$ with fermions interacting with gravity only (the vielbeins and the two kinds of the spin connection fields, the gauge fields of moments and the generators of the Lorentz transformations $S^{ab}$ and $\tilde{S}^{ab}$, respectively), Eq. (3.1), the explanation for all the assumptions of the *standard model* — for quarks and leptons, antiquarks and antileptons, for fermion families, for the vector gauge fields, for the scalar Higgs and Yukawa couplings — explaining also the phenomena like the existence of the dark matter [54], of the matter-antimatter asymmetry [4], offering correspondingly the next step beyond both standard models — cosmological one and the one of the elementary fields, Sect. 3.1.1. This theory predicts the fourth family to the observed three, Sect. 3.1.1, and the new scalar fields, some of those which explains the properties of the observed Higgs and Yukawa couplings, Sect. 3.1.1, and which will be observed at the LHC and other experiments in the future. This theory predicts also the existence of the stable fifth family, manifesting the dark matter and with the "new nuclear" force among the hadrons of these much heavier families, Sect. 3.1.1.

To these achievements the present study adds the recognition that the creation operators for one fermion states are in Clifford space already second quantized, and that the creation operators for any $n$ fermion second quantized vectors are products of one fermion creation operators, operating on the empty vacuum state. The *spin-charge-family* theory namely describes all the internal degrees of freedom of fermions in Clifford space — spins and charges.

There is in this theory no need for the existence of the negative energy states filled with fermions.

The most severe among the open problems of the *spin-charge-family* theory is the quantization of gravity gauge fields, although the *spin-charge-family* theory is explaining the phenomena in the low energy regime where all the vector and scalar gauge fields can be quantized in the known procedure. There are also other open problems, some of them needing only time to be solved, presented in Sect. 3.1.1.

The second quantization of "fermions" with the internal degrees of freedom described in Grassmann space might help to understand better the properties of scalars and vectors in the *spin-charge-family* theory.

Let us conclude with a question: Could "fermions" with integer spins and charges in adjoint representations be an acceptable possibility and no requirement of Eq. (3.69) needed?

## 3.5 APPENDIX: Creation and annihilation operators in Grassmann and Clifford space for $d = 4n$

We discuss in Subsect. 3.3 mainly cases with $d = 2(2n+1)$, since if assuming no conserved charges in the fundamental theory with fermions, which carry only spins and interact with only the gravity — as the *spin-charge-family* theory assumes — the dimensions $4n$, $n$ is positive integer, as well as all odd dimensions, are excluded under the requirement of mass protection [77].

Let us nevertheless add in this appendix comments on the second quantization procedure in $d = 4n$ spaces.

**i. Grassmann space**

In Eq. (3.51) we define in Grassmann space a possible starting creation operator for $d = 2(2n+1)$ spaces. In $d = 4n$ we correspondingly start with the state

$$
\begin{aligned}
|\phi^1_1> &= b^{\theta 1\dagger}_1 |1>\,,\\
b^{\theta 1\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,,
\end{aligned}
\tag{3.99}
$$

generated by the creation operator $b^{\theta 1\dagger}_1$, which is, as it ought to be — like in the $d = 2(2n+1)$ case — of an odd Grassmann character to fulfill the anticommutation relations for fermions, Eq. (3.62). Again the rest of states, belonging to the same

Lorentz representation, follow from the starting state by the application of the operators $\mathbf{S}^{cf}$, which do not belong to the Cartan subalgebra operators. Their annihilation partners follow by Hermitian conjugation.

One finds therefore for the (chosen) starting creation and the corresponding annihilation operator

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 1}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3})\,,\\
& d=4n\,.
\end{aligned} \tag{3.100}$$

The application of $\mathbf{S}^{01}$, for example, generates

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\,(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\,\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 1}_2 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-2}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}\,(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^3}\frac{\partial}{\partial\theta^0}-i\frac{\partial}{\partial\theta^2}\frac{\partial}{\partial\theta^1})\,.
\end{aligned} \tag{3.101}$$

There is the additional group of creation and annihilation operators in $d=4n$, which follows from the starting creation operator $\hat{b}^{\theta 2\dagger}_1$

$$\begin{aligned}
\hat{b}^{\theta 2\dagger}_1 &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots(\theta^{d-3}+i\theta^{d-2})\,\theta^{d-1}\theta^d\,,\\
\hat{b}^{\theta 2}_1 &= (\hat{b}^{\theta 2\dagger}_1)^\dagger = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,\frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}}-i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0}+\frac{\partial}{\partial\theta^3})\,,\\
& \text{for } d=4n\,.
\end{aligned} \tag{3.102}$$

All the rest of creation operators follow from the starting creation operator of each of the two groups by the (left) application of products of $\mathbf{S}^{ab}$

$$\begin{aligned}
\hat{b}^{\theta k\dagger}_i &\propto \mathbf{S}^{ab}\cdots\mathbf{S}^{ef}\hat{b}^{\theta k\dagger}_1\,,\\
\hat{b}^{\theta k}_i &= (\hat{b}^{\theta 2\dagger}_i)^\dagger\,,\quad k=1,2\,.
\end{aligned} \tag{3.103}$$

*Only creation and annihilation operators with an odd Grassmann character, fulfill, applied on the vacuum state $|1>$, the anticommutation relations required for fermions,* Eq. (3.54).

**i. Clifford space**

In Eq. (3.73) we define in Clifford space a possible starting creation operator for $d=2(2n+1)$ spaces. In $d=4n$ we correspondingly start with the state with an odd number of nilpotents and with one projector

$$\begin{aligned}
|\psi^1_1> &= \hat{b}^{1\dagger}_1|\psi_{oc}>\,,\\
\hat{b}^{1\dagger}_1 &:= \overset{03}{(+i)}\overset{12}{(+)}\overset{35}{(+)}\cdots\overset{d-3\;d-2}{(+)}\overset{d-1\;d}{[+]}\,,\\
\hat{b}^{1}_1 &= (\hat{b}^{1\dagger}_1)^\dagger = \overset{d-1\;d}{[+]}\overset{d-3\;d-2}{(-)}\cdots\overset{35}{(-)}\overset{12}{(-)}\overset{01}{(-i)}
\end{aligned} \tag{3.104}$$

All the other creation operators, creating all the members of the representation of this particular family, are obtainable by the application of products of $S^{ab}$ on this creation operator from the left hand side. There are $2^{\frac{d}{2}-1}$ members of each family. All the other families follows from the starting one by the application of products of $\tilde{S}^{ab}$. There are $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ members each.

A general creation operator in $d = 4n$ follows by the application of $S^{ab}$ and $\tilde{S}^{ab}$ on the starting creation operator of Eq. (3.104) and the corresponding annihilation operator is its Hermitian conjugated value.

Correspondingly we define $\hat{b}^{\alpha\dagger}_{i}$ (up to a constant) to be

$$\begin{aligned}
\hat{b}^{\alpha\dagger}_{i} &\propto \tilde{S}^{ab}\cdots\tilde{S}^{ef}S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_{1} \\
&\propto S^{mn}\cdots S^{pr}\hat{b}^{1\dagger}_{1}S^{ab}\cdots S^{ef}\,, \\
\hat{b}^{\alpha}_{i} = (\hat{b}^{\alpha\dagger}_{i})^{\dagger} &\propto S^{ef}\cdots S^{ab}\hat{b}^{1}_{1}S^{pr}\cdots S^{mn}\,, \\
& d = 4n\,.
\end{aligned} \tag{3.105}$$

These creation and annihilation operators — again of an odd Clifford character in $4n$ — fulfill the anticommutation relations of Eq. (3.83), if applied on the vacuum state of Eq. (3.79),

$$\begin{aligned}
|\psi_{oc}> &= \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \cdots|1>\,, \\
& d = 4n,
\end{aligned} \tag{3.106}$$

$n$ is a positive integer. There are $2^{\frac{d}{2}-1}$ summands, since we step by step replace all possible pairs of $\stackrel{ab}{[-]}\cdots\stackrel{ef}{[-]}$ in the starting part $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}$ into $\stackrel{ab}{[+]}\cdots\stackrel{ef}{[+]}$ and include new terms into the vacuum state so that the last $2n+1$ summand has for $d = 4n$ also the factor $\stackrel{d-1\,d}{[+]}$ in the starting term $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}$ changed into $\stackrel{d-1\,d}{[-]}$. The vacuum state has then the normalization factor $1/\sqrt{2^{d/2-1}}$.

## 3.6 APPENDIX: Lorentz algebra and representations in Grassmann and Clifford space

The Lorentz transformations of vector components $\theta^a$, $\gamma^a$, or $\tilde{\gamma}^a$ — usable for the description of the internal degrees of freedom of fermion fields obeying in the second quantization the anticommutation relations for fermions — and of vector components $x^a$, which are real (ordinary) commuting coordinates, $\theta'^a = \Lambda^a{}_b\,\theta^b$, $\gamma'^a = \Lambda^a{}_b\,\gamma^b$, $\tilde{\gamma}'^a = \Lambda^a{}_b\,\tilde{\gamma}^b$ and $x^a = \Lambda^a{}_b\,x^b$, leave forms $a_{a_1 a_2\dots a_i}\ \theta^{a_1}\theta^{a_2}\dots\theta^{a_i}$, $a_{a_1 a_2\dots a_i}\ \gamma^{a_1}\gamma^{a_2}\dots\gamma^{a_i}$, $a_{a_1 a_2\dots a_i}\ \tilde{\gamma}^{a_1}\tilde{\gamma}^{a_2}\dots\tilde{\gamma}^{a_i}$ and $b_{a_1 a_2\dots a_i}\, x^{a_1}x^{a_2}\dots x^{a_i}$, $i = (1,\dots,d)$, invariant.

While $b_{a_1 a_2\dots a_i}$ $(= \eta_{a_1 b_1}\eta_{a_2 b_2}\cdots\eta_{a_i b_i}\ b^{b_1 b_2\dots b_i})$ is a symmetric tensor field, $a_{a_1 a_2\dots a_i}$ $(= \eta_{a_1 b_1}\eta_{a_2 b_2}\cdots\eta_{a_i b_i}\ a^{b_1 b_2\dots b_i})$ are antisymmetric *Kalb-Ramond* fields.

The requirements: $x'^{a} x'^{b}\eta_{ab} = x^{c}\, x^{d}\eta_{cd}, \theta'^{a}\theta'^{b}\varepsilon_{ab} = \theta^{c}\theta^{d}\varepsilon_{cd}, \gamma'^{a}\gamma'^{b}\varepsilon_{ab} = \gamma^{c}\gamma^{d}\varepsilon_{cd}$ and $\tilde{\gamma}'^{a}\tilde{\gamma}'^{b}\varepsilon_{ab} = \tilde{\gamma}^{c}\tilde{\gamma}^{d}\varepsilon_{cd}$ lead to $\Lambda^{a}{}_{b}\,\Lambda^{c}{}_{d}\,\eta_{ac} = \eta_{bd}$. Here $\eta^{ab}$ (in our case $\eta^{ab} = diag(1, -1, -1, \ldots, -1)$) is the metric tensor lowering the indexes of vectors ($\{x^{a}\} = \eta^{ab}x_{b}, \{\theta^{a}\} = \eta^{ab}\,\theta_{b}, \{\gamma^{a}\} = \eta^{ab}\,\gamma_{b}$ and $\{\tilde{\gamma}^{a}\} = \eta^{ab}\,\tilde{\gamma}_{b}$) and $\varepsilon_{ab}$ is the antisymmetric tensor. An infinitesimal Lorentz transformation for the case with $\det\Lambda = 1, \Lambda^{0}{}_{0} \geq 0$ can be written as $\Lambda^{a}{}_{b} = \delta^{a}_{b} + \omega^{a}{}_{b}$, where $\omega^{a}{}_{b} + \omega_{b}{}^{a} = 0$.

In Eqs. (3.4, 3.8) the commutation relations among the above objects are presented.

### 3.6.1 Lorentz properties of basic vectors

What follows is taken from Ref. [2] and Ref. [9], Appendix B.

Let us first repeat some properties of the anticommuting Grassmann and Clifford coordinates, taking into account Eqs. (3.3,3.4). An infinitesimal Lorentz transformation of the proper ortochronous Lorentz group is then

$$\begin{aligned}
\delta\theta^{c} &= -\frac{i}{2}\omega_{ab}\mathbf{S}^{ab}\theta^{c} = \omega^{c}{}_{a}\theta^{a}\,,\\
\delta\gamma^{c} &= -\frac{i}{2}\omega_{ab}S^{ab}\gamma^{c} = \omega^{c}{}_{a}\gamma^{a}\,,\\
\delta\tilde{\gamma}^{c} &= -\frac{i}{2}\omega_{ab}\tilde{S}^{ab}\tilde{\gamma}^{c} = \omega^{c}{}_{a}\tilde{\gamma}^{a}\,,\\
\delta x^{c} &= -\frac{i}{2}\omega_{ab}L^{ab}x^{c} = \omega^{c}{}_{a}x^{a}\,,
\end{aligned} \tag{3.107}$$

where $\omega_{ab}$ are parameters of a transformation and $\gamma^{a}$ and $\tilde{\gamma}^{a}$ are expressible by $\theta^{a}$ and $\frac{\partial}{\partial\theta_{a}}$ in Eqs. (3.3,3.4).

Let us write the operator of finite Lorentz transformations as follows

$$\mathbf{S} = e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,, \tag{3.108}$$

$\mathbf{S}^{ab}$ have to be replaced by $S^{ab}$ and $\tilde{S}^{ab}$ in the Clifford case. We see that the Grassmann $\theta^{a}$ and the ordinary $x^{a}$ coordinates and the Clifford objects $\gamma^{a}$ and $\tilde{\gamma}^{a}$ transform as vectors

$$\begin{aligned}
\theta'^{c} &= e^{-\frac{i}{2}\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\,\theta^{c}\,e^{\frac{i}{2}\,\omega_{ab}(\mathbf{S}^{ab}+L^{ab})}\\
&= \theta^{c} - \frac{i}{2}\omega_{ab}\{\mathbf{S}^{ab}, \theta^{c}\}_{-} + \cdots = \theta^{c} + \omega^{c}{}_{a}\theta^{a} + \cdots = \Lambda^{c}{}_{a}\theta^{a}\,,\\
x'^{c} &= \Lambda^{c}{}_{a}x^{a}\,, \quad \gamma'^{c} = \Lambda^{c}{}_{a}\gamma^{a}\,, \quad \tilde{\gamma}'^{c} = \Lambda^{c}{}_{a}\tilde{\gamma}^{a}\,.
\end{aligned} \tag{3.109}$$

Correspondingly one finds that compositions like $\gamma^{a}p_{a}$ and $\tilde{\gamma}^{a}p_{a}$, here $p_{a}$ are $p^{x}_{a}$ ($= i\frac{\partial}{\partial x^{a}}$), transform as scalars (remaining invariants), while $S^{ab}\,\omega_{abc}$ and $\tilde{S}^{ab}\,\tilde{\omega}_{abc}$ transform as vectors. Objects like $R = \frac{1}{2}\, f^{\alpha[a}f^{\beta b]}\,(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^{c}{}_{b\beta})$ and $\tilde{R} = \frac{1}{2}\, f^{\alpha[a}f^{\beta b]}\,(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\tilde{\omega}^{c}{}_{b\beta})$ from Eq. (3.1) transform with respect to the Lorentz transformations as scalars.

Making a choice of the Cartan subalgebra set of the algebra $\mathbf{S}^{ab}$, $S^{ab}$ and $\tilde{S}^{ab}$, Eqs. (3.2, 3.5, 3.7),

$$\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,,
\end{aligned} \tag{3.110}$$

one can arrange the basic vectors so that they are eigenstates of the Cartan subalgebra, belonging to representations of $\mathbf{S}^{ab}$, or of $S^{ab}$ and $\tilde{S}^{ab}$, with $ab$ from Eq (3.110).

## 3.7 APPENDIX: Technique to generate spinor representations in terms of Clifford algebra objects

Here we briefly repeat the main points of the technique for generating spinor representations from Clifford algebra objects, following Ref. [2,47]. We advise the reader to look for details and proofs in these references. No requirements for the second quantization is taken into account.

We assume the objects $\gamma^a$, Eq. (3.4), which fulfill the Clifford algebra relations of Eq. (3.2), $\{\gamma^a, \gamma^b\}_+ = I \cdot 2\eta^{ab}$ , for $a, b \in \{0, 1, 2, 3, 5, \cdots, d\}$ , for any $d$, even or odd. I is the unit element in the Clifford algebra, while $\{\gamma^a, \gamma^b\}_\pm = \gamma^a\gamma^b \pm \gamma^b\gamma^a$.

The "Hermiticity" property for $\gamma^a$'s and $\tilde{\gamma}^a$'s, Eq. (3.25), follows from Eq. (3.18), $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$ , leading to $\gamma^{a\dagger}\gamma^a = I$, $\tilde{\gamma}^{a\dagger}\tilde{\gamma}^a = I$.

The Clifford algebra objects $S^{ab}$ close the Lie algebra of the Lorentz group $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$, Eq. (3.7). One finds from Eq.(3.25) that $(S^{ab})^\dagger = \eta^{aa}\eta^{bb}S^{ab}$ and that $\{S^{ab}, S^{ac}\}_+ = \frac{1}{2}\eta^{aa}\eta^{bc}$.

Recognizing that two Clifford algebra objects $(S^{ab}, S^{cd})$ with all indexes different commute, we select (out of many possibilities) the Cartan subalgebra set of the algebra of the Lorentz group of Eq. (3.110)

Let us present the operators of subgroups of the $SO(13+1)$ group

$$\vec{N}_\pm (= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}) , \tag{3.111}$$

$$\begin{aligned} \vec{\tau}^1 &:= \frac{1}{2}(S^{58} - S^{67},\, S^{57} + S^{68},\, S^{56} - S^{78}) ,\\ \vec{\tau}^2 &:= \frac{1}{2}(S^{58} + S^{67},\, S^{57} - S^{68},\, S^{56} + S^{78}) , \end{aligned} \tag{3.112}$$

$$\begin{aligned} \vec{\tau}^3 := &\ \frac{1}{2}\{S^{9\,12} - S^{10\,11} , S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12},\\ &S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14} , S^{11\,14} - S^{12\,13} ,\\ &S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\},\\ \tau^4 :&= -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}) . \end{aligned} \tag{3.113}$$

$$Y := \tau^4 + \tau^{23} , \quad Y' := -\tau^4 \tan^2\vartheta_2 + \tau^{23} , \quad Q := \tau^{13} + Y , \quad Q' := -Y \tan^2\vartheta_1 + \tau^{13} . \tag{3.114}$$

The equivalent expressions for the group $\widetilde{SO}(13,1)$ follows from the above one, if replacing $S^{ab}$ by $\tilde{S}^{ab}$.

To make the technique simple, we introduce the graphic representation, [47], Eq. (3.70),

$$\begin{aligned}
\stackrel{ab}{(k)}: &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
\stackrel{ab}{[k]}: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,
\end{aligned}$$

where $k^2 = \eta^{aa}\eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (Eqs. (3.4, 3.18)) and the definition of $S^{ab}$ (Eq. (3.2)) that if one multiplies from the left hand side by $S^{ab}$ the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, it follows that, Eq. (3.72), $S^{ab}\stackrel{ab}{(k)}= \frac{1}{2}k\stackrel{ab}{(k)}$, $S^{ab}\stackrel{ab}{[k]}= \frac{1}{2}k\stackrel{ab}{[k]}$. This means that $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$ acting from the left hand side on the vacuum state $|\psi_{oc}\rangle$), Eqs. (3.79, 3.106) for $d = 2(2n+1)$ and $d = 4n$ respectively, are eigenvectors of $S^{ab}$.

We further find

$$\begin{aligned}
\gamma^a \stackrel{ab}{(k)} &= \eta^{aa} \stackrel{ab}{[-k]},\\
\gamma^b \stackrel{ab}{(k)} &= -ik \stackrel{ab}{[-k]},\\
\gamma^a \stackrel{ab}{[k]} &= \stackrel{ab}{(-k)},\\
\gamma^b \stackrel{ab}{[k]} &= -ik\eta^{aa} \stackrel{ab}{(-k)}\;.
\end{aligned}\tag{3.115}$$

It follows that $S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{(k)}= -\frac{i}{2}\eta^{aa}\eta^{cc}\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}$, $S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{[k]}= \frac{i}{2}\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}$, $S^{ac}\stackrel{ab}{(k)}\stackrel{cd}{[k]}= -\frac{i}{2}\eta^{aa}\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}$, $S^{ac}\stackrel{ab}{[k]}\stackrel{cd}{(k)}= \frac{i}{2}\eta^{cc}\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}$.

It is useful to deduce the following relations

$$\begin{array}{llll}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)}= \eta^{aa}\stackrel{ab}{[k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(k)}= \eta^{aa}\stackrel{ab}{[-k]}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{(-k)}= 0\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[k]}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{[-k]}=\stackrel{ab}{[-k]}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)}=\stackrel{ab}{(k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[k]}=\stackrel{ab}{(-k)}\,, & \stackrel{ab}{(-k)}\stackrel{ab}{[-k]}= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(k)}= 0\,, & \stackrel{ab}{[-k]}\stackrel{ab}{(-k)}=\stackrel{ab}{(-k)}\;.
\end{array}\tag{3.116}$$

We recognize in the first equation of the first row and the first equation of the second row the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$, respectively.

*Whenever the Clifford algebra objects apply from the left hand side, they always transform* $\stackrel{ab}{(k)}$ *to* $\stackrel{ab}{[-k]}$, *never to* $\stackrel{ab}{[k]}$, *and similarly* $\stackrel{ab}{[k]}$ *to* $\stackrel{ab}{(-k)}$, *never to* $\stackrel{ab}{(k)}$.

We define in Eq. (3.79, 3.106) the vacuum state $|\psi_{oc}>$ so that one finds

$$
\begin{aligned}
&< \stackrel{ab}{(k)}^{\dagger} \stackrel{ab}{(k)} >= 1\,,\\
&< \stackrel{ab}{[k]}^{\dagger} \stackrel{ab}{[k]} >= 1\,.
\end{aligned}
\tag{3.117}
$$

Taking the above equations into account it is easy to find a Weyl spinor irreducible representation for $d$-dimensional space, with $d$ even or odd. (We advise the reader to see Ref. [2,47] in particular for $d$ odd.)

For $d$ even, we simply set the starting state as a product of $d/2$, let us say, only nilpotents $\stackrel{ab}{(k)}$ for $d = 2(2n+1)$, Eq. (3.73), or nilpotents and one projector, Eq. (3.104), for $d = 4n$, one for each $S^{ab}$ of the Cartan subalgebra elements (Eq. (3.110)), applying it on the vacuum state, Eqs. (3.79, 3.106). Then the generators $S^{ab}$, which do not belong to the Cartan subalgebra, applied to the starting state from the left hand side, generate all the members of one Weyl spinor.

$$
\begin{aligned}
&\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{(k_{d-1\;d-2})}\;|\psi_{oc}>,\\
&\vdots\\
&\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[-k_{d-1\;d-2}]}\;|\psi_{oc}>,\\
&\text{for } d = 2(2n+1)\,, \quad n = \text{positive integer}\,.
\end{aligned}
\tag{3.118}
$$

$$
\begin{aligned}
&\stackrel{0d}{(k_{0d})}\stackrel{12}{(k_{12})}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\;|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{[-k_{12}]}\stackrel{35}{(k_{35})}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\;|\psi_{oc}>,\\
&\stackrel{0d}{[-k_{0d}]}\stackrel{12}{(k_{12})}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\;|\psi_{oc}>,\\
&\vdots\\
&\stackrel{od}{(k_{0d})}\stackrel{12}{[-k_{12}]}\stackrel{35}{[-k_{35}]}\cdots\stackrel{d-1\;d-2}{[k_{d-1\;d-2}]}\;|\psi_{oc}>,\\
&\text{for } d = 4n\,, \quad n = \text{positive integer}\,.
\end{aligned}
\tag{3.119}
$$

### 3.7.1 Technique to generate "families" of spinor representations in terms of Clifford algebra objects

We found in this paper that for $d$ even there are $2^{d/2-1}$ "family members" and $2^{d/2-1}$ "families" of spinors, which can be second quantized. (The reader is advised to see also Refs. [2,71,47,48,72,9].) We shall here pay attention on only even $d$.

One Weyl representation forms a left ideal with respect to the multiplication with the Clifford algebra objects. We proved in Refs. ([9,48], and the references therein) that there is the application of the Clifford algebra object from the right hand side, which generates "families" of spinors.

Right multiplication with the Clifford algebra objects namely transforms the state with the quantum numbers of one "family member" belonging to one "family" into the state of the same "family member" (into the same state with respect to the generators $S^{ab}$ when the multiplication from the left hand side is performed) of another "family".

We defined in Ref.[2,48] the Clifford algebra objects $\tilde{\gamma}^a$'s as operations which operate formally from the left hand side (as $\gamma^a$'s do) on any Clifford algebra object A as follows, Eq. (3.69),

$$\tilde{\gamma}^a A = i(-)^{(A)} A\gamma^a \,, \tag{3.120}$$

with $(-)^{(A)} = -1$, if A is an odd Clifford algebra object and $(-)^{(A)} = 1$, if A is an even Clifford algebra object.

Then it follows, in accordance with Eq. (3.4), that $\tilde{\gamma}^a$ obey the same Clifford algebra relation as $\gamma^a$.

$$(\tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a)A = -ii((-)^{(A)})^2 A(\gamma^a\gamma^b + \gamma^b\gamma^a) = I\cdot 2.\eta^{ab}A \tag{3.121}$$

and that $\tilde{\gamma}^a$ and $\gamma^a$ anticommute

$$(\tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a)A = i(-)^{(A)}(-\gamma^b A\gamma^a + \gamma^b A\gamma^a) = 0\,. \tag{3.122}$$

We may write

$$\{\tilde{\gamma}^a, \gamma^b\}_+ = 0, \quad \text{while} \quad \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = I\cdot 2\eta^{ab}\,. \tag{3.123}$$

One accordingly finds

$$\begin{aligned}
\tilde{\gamma}^a \overset{ab}{(k)} &: = -i \overset{ab}{(k)} \gamma^a = -i\eta^{aa} \overset{ab}{[k]}\,,\\
\tilde{\gamma}^b \overset{ab}{(k)} &: = -i \overset{ab}{(k)} \gamma^b = -k \overset{ab}{[k]}\,,\\
\tilde{\gamma}^a \overset{ab}{[k]} &: = \;\; i \overset{ab}{[k]} \gamma^a = \;\; i \overset{ab}{(k)}\,,\\
\tilde{\gamma}^b \overset{ab}{[k]} &: = \;\; i \overset{ab}{[k]} \gamma^b = -k\eta^{aa} \overset{ab}{(k)}\;\,.
\end{aligned} \tag{3.124}$$

If we define, Eq. (3.2),

$$\tilde{S}^{ab} = \frac{i}{4}\,[\tilde{\gamma}^a, \tilde{\gamma}^b] = \frac{i}{4}\,\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_- = \frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,, \tag{3.125}$$

it follows

$$\tilde{S}^{ab}A = A\frac{1}{4}(\gamma^b\gamma^a - \gamma^a\gamma^b)\,, \tag{3.126}$$

manifesting accordingly that $\tilde{S}^{ab}$ fulfill the Lorentz algebra relation as $S^{ab}$ do. Taking into account Eq. (3.69), we further find

$$\{\tilde{S}^{ab}, S^{ab}\}_- = 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,, \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0\,. \tag{3.127}$$

One also finds

$$
\begin{aligned}
&\{\tilde{S}^{ab}, \Gamma\}_- = 0\,, \quad \{\tilde{\gamma}^a, \Gamma\}_- = 0\,, \quad \{\tilde{S}^{ab}, \tilde{\Gamma}\}_- = 0\,, \quad \text{for } d \text{ even}\,,\\
&\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a)\,, \quad \text{if} \quad d = 2n\,,\\
&\tilde{\Gamma}^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\tilde{\gamma}^a)\,, \quad \text{if} \quad d = 2n\,,
\end{aligned}
\tag{3.128}
$$

where handedness $\Gamma$ ($\{\Gamma, S^{ab}\}_- = 0$) is a Casimir of the Lorentz group, which means that in $d$ even transformation of one "family" into another with either $\tilde{S}^{ab}$ or $\tilde{\gamma}^a$ leaves handedness $\Gamma$ unchanged.

We advise the reader to read [2] where the two kinds of Clifford algebra objects follow as two different superpositions of a Grassmann coordinate and its conjugate momentum.

Below some useful relations [71,72] are presented

$$
\begin{aligned}
&N^{\pm}_{+} = N^1_{+} \pm i\, N^2_{+} = -\stackrel{03}{(\mp i)}\stackrel{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^1_{-} \pm i\, N^2_{-} = \stackrel{03}{(\pm i)}\stackrel{12}{(\pm)}\,,\\
&\tilde{N}^{\pm}_{+} = -\stackrel{03}{\tilde{(\mp i)}}\stackrel{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \stackrel{03}{\tilde{(\pm i)}}\stackrel{12}{\tilde{(\pm)}}\,,\\
&\tau^{1\pm} = (\mp)\;\stackrel{56}{(\pm)}\stackrel{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\;\stackrel{56}{(\mp)}\stackrel{78}{(\mp)}\,,\\
&\tilde{\tau}^{1\pm} = (\mp)\;\stackrel{56}{\tilde{(\pm)}}\stackrel{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\;\stackrel{56}{\tilde{(\mp)}}\stackrel{78}{\tilde{(\mp)}}\;.
\end{aligned}
\tag{3.129}
$$

$$
\begin{aligned}
\tilde{S}^{ab}\stackrel{ab}{(k)} &= \frac{k}{2}\stackrel{ab}{(k)}\,,\\
\tilde{S}^{ab}\stackrel{ab}{[k]} &= -\frac{k}{2}\stackrel{ab}{[k]}\,,\\
\tilde{S}^{ac}\stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc}\stackrel{ab}{[k]}\stackrel{cd}{[k]}\,,\\
\tilde{S}^{ac}\stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2}\stackrel{ab}{(k)}\stackrel{cd}{(k)}\,,\\
\tilde{S}^{ac}\stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\stackrel{ab}{[k]}\stackrel{cd}{(k)}\,,\\
\tilde{S}^{ac}\stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc}\stackrel{ab}{(k)}\stackrel{cd}{[k]}\;.
\end{aligned}
\tag{3.130}
$$

We transform the state of one "family" to the state of another "family" by the application of $\tilde{S}^{ac}$ (formally from the left hand side) on a state of the first "family" for a chosen $a, c$. To transform all the states of one "family" into states of another "family", we apply $\tilde{S}^{ac}$ to each state of the starting "family". It is, of course, sufficient to apply $\tilde{S}^{ac}$ to only one state of a "family" and then use generators of the Lorentz group ($S^{ab}$) to generate all the states of one Dirac spinor $d$-dimensional space.

One must notice that nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$ are "eigenvectors"not only of the Cartan subalgebra $S^{ab}$ but also of $\tilde{S}^{ab}$. Accordingly only $\tilde{S}^{ac}$, which do not carry the Cartan subalgebra indices, cause the transition from one "family" to another "family".

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# 4 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space — New Way of Second Quantization of Fermions — Part I ⋆

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**Abstract.** Both algebras, Clifford and Grassmann, offer the second quantized fermions [1–3] without postulating the second quantization conditions of Dirac [13]. But while fermions with the internal degrees of freedom described by the Clifford algebras manifest the half integer spins — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the Grassmann ”fermions” manifest integer spins. In Part I properties of the second quantized integer spins ”fermions” in Grassmann space are presented. In Part II the conditions are discussed under which the Clifford algebra offers the appearance of families of the second quantized fermions.

**Povzetek.** Avtorja sta v članku [3] pokazala, da ponudita obe algebri — Cliffordova in Grassmannova — razlago za Diracove postulate druge kvantizacije fermionov [1–3], saj imajo vektorji v obeh prostorih vse lastnosti, ki jih zahteva Diracov pogoj za drugo kvantizacijo [13]. Clanek razloži v prvem delu tega prispevka drugo kvantizacijo v Grassmannovem prostoru. Pri tem opisu nosijo“fermioni” celoštevilčni spin in naboje, kadar je prostor šest razsežen ali več, v adjungirani upodobitvi. Avtorja demonstrirata lastnosti teh “fermionov” na primeru šest razsežnega prostora. Spin v peti in šesti dimenziji (se po zlomitvi simetrije) “vidi” v $(3+1)$-razsežnem prostoru kot naboj “fermiona”. V drugem delu obravnavata lastnosti fermionov s polštevilčnimi spini v Cliffordovi algebri.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

## 4.1 Introduction

It is demonstrated in this paper how does the Grassmann algebra — in Part I — and the two kinds of the Clifford algebras — in Part II — take care of the second quantization of fermions without postulating anticommutation relations [13].

⋆ Talk presented by N.S. Mankoč Borštnik

In d-dimensional Grassmann space of anticommuting coordinates $\theta^a$'s, $i = (0, 1, 2, 3, 5, \cdots, d)$, there are $2^d$ operators ("vectors"), which are superposition of products of $\theta^a$. One can arrange them into irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to $\theta^a$'s, $\frac{\partial}{\partial \theta_a}$'s, which are Hermitian conjugated to $\theta^a$'s [3], ($\theta^{a\dagger} = \eta^{aa} \frac{\partial}{\partial \theta_a}$, $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$, which again form $2^d$ operators ("vectors"). Grassmann space offers correspondingly $2 \cdot 2^d$ degrees of freedom.

There are two kinds of the Clifford operators ("vectors"), which are expressible with $\theta^a$ and $\frac{\partial}{\partial \theta_a}$ — $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$ [2,4,5]. Each of these two kinds of the Clifford algebra objects has $2^d$ operators ("vectors"), together again $2 \cdot 2^d$ degrees of freedom. The Grassmann and each of the two Clifford algebras split into odd and even part with respect to the odd and even number of $\theta^a$'s, $\frac{\partial}{\partial \theta_a}$'s, $\gamma^a$'s, $\tilde{\gamma}^a$'s. There is the odd algebra in all three cases which fulfills the second quantized anticommutation relations without postulating them [13].

We present in Sect. 4.2 properties of the Grassmann odd anticommuting algebra and even commuting algebra of the corresponding creation and annihilation operators representing the second quantized "fermion" fields, manifesting in the Grassmann case an integer spin, and offering in d-dimensional space, $d > (3+1)$, the description of the corresponding charges in adjoint representations. We follow in this paper to some extent the Ref. [3].

In Part II we present in equivalent section properties of the two kinds of the Clifford algebras and discuss conditions under which operators of the two Clifford algebras demonstrate the anticommutation relations required for the second quantized fermion fields, this way with the half integer spin, offering in d-dimensional space, $d \geq (3+1)$, the description of charges, as well as the appearance of families of fermions [3], both needed to describe the properties of the observed quarks and leptons and antiquarks and antileptons, explaining the appearance of families.

In Sect. 4.3 we comment what we have learned from the second quantized "fermion" fields with integer spin when internal degrees of freedom is described in Grassmann space and compare these recognitions with the recognitions, which the Clifford algebra is offering, discussions on which appears in Part II. We discuss as well a possible action for such an integer spin "fermions" and the corresponding equations of motion, both taken from [3], which are needed that the theory would have any prediction power.

The Clifford algebra offers in even d-dimensional spaces, $d \geq (13+1)$ indeed, the description of the internal degrees of freedom for the second quantized fermions with the half integer spins, explaining all the assumptions of the *standard model*: The appearance of charges of the observed quarks and leptons and their families, as well as the appearance of the dark matter, of the matter/antimatter asymmetry, offering several predictions [1,2,6–12].

## 4.2 Second quantized "fermions" in Grassmann space

In Grassmann d-dimensional space there are d anticommuting operators $\theta^{a_i}$, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to $\theta^a$, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2\cdot 2^d$ operators, the half of which are superposition of products of $\theta^a$ and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a,b) = (0,1,2,3,5,\cdots,d)\,. \tag{4.1}$$

Defining [3]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,,$$
it follows
$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,, \tag{4.2}$$

The signature $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$ is assumed.

One can arrange products of $\theta^a$ into $2^d$ irreducible representations with respect to the Lorentz group with the generators [2]

$$\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{ab}\mathbf{S}^{ab}\,. \tag{4.3}$$

Half of the representations have an odd Grassmann character, those which are superposition of odd products of $\theta^a$ and half have an even Grassmann character, those which are superposition of even products of $\theta^a$.

Since $\mathbf{S}^{ab}$ do not change the character of operators ("vectors"), that is the oddness and evenness of operators, all the members of one irreducible representation have the same Grassmann character. Different representations, either even or odd, are not reachable by $\mathbf{S}^{ab}$.

The Hermitian conjugated $2^d$ representations are reachable, due to Eq. (4.2), from the $2^d$ representations of $\theta^a$'s.

It is useful to make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra. We make the ordinary choice

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,, \tag{4.4}$$

and choose the irreducible representations of the Lorentz group to be the "eigenvectors" of the Cartan subalgebra.

$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) = k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,$$
$$\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) = 0\,. \tag{4.5}$$

Let us point out that the Grassmann "vectors" have an integer spin. Making a choice of $\eta^{aa} = 1,-1,-1,\ldots,-1$, the eigenvectors of $\mathbf{S}^{03}$, $\frac{1}{\sqrt{2}}\,(\theta^0 \mp \theta^3)$, have $k = \pm i$, respectively, all the others have $k = \pm 1$.

"Vectors" are normalized, up to a phase, in accordance with Eq. (4.21) of App. 4.4. Lorentz transformations change the Cartan subalgebra, correspondingly also the "eigenvectors" of the Cartan subalgebra change, since the choice of the Cartan subalgebra depends on the Lorentz frame.

The Hermitian conjugated representations of (odd and even) products of $\theta^a$ are obtainable according to Eq. (4.2).

$$\begin{aligned}
\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)^\dagger &= \eta^{aa}\,\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_a} + \frac{\eta^{aa}}{i(-k)}\frac{\partial}{\partial\theta_b})\,,\\
\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\,\theta^a\theta^b)^\dagger &= \frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\,\frac{\partial}{\partial\theta_a}\,\frac{\partial}{\partial\theta_b})\,.
\end{aligned} \tag{4.6}$$

### 4.2.1 Properties of Grassmann "vectors"

$2^{d-1}$ odd and $2^{d-1}$ even Grassmann operators, which are superposition of odd and even products of $\theta^a$'s are well separated from their $2^{d-1}$ odd and $2^{d-1}$ even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, Eq. (4.6) [1].

To make discussions concrete let us start with illustrating properties of the representations in Grassmann space in $d = (5+1)$-dimensional space. Table 4.1 represents two decuplets, which are "egenvectors" of the Cartan subalgbra ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{5,6}$), Eq. (4.4), of the Lorentz algebra $\mathbf{S}^{ab}$. The two decouplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line of both decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the "eigenvectors" of $\mathbf{S}^{03}$ and $\mathbf{S}^{12}$ alone, while the "eigenvactor" of $\mathbf{S}^{56}$ has, as a "spectator", the "eigenvalue" either $+1$ (the first triplet in both decouplets) or $-1$ (the second triplet in both decuplets). Each of the two decuplets contains also one fourplet (($7^{th}, 8^{th}, 9^{th}, 10^{th}$) lines in each of the two decuplets (Table II in Ref. [2])).

Paying attention on the eigenvectors of $\mathbf{S}^{03}$ alone one recognizes as well even and odd representations of $SO(1,1)$: $\theta^0\theta^3$ (Table II in Ref. [2] includes instead $1 \pm \theta^0\theta^3$) and $\theta^0 \pm \theta^3$, respectively.

The Hermitian conjugated "vectors" follow by using Eq. (4.6) and is for the first "vector" of Table 4.1 equal to $(-)^2(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$. One correspondingly finds that when $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ applies on $(\frac{1}{\sqrt{2}})^3(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)$ the result is identity. Application of $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})\,(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})\,(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ on all the rest of "vectors" of the decuplet I as well as on all the "vectors" of the decuplet II gives zero. "Vectors" are orthonormalized with respect to Eq. (4.21). Let us notice that $\frac{\partial}{\partial\theta_a}$ on a "state"

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[1] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case. In the Grassmann case Hermitian conjugated operators follow by taking into account Eq. (4.2). In the Clifford case $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is proportional to $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

| I | i | decuplet of "eigenvectors" | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{(5+1)}$ | $\Gamma^{(3+1)}$ |
|---|---|---|---|---|---|---|---|
| | 1 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$ | $i$ | 1 | 1 | 1 | 1 |
| | 2 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$ | 0 | 0 | 1 | 1 | 1 |
| | 3 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$ | $-i$ | $-1$ | 1 | 1 | 1 |
| | 4 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$ | $i$ | $-1$ | $-1$ | 1 | $-1$ |
| | 5 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$ | 0 | 0 | $-1$ | 1 | $-1$ |
| | 6 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$ | $-i$ | 1 | $-1$ | 1 | $-1$ |
| | 7 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$ | $i$ | 0 | 0 | 1 | 0 |
| | 8 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$ | $-i$ | 0 | 0 | 1 | 0 |
| | 9 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$ | 0 | 1 | 0 | 1 | 0 |
| | 10 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$ | 0 | $-1$ | 0 | 1 | 0 |

| II | i | decuplet of "eigenvectors" | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\gamma^{(5+1)}$ | $\gamma^{(3+1)}$ |
|---|---|---|---|---|---|---|---|
| | 1 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$ | $-i$ | 1 | 1 | $-1$ | $-1$ |
| | 2 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$ | 0 | 0 | 1 | $-1$ | $-1$ |
| | 3 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$ | $i$ | $-1$ | 1 | $-1$ | $-1$ |
| | 4 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$ | $-i$ | $-1$ | $-1$ | $-1$ | 1 |
| | 5 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$ | 0 | 0 | $-1$ | $-1$ | 1 |
| | 6 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$ | $i$ | 1 | $-1$ | $-1$ | 1 |
| | 7 | $\frac{1}{\sqrt{2}}(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$ | $-i$ | 0 | 0 | $-1$ | 0 |
| | 8 | $\frac{1}{\sqrt{2}}(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$ | $i$ | 0 | 0 | $-1$ | 0 |
| | 9 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$ | 0 | 1 | 0 | $-1$ | 0 |
| | 10 | $\frac{1}{\sqrt{2}}(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$ | 0 | $-1$ | 0 | $-1$ | 0 |

**Table 4.1.** The two decouplets, the largest odd "eigenvectors" of the Cartan subalgebra, Eq. (4.4), ($\mathbf{S}^{03}, \mathbf{S}^{12}$, $\mathbf{S}^{56}$, for $SO(5,1)$) of the Lorentz algebra in Grassmann $(5+1)$-dimensional space, forming two irreducible representations, are presented. Table is partly taken from Ref. [3]. "Vectors" within each decuplet are reachable from any member by $\mathbf{S}^{ab}$'s and are decoupled from another decouplet. The two operators of handedness, $\Gamma^{(d-1)+1}$ for $d = (5,4)$ are invariants of the Lorentz algebra, Eq. (4.23).

which is just an identity, $| I >$, gives zero, $\frac{\partial}{\partial\theta_a} | I >= 0$, while $\theta^a | I >$, or any superposition of products of $\theta^a$'s applied on $| I >$, gives the "vector" back.

The two by $\mathbf{S}^{ab}$ decoupled Grassmann decouplets of Table 4.1 are the largest two irreducible representations of odd products of $\theta^a$'s. There are 12 additional Grassmann odd "vectors", arranged into irreducible representation, ($\frac{1}{2}(\theta^0 \mp \theta^3)(1 \pm \theta^1\theta^2\theta^5\theta^6)$, $\frac{1}{2}(\theta^1 \pm i\theta^2)(1 \pm \theta^0\theta^3\theta^5\theta^6)$, $\frac{1}{2}(\theta^5 \pm i\theta^6)(1 \pm \theta^0\theta^3\theta^1\theta^2)$).

And there are 32 Grassmann "vectors" arranged into irreducible representations, which are superposition of even products of $\theta^a$'s.

### 4.2.2 Second quantized "Grassmann fermions" and bosons

It is not difficult to see that Grassmann "vectors" of an odd Grassmann character — odd products of superposition of $\theta^a$'s — anticommute among themselves and so do odd products of superposition of $\frac{\partial}{\partial\theta^a}$'s, while equivalent even products commute.

Defining the vacuum state in the Grassmann case as $|1>$ [3] [2], one easily sees that application of products of superposition of $\theta^a$'s on $|1>$ gives nonzero

[2] We shall see in Part II that the vacuum states are for both kinds of the Clifford algebra objects, $\gamma^a$'s and $\tilde{\gamma}^a$'s, the sums of products of projectors.

contribution, while application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on $|1>$ gives zero.

Application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on the corresponding Hermitian conjugated partners, which are products of superposition of $\theta^a$'s, leads to identity for either even or odd Grassmann character [3].

All these algebras of an odd character, the Grassmann one and the Clifford two, offer the description of the anticommuting second quantized fields, as postulated by Dirac. But the Grassmann "fermions" carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

**a.** *Grassmann anticommuting "vectors" with integer spins*

Let us first study properties of Grassmann odd "vectors".

Let us use in $d = 2(2n+1)$, $n$ is a positive integer, for the starting Grassmann odd "vector" — in $d = (5+1)$ this is the first "vector" on Table 4.1 — the notation

$$\begin{aligned}\hat{b}^{\theta 1\dagger}_{1} &:= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0 \pm \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\ (\hat{b}^{\theta 1\dagger}_{1})^{\dagger} = \hat{b}^{\theta 1}_{1} &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}})\,.\end{aligned} \tag{4.7}$$

$\hat{b}^{\theta 1}_{1}$ is the Hermitian conjugate $(\hat{b}^{\theta 1\dagger}_{1})^{\dagger}$.

In the case of $d = 4n$, $n$ is a positive integer, the starting Grassmann odd "vectors" of one Lorentz irreducible representation, and correspondingly the creation operator must be of the kind

$$b^{\theta 1\dagger}_{1} := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^{d} \tag{4.8}$$

All the rest of "vectors" belonging to the same irreducible representation follow by the application of $\mathbf{S}^{ab}$. We denote them by $\hat{b}^{\theta k\dagger}_{1}$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_{1}$.

Let those "vectors" belonging to different irreducible representations be denoted by $\hat{b}^{\theta k\dagger}_{j}$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_{j} = (\hat{b}^{\theta k\dagger}_{j})^{\dagger}$.

From Sect. 4.2.1 we derive

$$\begin{aligned}\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{+}|\,1> &= \delta_{ij}\,\delta^{kl}\,|\,1>\,,\\ \{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l}_{j}\}_{+}|\,1> &= 0\,|\,1>\,,\\ \{\hat{b}^{\theta k\dagger}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{+}\,|\,1> &= 0\,|\,1>\,,\\ \hat{b}^{\theta k}_{j}\,|\,1> &= 0\,|\,1>\;.\end{aligned} \tag{4.9}$$

[3] The Clifford case requires more detailed analyses, as we shall see in Part II: Clifford odd "vectors" of each of the two Clifford algebras anticommute with all the members of the same irreducible representation and so do anticommute among themselves their Hermitian conjugated partners. One must, however, introduce the family quantum numbers in order that anticommutator of a "vector" only with its Hermitian conjugated parter gives a nonzero contribution.

These anticommutation relations are just the relations among creation and annihilation operators required by Dirac [13] for fermions. Fermion states correspondingly follow by the application off creation operators on the vacuum state $\mid 1>$.

$$|\phi^k_{ib}> = \hat{b}^{\theta k\dagger}_i \mid 1> \tag{4.10}$$

But Grassmann "fermions" have an integer spin — this follows from Eq. (4.5), and is demonstrated on Table 4.1 — and not half integer spin as it is the case for the so far observed fermions.

**b**. *Grassmann commuting "vectors" with integer spins*

Grassmann even "vectors" commute, and not anticommute as it is the case for the Grassmann odd "vectors". Let us use in the Grassmann even case, that is in the case of even number of $\theta^a$'s, and correspondingly of the commuting "vectors", in $d = 2(2n+1)$ the notation

$$\hat{a}^{\theta 1\dagger}_j = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d. \tag{4.11}$$

Again the rest of "vectors", belonging to the same Lorentz irreducible representation, follow by the application of $\mathbf{S}^{ab}$. The Hermitian conjugated partner of $\hat{a}^{\theta 1\dagger}_1$ is $\hat{a}^{\theta 1}_1 = (\hat{a}^{\theta 1\dagger}_1)^\dagger$

$$\hat{a}^{\theta 1}_1 = (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} \frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}} - i\frac{\partial}{\partial\theta^{d-2}})\cdots(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3}). \tag{4.12}$$

*Let us noticed, that the "vector" identity, 1, is not allowed, since the Hermitian conjugated "vector" of the identity is the identity back.* Then the last requirement of Eq.(4.9) for the commutation relations in the case of Grassmann even "vectors", instead of the anticommutation relations in the case of Grassmann odd "vectors", presented in Eq. (4.9), could not be fulfilled.

If $\hat{a}^{\theta k}_j$ represents a Grassmann even operator, then one obtains, with the index $j$ denoting different irreducible representations and the index $k$ denoting a particular member of the $j^{th}$ irreducible representations, taking into account Sect. 4.2.1, the relations

$$\begin{aligned}
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k'\dagger}_j\}_- \mid 1> &= \delta_{ij}\,\delta_{kl} \mid 1>,\\
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta l}_j\}_- \mid 1> &= 0 \mid 1>,\\
\{\hat{a}^{\theta k\dagger}_i, \hat{a}^{\theta k'\dagger}_j\}_- \mid 1> &= 0 \mid 1>,\\
\hat{a}^{\theta k}_j \mid 1> &= 0 \mid 1>,\\
\hat{a}^{\theta k\dagger}_i \mid 1> &= |\phi^k_{ia}> .
\end{aligned} \tag{4.13}$$

**c**. *Action for free massless Grassmann "fermions" with integer spin* [3]

To obtain the equations of motion for at least noninteracting Grassmann massless "fermions" the corresponding Lorentz invariant action for a free massless "fermions" must be proposed. We follow here the suggestion from Ref. [3].

$$\mathcal{A}_G = \int d^dx\, d^d\theta\, \omega\{\phi^\dagger(1 - 2\theta^0\frac{\partial}{\partial\theta^0})\,\frac{1}{2}\,\theta^a p_a\phi\} + h.c.. \tag{4.14}$$

We use the integral over $\theta^a$ coordinates with the weight function $\omega$ from Eq. (4.21, 4.22). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator $\gamma^0_G$ ($\gamma^a_G$ $= (1 - 2\theta^a \frac{\partial}{\partial \theta^a})$), which takes care of the Lorentz invariance. Namely

$$
\begin{aligned}
\mathbf{S}^{ab\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\,\mathbf{S}^{ab}\,,\\
\mathbf{S}^{\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\,\mathbf{S}^{-1}\,,\\
\mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(\mathbf{L}^{ab} + \mathbf{S}^{ab})}\,,
\end{aligned}
\tag{4.15}
$$

while $\theta^a$, $\frac{\partial}{\partial \theta_a}$ and $p^a$ transform as Lorentz vectors. The equations of motion follow from the action, Eq. (4.14),

$$
\frac{1}{2}\,\gamma^0_G\,(\theta^a - \frac{\partial}{\partial \theta^a})\,p_a\,|\phi> \, = 0\,,
\tag{4.16}
$$

as well as the Klein-Gordon equation, $\gamma^0_G\,(\theta^a - \frac{\partial}{\partial \theta^a})\,p_a\,\gamma^0_G\,(\theta^b - \frac{\partial}{\partial \theta^a})\,p_b\,|\phi>= 0$, leading to

$$
\{\theta^a p_a, \frac{\partial}{\partial \theta_b}\,p_b\}_+ \; = p^a p_a = 0\,.
\tag{4.17}
$$

From the Lagrange density, presented in Eq. (4.14), using Eq. (4.2), and the relations $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$, $\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$, it follows, up to the surface term,

$$
\begin{aligned}
\mathcal{L}_G \; &= -i\frac{1}{2}\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,(\hat{p}_a \phi)\\
&= -i\frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\phi\; - \hat{p}_a\phi^\dagger\;\gamma^0_G\,\tilde{\gamma}^a\,\phi\,\}.
\end{aligned}
\tag{4.18}
$$

One correspondingly finds equations of motion

$$
\begin{aligned}
\frac{\partial \mathcal{L}_G}{\partial \phi^\dagger} - \hat{p}_a\,\frac{\partial \mathcal{L}_G}{\partial \hat{p}_a \phi^\dagger} &= 0 = \frac{-i}{2}\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\,\phi\,,\\
\frac{\partial \mathcal{L}_G}{\partial \phi} - \hat{p}_a\,\frac{\partial \mathcal{L}_G}{\partial (\hat{p}_a \phi)} &= 0 = \frac{i}{2}\hat{p}_a\,\phi^\dagger \gamma^0_G\,\tilde{\gamma}^a\,,
\end{aligned}
\tag{4.19}
$$

The eigenstates of Eq. (4.16, 4.19) for free massless "fermions" are superposition of states $|\phi^k_i>$, describing their internal degrees of freedom, with coefficients depending on momentum $p^a$, $a = (0, 1, 2, 3, 5, \dots, d)$ of the plane wave solution $e^{-ip_a x^a}$

$$
|\phi^k_{sp}> = \sum_i c^k{}_{spi}\,\hat{b}^{\theta k\dagger}_i\,|1>e^{-ip_a x^a}\,,
\tag{4.20}
$$

with $s$ representing different solutions of the equations of motion, and, since they are orthogonalized, they fulfill the relation $<\phi^k_{sp}|\phi^{k'}_{s'p'}>= \delta_{kk'}\,\delta_{ss'}\,\delta^{pp'}$, where we assumed the discretization of momenta.

One of the plane wave massless solutions of these equations, in $d = (5+1)$, for $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, the positive energy $p^0 = |p^0|$, the spin $\frac{1}{2}$ and the charge $\frac{1}{2}$, from the point of view of $d = (3+1)$, for example, is $\hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p}) =$ $\beta\{(\frac{1}{\sqrt{2}})^3\,(\theta^0 - \theta^3)(\theta^1 + i\theta^2) - \frac{2(|p^0|-|p^3|)}{p^1 - ip^2}\,(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3 + i\theta^1\theta^2)\{\overset{56}{(+)}\}\,e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}$, $\beta$ is the normalization factor.

The corresponding state follows by the application of the creation operator $\hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p})$ on the vacuum state $|\,1>$, $|\phi_{\frac{1}{2},\frac{1}{2}}> = \hat{b}^{\theta\dagger}_{\frac{1}{2},\frac{1}{2}}(\vec{p})\,|\,1>$. More solutions can be found in [3] and the references therein.

## 4.3 Conclusions

We learn in this paper, in Part I, that products of superposition of $\theta^a$'s, Eqs. (4.7, 4.5), exist, which together with their Hermitian conjugated partners, Eqs. (4.7, 4.6), fulfill all the requirements for the anticommutation relations for Dirac fermions. No postulation of anticommutation relations is needed. If using products of superposition of $\theta^a$'s as creation operators to describe the internal degrees of freedom of "Grassmann fermions", these "fermions" carry the integer spin, and in spaces $d \geq 5$ the corresponding charges belong to adjoint representations. No families appear in this case, that means that there is no available operators, which would connect different irreducible representations of the Lorentz group (without breaking symmetries).

The presented Lorentz invariant action leads to the equations of motion for free massless "Grassmann fermions" [3].

No elementary fermions with these properties have been observed. The interaction of such "Grassmann fermions" [3] with the corresponding gauge fields could tell more about the possibility whether or not these "Grassmann fermions" exist in nature, not yet observed.

In Part II two kinds of operators are studied; There are namely two kinds of the Clifford algebra objects, $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$, which anticommute, $\{\gamma^a, \tilde{\gamma}^a\}_+ = 0$, and correspondingly form two kinds of independent representations.

Each of these two kinds of independent representations can be arranged into irreducible representations with respect to the two Lorentz generators — $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$. All the Clifford irreducible representations of any of the two kinds of algebras are independent and completely disconnected.

The Dirac action in d-dimensional space for free massless fermions — $\mathcal{A} = \int\, d^dx\,\frac{1}{2}\,(\psi^\dagger\gamma^0\,\gamma^a\,p_a\psi) + \text{h.c.}$ (or $\mathcal{A} = \int\, d^dx\,\frac{1}{2}\,(\psi^\dagger\tilde{\gamma}^0\,\tilde{\gamma}^a\,p_a\psi) + \text{h.c.}$ ) — leads to equations of motions, which have the solutions in both kinds of algebras for either even or odd Clifford character, that is for an even or odd products of the superposition of $\gamma^a$ in one kind and $\tilde{\gamma}^a$ in another kind of the Clifford algebra objects.

Although the "vectors" of one irreducible representation of an odd Clifford algebra character, anticommute among themselves and so do their Hermitian conjugated partners in each of the two kinds of the Clifford algebras, the anticommutation relations among creation and annihilation operators in each of the two

Clifford algebras separately, do not fulfill the requirement, that only the Hermitian conjugated partner of the creation operator gives nonzero contribution.

The decision, the postulate, that only one kind of the Clifford algebra objects — let say $\gamma^a$ — is used to describe the internal space of fermions, while the second kind — $\tilde{\gamma}^a$ in this case — which does not contribute to description of the internal space of fermions, determines quantum numbers of the irreducible representations of the $S^{ab}$, solves both problems: **a.** Different irreducible representations with respect to $S^{ab}$ carry now different "family" quantum numbers determined by $\frac{d}{2}$ commuting operators among $\tilde{S}^{ab}$. **b.** Creation operators and their Hermitian conjugated partners, which are odd products of superpositions of $\gamma^a$, fulfill all the requirements which Dirac postulated for fermions.

## 4.4 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $< \mathbf{B}|\theta >< \mathbf{C}|\theta >$, $< \mathbf{B}|\theta >=< \theta|\mathbf{B} >^{\dagger}$, $< \mathbf{b}|\theta >= \sum_{k=0}^{d} b_{a_1 \ldots a_k} \theta^{a_1} \cdots \theta^{a_k}$, by requiring

$$
\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a \theta^a = 1\,, \quad \int d^d\theta\; \theta^0 \theta^1 \cdots \theta^d = 1\,,\\
&d^d\theta = d\theta^d \ldots d\theta^0\,, \quad \omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial \theta_k} + \theta^k)\,,
\end{aligned} \tag{4.21}
$$

with $\frac{\partial}{\partial \theta_a} \theta^c = \eta^{ac}$. We shall use the weight function [2] $\omega = \Pi_{k=0}^{d}(\frac{\partial}{\partial \theta_k} + \theta^k)$ to define the scalar product in Grassmann space $< \mathbf{B}|\mathbf{C} >$

$$
< \mathbf{B}|\mathbf{C} > = \int d^d\theta^a \;\, \omega < \mathbf{B}|\theta >< \theta|\mathbf{C} >= \sum_{k=0}^{d} \int b^*_{b_1 \ldots b_k} c_{b_1 \ldots b_k}\,. \tag{4.22}
$$

To define norms in Clifford space Eq. (4.21) can be used as well.

## 4.5 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SOd$, with the infinitesimal generators of the Lorentz group $S^{ab}$, defined as

$$
\Gamma^{(d)} = \alpha \varepsilon_{a_1 a_2 \ldots a_{d-1}\, a_d}\, S^{a_1 a_2} \cdot S^{a_3 a_4} \cdots S^{a_{d-1} a_d}\,, \tag{4.23}
$$

with $\alpha$, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case $S^{ab}$ is defind in Eq. (4.3), while in the Clifford case Eq. (4.23) simplifies, if we take into account that $S^{ab}|_{a \neq b} = \frac{i}{2} \gamma^a \gamma^b$ and $\tilde{S}^{ab}|_{a \neq b} = \frac{i}{2} \tilde{\gamma}^a \tilde{\gamma}^b$, as follows

$$
\begin{aligned}
\Gamma^{(d)} :&= (i)^{d/2} \quad\;\; \prod_a \quad (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if} \quad d = 2n,\\
\Gamma^{(d)} :&= (i)^{(d-1)/2} \prod_a \quad (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if} \quad d = 2n+1\,.
\end{aligned} \tag{4.24}
$$

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# 5 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space — New Way of Second Quantization of Fermions — Part II *

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**Abstract.** We discuss in Part I and Part II of this paper the possibility to present internal part of degrees of freedom of the second quantized fermions in Grassmann space — in Part I — and in Clifford space — Part II [1–3]. They both offer description for second quantized fermions [3]. It is no need in either of these algebras to postulate the second quantization relations as Dirac [13], since both algebras by themselves offer the appropriate anticommutation relations. But while fermions with the internal degrees of freedom described by the Clifford algebras manifest the half integer spins and charges in the fundamental representations — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the "Grassmann fermions" manifest integer spins. In Part II we discuss properties of the two kinds of the Clifford algebra objects — both expressible with the Grassmann coordinates, $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ and $\tilde{\gamma}^a = i\,(\theta^{a_i} - \frac{\partial}{\partial\theta_a})$ [2,4,5], $\{\gamma^a\,, \tilde{\gamma}^b\}_+ = 0$ — and conditions under which the members of the irreducible representation of the Lorentz algebra carry the family quantum numbers.

**Povzetek.** Drugi del tega prispevka obravnava obstoj dveh neodvisnih vektorskih prostorov v Cliffordovi algebri, ki sta skupaj ekvivalentna prostoru, ki ga določa Grassmanova algebra. Vsak od vektorskih prostorov v Cliffordovi algebri ponudi kreacijske in anihilacijske operatorje, ki določajo na vakuumskem stanju, ki je vsota produktov anihilacijskih operatorjev na kreacijskih operatotorjih, stanja fermionov s spinom $\frac{1}{2}$ in so rešitve Weylove enačbe. Avtorja postavita zahtevo, da samo ena od obeh Cliffordovih algeber določa vektorski prostor fermionov, druga pa opremi nerazcepne upodobitve Lorentzove grupe v prostoru prve s kantnim številom družine. Zahteva zagotovi, da zadostijo kreacijski in anihilacijski operatorji Diracovim postulatom za fermione v drugi kvantizaciji.

Keywords:Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

* Talk presented by N.S. Mankoč Borštnik

## 5.1 Introduction

In Part I of this paper the properties of ”Grassmann fermions” of integer spins are presented. Let us repeat: In d-dimensional Grassmann space of anticommuting coordinates $\theta^a$'s, $i = (1, .., d)$, there are $2^d$ ”vectors”, which are superposition of products of $\theta^a$'s. One can arrange them into irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to $\theta^a$'s, $\frac{\partial}{\partial \theta_a}$'s, which again form $2^d$ ”vectors”, representing Hermitian conjugated partners to the members of the irreducible representations of $\theta^a$'s, Eq. (6) of Part I. Grassmann coordinates offer correspondingly $2 \cdot 2^d$ vectors.

Taking superposition of products of $\theta^a$'s as creation operators and their Hermitian conjugated partners as annihilation operators, the creation and annihilation operators fulfill, applied on a simple vacuum state $|1>$, the anticommutation relations required for the second quantized fermions, if the unity is not included. The ”Grassmann fermions” of an odd products of $\theta^a$'s carry integer spins and the charges in adjoint representations. There are no elementary fermions with integer spin observed so far.

In this Part II the properties of the two kinds of the Clifford algebras objects, $\gamma^a$'s and $\tilde{\gamma}^a$'s, both expressible with $\theta^a$'s and $\frac{\partial}{\partial \theta_a}$'s ($\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$ [2,4,5]), are presented and the conditions discussed, which limit the space of Clifford ”vectors”, so that the Clifford algebra ”vectors” of each irreducible representation of the corresponding Lorentz algebra of this limited space are equipped by the family quantum numbers. This limited space of the Clifford algebra ”vectors”, when used to describe the internal degrees of freedom of (the second quantized) fermions, explain the anticommutation relations postulated by Dirac [13].

These *Clifford second quantized fermions* enable the descriptions for not only spins and all the charges of the observed quarks and leptons, but also for their families.

We present in Sect. 5.2 properties of the Clifford algebra ”vectors” in the space of d $\gamma^a$'s and d $\tilde{\gamma}^a$'s and discuss conditions, under which operators of these two kinds of the Clifford algebra objects demonstrate by themselves the anticommutation relations required for the second quantized ”fermions”, manifesting the half integer spins, offering the explanation for the spin and charges of the observed quarks and leptons and anti-quarks and anti-leptons and also for their families, Refs. [1,2,6–12,3].

In Sect. 5.3 we comment on what we have learned from the second quantized ”Grassmann fermions”, carrying the integer spins and (from the point of view of $d = (3+1)$) the charges in the adjoint representations and compare these recognitions with the recognitions, which the Clifford algebra is offering for description of the fermions, appearing on families, with half integer spins and charges in the fundamental representations [1,2,6–11,3].

## 5.2 Second quantized fermions in Clifford space

We learn in Part I that in d-dimensional space of anticommuting Grassmann coordinates (and of their Hermitian conjugated partners — derivatives), Eqs. (2,6) of Part I, there exist two kinds of the Clifford coordinates (operators) — $\gamma^a$ and $\tilde{\gamma}^a$ — which are expressible in terms of $\theta^a$ and their conjugate momentum $p^{\theta a} = i\,\frac{\partial}{\partial \theta_a}$ [2].

$$\begin{aligned}
\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,\\
\theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{5.1}$$

offering together $2\cdot 2^d$ operators: $2^d$ of those which are products of $\gamma^a$ and $2^d$ of those which are products of $\tilde{\gamma}^a$.

Taking into account Eqs. (1, 2) of Part I, ($\{\theta^a, \theta^b\}_+ = 0$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, $\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}$, $\theta^{a\dagger} = \eta^{aa}\,\frac{\partial}{\partial\theta_a}$ and $(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a$), one finds

$$\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\
&\quad (\gamma^a)^\dagger = \eta^{a}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,,
\end{aligned} \tag{5.2}$$

with $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

It follows for the generators of the Lorentz algebra of each of the two kinds of the Clifford algebra operators, $S^{ab}$ and $\tilde{S}^{ab}$, that:

$$\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,\\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,,\\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,, \quad \{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- = i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,,\\
\{S^{ab}, \tilde{\gamma}^c\}_- &= 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,,
\end{aligned} \tag{5.3}$$

where $\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})$, Eq. (3) of Part I.

Let us make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra for each of the two kinds of the operators of the Clifford algebra, $S^{ab}$ and $\tilde{S}^{ab}$,

$$\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,.
\end{aligned} \tag{5.4}$$

The two kinds of the Lorentz algebras, the one generated by $\gamma^a$ and the other by $\tilde{\gamma}^a$, are obviously completely independent. We make a choice of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebras of Eq. (5.4), and take into account Eq. (5.2),

$$
\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned}
\tag{5.5}
$$

The Clifford "vectors" of both kinds are normalized, up to a phase, with respect to Eq. (4.21) of App. 4.4. Both have half integer spin. The "eigenvalues" of the operator $S^{03}$, for example, for the "vector" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$ are equal to $\pm\,\frac{i}{2}$, respectively, for the "vector" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm\,\frac{i}{2}$, respectively, while all the rest "vectors" have "eigenvalues" $\pm\,\frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}\,(\tilde{\gamma^0} \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm\,\frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm\,\frac{i}{2}$, respectively, while all the rest "vectors" have $k = \pm\,\frac{1}{2}$.

To make discussions easier let us introduce the notation for the "eigenvectors" of the two Cartan subalgebras, Eq. (5.4), Ref. [4,2].

$$
\begin{aligned}
&\stackrel{ab}{(k)}:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, && \stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\stackrel{ab}{(-k)}\,, && (\stackrel{ab}{(k)})^2 = 0\,,\\
&\stackrel{ab}{[k]}:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, && \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, && (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{(\tilde{k})}:= \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, && \stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa}\stackrel{ab}{(\tilde{-k})}\,, && (\stackrel{ab}{(\tilde{k})})^2 = 0\,,\\
&\stackrel{ab}{[\tilde{k}]}:= \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, && \stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,, && (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,,
\end{aligned}
\tag{5.6}
$$

with $k^2 = \eta^{aa}\eta^{bb}$. Let us point out that the eigenvectors of the Cartan subalgebras are either the nilpotents — $(\stackrel{ab}{(k)})^2 = 0$ and $(\stackrel{ab}{(\tilde{k})})^2 = 0$ — or projectors — $(\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}$ and $(\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}$.

Representations of $\gamma^a$ and representations of $\tilde{\gamma}^a$ are completely independent, each with $2^{\frac{d}{2}-1}$ members in $2 \cdot 2 \cdot 2^{\frac{d}{2}-1}$ representations.

### 5.2.1 Properties of Clifford vectors

$2^{d-1}$ odd and $2^{d-1}$ even Grassmann operators, which are superposition of odd and even products of $\theta^a$'s, are well distinguishable from their $2^{d-1}$ odd and $2^{d-1}$ even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, Eq. (6) in Part I.

In the Clifford case (of either $\gamma^a$'s or $\tilde{\gamma^a}$'s) the "vectors", made of products of nilpotents ($\stackrel{ab}{(k)}$ or $\stackrel{ab}{(\tilde{k})}$) and projectors ($\stackrel{ab}{[k]}$ or $\stackrel{ab}{[\tilde{k}]}$), Eq. (5.6), which each of them are "eigenvectors" of one of the member of the Cartan subalgebra of one of the two kinds, Eq. (5.4), the relations among "vectors" and their Hermitian conjugated partners are less transparent (although easy to be evaluated). This can be noticed in Eq. (5.6), since $\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is $\eta^{aa}\ \frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

Let us recognize the properties of the nilpotents and projectors. The relations are taken from Ref. [6].

$$\overset{ab}{(k)}\overset{ab}{(k)} = 0\,, \qquad \overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\ \overset{ab}{[k]}\,, \qquad \overset{ab}{[k]}\overset{ab}{[k]} = \overset{ab}{[k]}\,, \qquad \overset{ab}{[k]}\overset{ab}{[-k]} = 0\,,$$
$$\overset{ab}{(k)}\overset{ab}{[k]} = 0\,, \qquad \overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}\,, \qquad \overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}\,, \qquad \overset{ab}{[k]}\overset{ab}{(-k)} = 0\,. \tag{5.7}$$

The same relations are valid also if one replaces $\overset{ab}{(k)}$ with $\overset{ab}{(\tilde{k})}$ and $\overset{ab}{[k]}$ with $\overset{ab}{[\tilde{k}]}$.

We illustrate properties of "vectors" of the Clifford algebra of $\gamma^a$'s on irreducible representations of the Lorentz group $SO(5,1)$ and their subgroups $SO(3,1)$ and $SO(1,1)$, presented in Table 5.1, for the case of $\tilde{\gamma}^a$'s all $\overset{ab}{(k)}$'s have to be replaced by $\overset{ab}{(\tilde{k})}$'s and all $\overset{ab}{[k]}$ by $\overset{ab}{[\tilde{k}]}$'s.

| odd I | i | quadruplet a<br>03 12 56 | quadruplet b<br>03 12 56 | quadruplet c<br>03 12 56 | quadruplet d<br>03 12 56 | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{(5+1)}$ | $\Gamma^{(3+1)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 03 12 56<br>(+i)(+)(+) | 03 12 56<br>[+i][+](+) | 03 12 56<br>[+i](+)[+] | 03 12 56<br>(+i)[+][+] | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| | 2 | [−i][−](+) | (−i)(−)(+) | (−i)[−][+] | [−i](−)[+] | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| | 3 | [−i](+)[−] | (−i)[+][−] | (−i)(+)(−) | [−i][+](−) | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | −1 |
| | 4 | (+i)[−][−] | [+i](−)[−] | [+i][−](−) | (+i)(−)(−) | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | −1 |
| odd II | i | quadruplet a<br>03 12 56 | quadruplet b<br>03 12 56 | quadruplet c<br>03 12 56 | quadruplet d<br>03 12 56 | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{(5+1)}$ | $\Gamma^{(3+1)}$ |
| | 1 | (−i)(+)(+) | [−i][+](+) | [−i](+)[+] | (−i)[+][+] | $-\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | −1 | −1 |
| | 2 | [+i][−](+) | (+i)(−)(+) | (+i)[−][+] | [+i](−)[+] | $\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | −1 | −1 |
| | 3 | [+i](+)[−] | (+i)[+][−] | (+i)(+)(−) | [+i][+](−) | $\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | 1 |
| | 4 | (−i)[−][−] | [−i](−)[−] | [−i][−](−) | (−i)(−)(−) | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | 1 |
| even I | i | quadruplet a<br>03 12 56 | quadruplet b<br>03 12 56 | quadruplet c<br>03 12 56 | quadruplet d<br>03 12 56 | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{(5+1)}$ | $\Gamma^{(3+1)}$ |
| | 1 | [−i](+)(+) | (−i)[+](+) | [−i][+][+] | (−i)(+)[+] | $-\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | −1 | −1 |
| | 2 | (+i)[−](+) | [+i](−)(+) | (+i)(−)[+] | [+i][−][+] | $\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | −1 | −1 |
| | 3 | (+i)(+)[−] | [+i][+][−] | (+i)[+](−) | [+i](+)(−) | $\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | 1 |
| | 4 | [−i][−][−] | (−i)(−)[−] | [−i](−)(−) | (−i)[−](−) | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | −1 | 1 |
| even II | i | quadruplet a<br>03 12 56 | quadruplet b<br>03 12 56 | quadruplet c<br>03 12 56 | quadruplet d<br>03 12 56 | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{(5+1)}$ | $\Gamma^{(3+1)}$ |
| | 1 | [+i](+)(+) | (+i)[+](+) | [+i][+][+] | (+i)(+)[+] | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| | 2 | (−i)[−](+) | [−i](−)(+) | (−i)(−)[+] | [−i][−][+] | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| | 3 | (−i)(+)[−] | [−i][+][−] | (−i)[+](−) | [−i](+)(−) | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | −1 |
| | 4 | [+i][−][−] | (+i)(−)[−] | [+i](−)(−) | (+i)[−](−) | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 1 | −1 |

**Table 5.1.** $2^d = 64$ "eigenvectors" of the Cartan subalgebra, Eq. (5.4), of the Clifford $\gamma^a$ algebra in $d = (5+1)$ are presented, divided into four groups of four irreducible representations. Two of four groups have an odd number of $\gamma^a$'s. "Vectors" in the *odd I* part have Hermitian conjugated partners among "vectors" of the *odd II* part, and the opposite. The two groups with the even number of $\gamma^a$'s, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. Numbers — 03 12 56 — explain the indexes of the corresponding Cartan subalgebra. Equivalent table for $\tilde{\gamma}^a$'s follow by replacing all $\overset{ab}{(k)}$ by $\overset{ab}{(\tilde{k})}$ and $\overset{ab}{[k]}$ by $\overset{ab}{[\tilde{k}]}$.

There are in the $\gamma^a$ part of the Clifford algebra "vectors" twice $2^{\frac{6}{2}-1} = 4$ odd irreducible representations, each representation with $2^{\frac{6}{2}-1} = 4$ members and twice 4 even irreducible representations with 4 members, as presented in Table 5.1. The representations for the $\tilde{\gamma}^a$ sector follow from Table 5.1, if one replaces $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[k]}$ with $\stackrel{ab}{[\tilde{k}]}$.

Hermitian conjugation transforms $2^{\frac{d}{2}-1}$ Clifford *odd* representations with $2^{\frac{d}{2}-1}$ members, into $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ Hermitian conjugated partners for each kind of the two kinds of the Clifford algebra operators — $\gamma^a$ and $\tilde{\gamma}^a$. Hermitian conjugated partners of one Lorentz irreducible representation with $2^{\frac{d}{2}-1}$ members, however, belong to $2^{\frac{d}{2}-1}$ Lorentz irreducible representations: The first column of the four representations in the *odd I* part has the corresponding Hermitian conjugated partners in the fourth line of the *odd II*, for example.

In Table 5.2 only one *quadruplet* is presented, the *quadruplet a* from Table 5.1, together with the corresponding Hermitian conjugated partner. All the "vectors" of the quadruplet are orthogonal among themselves and so are also the "vectors" of the Hermitian conjugated partners. The product of each of the Hermitian conjugated partner with its "vector" gives $\stackrel{03}{[-i]}\stackrel{12}{[-1]}(\stackrel{56}{[-1]}$. For the first "vector" one finds: $\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)} \cdot \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}=\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{[-1]}$. This follows by taking into account Eq. (5.7).

If we denote by $\hat{b}^{m\dagger}_f$, with $f = 1$ and $m = (1, 2, 3, 4)$, the first four "vectors" of Table 5.2, and their Hermitian conjugated partners by $(\hat{b}^{m\dagger}_f)^\dagger = \hat{b}^m_f$, with $f = 1$ and $m = (1, 2, 3, 4)$, we can write

$$\begin{aligned} \hat{b}^{m'}_f \cdot \hat{b}^{m\dagger}_f &= \delta^{mm'} \; \stackrel{03}{[-i]}\stackrel{12}{[-1]}(\stackrel{56}{[-1]}\,, \\ &\text{for } f = 1 \;\text{ and all}\,(m, m')\,. \end{aligned} \tag{5.8}$$

One easily checks, taking into account Eq. (5.7), that *quadruplets (a,b,c,d)* of the irreducible representation *odd I* fulfill the equivalent relations, only the products of Hermitian conjugated partner $m$ with its "vector" $m$ change: It follows that $\hat{b}^{m'}_f \cdot \hat{b}^{m\dagger}_f = \delta^{mm'}\,(\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{[-1]}\,, \stackrel{03}{[+i]}\stackrel{12}{[+1]}\stackrel{56}{[-1]}\,, \stackrel{03}{[+i]}\stackrel{12}{[-1]}\stackrel{56}{[+1]}\,, \stackrel{03}{[-i]}\stackrel{12}{[+1]}\stackrel{56}{[+1]}\,)$ for $f = (1, 2, 3, 4)$, respectively. All these "vectors", which are products of $\hat{b}^m_f \cdot \hat{b}^{m\dagger}_f$, are products of selfadjoint projectors only, having an even Clifford character.

One can check for $d = (5 + 1)$, using Eq. (5.7), that it follows.

$$\begin{aligned} &\hat{b}^m_f \cdot \hat{b}^{m'}_{f'} = 0\,, \\ &\hat{b}^{m\dagger}_f \cdot \hat{b}^{m'\dagger}_{f'} = 0\,, \\ &\hat{b}^m_f \cdot \hat{b}^{m'\dagger}_f = \delta^{mm'}\,|\psi_{oc}>\,, \quad \text{for a chosen } f\,, \\ &\hat{b}^{m\dagger}_f\,|\psi_{oc}>= |\psi^m_f>\,, \\ &\hat{b}^m_f\,|\psi_{oc}>= 0\,, \end{aligned} \tag{5.9}$$

for all $(f, f')$ and all $(m, m')$ of Clifford odd Lorentz irreducible representations, with the normalized vacuum state $|\psi_{oc} >= \frac{1}{\sqrt{2^{\frac{6}{2}-1}}} (\stackrel{03}{[-i]}\stackrel{12}{[-1]}\stackrel{56}{[-1]} + (\stackrel{03}{[+i]}\stackrel{12}{[+1]}\stackrel{56}{[-1]}$ $+(\stackrel{03}{[+i]}\stackrel{12}{[-1]}\stackrel{56}{[+1]} + (\stackrel{03}{[-i]}\stackrel{12}{[+1]}\stackrel{56}{[+1]})$.

The generalization of these recognitions to any even $d$, if $d$ is either $d = 2(2n+1)$ or $d = 4n$, $n$ is a positive integer, is straightforward. We shall do this in Subsect. 5.2.3).

| $i$ | quadruplet $a$ | Her. con. quadruplet $a$ |
|---|---|---|
| 1 | $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$ | $\stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)}$ |
| 2 | $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}$ | $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}$ |
| 3 | $\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]}$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\stackrel{56}{[-]}$ |
| 4 | $\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$ | $\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{[-]}$ |

**Table 5.2.** The *quadruplet a* of the irreducible representation *odd I*, from Table 5.1, $d = (5+1)$, together with the Hermitian conjugated partner is presented. Each member of the *quadruplet a* is a product of nilpotents and projectors, which are the "eigenvectors" of the Cartan subalgebra, Eq. (5.4), of the Clifford $\gamma^a$ algebra.

Let us noticed that all the vectors of the first column, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet a* of *even I*, give the vectors of the first column, *odd I*, back, Eq. (5.7). The vectors of the second column, *quadruplet b*, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet b*, *even I*, give the vectors of the second column back. This also happens to the third column, *quadruplet c*, *odd I*, when applied on the selfadjoint "vector" of the *quadruplet c*, *even I*, and to the fourth column, *quadruplet d*, *odd I*, when applied on the self adjoint vector of the *quadruplet d even I*. Similar properties follow when the columns of *odd II* apply on the corresponding selfadjoint operators of *even II*.

Let us notice also that all the annihilation operators anticommute among themselves, $\{\hat{b}^{m'}_{f}, \hat{b}^{m}_{f'}\}_+ = 0$, the same is true for creation operators, $\{\hat{b}^{m'\dagger}_{f}, \hat{b}^{m\dagger}_{f}\}_+ = 0$, while $\{\hat{b}^{m'}_{f'}, \hat{b}^{m\dagger}_{f}\}_+|_{f'=f} = \delta^{mm'}|\psi_{oc}>$ is valid only for $f' = f$ and not for the rest members of particular family to which $\hat{b}^{m'}_{f}$ belong [1].

In any even dimensional space there is in any Clifford even irreducible representation of the corresponding Lorentz algebra of the two kinds of Clifford "vectors" (defined by either $\gamma^a$'s or $\tilde{\gamma}^a$'s) one member, which is the product of $\frac{d}{2}$ selfadjoint projectors $(1 + \frac{i}{k}\gamma^a\gamma^b)$. Correspondingly the whole "vector" is self-

[1] Anticommutator $\{\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}, \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)}\}_+ = -\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$, for example, and applied on the first summand of $|\psi_{oc}>$ gives this Clifford even creation operator $-\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[-]}$ back, which can be found in Table 5.1 among *even I* in the third line of the column *quadruplet a*, while $\stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(-)}$ appears in the third line of *quadruplet d* in *odd II* and $\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}$ appears in the first line of *quadruplet a* in *odd I* of the same table.

adjoint. In Table 5.1 there are in *even I* representations of Clifford even "vectors" four "vectors" ($m = (4, 3, 1, 2)$ of *quadruplets (a,b,c,d)*, respectively), which can be obtained as well from the application of the annihilation operator $\hat{b}^{m'}_{f}$ (*odd II*) on its creation partner $\hat{b}^{m\dagger}_{f}$ (*odd I*), for each irreducible representation f separately.

The selfadjoint even "vectors" appear also in *even II* sector, belonging as well to different irreducible representations of the Lorentz group (in the *quadruplets (a,b,c,d)* they carry the family member number $m = (4, 3, 1, 2)$, respectively). All the Clifford even "vectors" of the same irreducible Lorentz representation, applied on their selfadjoint "vector", gives these "vectors" back.

All the Clifford even representations follow from the products of the Clifford odd "vectors",

Equivalent Clifford even representations as in the space of $\gamma^a$'s appear also in the space of $\tilde{\gamma}^a$'s.

### 5.2.2 Second quantized "Clifford fermions"

We learned in Subsect. 5.2.1 that:

**a.** The two vector spaces, the one spanned by $\gamma^a$'s and the second one spanned by $\tilde{\gamma}^a$'s, are completely independent vector spaces, each with $2^d$ "vectors". The Clifford odd "vectors" (the superposition of products of odd numbers of $\gamma^a$'s or $\tilde{\gamma}^a$'s, respectively) can be arranged for each kind of the Clifford algebras as twice $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group.

The Clifford even part (made of superposition of products of even numbers of $\gamma^a$'s and $\tilde{\gamma}^a$'s, respectively) splits again into twice $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group. **b.** The two groups of the Clifford odd parts (of each of the two kinds) of "vectors", each with $2^{\frac{d}{2}-1}$ irreducible representations of $2^{\frac{d}{2}-1}$ members, are Hermitian conjugated to each other.

**b.i.** The members of one irreducible representation share all the quantum numbers (determined by the members of the Cartan sublagebra (of either $S^{ab}$ or $\tilde{S}^{ab}$) with the corresponding members of another irreducible representations. The same is true also for their Hermitian conjugated partners.

**b.ii.** The $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations are orthogonal and so are orthogonal their corresponding Hermitian conjugated partners.

**b.iii.** Making a choice of "vectors" and denoting them by $\hat{b}^{m\dagger}_{f}$, (where f denotes different irreducible representations and m a member in the representation f), and their Hermitian conjugate partners by $\hat{b}^{m}_{f} = (\hat{b}^{m\dagger}_{f})^{\dagger}$, while choosing the vacuum state $|\psi_{oc}>$ as the sum of all the products of $\hat{b}^{m}_{f} \cdot \hat{b}^{m\dagger}_{f}$ for all $f = (1, 2, \cdots, 2^{\frac{d}{2}-1})$, we end up with Eq. (5.9), valid for superposition of odd products of either $\gamma^a$'s or $\tilde{\gamma}^a$'s, each in its own "vector space".

**b.iv.** The Clifford odd creation and annihilation operators of any irreducible representation f obey the anticommutation relations, postulated by Dirac for fermions. However (as we learn in Subsect. 5.2.1), there exist among annihilation operators $2^{\frac{d}{2}-1} - 1$ members of the same irreducible representation of annihilation operators, to which the particular Hermitian conjugated partner $\hat{b}^{m}_{f}$ (of a particular creation operator $\hat{b}^{m\dagger}_{f}$) belong (obviously obtainable by the generators of the

Lorentz transformations, $S^{ab}$ or $\tilde{S}^{ab}$, respectively), the anticommutators of which with the creation operator $\hat{b}_f^{m\dagger}$ gives one of the $2^{\frac{d}{2}-1}$ members (In Table 5.1 one gets *quadruplets (a,b,c,d)* of *even I*, if one chooses $\hat{b}_f^{m\dagger}$ from *odd I* — otherwise one would get one member of *even II* — which does not belong to self adjoint operators).

**c.** There are the same number of the Clifford even irreducible representations — twice $2^{\frac{d}{2}-1}$, each with $2^{\frac{d}{2}-1}$ number of members — as in the case of the odd irreducible representations. While in the case of the odd irreducible representations the two groups of $2^{\frac{d}{2}-1}$ representations, each with $2^{\frac{d}{2}-1}$ members, are Hermitian conjugated to each other, the Hermitian conjugated partners appear in the even case within each of the two groups separately.

**c.i.** The members of one irreducible representation share all the quantum numbers (determined by the members of the Cartan sublagebra (of either $S^{ab}$ or $\tilde{S}^{ab}$) with the corresponding members of another irreducible representations.

**c.ii.** Only $2^{\frac{d}{2}-1}-1$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations of each of the two groups are orthogonal to each other, while their application on the member which is the product of the projectors only, gives the same member back. All the members of one irreducible representation are orthogonal to all the members of another representation and to all the members of all the representations of another group.

**c.iii.** All the Clifford even "vectors" can be expressed as the products of the Clifford odd "vectors".

*The creation and annihilation operators of an odd Clifford algebras of both kinds, of either $\gamma^a$'s or $\tilde{\gamma}^a$'s, would obviously obey the anticommutation relations for the second quantized fermions, postulated by Dirac, provided that each of the irreducible representations would carry a different quantum number.*

But we know that a particular member $m$ of all the irreducible representations have the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either $\gamma^a$'s or $\tilde{\gamma}^a$'s) Eq. (5.6).

The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces, let us make a choice of $\tilde{\gamma}^a$'s, and use these operators to define the "family" quantum number for the irreducible representation of the vector space of $\gamma^a$'s, keeping the relations of Eq. (5.2) unchanged: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a$, $(a, b) = (0, 1, 2, 3, 5, \cdots, d)$.

We therefore *postulate*:

Let $\tilde{\gamma}^a$'s operate on $\gamma^a$'s as follows [5,2,10,11,5,3]

$$\tilde{\gamma}^a B(\gamma^a) = (-)^B \, i \, B\gamma^a \, , \tag{5.10}$$

with $(-)^B = -1$, if $B$ is an odd product of $\gamma^a$'s, otherwise $(-)^B = 1$ [5].

The vector space of $\tilde{\gamma}^a$'s have correspondingly no meaning any longer, it is "frozen out". (No vector space of $\tilde{\gamma}^a$'s can be taken into account any longer).

Taking into account Eq. (5.10) we can check that

**a.** Relations of Eq. (5.2) remain unchanged.

**b.** Relations of Eq. (5.6) remain unchanged.

**c.** The eigenvalues besides of the operators $S^{ab}$ also of $\tilde{S}^{ab}$ on nilpotents and projectors of $\gamma^a$'s can be calculated, leading to

$$
\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}, &\qquad \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)},\\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}, &\qquad \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]},
\end{aligned} \tag{5.11}
$$

demonstrating that the eigenvalues of $S^{ab}$ on nilpotents and projectors of $\gamma^a$'s differ from the eigenvalues of $\tilde{S}^{ab}$, so that $\tilde{S}^{ab}$ can be used to denote irreducible representations of $S^{ab}$ with the "family" quantum number.

**d.** We further recognize that $\gamma^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$
\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]},\; \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]},\; \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)},\; \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)},\\
&\tilde{\gamma^a} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]},\; \tilde{\gamma^b} \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]},\; \tilde{\gamma^a} \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)},\; \tilde{\gamma^b} \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}
\end{aligned} \tag{5.12}
$$

**e.** One finds, using Eq. (5.10),

$$
\begin{aligned}
&\stackrel{ab}{(\tilde{k})} \stackrel{ab}{(k)} = 0, \quad \stackrel{ab}{(\tilde{-k})} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad \stackrel{ab}{(\tilde{k})} \stackrel{ab}{[k]} = i \stackrel{ab}{(k)}, \quad \stackrel{ab}{(\tilde{k})} \stackrel{ab}{[-k]} = 0,\\
&\stackrel{ab}{[\tilde{k}]} \stackrel{ab}{(k)} = \stackrel{ab}{(k)}, \quad \stackrel{ab}{[\tilde{-k}]} \stackrel{ab}{(k)} = 0, \quad \stackrel{ab}{[\tilde{k}]} \stackrel{ab}{[k]} = 0, \quad \stackrel{ab}{[\tilde{-k}]} \stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,.
\end{aligned} \tag{5.13}
$$

**f.** From Eq. (5.12) it follows

$$
\begin{aligned}
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}, &\qquad \tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[k]}\stackrel{cd}{[k]},\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}, &\qquad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2} \stackrel{ab}{(k)}\stackrel{cd}{(k)},\\
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}, &\qquad \tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[k]}\stackrel{cd}{(k)},\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}, &\qquad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(k)}\stackrel{cd}{[k]}\,.
\end{aligned} \tag{5.14}
$$

**g.** Each irreducible representation of the *odd I* has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (5.4). Correspondingly the creation and annihilation operators fulfill the anticommutation relations of Dirac fermions, without postulating them.

$$
\begin{aligned}
\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_+ \,|\psi_{oc}> &= \delta^{mm'}\,\delta_{ff'}\,|\psi_{oc}>,\\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_+ \,|\psi_{oc}> &= 0\,|\psi_{oc}>,\\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m'\dagger}_{f'}\}_+ \,|\psi_{oc}> &= 0\,|\psi_{oc}>,\\
\hat{b}^{m\dagger}_f \,|\psi_{oc}> &= |\psi^m_f>,\\
\hat{b}^m_f \,|\psi_{oc}> &= 0\,|\psi_{oc}>,
\end{aligned} \tag{5.15}
$$

with $(m, m')$ denoting the "family" member and $(f, f')$ denoting "families".

**h.** The vacuum state for the vector space determined by $\gamma^a$'s remains unchanged $|\psi_{oc}>$, Eq. (80) of Ref. [3].

$$|\psi_{oc}> = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]} + \cdots |1>,$$
$$\text{for } d = 2(2n+1),$$
$$|\psi_{oc}> = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]} + \cdots |1>,$$
$$\text{for } d = 4n, \tag{5.16}$$

$n$ is a positive integer.

**i.** Taking into account relation among $\theta^a$ in Eq. (5.1) it follows from Eq. (5.10), since $\tilde{\gamma}^a \cdot 1 = i\gamma^a$

$$\theta^a = \gamma^a, \quad \frac{\partial}{\partial\theta_a} = 0. \tag{5.17}$$

The Hermitian conjugated part of the space in the Grassmann case "freezed out" together with the "vector" space of $\tilde{\gamma}^a$'s.

### 5.2.3 Second quantization of "Clifford fermions" with families in any d

Let us generalize what we learned in Subsect. 5.2.2 to any dimension $d$, with the vector space determined by $\gamma^a$'s, while $\tilde{\gamma}^a$'s define the family quantum numbers of each creation operator $\hat{b}^{m\dagger}_f$, which is the product of nilpotents and projectors, Eq. (5.6).

Let us make a choice of the starting creation operator $\hat{b}^{1\dagger}_1$ of an odd Clifford character and their Hermitian conjugated partner in $d = 2(2n+1)$ as follows

$$\hat{b}^{1\dagger}_1 := \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\stackrel{d-1\,d}{(+)},$$
$$\hat{b}^{1}_1 \;= (\hat{b}^{1\dagger}_1)^\dagger = \stackrel{d-1\,d}{(-)}\stackrel{d-3\,d-2}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}. \tag{5.18}$$

All the rest "vectors", belonging to the same Lorentz representation, follow by the application of the Lorentz generators $S^{ab}$'s.

The representations with different "family" quantum numbers are reachable by $\tilde{S}^{ab}$, since, according to Eq. (5.14), we recognize that $\tilde{S}^{ac}$ transforms two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into two projectors $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$, without changing $k$ ($\tilde{S}^{ac}$ transforms $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$ into $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$, as well as $\stackrel{ab}{[k]}\stackrel{cd}{(k)}$ into $\stackrel{ab}{(k)}\stackrel{cd}{[k]}$). All the "family" members are reachable from one member of a new family also by the application of $S^{ab}$'s from any of the family members of a particular family.

In this way, by starting with the creation operator $\hat{b}^{1\dagger}_1$, Eq. (5.18), $2^{\frac{d}{2}-1}$ "families" each with $2^{\frac{d}{2}-1}$ "family" members follow. (In the *odd I* part of Table 5.1 we correspondingly recognize four representations with the "family" quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}) = [(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2}), (-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2}), (\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})]$, respectively, for $d = (5+1)$.)

The corresponding annihilation operators, that is the Hermitian conjugated partners of $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members, following from the starting creation operator $\hat{b}^{1\dagger}_{1}$, can be obtained besides with the Hermitian conjugation also by the application of $\tilde{\gamma}^a \gamma^a$ on any member of any "family" of the Clifford odd creation operators. (The application of $\tilde{\gamma}^0 \gamma^0$ on $\hat{b}^{1\dagger}_{1}$ leads to $\hat{b}^{1}_{1}$), all the rest $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ annihilation operators follow by the application of $S^{ab}$ and $\tilde{S}^{ab}$ on $\hat{b}^{1}_{1}$). (Table 5.1 represents in the *odd II* part the annihilation operators to the creation partners of the *odd I* part.)

*The creation and annihilation operators of an odd Clifford character, expressed by nilpotents and projectors of* $\gamma^a$*'s, obey the anticommutation relations of* Eq. (5.15), *without postulating the second quantized anticommutation relations*.

The even partners of the Clifford odd creation and annihilation operators follow by either the application of $\gamma^a$ on the creation operators, leading to $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members, or with the application of $\tilde{\gamma}^a$ on the creation operators, leading to another group of the Clifford even operators, again with the $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members.

It is not difficult to recognize, that each of the Clifford even "families", obtained by the application of $\gamma^a$ on the creation operators contains one selfadjoint operator, which is the product of projectors only, determining the vacuum state, Eq. (5.16). (Table 5.1 represents in the *even I* part these four selfadjoint operators, together with the rest of $(2^{\frac{6}{2}-1} - 1) \cdot 2^{\frac{6}{2}-1}$ Clifford even operators.)

The second Clifford even group of $2^{\frac{d}{2}-1}$ "families" with $2^{\frac{d}{2}-1}$ members, which follows by the application of $\gamma^a$ on the annihilation operators, has again $2^{\frac{d}{2}-1}$ selfadjoint operators, which would determine the vacuum state, if the annihilation and the creation operators would exchange their roles. (Table 5.1 represents in the *even II* part the second group of even operators, with $\cdot 2^{\frac{6}{2}-1}$ selfadjoint operators, together with the rest of $(2^{\frac{d}{2}-1} - 1) \cdot 2^{\frac{d}{2}-1}$ Clifford even operators.)

### 5.2.4 *Action for free massless Clifford "fermions" with half integer spin*

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger} \gamma^0 \, \gamma^a p_a \psi) + h.c. \, , \tag{5.19}$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a p_a |\psi> = 0 \, , \tag{5.20}$$

which fulfill also the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi> = 0 \, , \tag{5.21}$$

for each of the basic states $|\psi^m_f>$. $\gamma^0$ appears in the action to take care of the Lorentz invariance of the action.

Solutions of Eq. (5.20) are for free massless "fermions" superposition of $\hat{b}^{m\dagger}_{f}$, for a chosen "family" $f$, describing internal degrees of freedom, with coefficients depending on momentum $p^a$, $a = (0, 1, 2, 3, 5, \ldots, d)$ of the plane wave solution $e^{-ip_a x^a}$

$$|\phi^s_{fp}> = \sum_m c^{ms}{}_{fp}\, \hat{b}^{m\dagger}_{f}\, e^{-ip_a x^a}\, |\psi_{oc}>,$$
$$\hat{b}^{s\dagger}_{fp} = \sum_m c^{ms}{}_{fp}\, \hat{b}^{m\dagger}_{f}\, e^{-ip_a x^a}\,, \tag{5.22}$$

$s$ represents different solutions of the equations of motion, and, since they are orthonormalized, they fulfill the relation $< \phi^s_{fp}|\phi^{s'}_{f'p'} >= \delta_{ss'}\,\delta_{ff'}\,\delta^{pp'}$, where we assumed the discretization of momenta $p^a$.

### 5.2.5 *Solutions for $n$ free massless Clifford "fermions" with half integer spin with the family quantum number*

The number of creation operators $\hat{b}^{s\dagger}_{fp}$ in $d$-dimensional space is

$$2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1} \tag{5.23}$$

for a chosen momentum $p^a$, due to the number of families and number of members in each family, respectively. They all anticommute, fulfilling with the annihilation operators Eq. (5.15) ([3] andreferences therein).

When we discus more then one "fermion", we must keep in mind that the number of creation operators for a particular momentum is

$$2^{2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}}, \tag{5.24}$$

since each state can be either fulfilled by a fermion or empty. Since the momentum can be any and the solutions of different momentum are, in the discretized case, orthogonal, the number of states is correspondingly infinite.

Since the states are for different momentum orthogonal, the creation and annihilation operators fulfill the anticommutation relations of Eq. (5.15) for each momentum $p^a$.

$$\{\hat{b}^s_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_+\, |\psi_{oc}> = \delta^{ss'}\,\delta_{ff'}\,\delta_{pp'}\, |\psi_{oc}>,$$
$$\{\hat{b}^s_{fp}, \hat{b}^{s'}_{f'p'}\}_+\, |\psi_{oc}> = 0\, |\psi_{oc}>,$$
$$\{\hat{b}^{s\dagger}_{fp}, \hat{b}^{s'\dagger}_{f'p'}\}_+\, |\psi_{oc}> = 0\, |\psi_{oc}>,$$
$$\hat{b}^{s\dagger}_{fp}\, |\psi_{oc}> = |\psi^s_{fp}>,$$
$$\hat{b}^s_{fp}\, |\psi_{oc}> = 0\, |\psi_{oc}>\ . \tag{5.25}$$

In Ref. [3], Eqs. (47, 65, 87), discuss properties of the $n$ fermion states.

## 5.3 Conclusions

We learn in Part I of this paper, that odd products of superposition of $\theta^a$'s, Eqs. (7,6) in Part I, exist, which together with their Hermitian conjugated partners, fulfill all the requirements for the anticommutation relations for the Dirac fermions. There is no need to postulate the anticommutation relations. However, *these "fermions" carry the integer spin and the corresponding charges originating in* $d \geq 5$ *belong to adjoint representations*. No families appear in this case, that means that there is no available operators, which would connect different irreducible representations of the Lorentz group.

In Part II we learn that the Grassmann space offers two kinds of the Clifford operators — $\gamma^a$'s and $\tilde{\gamma}^a$'s. Both kinds of the Clifford objects define two kinds of independent Clifford spaces. "Vectors" of an odd products of $\gamma^a$'s or $\tilde{\gamma}^a$'s, respectively, carry the half integer spins and charges, originating in $d \geq 5$, in fundamental representations. Both kinds of odd Clifford "vectors" together offer two times $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ creation operators and two times $2^{\frac{d}{2}-1} \cdot 2^{\frac{d}{2}-1}$ annihilation operators. The Clifford odd creation and annihilation operators of both kinds of the Clifford spaces for each of the corresponding irreducible Lorentz representations separately fulfill the anticommutation relations for the Dirac fermions – without postulating them.

To achieve that at least in one of the two groups of the Clifford odd creation and annihilation operators fulfill all the requirements for the Dirac fermions also when different irreducible representations are taken into account, the "family" quantum number must be introduced for any of the irreducible representation.

To achieve this we "sacrifice" one of the two kinds of the Clifford vector spaces — the one determined by $\tilde{\gamma}^a$'s — and use the corresponding $\tilde{S}^{ab}$'s to define the "family" quantum number for each irreducible representation of $S^{ab}$. The creation operators $\hat{b}^{m\dagger}_{f}$ and the annihilation operators $\hat{b}^{m'}_{f'}$ — $(f, f')$ determine now family quantum numbers and $(m, m')$ determine family members quantum numbers — fulfill the anticommutation relations of Eq. (5.15). The solutions of equations of motion for free massless fermions, Eq. (5.20), for a particular momentum $p^a$ fulfill correspondingly the anticommutation relations of Eq. (5.25).

Solutions of equation of motion of different moments $p^a$ obviously anticommute, due to the fact that the creation and annihillation operators fullfil the anticommutation relations of of Eq. (5.15). *There is no need to postulate anticommutation relations as Dirac did for the second quantized fermions.*

*The Clifford algebra by itself, including "families", explains the Dirac assumption for second quantized fermions with the half integer spins and the charges in the fundamental representations, if charges origin in* $d \geq 5$ .

The reduction of the Clifford space, defined with two completely independent operators $\gamma^a$'s and $\tilde{\gamma}^a$'s, into the space spanned by $\gamma^a$'s only has as the consequencethat $\theta^a$"s become $\gamma^a$'s, while their Hermitian conjugated partners do not exist any longer.

While in Grassmann space the Grassmann odd "vectors" fulfill the anticommutation relations for "fermions" with integer spins and charges in the adjoint representations (originating in $d \geq 5$), and the Grassmann even "vectors" commute, with the vacuum state in both cases, which is just the identity, the Clifford even "vectors" are used to determine the (rather complicated) vacuum state.

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# 6 Deriving Locality From Diffeomorphism Symmetry in a Fiber Bundle Formalism ⋆

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**Abstract.** We normally assume that a quantum field theory should have an action of the form $S[\phi] = \int \mathcal{L}\sqrt{g}d^4x$, and we say that with this form the action is local. In the present work we however do not assume locality, but rather derive it. The point of departure for this derivation of locality, is a diffeomorphism symmetric, very general action S which is Taylor expandable as a functional. We are moreover only interested in long distance physics, compared to the fundamental scale. We already published - in a somewhat hidden way - such an argument in reference [2], but here we extract this derivation as the main point, and further formulate it in fiber bundle notation.

**Povzetek.** Običajno privzamejo, da ima akcija v kvantni teoriji polja obliko $S[\phi] = \int \mathcal{L}\sqrt{g}d^4x$ in rečejo, da je akcija v tej obliki lokalna. Avtorja v tem prispevku ne predpostavita lokalnosti, ampak jo izpeljeta. Izhodišče je zelo splošna akcija S z difeomorfno simetrijo. Akcija se da razviti v Taylorjevo vrsto kot funkcional. Zanimajo ju lastnosti te akcije pri velikih razdaljah. To izpelijavo sta na kratko že objavila v referenci [2], tukaj pa je osrednja točka prispevka.

Keywords: Deriving locality, fiber bundles

## 6.1 Introduction

In a generic physical model, the property of locality is usually taken for granted. Its actual meaning is seldom discussed at great length, and this is even more true for nonlocality. The unreflected assumption that locality is fundamental, is reflected in the locality of the laws of nature, as well as in the continuity equation which tells that there are no jumps!

We usually think of locality in terms of information being localized, propagating from one spacetime point to another by at most the speed of light. Another way of expressing it, is that all cause-and-effect relations are limited by the speed of light. Thus, an experiment in one place is not supposed to have an immediate influence on an experiment in an other place, this is also true for effects like the butterfly effect, because they take time.

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A theory is local when every degree of freedom is assigned a spatio-temporal site $x_\mu$. That means that all interactions take place in *one* spacetime point, implying that there is a system for assigning one site $x_\mu$ to each degree of freedom. In a local theory the action S can be factorized, $S = S_1 + S_2 + ...$, such that each contribution only depends on the fields in limited regions of $x_\mu$-spacetime. This locality concept is moreover coordinate choice dependent,

$$S = \int \mathcal{L}(x) d^4x \tag{6.1}$$

In the present work we do not assume that locality is fundamental, on the contrary, our goal is to derive locality. Our point of departure is an analytic and diffeomorphic symmetric action, using fiber bundle formalism. The philosophical framework of this approach is Random Dynamics [1], which postulates that at a fundamental level, there is a "world machinery", meaning a very general, random mathematical structure, which merely contains non-identical elements and some set-theoretical notions. From this "world machinery", differentiability, space and time [2], diffeomorphism symmetry [3], locality, and eventually all other physical concepts, are to be derived.

But even after locality has been derived, some smeared out left-over nonlocal effects remain, showing up in coupling constants (which feel an average over spacetime, and also depend on such averages). This remaining (mild) nonlocality is moreover supported by the Multiple Point Principle (MPP) [4].

The "locality" that we want to derive comes about by formulating the action as an integral over spacetime of a Lagrangian density

$$\mathcal{L}(\phi(x), \partial\phi(x)/\partial x)$$

which only depends on the fields - such as $\phi$ - and their (up to finite order) derivatives taken with values of the spacetime point $x$. Our starting point is a diffeomorphic symmetric action $S[\phi]$, a fiber bundle of dimension $\geq 4$, and the idea that we can get genuine locality (not super locality) along p = 4-dimensional p-surfaces. When you have a field configuration on your fiber bundle (a compact fiber bundle that you can integrate over), it is namely in the spirit of fields that you can only make various local functions of them.

## 6.2 Diffeomorphisms

A diffeomorphism is an isomorphism $\psi$ on a smooth manifold $\mathcal{M}$ (thus preserving the structure of $\mathcal{M}$); and the group of diffeomorphisms on $\mathcal{M}$ is the set of such mappings,

$$\mathcal{D}(\mathcal{M}) = \{\psi : \mathcal{M} \to \mathcal{M}\}$$

Every diffeomorphism is a homeomorphism, but not every homeomorphism is a diffeomorphism. To be diffeomorphic is a much stronger condition, since for a mapping $\psi$ to be diffeomorphic, both $\psi$ and its inverse need to be differentiable; while to be homeomorphic it suffices that $\psi$ and its inverse are both continuous.

We can always define a coordinate system on the manifold, whereby we identify the points on the manifold in terms of the coordinates. Diffeomorphisms can be perceived as synonymous with reparametrizations, which in local coordinates $x^{\mu}$ are analytic or at least smooth maps,

$$\psi : x^{\mu} \to x^{\mu} + \eta^{\mu} = x'^{\mu}, \tag{6.2}$$

and in changing coordinates in $\mathcal{M}$ we establish the transformation rules between two different coordinate systems. We can introduce a new coordinate system and transform to it, or we can just keep our first coordinate system, and then intoduce a diffeomorphism

$$\phi : \mathcal{M} \to \mathcal{M}$$

which is the same as "moving" the points on the manifold, after which we have to evaluate the coordinates of the new points. We then think of diffeomorphism invariance as reparametrization invariance, but the diffeomorphic symmetry exists whether we use coordinates or not.

Our first challenge is to formulate what diffeomorphism symmetry means in the case of a functional over the fields on $\mathcal{M}$.

## 6.3 Analyticity

We do not assume that our initial action is local, so it may in principle comprehend nonlocal terms of the form $f(x, y)$, which depend on more than one spacetime point.

An analytic function is locally given by a convergent power series. There is however no demand that a power series must depend on a single variable, it can just as well be an infinite series of the form

$$f(x, y, \cdots, z) = \sum_{j_x, j_y, \cdots, j_z = 0}^{\infty} a_{j_x, j_y, \cdots, j_z} (x - C_x)^{j_x} (y - C_y)^{j_y} \cdots (y - C_z)^{j_z} \tag{6.3}$$

where $j_x, j_y, \cdots$ are natural numbers, $a_{j_x, j_y, \cdots, j_z}$ and $x, y, .., z$ are variables, and $C_j$ constitute the 'center'.

The theory of such series is trickier than for single-variable series, with more complicated regions of convergence, but for example the power series $\sum_{n=0}^{\infty} x_1^n x_2^n$ is absolutely convergent in the set $\{(x_1, x_2) : x_1 x_2 < 1\}$ between two hyperbolas. This means that a nonlocal function can be analytic, implying that analyticity does not in itself guarantee locality. Our action, however, is not only analytic but also diffeomorphic symmetric, and we want to prove that this is enough to make it local. It remains to formulate analyticity in the case of a functional over the fields on $\mathcal{M}$.

## 6.4 The action

In our action $S[\phi]$, the function $\phi$ is defined over the manifold $\mathcal{M}$. For a field on a manifold, a value in one single point has no signification (because it is of Lebesgue measure = 0), in the sense that in a continuous field a given point can always be replaced by some small integral piece, so these fields must be integrated over. The integration is taken over such small integral pieces, and the final, generic action is some complicated combination of these integrals. Moreover, if you have diffeomorphism symmetry, you cannot have boundaries on your integrals, so every integral must be taken over the whole space (which is assumed to be a connected manifold). Our action must thus be a function of integrals over the entire space.

By loosely identifying a point on $\mathcal{M}$ with its coordinates $x_j$, the summing over the various j=1,2,...,N gets replaced by integrals over the coordinate variable sets on the manifold, and the Taylor expansion of the functional for a single function $\phi(x)$ is then defined as

$$S[\phi] = S[\phi = 0] + \int \frac{\delta S}{\delta\phi(x)}\phi(x) d^d x + \frac{1}{2!}\int\int \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\phi(x)\phi(y) d^d x d^d y + \cdots = \tag{6.4}$$

$$= \sum_n \frac{1}{n!}\int \overset{n}{\cdots} \int \frac{\delta^n S}{\delta\phi(x_1)\cdots\delta\phi(x_n)}|(\phi = 0)\phi(x_1)\cdots\phi(x_n) d^d x_1 \cdots d^d x_n \tag{6.5}$$

Initially we only consider a subset of diffeomorphism transformations that leave the local d-volume invariant, i.e. a subset of diffeomorphism transformations $x \to x'(x)$ such that

$$\det\left(\frac{\partial x'^\mu(x)}{\partial x^\nu}\right) = 1, \tag{6.6}$$

which means that the d-volume of spacetime

$$\epsilon_{\mu\nu\cdots\tau} dx^\mu dx^\nu \cdots dx^\tau \tag{6.7}$$

remains invariant under this subset of diffeomorphism transformations.

In the second step, we include "pseudoscalar" fields $P(x)$ that have other transformation properties than mere scalars under diffeomorphism transformations, transforming as

$$P(x) \to P(x')\left(\frac{\partial x'^\mu(x)}{\partial x^\nu}\right) \tag{6.8}$$

Imposing the diffeomorphism symmetry coordinate shift $x \to x'$, preserving the integral $d^d x = d^d x'$ on the Taylor expansion

$$S[\phi] \sim \sum_{n=0,1,..} \phi(x_1)\cdots\phi(x_n) d^d x_1 \cdots d^d x_n \tag{6.9}$$

leads to the requirement that the coefficients, i.e. the derivatives

$$\frac{\delta S}{\delta\phi(x_1)\cdots\delta\phi(x_n)} \tag{6.10}$$

must be invariant in a similar way under diffeomorphism transformations, except if some of the $x_j$'s are infinitely close or coinciding, which implies that we get terms like (6.9) multiplied with δ functions $\delta(x_j - x_k)$. When including the terms coming from $\delta$ function forms at the coincidences, we however also get integrals over various products of the fields and their derivatives, though the integrals allowed by the diffeomorphism symmetry turn out to be such that the quantities getting integrated in the appearing integrals, are all time "pseudoscalars", like the $P(x)$'s themselves.

This implies that the total Taylor expansion runs out to be a **function** of a lot of integrals over various pseudoscalars, which can be formed in terms of the various variables/fields in the theory.

With the condition of diffeomorhism invariance, these integrals must thus be pseudoscalar (i.e. transform as $\sqrt{g}$), resulting in an action which by definition is pseudoscalar, that is, a function (not a functional) of all the various pseudoscalar terms that we can construct, and we can make differentiation through all these pseudoscalar contributions.

We can write the functional derivative of S as a partial derivative of S with respect to all the different integrals over the whole space summed over, multiplied by the functional derivatives of the latter,

$$\frac{\delta[S]}{\delta\xi(x)} = \sum_k \frac{\partial S}{\partial V^k}\frac{\delta V^k}{\delta\xi(x)} = \frac{\delta S_{eff}}{\delta\xi(x)} \tag{6.11}$$

where

$$S_{eff} = \int_M \sum_k \frac{\partial S}{\partial V^k}\mathcal{P}_k(x)d^4x, \tag{6.12}$$

and $\mathcal{P}_k(m) \approx \mathcal{L}_k(x)\sqrt{g(x)} = \mathcal{L}_k(m)\sqrt{g(m)}$ are pseudoscalars, $\mathcal{L}_k(m)$ are Lagrangian densities, and $V^k$ are the integrals

$$V^k = \int \mathcal{P}_k(m)dm, \tag{6.13}$$

where $m \in \mathcal{M}$ are spacetime points/events. In local coordinates $x^\mu$ on $\mathcal{M}$, $x^\mu(m)$ are the coordinates of the event $m$, and $d^4x$ is a measure in the coordinates $x^\mu$, such that $d^4x = dx^1dx^2\cdots dx^d \equiv dm$.
In general relativity the transformation property of the metric tensor field $g_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu}(x) = \frac{\partial\psi^\eta}{\partial x^\mu}\frac{\partial\psi^\rho}{\partial x^\nu}g_{\eta\rho}(\psi(x)), \tag{6.14}$$

we infer that by having terms like a second order tensor field

$$\mathbf{g} = g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}, \tag{6.15}$$

we can formulate a theory with effective locality (and no super-locality), there will however still be a certain nonlocality, because with the assumption of local diffeomorphism symmetry, the effective action comes out as

$$S_{eff} = \int \mathcal{L}_{(pseudoscalar)}dx^1 \wedge dx^2 \wedge \cdots dx^d \tag{6.16}$$

although with a very important detail: The coupling constants or coefficients become complicated integrals over the whole manifold/base space $\mathcal{M}$, i.e. over all spacetime including both future and past. So in this restricted sense the resulting theory is still non-local, although the non-locality only comes in via the coupling constants.

In our Taylor expansion philosophy, there is an interesting point: If we just take the usual inverse metric tensor field $g^{\mu\nu}$ as the field provididng the indices to be contracted with the ones from the derivative, on say the scalar fields $\phi(x)$, we cannot obtain in a Taylor expansion that only provides **polynomials**, the needed pseudoscalar factor $\sqrt{g(x)}$, where $g(x) = det(g_{\mu\nu})$. The reason is that the square root is singular - at zero field - thus this conventional model of general relativity does not work for our philosophy, even in the case of a purely bosonic theory. So even for the purely bosonic theory - wherein one a priori expects that the gravity based on just the metric tensor would be o.k. - we **cannot** obtain in our Taylor expansion in the usual action form, because of the square root singularity is needed. In fact one can easily see that all polynomially constructed pseudoscalar-like field combinations based on the metric tensor alone, obtain transformation rules of the form of being multiplied with an **even** power of the det $\frac{\partial x^\mu}{\partial x'^\nu}$ factor. But for the construction of a diffeomorphism invariant integral we need an odd power, namely the "pseudoscalar" replacing $\sqrt{g(x)}$ which transforms with this determinant of the partial derivative matrix to **first power** only.

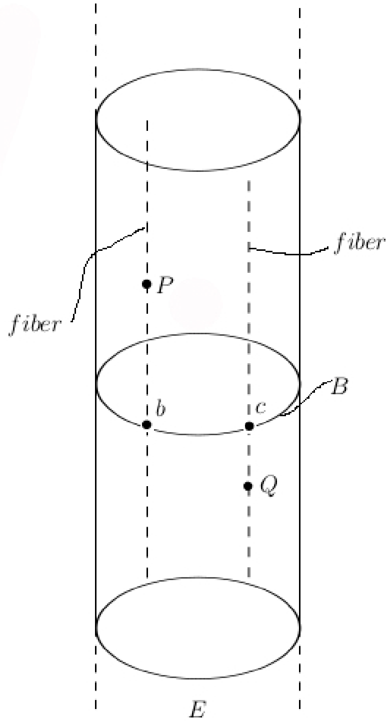
One could thus claim that we have a (kind of) prediction that there should be vier beins (or some replacement for vierbeins) rather than the simple metric tensor in the theory where locality is obtained in the spirit of our derivation. And this claim is of course only of much interest in the case where we ignore the fermions, because with fermions we need the vierbein anyway.

As a consequence of this consideration we should say that the typical example of fields to be used for illustrating our model for deriving locality, should have a vierbein among the fundamental fields.

## 6.5 Fiber bundles

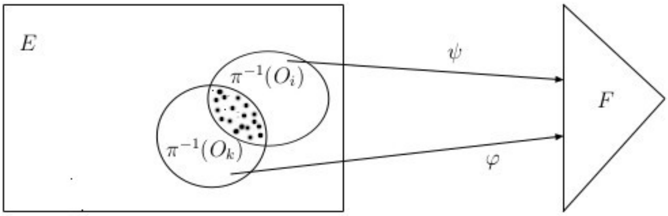
A fiber bundle is often simply written as $(E, B, F, \pi)$, where $E$, $B$ and $F$ are topological spaces, and $\pi$ is a continuous map. $E$ is known as the total space of the fiber bundle, $B$ as the base space, $F$ is the fiber, and in small regions of $E$, the fiber bundle $\pi : E \rightarrow B$ behaves like a map from $B \times F$ to $B$. This is a local relation that is not necessarily globally valid.
A simple example of a fiber bundle is the $S_1 \times \mathbb{R}$ surface of an infinitely long cylinder, which by definition is a differentiable manifold. Here the total space $(E)$ is the entire cylinder, the base space $(B)$ is the circle $S_1$ running around the cylinder, and the fiber $(F)$ is $\mathbb{R}$.

The fiber that runs through $b \in B$ is called a fiber over $b$, and is formally defined as the pre-image $\pi^{-1}(b) \in E$, which is diffeomorphic to $F$. The fiber over $c \in B$ is a different fiber, but the fibers are all diffeomorphic to each other. It is this set that constitutes a fiber "bundle", in the sense that while there is only one total space $E$ and one base space $B$, there is a whole set of fibers; we say that $E$ is a fiber bundle of $F$ over $B$.




In the case of the cylinder, the pre-image $\pi^{-1}(B)$ of $B$ is trivial, i.e. it is the entire total space, $\pi^{-1}(B) = B \times F = E$, so the topology of the total space $E$ is the usual topology on a 2-dimensional manifold. In the general case, however, it is only the pre-image of an open set $O$ in the base space that is (locally) trivial, i.e. $\pi^{-1}(O) \sim B \times F$.

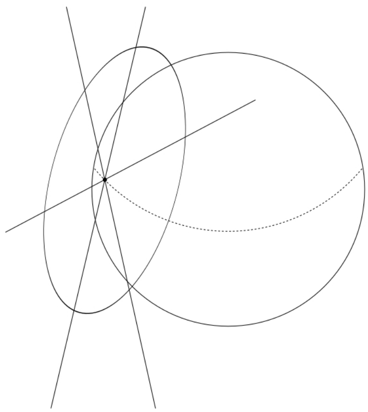
Now, since $E$ is locally diffeomorphic to a product space, a point $p$ in $E$ can be written as $(b, f)$ where $b \in B$ and $f \in F$, and $\pi$: $(b, f) \to b$. As we take $b$ back to $E$ under $\pi^{-1}$, the pre-image $\pi^{-1}(b)$ of $b$ is not a point, but a subset of $E$ that is diffeomorphic to $F$ under $\pi$. Around any point in $B$, we can moreover find at least one neighbourhood $O_i \subset B$ such that its pre-image $\pi^{-1}(O_i)$ is trivial, i.e. diffeomorphic to $O_i \times F$. There is thus a mapping $(\psi, \pi) : \pi^{-1}(O_i) \to O_i \times F$, where $\psi : \pi^{-1}(O_i) \to F$ is a homeomorphism.
A different open set $O_k \subset B$ will have different pre-image in $E$, and projected on $F$ by a different homeomorphism $\phi : \pi^{-1}(O_k) \to F$. Since $\pi^{-1}(O_i)$ and $\pi^{-1}(O_k)$ are connected to $F$ by $\psi$ and $\phi$, respectively, the intersection $\pi^{-1}(O_i) \cap \pi^{-1}(O_k)$ is also connected to $F$ by these two homeomorphisms, from which we can construct the diffeomorphisms $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1} : F \to F$. Such diffeomorphisms define the structure group $G(F)$, which is a subset of the group of diffeomorphisms $D(F)$ on $F$. Every transition function of the fiber bundle must belong to the structure group.




In the case of the cylinder, $G(F)$ is the identity element, $G(F) = I$. On the Möbius band, most of the transition functions can be identified with the identity, but at least one of them must be negative, i.e. G(Möbius) $\sim \{I, -I\}$.

## 6.6 Analyticity in fiber bundle notation

As a visualisation of the relation between fiber bundles and tangent spaces, consider all of the unit tangent vectors on the sphere. Over every point in $S^2$, there is a circle of unit tangent vectors all of which constitute a principal bundle $E$ on the sphere with the circle $S^1$ as fiber, and every tangent vector projects to its base point in $S^2$, giving the map $\pi : E \to S^2$.



For our purpose, we identify the 4-dimensional spacetime manifold as the basis space, i.e. $B = \mathcal{M}_{(\text{spacetime})}$, and the total space is then identified as the p-dimensional space $E = B \times F$.

We say that the functional $S[\phi]$, $\phi : B \to E$, is "analytic" provided we have a convergent Taylor expansion

$$S[\phi] = S[\phi_0] + \int \frac{\delta S}{\delta \phi^i(m)} (\phi^i(m) - \phi_0^i(m)) dm + \cdots \tag{6.17}$$

where $m \in B$, and

$$\frac{\delta S}{\delta \phi(m)} \sim \frac{\delta S}{\delta \phi^i(m)} d\phi^i = dS \in [\mathtt{cotangent\ \ space\ \ for\ \ F}] \tag{6.18}$$

Locally in B-space you have coordinates, so we define $\Delta\phi^i(m)$ as

$$\Delta\phi^i(m')|_{(at\ m)} = (\phi^i(m') - \phi^i_0(m'))\frac{\partial}{\partial\phi^i(m)} \tag{6.19}$$

which is a tangent vector, and $\delta S/\delta\phi^i(m)$ is the basis of the tangent space. In terms of the coordinates $x^\mu = (x^1, x^2, \cdots)$, the basis in the tangent space is

$$\frac{\partial}{\partial x^\mu} = \tau^\mu, \tag{6.20}$$

and the tangent vectors $t_\mu \tau^\mu \in T$, where T is the tangent vector space.

The $p = 4$ surfaces should run through the d-dimensional E-space so as to have their 4-dimensional tangent vector space embedded (naturally) in the tangent space of B at the point m, in such a way that it is just the one that is spanned by the four tangent vectors $V_a = V^\mu_a \delta/\delta x^\mu$. Now

$$\frac{\delta S}{\delta\phi^i(m)}\Delta\phi^i(m)|_{(at\ m)} = \frac{\delta S}{\delta\phi^i(m)} d\phi^i \Delta\phi(m')|_{(at\ m')} = dS(m)\Delta\phi(m)|_{(at\ m)} \tag{6.21}$$

where

$$d\phi^i \frac{\delta}{\delta\phi^k(m)} = \delta^i_k, \tag{6.22}$$

In order to express this in the language of functionals, we expand S around $\phi_0$:

$$S[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\delta^n S[\phi_0]}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)..\delta\phi^{i_n}(m_n)} \left(\phi^{i_1}(m_1) - \phi^{i_1}_0(m_1)\right) \times \tag{6.23}$$

$$\times \left(\phi^{i_2}(m_2) - \phi^{i_2}_0(m_2)\right) \cdots \left(\phi^{i_n}(m_n) - \phi^{i_n}_0(m_n)\right) dm_1 \cdots dm_n \tag{6.24}$$

and then define a dual function to the tangent space vector, i.e. a covector, $\mathcal{D}\phi^i(m)$, by using the tangent space basis vectors

$$\frac{\delta}{\delta\phi^j(m')} \tag{6.25}$$

to define the number

$$\langle \mathcal{D}\phi^i(m) | \frac{\delta}{\delta\phi^j(m')} \rangle = \delta^i_j \delta(m - m') \tag{6.26}$$

One tests $\delta(m - m')$ by a test function $K(m)$,

$$K(m)\mathcal{D}\phi^i(m)\frac{\delta}{\delta\phi^i(m')} = \delta^i_j K(m') \tag{6.27}$$

Is this the right definition of $\mathcal{D}\phi^i(m)$? $\mathcal{D}\phi^i(m)$ should be in the dual space of the functional tangent space in which the basis vectors are $\delta/\delta\phi^j(m')$. So $\mathcal{D}\phi^i(m)$ is defined by defining its action

$$\langle \mathcal{D}\phi^i(m) | \frac{\delta}{\delta\phi^j(m')} \rangle = \delta^i_j \delta(m - m') \tag{6.28}$$

Inserting a product of $n$ of these delta-functions in the action, we obtain

$$S[\phi] = \sum_{n=0}^{\infty} \int \overset{n}{\cdots} \int \frac{1}{n!} \frac{\delta^n S[\phi_0]}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)..\delta\phi^{i_n}(m_n)}$$
$$\langle \mathcal{D}\phi^{i_1}(m_1) | \frac{\delta}{\delta\phi^{j_1}(m'_1)} \rangle \cdots \langle \mathcal{D}\phi^{i}_{n}(m_n) | \frac{\delta}{\delta\phi^{j_n}(m'_n)} \rangle \cdot$$
$$\cdot(\phi^{i_1}(m'_1) - \phi_0^{i_1}(m'_1))(\phi^{i_2}(m'_2) - \phi_0^{i_2}(m'_2)) \cdots (\phi^{i_n}(m'_n) - \phi_0^{i_n}(m'_n)) dm'_1 \cdots dm'_n \tag{6.29}$$

Now define

$$\Delta\phi = \int (\phi^j(m') - \phi_0^j(m')) \frac{\delta}{\delta\phi^j(m')} dm' \tag{6.30}$$

we get

$$S[\phi] = \sum_{n=0}^{\infty} \int \overset{n}{\cdots} \int \frac{1}{n!} \frac{\delta^n S[\phi_0]}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)..\delta\phi^{i_n}(m_n)}$$
$$\mathcal{D}\phi^{i_1}(m_1) \otimes \cdots \mathcal{D}\phi^{i_n}(m_n) \Delta\phi \otimes \cdots \otimes \Delta\phi =$$
$$= \sum_0^{\infty} \delta^{\otimes} S(\Delta\phi)^{\otimes n} \tag{6.31}$$

Actually the simple requirement that (6.29) should be constant when the arguments $m_i$ are all different, is only true if we restrict the diffeomorphisms by which we shuffle them around, to those transformations that keep the $\det(\frac{\partial x^j}{\partial x'})$ equal to unity. For more general diffeomorphisms we have to modify the functional derivatives for volume-non-preserving diffeomorphisms, by inserting density factors that are constant rather than simply derivatives. This is achieved by multiplying the functional partial derivatives by "pseudoscalar" correction factors, like (6.13).

## 6.7 Diffeomorphisms of the action

Let us consider a symmetry under a group of bundle maps $f$:

$$f : E \to E \quad \text{bijective; and } f \circ \pi = \pi \circ f \quad \text{is a requirement for bundle map} \tag{6.32}$$

This induces a transformation on $\mathcal{M}$, $\tilde{f}(\mathcal{M}) \to \mathcal{M}$ so that if

$$\pi \circ f(e) = \tilde{f}(e) \Leftarrow e \in F(m) \tag{6.33}$$

$$\tilde{f}(m) = m', \text{then if } \pi \circ f(e) = m' \Leftrightarrow \pi(e) = m, \tag{6.34}$$

the symmetry transforms $\phi_1 \to \phi_2$ where

- $f : E \to E$
- $\tilde{f} : \mathcal{M} \to \mathcal{M}$ (defined from $f$ when bundle map).
- $\pi \circ f = \tilde{f} \circ \pi$,

i.e. $f \circ \pi = \pi \circ f \Rightarrow$ the fiber $F$ over say $\pi^{-1}(m)$ is mapped onto/into itself, where $F$ is a function $F(V_1, ....V_k, \phi_0^{i_1}|_{scalar}, ...)$. For $e \in \pi^{-1}(m), \pi \circ f(e) = f \circ \pi(e)$ is independent of $e$, except through $m$. So for each $m$ there is a map $f_m(e)$ inside the fiber on $m$,

$$\phi_2(e) = f_{\pi(e)}(\phi_1(\tilde{f}^{-1}(\pi(e)))) \tag{6.35}$$

A true diffeomorphism is defined by choosing an $\tilde{f}$ rather than $f$, and then deduce a $f$ according to semi-local rules like

$$g^{\sigma\tau} \to \frac{\partial x^{\sigma}}{\partial x^{\nu}} \frac{\partial x^{\tau}}{\partial x^{\mu}} g^{\nu\mu} \tag{6.36}$$

This is semi-local: it only depends on derivatives and values near or at $x$, and then going to $x$. Then we can almost choose $\tilde{f}$ freely and still get $\partial x^{\nu}/\partial x^{\rho}$, etc.

Assuming:

- that we have so much symmetry that all diffeomorphic maps $\tilde{f} : \mathcal{M} \to \mathcal{M}$ are achievable.
- that the full transformation $f : E \to E$ as far as the moving around on the fiber is concerned, i.e. $f_m$ for all $m \in \mathcal{M}$, is determined by derivations of $\tilde{f}$ in the neighbourhood of $m$,

then we can prove that we can choose some subset of $\tilde{f}$'s in the supposed symmetry group so that it follows that

$$\frac{\delta^n S}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)\cdots\delta\phi^{i_n}(m_n)} \tag{6.37}$$

must be the same even if one moves any of the $m_i$'s, except if this $m_i$ coincides with (up to infinitesimals) another $m$, say $m_j$.

This implies first that if we ignore any grouping of the $m_i$, i.e. if they are all different, then (6.37) is independent of the $m_i$'s. We should and could (if we think of true diffeomorphisms with usual tensors) also assume that

- we can arrange $\tilde{f}$ in such a way that the subsequent $f$ can locally "rotate" or "transform" indices (on the $\phi$'s) so that the pseudoscalars are not transformed, so there is only a dependence on $\phi^i(m_i)$.

With the assumption that we have a diffeomorphism invariant "expansion start field $\phi_0$" which is constant for scalars, and otherwise zero, the form $F(\phi_0^{i_1}, \cdots \phi_0^{i_n})$ becomes constant over the entire base space product, and so the derivative

$$\frac{\delta^n S}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)\cdots\delta\phi^{i_n}(m_n)} \tag{6.38}$$

is only allowed in the form $F(\phi_0^{i_1}(m_1)|_{(scalar)}, \cdots \phi_0^{i_n}(m_n)|_{(scalar)})$, where the scalars are the $\phi_0^{i_j}$-values corresponding to scalar components of the $\phi^{i_j}(m)$. Then only $\mathcal{D}\phi^i(m)$ with a scalar component will be relevant in

$$\delta^{\otimes n}S = \int \overset{n}{\cdots} \int \frac{\delta^n S}{\delta\phi^{i_1}(m_1)\cdots\delta\phi^{i_n}(m_n)} \mathcal{D}\phi^{i_1}(m_1)\cdots\mathcal{D}\phi^{i_n}(m_n)dm_1\cdots dm_n =$$
$$= \delta^{\otimes n}S|_{\substack{\text{(projected onto}\\ \text{the "scalar" component)}}} \int \mathcal{D}\phi^{i_1}_{(scalar)}(m_1)\cdots\mathcal{D}\phi^{i_n}_{(scalar)}(m_n)dm_1\cdots dm_n =$$
$$= V^n F(\phi_0^{i_1}|_{sc}, \phi_0^{i_2}|_{sc}, \cdots \phi_0^{i_n}|_{sc}) \quad (6.39)$$

The quantity (6.38) depends on a background field which a priori is a combination of all the fields in the theory, but for any fixed value of the fields it depends on n event-points $m_1$ , $m_2$,... $m_n$. Now in (6.39) this dependence on the set of the n $m$-values $(m_1, ..., m_n)$ gets intergrated over the $m$'s with a weighting by the duals of the partial derivatives called $\mathcal{D}(m_i)$, i.e. by a product of n such $\mathcal{D}(m_i)$'s. This contraction with such $\mathcal{D}$'s leads to (6.39).

It is with the simplification to the case of a constant start field $\phi_0$, that we get that (6.38) can only be constant - as long as the arguments $(m_1, m_2, ..., m_n)$ do not coincide. The background is that while we in the general case Taylor expand around start functions that are not necessarily diffeomorphism invariant, for the simplicity of the argument, we restrict ourselves to scalar or pseudoscalar fields $\phi$. Then by transformation with det$\{\frac{\partial x'^\mu}{\partial x^\nu}\} = 1$, we can argue that merely with invariance under the restricted diffeomorphisms, in order to keep the total action S invariant it must not change as the $m_i$'s move around on the base manifold B, unless some of the $m_i$'s are moved to a place with a different $\phi_0(m_i)$ value.

But inside a range with given value of the $(\phi_0(m_1), ..., \phi_0(m_n))$, the diffeomorphism invariance implies that the expression (6.38) should be unchanged by such moving around, which in its turn implies that the expression (6.38) can only be of the form $F(\phi_0(m_1), ..., \phi_0(m_n))$.

If we look for components with more complicated types of transformation, like vector components or tensor components, we could consider diffeomorphism transformations that are restricted in a different way, so that the components considered remain unchanged. The question is if the separate n $m_i$'s can still be moved around for such restricted diffeomorphisms, of course with the exception of coinciding points, since we naturally cannot move such points to different places with a continuous diffemorphism. Such an appropriate "moving around diffeomorphism" is in general easy to construct, because we do not require all vector components in a given direction to be unchanged under the diffeomorphism, but only certain components infinitesimally near the points $m_1$ , $m_2$ ,..., $m_n$ that we want to move. With the freedom to move as we please sufficiently far away from the n special points, it is easy to make the desired transformations, bringing the n $m_i$'s wherever we like, modulo coincidence. It means that for all these components of the fields, we can deduce the form $F(\phi_0^{i_1}(m_1), ..., \phi_{i_n}(m_n))$ for the expression (6.38). Here the function F could of course be a complicated function of its n arguments, and its form depends on the original "fundamental" action functional S. We just derived the existence of such an F.

To sum up, we argue that as a consequence of diffeomorphism invariance, the functional derivatives of the action S w.r.t. pseudoscalars, are constant over the basis space. To come through this argument, we simplify by only considering the case where the function from which we expand "$\phi_0$ ", is diffeomorphism invariant. That implies that all components which are not genuine scalars will be zero, because the field would otherwise tranform under the diffeomorphisms. But even the scalar fields have to be constant over the base space ( $\sim$ spacetime), and the pseudoscalars must be zero in order to keep the reparametrization invariance of the "expansion start function $\phi_0$". This assumption is all right if we have assumed that $S$ is "analytic" over the entire space of sections, so that we have Taylor expandability for all start functions, also for diffeomorphism invariant ones.

Let us note that the $\Delta\phi$ which of course are present in the Taylor expansion, have been left out in (6.39). Thus in (6.39) there is no dependence on the fields (=cross sections), apart for the dependence on the start field; but those are essentially taken to be zero. At least for a start field that is zero all over, (6.39) only depends on the action $S$, but not on any field configuration.

The quantity $\delta^{\otimes n}S$ in (6.39) thus is a tensor in function space in the sense that it is expanded on the $\mathcal{D}(m)$'s, it is in fact a product of $n$ such $\mathcal{D}(m)$, which in the last step in equation (6.39) is embedded in the definition of the $V$'s.

So for the separate points at which we differentiated, we only get non-zero functional differential quotients (6.38) for differentiation w.r.t. pseudoscalars, and then the differential quotients must be constant in the base space. We only took the true scalars in the start-function to be non-zero, and they are also constant over the base space. Now we get the integrals

$$\int_{\mathcal{M}} (\phi^i(m) - \phi_0^i(m))dm = V^i \tag{6.40}$$

where $i$ is "pseudoscalar". Our Taylor expansion then takes the form

$$S[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p=0}^{n} \binom{n}{p} f_{np}(V^1)^{n-p}(V^2)^p = F(V^1, V^2, \phi_0) \tag{6.41}$$

where $V^k = \int_{\mathcal{M}} P_k dm$ only depend on "pseudoscalar" components of $\phi^i(m)$, and we shall think of $\phi_0 = 0$.

There are also the cases where two or more of the $m_i$'s are infinitely close/coinciding. In such cases we however only get non-negligible contributions to $S[\phi]$ if we let the derivative

$$\frac{\delta^n S}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)\cdots\delta\phi^{i_n}(m_n)} \tag{6.42}$$

have factors $\delta(m_i - m_j)$. Derivatives of $\delta$-functions may also contribute, then the derivative (6.42) will have a series of terms classified by clusterings of the $m_i$'s. The number of ways of creating clusters corresponding to the partition $n = p_1 + p_2 + \cdots + p_l$ is

$$\binom{n}{p_1, p_2, \cdots, p_l} = \frac{n!}{p_1!p_2! \cdots, p_l!} \tag{6.43}$$

For each cluster with a number of say $p$ $m_i$-values, we need a $\delta$-function with $p-1$ delta functions $\delta(m_i - m_j)$ to compensate for $p-1$ of the $dm_i$ integrations, so that only one integration remains and gives us an all-over the spacetime manifold B integral

$$\int \frac{\delta^n S}{\delta\phi^{i_1}(m_1)\delta\phi^{i_2}(m_2)...\delta\phi^{i_n}(m_n)} dm_i$$
$$\propto \Pi_{(p-1 \text{ of the i-values})} \delta(m_i - m_j) \cdots \Pi_{(p \text{ of the i-values})} dm_i \cdots \quad (6.44)$$

When the $\phi^{i_k}$'s are pseudoscalar, the integral $\int_A \frac{\delta S}{\delta \phi^{i_k}} d^d x$ integrated over a region A, will be the same as the integral over the image $\tilde{f}(A)$ of this A, by a diffeomorphism $\tilde{f}$ of the base space $\mathcal{M} = B$. This corresponds to the diffeomorphically transformed quantity i.e.

$$\int_A \frac{\delta S}{\delta \phi^{ik}} d^d x = \int_{\tilde{f}(A)} (\text{with } \frac{\delta S}{\delta \phi^{ik}} d^d x \text{ transformed by the } \tilde{f}). \quad (6.45)$$

Now, the integral $\int_A d^d x$ actually gives the formal integral over the coordinates $x^l$ for the region A. By using an active diffeomorphism to push it around to $\tilde{f}(A)$, and also correspondingly transforming the coordinates, we get the same number of coordinate in all points that are related by the diffeomorphism - i.e. we get the same integral,

$$\int_A d^d x = \int_{\tilde{f}(A)} (d^d x \text{ transformed under } \tilde{f}). \quad (6.46)$$

We indeed see that in order to have diffeomorphism invariance, small regions (with $\delta S/\delta\phi^{(i-k)}$ removed) must transform in such a way that they are the same all over. Generalized this means that by taking the $\phi^{j_k}$ as pseudosacalars we indeed get the constancy.

## 6.8 Locality

The invariance under transformations that only transform f in the neighbourhood of one of the clusters, will only allow a non-zero contribution when the δ-functions of the cluster with the associated derivatives in the δ-functions eventually run out to extract a "pseudoscalar" component (of order p) from the product

$$(\phi^{i_k}(m_k) - \phi_0^{i_k}(m_k))(\phi^{i_l}(m_k) - \phi_0^{i_l}(m_k)) \cdots$$

that it is going to multiply.

So apart from the $S[\phi_0]$-term (though it best to just assume $S[\phi_0]$=S[0]), the only non-zero cluster-contributions are total spacetime integrals over "pseudoscalar" combinations of the fields, such as

$$\int \sqrt{g}(m) g^{\mu\nu}(m) \partial_\mu \phi(m) \partial_\nu \phi(m) dm \quad (6.47)$$

Here we could think of $\sqrt{g}$ as just a (fundamental) pseudoscalar field transforming under diffeomorphism symmetry with a determinant of the transformation partial derivatives,

$$\sqrt{g}(x) \to \det(\frac{\partial x^\nu}{\partial x^\mu})\sqrt{g}(x') \quad (6.48)$$

arranged in such a way that $\int \sqrt{g} dm$ is diffeomorphism invariant.

Everything in the Taylor expansion after choosing $\phi_0 = 0$ (by field redefinition) becomes expressed by means of all integrals over $\mathcal{M}$ of the type $V^k = \int_{\mathcal{M}} P_k dm$. For sufficiently high $n$, we can expect to get the same $P_k$ out of several of the clusters into which we partition such "big enough" $n$. In that case we might evaluate the

$$\frac{1}{n!}\binom{n}{p_1 \cdots p_l} \tag{6.49}$$

and count the possibilities, but it is not really needed because the weight coefficients for the term combination can only be obtained if we somehow know the fundamental action functional S. We have already seen that we apriori shall get a series of terms in which all powers and all products of such powers of the integrals $V^k = \int_{\mathcal{M}} P_k dm$ occur. That is to say, we get an expression of the form

$$S[\phi] = \sum_{k_1, k_2, \cdots k_q}^{\infty} C_{k_1 k_2 \cdots k_q} V^{k_1} V^{k_2} \cdots V^{k_q} \tag{6.50}$$

which in fact is the Taylor expansion for any function in the variables $(V^1, V^2, ..)$, provided one chooses the $C_{k_1 k_2 \cdots k_q}$ appropriately.

So all we have derived is that $S[\phi]$ is a function of these variables $(V^1, V^2, ..)$, but we do not know which function. The variables on which are all $\mathcal{M}$-integrals of the "pseudoscalar" field combinations $V^k$. Now we shall however follow our earlier work where we derived an effective locality.

The main use of the action is via the Euler-Lagrange equations. Suppose we have a field $\xi$ which can even be a component of some tensor field, or whatever; then the Euler-Lagrange equation for $\xi$ is

$$\frac{\delta S[\phi]}{\delta \xi(x)} = 0 \tag{6.51}$$

and now, since we derived S to be of the form $S[V^1, V^2, ..]$, we get

$$\frac{\delta S[\phi]}{\delta \xi(x)} = \sum_k \frac{\partial S[\phi]}{\partial V^k} \frac{\delta V^k}{\delta \xi(x)} = \tag{6.52}$$

$$= \sum_k \frac{\partial S[\phi]}{\partial V^k} \frac{\delta \mathcal{P}_k}{\delta \xi(x)} (\text{mod partial integration}) = \frac{\delta S_{eff}}{\delta \xi(x)} \tag{6.53}$$

where

$$S_{eff} = \int_{\mathcal{M}} \sum_k \frac{\partial S}{\partial V^k} \mathcal{P}_k(x) dm \tag{6.54}$$

which by construction is local, provided the coefficients $\partial S / \partial V^k$ do not depend on the $V^k$'s. But these $V^k$ are "constants" in the sense that they do not depend on space and time, i.e. not on $m \in \mathcal{M}$ (careful with double labeling with $m$ and $x$).

So we got locality except that the coupling constants via the $\partial S / \partial V^k$'s depend on integrals taken over all spacetime.

## 6.9 Conclusion

We have derived locality from an analytic and diffeomorphism symmetric, very general action $S$, except for the mildly non-local coupling constants. This non-locality would not be easy to observe, because it is difficult to establish the dependence of the effective action on the derivatives of the (over all spacetime) integrals $V_i$ of the coupling constants. The problem is that we do not know the original/fundamental action, and in any case, one would have to determine these derivatives by simple measurements (renormalizing the couplings). The only way out would be that we either manage to successfully guess the fundamental action $S$, or use the fact that there will appear a consistency problem in the integrals $V_i$, being integrals over a spacetime development on which the constants themselves depend (in a presumably complicated way).

Very mild assumptions about general properties of the fundamental action $S$, combined with such a consistency restriction, might however lead to interesting results. We indeed argued that our beloved "law of nature" Multiple point principle can be derived within such a consistency philosophy.

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# 7 How Compact Stars Challenge Our View About Dark Matter

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**Abstract.** It is by now well established that non-relativistic matter in the Universe is dominated by dark matter, the origin and nature of which still remains a mystery. Although the collisionless dark matter paradigm works very well at large distances, a few puzzles at galactic scales arise. These problems may be tackled assuming a self-interacting dark matter. If dark matter is accumulated inside a star it will modify its evolution and its properties, such as mass-to-radius profiles and frequency oscillation modes. Asteroseismology is a relatively new, powerful tool that allows us to constrain dark matter models, offering us complementary bounds to the results coming from other means, such as collider or direct searches. I will present here the main results we have obtained assuming that the dark matter particle is a boson, which inside a star is modelled as a Bose-Einstein condensate with a polytropic equation-of-state. We have computed i) the radial and non-radial oscillation modes of light clumps of dark matter made of ultra light repulsive scalar fields, and ii) the mass-to-radius profiles as well the frequencies of radial modes of admixed dark matter strange quark stars.

**Povzetek.** Temna snov poskrbi za večino snovi v vesolju. Kaj je temna snov zgolj ugibamo. Z običajno snovjo temna snov (skoraj) ne interagira. Avtor meni, da lahko nekaj izvemo o temni snovi iz opazovanja zvezd, v katerih se je nakopičila temna snov, ker se lastnosti takih zvezd razlikujejo od lastnosti običajnih zvezd — denimo v razmerju med maso in radijem in v frekvenci oscilacij. Avtor predstavi model, ki predpostavi, da tvorijo temno snov bozoni, ki se znotraj zvezde obnašajo kot Bose-Einsteinov kondenzat, opisljiv s politropno enačbo stanja. Izračunal je: i. Radialne in neradialne nihajne načine zelo lahkih gruč temne snovi iz skalarnih polj, z odbojno medsebojno interakcijo. ii. Masne profile v odvisnosti od radija in frekvence radialnih nihajnih načinov za zvezde, ki jih sestavljajo čudni kvarki s primesjo temne snovi.

Keywords: Composition of astrophysical objects; Asteroseismology; Self-interacting dark matter; Bose-Einstein condensates.
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## 7.1 Introduction

Since the pioneer work of F. Zwicky about the dynamics of the Coma galaxy cluster in the 30's [1], and the observations made by V. Rubin to determine the rotation curves of galaxies a few decades later [2], we are convinced that most of the non-relativistic matter in the Universe is dark, usually referred to as cold dark matter. In modern times current well-established data coming from many different sources confirm the existence of dark matter [3], although its nature and origin still remains a mystery. For a review on dark matter see [4,5], and for recent reviews on dark matter detection searches see [6–8].

Usually dark matter (DM) in the standard parametrization of the Big-Bang cosmological model is assumed to be made of weakly interacting massive particles, a conjecture which works very well at large (cosmological) scales ($\geq \mathrm{Mpc}$), but encounters several problems at smaller (galactic) scales, like the core-cusp problem, the diversity problem, the missing satellites problem and the too-big-to-fail problem [9]. These problems may be tackled in the context of self-interacting dark matter [10,11], as any cuspy feature will be smoothed out by the dark matter collisions. In addition, if dark matter consists of ultralight scalar particles with a mass $m \leq eV$, and with a small repulsive quartic self-interaction a Bose-Einstein condensate (BEC) may be formed with a long range correlation. This scenario has been proposed as a possible solution to the aforementioned problems at galactic scales [12–14].

Boson stars are star-like, self-gravitating bosonic configurations, where bosons are exclusively trapped in their own gravitational potentials. Boson stars have been studied in [15–23], see also [24–27] for Newtonian self-gravitating Bose-Einstein condensates. The maximum mass of bosons stars in non-interacting systems was found in [15,16], while in [17,18] it was shown that self-interactions can cause significant changes. In [20,21] the authors constrained the boson star parameter space using data from galaxy and galaxy cluster sizes.

Unlikely many other forms of matter, compact objects [28–30], which are formed at the end stages of stellar evolution, are unique probes to study the properties of matter under exceptionally extreme conditions. The matter inside such objects is characterized by ultra-high matter densities for which the usual classical description of stellar plasmas in terms of non-relativistic Newtonian fluids is inadequate. Therefore, such very dense compact objects are relativistic and as such, they are only properly described within the framework of Einstein's General Relativity (GR) [31].

Strange quark stars [32–37], at the moment hypothetical objects, can be viewed as ultra-compact NSs. Since quark matter is by assumption absolutely stable, it may be the true ground state of hadronic matter [38,39], and therefore this new class of relativistic compact objects has been proposed as an alternative to typical NSs. In fact strange quark stars may explain the observed super-luminous supernovae [40,41], which occur in about one out of every 1000 supernovae explosions, and which are more than 100 times more luminous than regular supernovae. One plausible explanation is that since quark stars are much more stable than NSs, they could explain the origin of the huge amount of energy released in super-luminous supernovae. Many works have been recently proposed to validate its existence in different astrophysical scenarios [42,43].

It is well-known that the properties of stars, such as mass and radius, depend crucially on the equation-of-state. Furthermore,the presence of DM inside a star is expected to influence the structure, the evolution as well as certain properties of the object, such as mass-to-radius profiles and frequency oscillation modes. Even if dark matter does not interact directly with normal matter, it can have significant gravitational effects on stellar objects DM that can influence evolution and structure of compact objects [44–58]. Given the recent advances in Helioseismology and Asteroseismology in general, studying the oscillations of stars and computing the frequency modes offer us the opportunity to probe the interior of the stars and learn more about the equation-of-state, since the precise values of the frequency modes are very sensitive to the thermodynamics of the internal structure of the star [59]. For previous works on radial oscillations of stars see [60–68] and references therein.

## 7.2 Impact of DM on strange quark stars

In the first part of the presentation we discuss the impact of bosonic self-interacting DM on properties of strange quark stars.

### 7.2.1 Mass-to-radius profiles

-Structure equations: We briefly review relativistic stars in General Relativity (GR). The starting point is Einstein's field equations without a cosmological constant

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{7.1}$$

where we have set Newton's constant equal to unity, $G = 1$, and in the exterior problem the matter energy momentum tensor vanishes. For matter we assume a perfect fluid with pressure $p$, energy density $\rho$ and an equation of state $p(\rho)$, while the energy momentum trace is given by $T = -\rho + 3p$. For the metric in the case of static spherically symmetric spacetimes we consider the following ansatz

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega^2 \tag{7.2}$$

with two unknown functions of the radial distance $f(r), g(r)$. For the exterior problem one obtains the well-known Schwarzschild solution [69]

$$f(r) = g(r)^{-1} = 1 - \frac{2M}{r} \tag{7.3}$$

where $M$ is the mass of the star. For the interior solution we introduce the function $m(r)$ instead of the function $g(r)$ defined as follows

$$g(r)^{-1} = 1 - \frac{2m(r)}{r} \tag{7.4}$$

so that upon matching the two solutions at the surface of the star we obtain $m(R) = M$, where $R$ is the radius of the star. The Tolman-Oppenheimer-Volkoff (TOV) equations for the interior solution of a relativistic star with a vanishing cosmological constant read [70,71]

$$m'(r) = 4\pi r^2 \rho(r) \tag{7.5}$$

$$p'(r) = -(p(r) + \rho(r)) \frac{m(r) + 4\pi p(r) r^3}{r^2(1 - \frac{2m(r)}{r})} \tag{7.6}$$

where the prime denotes differentiation with respect to r, and the equations are to be integrated with the initial conditions $m(r = 0) = 0$ and $p(r = 0) = p_c$, where $p_c$ is the central pressure. The radius of the star is determined requiring that the pressure vanishes at the surface, $p(R) = 0$, and the mass of the star is then given by $M = m(R)$.

-Two-fluid formalism: Now let us assume that the star consists of two fluids, namely strange matter (de-confined quarks) and dark matter with only gravitational interaction between them, and equations of state $p_s(\rho_s)$, $p_\chi(\rho_\chi)$ respectively. The total pressure and the total energy density of the system are given by $p = p_s + p_\chi$ and $\rho = \rho_s + \rho_\chi$ respectively. Since the energy momentum tensor of each fluid is separately conserved, the TOV equations in the two-fluid formalism for the interior solution of a relativistic star with a vanishing cosmological constant read [72,73]

$$m'(r) = 4\pi r^2 \rho(r) \tag{7.7}$$

$$p_s'(r) = -(p_s(r) + \rho_s(r)) \frac{m(r) + 4\pi p(r) r^3}{r^2(1 - \frac{2m(r)}{r})} \tag{7.8}$$

$$p_\chi'(r) = -(p_\chi(r) + \rho_\chi(r)) \frac{m(r) + 4\pi p(r) r^3}{r^2(1 - \frac{2m(r)}{r})} \tag{7.9}$$

In this case in order to integrate the TOV equations we need to specify the central values both for normal matter and for dark matter $p_s(0)$ and $p_\chi(0)$ respectively. So in the following we show the mass-radius diagram for a certain value of the constant $K = 2\pi l/m_\chi^3$ and for fixed dark matter fraction

$$\epsilon = \frac{p_\chi(0)}{p_s(0) + p_\chi(0)} \tag{7.10}$$

and we consider four cases, namely $\epsilon = 0.02, 0.035, 0.05, 0.09$. We have chosen these values in agreement with the current dark matter constraints obtained from stars like the Sun. Actually, as shown by several authors, even smaller amounts of DM (as a percentage of the total mass of the star) can have a quite visible impact on the structure of these stars [74–76]. As we discuss in this work even such small amounts of DM can change the $M - R$ relation of neutron stars.

-Equation-of-states: For the condensed dark matter we shall consider the equation of state obtained in [77], namely $P_\chi = K\rho_\chi^2$, where the constant $K = 2\pi l/m_\chi^3$ is given in terms of the mass of the dark matter particles $m_\chi$ and the scattering length $l$. In a dilute and cold gas only the binary collisions at low energy are relevant, and these collisions are characterized by the s-wave scattering length $l$ independently of the form of the two-body potential [77]. Therefore we can consider a short range repulsive delta-potential of the form

$$V(\vec{r}_1 - \vec{r}_2) = \frac{4\pi l}{m_\chi}\delta^{(3)}(\vec{r}_1 - \vec{r}_2) \tag{7.11}$$

which implies a dark matter self interaction cross section of the form $\sigma_\chi = 4\pi l^2$ [52,77]. Following previous studies we fix the scattering length to be $l = 1fm$ [52,77], and for $\sigma_\chi/m_\chi$ we apply the bounds discussed in the Introduction

$$0.45\frac{cm^2}{g} < \frac{\sigma_\chi}{m_\chi} < 1.5\frac{cm^2}{g} \tag{7.12}$$

which then implies the following range for the mass of the dark matter particle

$$0.05GeV < m_\chi < 0.16GeV \tag{7.13}$$

and thus for the constant K

$$\frac{4}{B} < K < \frac{150}{B} \tag{7.14}$$

where now the constant K is given in units of the bag constant. Our main results are shown in figures 7.1 and 7.2. In Fig. 7.1 we show the mass-to-radius profiles for $K = 4/B$ and for $\epsilon = 0.02, 0.05, 0.09$, while in Fig. 7.2 we show the profiles for $K = 150/B$ and for $\epsilon = 0.02, 0.035, 0.05$. The standard curve corresponding to no DM (in black) is shown as well for comparison reasons.

For strange matter we shall consider the simplest equation of state corresponding to a relativistic gas of de-confined quarks, known also as the MIT bag model [78,79]

$$p_s = \frac{1}{3}(\rho_s - 4B) \tag{7.15}$$

and the bag constant has been taken to be $B = (148MeV)^4$ [80].

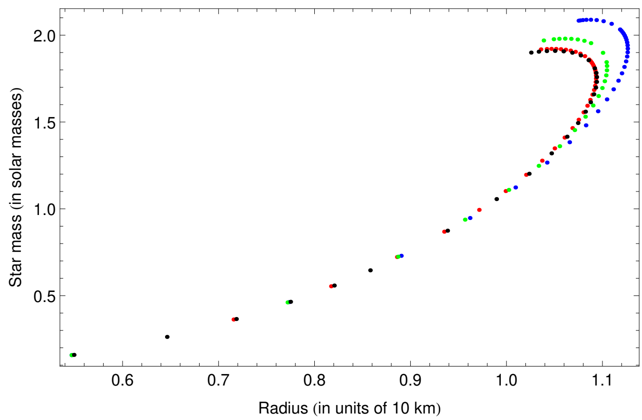


**Fig. 7.1.** Mass-to-radius profile for $K = 4/B$.

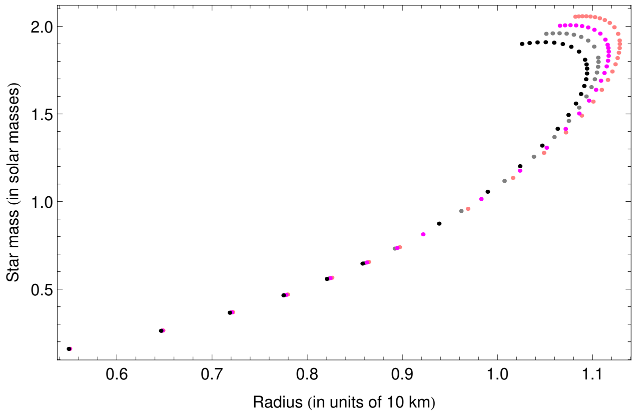


**Fig. 7.2.** Mass-to-radius profile for $K = 150/B$.

### 7.2.2 Radial oscillations

If $\delta r$ is the radial displacement and $\delta P$ is the perturbation of the pressure, the equations governing the dimensionless quantities $\xi = \delta r/r$ and $\eta = \delta P/P$ are the following [81,82]

$$\xi'(r) = -\frac{1}{r}\left(3\xi + \frac{\eta}{\gamma}\right) - \frac{P'}{P+\epsilon}\xi \tag{7.16}$$

$$\begin{aligned}\eta'(r) = \xi &\left[\omega^2 r \frac{P+\epsilon}{P} e^{\lambda - A} - \frac{4P'}{P} - 8\pi(P+\epsilon) r e^{\lambda} + \frac{r(P')^2}{P(P+\epsilon)}\right] \\ &+ \eta\left[-\frac{\epsilon P'}{P(P+\epsilon)} - 4\pi(P+\epsilon) r e^{\lambda}\right]\end{aligned} \tag{7.17}$$

where $e^{\lambda}, e^{A}$ are the two metric functions, $\omega$ is the frequency oscillation mode, and $\gamma$ is the relativistic adiabatic index defined to be

$$\gamma = \frac{dP}{d\epsilon}(1 + \epsilon/P) \tag{7.18}$$

The system of two coupled first order differential equations is supplemented with two boundary conditions, one at the center as $r \to 0$, and another at the surface $r = R$. The boundary conditions are obtained as follows: In the first equation, $\xi'(r)$ must be finite as $r \to 0$, and therefore we require that

$$\eta = -3\gamma\xi \tag{7.19}$$

must satisfied at the center. Moreover, in the second equation, $\eta'(r)$ must be finite at the surface as $\epsilon, P \to 0$ and therefore we demand that

$$\eta = \xi\left[-4 + (1 - 2M/R)^{-1}\left(-\frac{M}{R} - \frac{\omega^2 R^3}{M}\right)\right] \tag{7.20}$$

must satisfied at the surface, where we recall that $M, R$ are the mass and the radius of the star respectively. Using the shooting method we first compute the

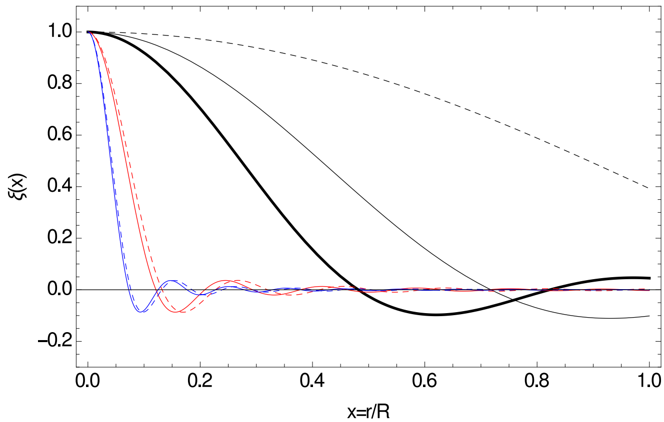


**Fig. 7.3.** Eigenfunctions $\xi$ vs $r/R$.

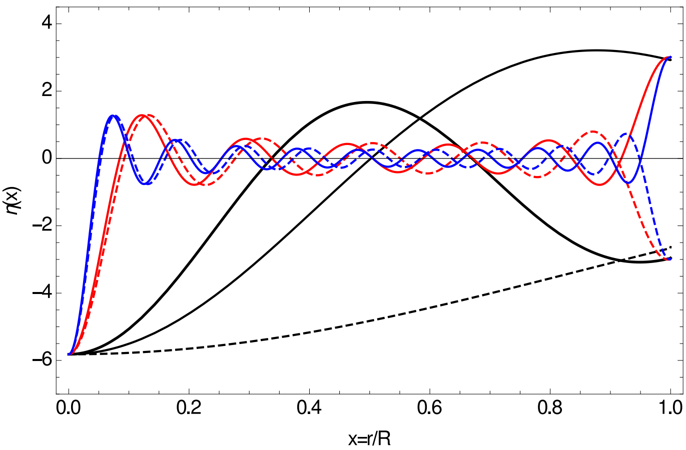


**Fig. 7.4.** Eigenfunctions $\eta$ vs $r/R$.

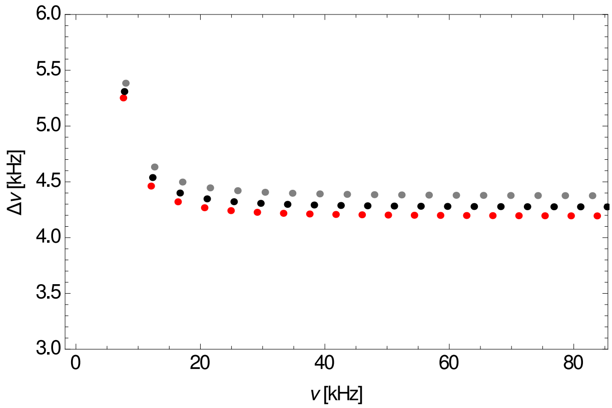


**Fig. 7.5.** Large frequency separation.

dimensionless quantity $\bar{\omega} = \omega t_0$ where $t_0 = 1ms$. Then the frequencies are computed by

$$\nu = \frac{\bar{\omega}}{2\pi}\ \mathrm{kHz} \tag{7.21}$$

Therefore, contrary to the previous hydrostatic equilibrium problem, which is an initial value problem, this is a Sturm-Liouville boundary value problem, and as such the frequency $\nu$ is allowed to take only particular values, the so-called eigenfrequencies $\nu_n$. Each $\nu_n$ corresponds to a specific radial oscillation mode of the star. Accordingly, each radial mode is identified by its $\nu_n$ and by an associated pair of eigenfunctions – the displacement perturbation $\xi_n(r)$ and the pressure perturbation $\eta_n(r)$. Our main results are shown in figures 7.3, 7.4 and 7.5. In particular, in Fig. 7.3 we show several eigenfunctions $\xi_n$ (n=0,1,2,10,11,18,19) versus normalized coordinate distance $r/R$, in Fig. 7.4 we show several eigenfunctions $\eta_n$ (same values of $n$), and in Fig. 7.5 we show the large frequency separation $\Delta\nu_n = \nu_{n+1} - \nu_n$ versus frequencies in kHz for 3 cases, namely i) no DM (red color), 5% of DM (black color) and 12% of DM (grey color).

## 7.3 Newtonian stars made of ultralight repulsive DM

Equation-of-state: The perturbative Lagrangian of a relativistic real scalar field $\phi$ is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{7.22}$$

where the scalar potential is of the form

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{24}\frac{m^2}{F^2}\phi^4 + ... \tag{7.23}$$

and where we consider renormalizable theories only, ignoring all higher order terms. In this work the scalar field is identified with any pseudo-Goldstone boson. The sign of the quartic self-interaction is taken to be positive since we assume a repulsive self-interaction for the Dark pseudo-Goldstone boson. Therefore, the model assumed here is characterized by two unknown mass scales, namely the mass of the scalar particle, $m$, as well as the decay constant, $F \gg m$, arising from the spontaneous breaking of some global symmetry. Unfortunately, it turns out that it is not easy to obtain scalar field models with a tiny mass and a repulsive force within known Particle Physics, although some attempts have been made [83]. In the following, without relying on concrete Particle Physics models, we shall assume that this is possible, and we shall study radial oscillations of objects made of Dark pseudo-Goldstone bosons.

The above scalar potential combined with the Gross-Pitaevskii equation [84–86], also known as non-linear Schrödinger equation, leads to the following equation-of-state for the ultralight pseudo-Goldstone boson [83,87]:

$$P(\epsilon) = K\epsilon^2 \tag{7.24}$$

where the constant K is computed to be [83,87]

$$K = \frac{1}{(2\Lambda)^4} \tag{7.25}$$

where a new mass scale $\Lambda \equiv \sqrt{mF}$ has been introduced.

-Hydrostatic equilibrium: Since the axion star is non-relativistic described by a polytropic EoS, to study the hydrostatic equilibrium one has to solve the non-relativistic version of the Tolman-Oppenheimer-Volkoff (TOV) equations [70,71]

$$m'(r) = 4\pi r^2 \epsilon(r) \tag{7.26}$$

for the mass function, and

$$P'(r) = -\epsilon(r)\frac{m(r)}{r^2} \tag{7.27}$$

for the pressure, where the prime denotes differentiation with respect to the radial coordinate $r$. Combining these two equations we can derive a single second order differential equation, known as the Lane-Emden equation [28]

$$\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) = -x^2\theta \tag{7.28}$$

with the initial conditions $\theta(0) = 1$ and $d\theta/dx(0) = 0$, where the new variables are defined as follows

$$x = \frac{r}{a} \tag{7.29}$$

and

$$\theta = \frac{\epsilon}{\epsilon_c} \tag{7.30}$$

with $\epsilon_c$ being the central energy density, while $a$ is given by $a = \sqrt{K/2\pi}$. It is easy to verify that the solution

$$\theta(x) = \frac{\sin(x)}{x} \tag{7.31}$$

satisfies both the Lane-Emden equation and the initial conditions. Therefore, the energy density as a function of the radial coordinate is given by

$$\epsilon(r) = \epsilon_c\frac{\sin(r/a)}{(r/a)}. \tag{7.32}$$

The above equation is valid for the radius varying from $r = 0$ until the first zero of the function $\epsilon(r)$, therefore the function $\epsilon(r)$ varies between $\epsilon_c$ and 0. Finally, the mass M and the radius R of the star are given by

$$M = 4\pi\epsilon_c a^3 \int_0^{\pi} dx x^2\theta(x) \tag{7.33}$$

$$R = \pi a \tag{7.34}$$

Clearly, only the mass of the star depends on the central energy density, while the radius is fixed. This happens only in the special case $n = 1$, whereas in general both M and R depend on the central energy density. This can also be seen in the mass-to-radius profile for Newtonian boson stars with repulsive forces as shown in the work of Chavanis and collaborators (see Fig. 2 of [26] and Fig. 4 of [27]).

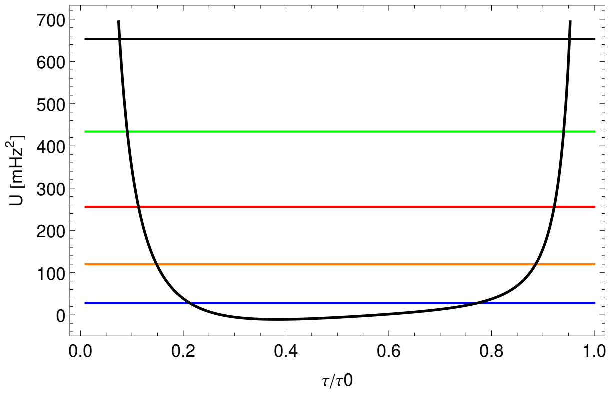


**Fig. 7.6.** Radial oscillations: Acoustic potential vs acoustic time.

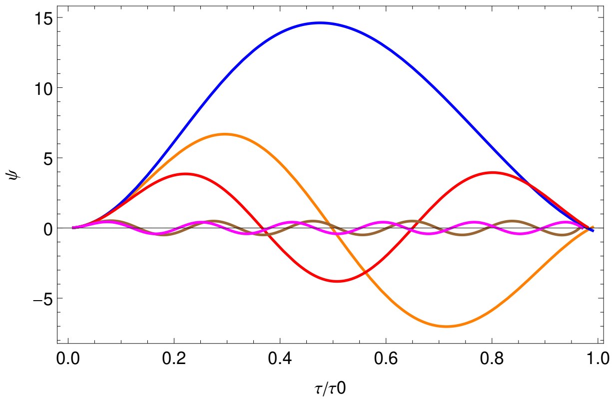


**Fig. 7.7.** Radial oscillations: Eigenfunctions vs acoustic time.

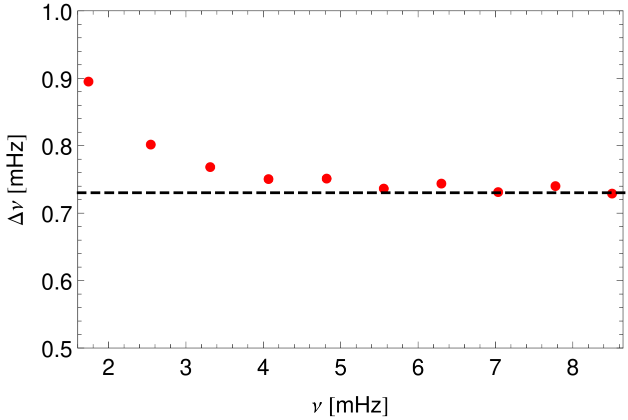


**Fig. 7.8.** Radial oscillations: Large frequency separation in mHz.

### 7.3.1 Radial oscillations

In the first part of the presentation we presented the first order system of two coupled equations for the perturbations of a pulsating star. Here, however, we prefer to work equivalently with a second order differential equation used in [66]

$$-f^2\zeta'' + G\zeta' + (H - \omega^2)\zeta = 0, \tag{7.35}$$

supplemented with the boundary conditions at the origin $r = 0$ and at the surface of the star $r = R$: $\zeta(r = 0) = 0$ and $\delta p(r = R) = 0$. In the previous equation $\zeta = r^2 e^{-A/2}\xi$, with $e^A \equiv g_{tt}$ being the temporal component of the metric tensor, while the background functions $f$, $G$ an $H$ are given by

$$f^2(r) = \frac{\gamma P e^{A-\lambda}}{P + \epsilon}, \tag{7.36}$$

$$G(r) = -\frac{f^2}{\gamma P}\left[\frac{\gamma P}{2}(\lambda + 3A) + (\gamma P)' - \frac{2\gamma P}{r}\right], \tag{7.37}$$

$$H(r) = -\frac{f^2}{\gamma P}\left[\frac{4P'}{r} + 8\pi P(P + \epsilon)e^{\lambda} - \frac{(P')^2}{P + \epsilon}\right], \tag{7.38}$$

and finally the perturbation of the pressure can be computed as

$$\delta p(r) = -\frac{e^{A/2}}{r^2}(\zeta P' + \gamma P \zeta'). \tag{7.39}$$

The Sturm-Liouville boundary value problem at hand can be treated equivalently as a quantum mechanical problem by recasting the second order differential equation for $\zeta$ into a Schrödinger-like equation [88] of the form

$$\frac{d^2\psi}{d\tau^2} + \left[\omega^2 - U(\tau)\right]\psi = 0, \tag{7.40}$$

where the new variables $\tau$ and $\psi$ are defined as acoustic radius $\tau = \int_0^r f^{-1}(z)dz$ and $\psi(\tau) = \zeta/u$. The effective potential is found to be

$$U = H + \frac{\Pi^2}{4} + \frac{f\Pi'}{2}, \tag{7.41}$$

where the function $\Pi$ is given by $\Pi = -f' - G/f$ while $u$ is determined by the condition $u'/u = -\Pi/(2f)$.

The acoustic potential with the first 5 eigenvalues, the corresponding eigenfunctions as well as the large frequency separation in $mHz$ are shown in the figures 7.6, 7.7 and 7.8, respectively.

### 7.3.2 Nonradial oscillations

Linear adiabatic acoustic perturbations in the Cowling approximation [89], where the perturbations of the gravitational potential are neglected, are described by the following equation [90]

$$\zeta''(r) + \left(\frac{2}{r} + \frac{2\epsilon'(r)}{\epsilon(r)}\right)\zeta'(r) + \left(\frac{\omega_{n,l}^2}{c_s^2} - \frac{l(l+1)}{r^2}\right)\zeta(r) = 0 \tag{7.42}$$

where $c_s$ is the speed of sound defined by $c_s^2 = dP/d\epsilon$, $\omega_{n,l}$ $(= 2\pi\nu_{n,l})$ are the discrete eigenvalues, $l > 0$ is the degree of angular momentum (or degree of the mode), and $n = 0, 1, 2, ...$ is the overtone number (or radial mode).

The Sturm-Liouville boundary value problem at hand can be treated equivalently as a quantum mechanical problem by recasting the second order differential equation for $\zeta$ into a Schrödinger-like equation [88,91,92] of the form

$$\frac{d^2\psi}{d\tau^2} + \left[\omega^2 - U_l(\tau)\right]\psi = 0. \tag{7.43}$$

Introducing the functions

$$A(r) = \frac{2}{r} + \frac{2\epsilon'(r)}{\epsilon(r)}, \tag{7.44}$$

$$H_l(r) = c_s(r)^2\frac{l(l+1)}{r^2}, \tag{7.45}$$

and

$$P(r) = A(r)c_s(r) - c_s'(r). \tag{7.46}$$

The new variables $\tau$ and $\psi$ are defined as follows

$$\psi(r) = \frac{\zeta(r)}{u(r)} \tag{7.47}$$

where $u$ satisfies the condition $u'/u = -P/(2c_s)$, and $\tau$ is the acoustic time

$$\tau = \int_0^r c_s^{-1}(z)dz. \tag{7.48}$$

Finally, the effective potential is found to be

$$U_l(r) = H_l(r) + \left(\frac{P(r)}{2}\right)^2 + \frac{c_s(r)P'(r)}{2}, \tag{7.49}$$

and we thus obtain the effective potential as a function of the acoustic time in parametric form $\tau(r), U_l(r)$.

The acoustic potential with the first 7 eigenvalues, the corresponding eigenfunctions as well as the large frequency separation in $mHz$ for $l = 2$ are shown in the figures 7.9, 7.10 and 7.11, respectively.

## 7.4 Conclusions

In this presentation we have presented results of our work on properties of self-interacting scalar field dark matter in two respects. In particular, in the first part we studied the impact of dark matter on the mass-to-radius profiles as well as on the radial oscillation modes of non-rotating, spherically symmetric strange quark stars in which dark matter is accumulated. Then, in the second part we studied radial and non-radial oscillations of self-gravitating bosonic (star-like) configurations.

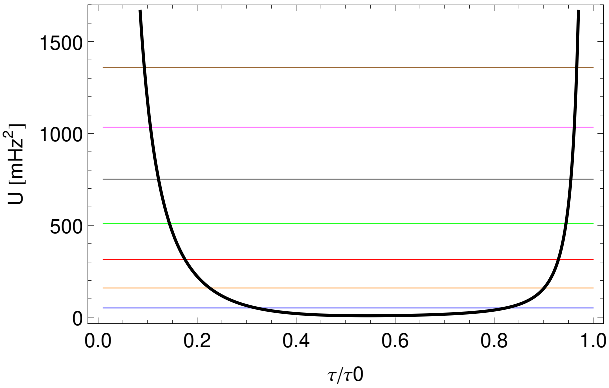


**Fig. 7.9.** Non radial oscillations: Acoustic potential vs acoustic time for $l = 2$.

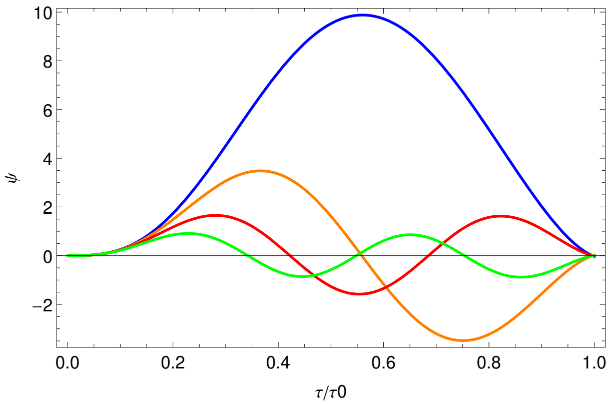


**Fig. 7.10.** Non radial oscillations: Eigenfunctions vs acoustic time for $l = 2$.

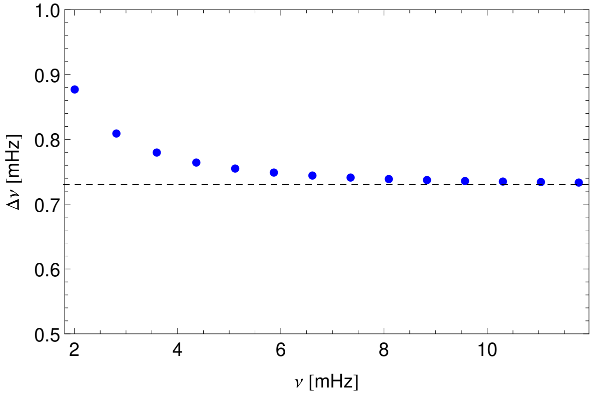


**Fig. 7.11.** Non radial oscillations: Large frequency separation in mHz for $l = 2$.

Strange stars are hypothetical compact objects that are supposed to be much more stable than neutron stars, and thus could explain the super luminous supernovae. For the star interior problem we have solved numerically the Tolman-

Oppenheimer-Volkoff equations in the two-fluid formalism. For strange matter we have assumed the simplest version of the MIT bag model (radiation plus the bag constant), while if dark matter is modelled inside the star as a BEC, it can be described by a polytropic equation of state with index $n = 1$. We have shown the mass-radius diagram assuming that strange stars are made of up of $(5 - 10)$ % of dark matter. We conclude that if strange stars do exist, and if they accumulate dark matter, our findings limit in a certain way the radius and the mass of these compact objects.

After that we studied the radial oscillations of dark matter admixed strange stars. Integrating numerically the equations for the perturbations we solved the corresponding boundary value problem to compute the first 11 frequency radial modes for three stars with the same mass and radius, but with different dark matter amounts. The large frequency separation were computed as well, and we showed them for all three models in the same plot for comparison reasons so that the impact of dark matter could be inferred.

In the second part we studied radial oscillations of Dark BEC stars made of ultralight repulsive scalar particles in the Fermi-Thomas approximation. Using the known background solution to the Lane-Emden equation for a Newtonian polytropic star with index $n = 1$ we solved the Sturm-Liouville boundary value problem for the perturbation with the shooting method. We have computed the fundamental as well as several excited modes for two different star masses, and we have shown graphically i) several eigenfunctions corresponding to the first three and two highly excited oscillation modes, and ii) how the large frequency difference varies with the frequencies themselves. In addition, we have reformulated the boundary value problem equivalently by writing down a Schrödinder-like equation, and we have shown the effective potential together with the first five values of $\omega^2$.

Finally, we have studied non-radial oscillations of bosonic configurations made of ultralight repulsive scalar particles in the Cowling approximation. For three different values of the angular degree $l = 1, 2, 3$ we have computed the lowest frequencies, several associated eigenfunctions, and the effective potential in the equivalent description in terms of a Schrödinder-like equation. The large frequency separations are shown as well. In all three cases, like in the radial oscillation case, for the higher excited modes the large separation tends to a constant determined entirely by the mass scale $\Lambda = \sqrt{mF}$, where $m$ is the mass of the scalar field and $F$ is a high mass scale that determines the self-interaction coupling constant in the scalar potential.

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# 8 Phenomenological Studies of Models With a Pseudo Nambu Goldstone Boson

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**Abstract.** We discuss how to probe a class of models where the Standard Model-like Higgs boson is identified with a pseudo Nambu Goldstone Boson (pNGB) associated with the spontaneous breaking of a global symmetry. We focus on the minimal version of such models. There SO(5) symmetry is broken to SO(4) so that four pNGBs appear which corresponds to the Higgs doublet in the SM. In order to probe such a model, double Higgs production process is found to be quite powerful. It is shown that the production cross section of this process has a model specific behavior so that we can distinguish different new physics scenarios.

**Povzetek.** Avtor študira model, ki uporabi za opis Higgsovega skalarja psevdo Nambu-Goldstonove bozone, ki nastanejo pri spontani zlomitvi globalne simetrije SO(5) v SO(4). Štirje psevdo Nambu-Goldstonovi bozoni ustrezajo Higgsovemu dubletu standardnega modela. Avtor pokaže, da je presek za nastanek teh dveh Higgsovih bozonov odvisen od parametrov realizacije modela, s čimer se spremenijo tudi napovedi modela.

Keywords: Hierarchy problem, composite Higgs model, double Higgs production processes

## 8.1 Introduction

In 2012, the Higgs boson was discovered at the LHC experiments, and the Standard model (SM) is then experimentally established. However, the SM has several serious problems. There is a well-known theoretical problem known as a gauge hierarchy problem. If there is a unified theory including gravity, the unified theory is considered to be realized at around the Planck scale $\sim 10^{19}$ GeV. On the other hand, the typical energy scale of the SM (electroweak theory) is the scale of the Higgs vacuum expectation value, 175 GeV. There is a colossal hierarchy between these two energy scales. It is an origin of gauge hierarchy problem. The so-called gauge hierarchy problem is a mixture of two kinds of problems. One is how to set appropriate values for the model parameters of order of 100 GeV if the fundamental scale is of the order of $10^{19}$GeV at the boundary. The naive expectation in such a case is that all the parameters with a mass dimension are set to be $\mathcal{O}(10^{19})$ GeV.

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Even if we can set the value of parameters to be $\mathcal{O}(100)$ GeV by some mechanisms, the parameters can be affected by radiative corrections, and they may be as large as $\mathcal{O}(10^{19})$ GeV. In order to avoid it, we need a fine-tuning or some mechanisms to cancel such a huge radiative correction. It is the second type of problem.

In order to address the hierarchy problem, several excellent mechanisms are proposed in the literature. For example, supersymmetry provides a cancellation between a bosonic loop and fermionic loop. Therefore quadratic divergences in scalar mass parameter disappear (The second type of problem is solved). The first type of problem in the SUSY model is known as $\mu$-problem[1], and there are many attempts to solve it(*e.g.* [2]). There are other many ideas such as the gauge Higgs unification scenario, models with the classical conformal invariance, and so on. In a model where the Higgs boson is identified with a pNGB associated with some global symmetry breaking, the Higgs mass parameter is naturally set to be much smaller than the fundamental scale (A solution to the first problem). Also, such pNGB can be sometimes considered as a composite state by the analogy of the pion which can be treated as a pNGB associated with chiral symmetry breaking. In such a scenario, the cut-off scale of the model is lowered to be $\mathcal{O}(10)$ TeV[1] and the second problem can become milder.

In this talk, based on Ref. [3], we focus on the minimal version of such a scenario with pNGB, so-called Minimal Composite Higgs models (MCHMs)[4] and we study phenomenology in MCHMs. We have found that the double Higgs production process is interesting and powerful to probe MCHMs.

## 8.2 Model

Since there are four real degrees of freedom in a SU(2) doublet, the minimal setup of models where the SM-like Higgs doublet is identified with pNGBs contains four pNGBs. It means that the breaking pattern of global symmetry should include four broken generators. One of the minimal breaking patterns is SO(5) to SO(4). We here consider SO(5)$\times$ U(1)$_X$ $\to$ SO(4)$\times$ U(1)$_X$ model. The breaking occurs at the scale $f$. Associated with this symmetry breaking, four NGBs appears. Since the SU(2)$_L\times$U(1)$_Y$ subgroup in SO(4)$\times$U(1)$_X$ is gauged, the global symmetry is explicitly broken by the gauge coupling. Also, the matter fermions in the SM cannot compose SO(5) multiplet so that the SM Yukawa interactions explicitly break the global symmetry too. Because of these explicit breaking effects, the NGBs become pNGBs, and they get smaller mass compared to the symmetry breaking scale.

The gauge interactions in the low energy effective theory are completely determined by the breaking pattern of the global symmetry, while the Yukawa interactions depend on what representations of SO(5) the SM fermions are embedded into. In this review, we consider three cases; all the SM fermions are embedded into 4 dimensional representations (MCHM4), 5 dimensional representations (MCHM5) and 14 dimensional representations (MCHM14) of SO(5). In

1 It means that there can be some intermediate theory which may include a strong dynamics before appearing the final unified theory.

each model, the effective Lagrangian for fermion interactions with the SM-like Higgs boson is given by

$$
\begin{aligned}
\mathcal{L}^{\text{matter}}_{\text{MCHM4}} = & \sum_{r=q,\ell,u,d,e} \overline{\Psi}^{(4)}_r \not{p} \left[\Pi^r_0 + \Pi^r_1 \Gamma^i \Sigma_i\right] \Psi^{(4)}_r + \overline{\Psi}^{(4)}_q \left[M^t_0 + M^t_1 \Gamma^i \Sigma_i\right] \Psi^{(4)}_t \\
& + \overline{\Psi}^{(4)}_q \left[M^b_0 + M^b_1 \Gamma^i \Sigma_i\right] \Psi^{(4)}_b + \overline{\Psi}^{(4)}_\ell \left[M^\tau_0 + M^\tau_1 \Gamma^i \Sigma_i\right] \Psi^{(4)}_\tau + \text{h.c.}\,, \\
\mathcal{L}^{\text{matter}}_{\text{MCHM5}} = & \sum_{r=q,\ell,u,d,e} \overline{\Psi}^{(5)}_r \left[\not{p}\Pi^r_0 + \Sigma^\dagger \not{p} \Pi^r_1 \Sigma\right] \Psi^{(5)}_r + \overline{\Psi}^{(5)}_q \left[M^t_0 + \Sigma^\dagger M^t_1 \Sigma\right] \Psi^{(5)}_t \\
& + \overline{\Psi}^{(5)}_q \left[M^b_0 + \Sigma^\dagger M^b_1 \Sigma\right] \Psi^{(5)}_b + \overline{\Psi}^{(5)}_\ell \left[M^\tau_0 + \Sigma^\dagger M^\tau_1 \Sigma\right] \Psi^{(5)}_\tau + \text{h.c.}\,, \\
\mathcal{L}^{\text{matter}}_{\text{MCHM14}} = & \sum_{r=q,\ell,u,d,e} \Bigg[ \overline{\Psi}^{(14)}_r \not{p} \Pi^r_0 \Psi^{(14)}_r \\
& \qquad + (\Sigma \overline{\Psi}^{(14)}_r) \not{p} \Pi^r_1 (\Psi^{(14)}_r \Sigma^\dagger) + (\Sigma \overline{\Psi}^{(14)}_r \Sigma^\dagger) \not{p} \Pi^r_2 (\Sigma \Psi^{(14)}_r \Sigma^\dagger) \Bigg] \\
& + \overline{\Psi}^{(14)}_q M^t_0 \Psi^{(14)}_t + (\Sigma \overline{\Psi}^{(14)}_q) M^t_1 (\Psi^{(14)}_t \Sigma^\dagger) + (\Sigma \overline{\Psi}^{(14)}_q \Sigma^\dagger) M^t_2 (\Sigma \Psi^{(14)}_t \Sigma^\dagger) \\
& + \overline{\Psi}^{(14)}_q M^b_0 \Psi^{(14)}_b + (\Sigma \overline{\Psi}^{(14)}_q) M^b_1 (\Psi^{(14)}_b \Sigma^\dagger) + (\Sigma \overline{\Psi}^{(14)}_q \Sigma^\dagger) M^b_2 (\Sigma \Psi^{(14)}_b \Sigma^\dagger) \\
& + \overline{\Psi}^{(14)}_\ell M^\tau_0 \Psi^{(14)}_{\tau_R} + (\Sigma \overline{\Psi}^{(14)}_\ell) M^\tau_1 (\Psi^{(14)}_\tau \Sigma^\dagger) + (\Sigma \overline{\Psi}^{(14)}_\ell \Sigma^\dagger) M^\tau_2 (\Sigma \Psi^{(14)}_\tau \Sigma^\dagger) \\
& + \text{h.c.}\,,
\end{aligned} \tag{8.1}
$$

where $\Sigma$ is given by

$$
\Sigma = \frac{\sin(h/f)}{h}(h^1, h^2, h^3, h^4, h\cot(h/f))\,, \quad h = \sqrt{h^a h^a}\,, \tag{8.2}
$$

with $h^a$ being the pNGBs. Also $\Psi^{(R)}_r$ denotes the R-dimensional representation into which the SM matter fermion $r = q, u, d, \ell, e$ is embedded, $\Gamma_i$ are the gamma matrices in SO(5), and $\Pi$'s and $M$'s are the form factor. In the following, we only focus on the third generation quarks and leptons.

In Table 8.1, deviations in the Higgs couplings from the SM predictions are summarized. All the deviations depend on a model parameter $\xi \equiv v^2/f^2$ where $v$ is the vev of the Higgs boson, and $f$ is the scale where the global symmetry is broken. In the table, we use the scale factors $\kappa_a$, which are defined by $\kappa_a \equiv g_a/g_a^{\text{SM}}$, where $g_a$ denote the coupling constants of the Higgs boson coupling with the weak gauge bosons $V = W$ and $Z$, matter fermions and the Higgs boson itself as $a = hVV$, $htt$, $hbb$, and $hhh$. For $\kappa_{hVV}$, $\kappa_{htt}$ and $\kappa_{hbb}$, the abbreviations $\kappa_V$, $\kappa_t$ and $\kappa_b$, respectively are used. For $hhVV$ couplings, we use the parameter $c_{hhVV} = g_{hhVV}/g^{\text{SM}}_{hhVV}$. In the effective theories of the MCHMs, there are new dimension five operators of two Higgs bosons and two fermions such as $hht\bar{t}$. The coupling constant for $hht\bar{t}$ is parameterised as $g_{hhtt} = c_{hhtt} m_t/(2v^2)$. The four point interactions such as $hhVV$ and $hht\bar{t}$ also play important roles in our analysis.

| Model | $\kappa_V$ | $c_{hhVV}$ | $\kappa_{hhh}$ | $\kappa_t$ | $\kappa_b$ | $\kappa_\tau$ | $c_{hhtt}$ |
|---|---|---|---|---|---|---|---|
| MCHM4 | $1-\frac{1}{2}\xi$ | $1-2\xi$ | $1-\frac{1}{2}\xi$ | $1-\frac{1}{2}\xi$ | $1-\frac{1}{2}\xi$ | $1-\frac{1}{2}\xi$ | $-\xi$ |
| MCHM5 | $1-\frac{1}{2}\xi$ | $1-2\xi$ | $1-\frac{3}{2}\xi$ | $1-\frac{3}{2}\xi$ | $1-\frac{3}{2}\xi$ | $1-\frac{3}{2}\xi$ | $-4\xi$ |
| MCHM14 | $1-\frac{1}{2}\xi$ | $1-2\xi$ | $1-\frac{3}{2}\xi$ | $1-\frac{9M_1^t+64M_2^t}{6M_1^t+16M_2^t}\xi$ | $1-\frac{3}{2}\xi$ | $1-\frac{3}{2}\xi$ | $-\frac{4(3M_1^t+23M_2^t)}{3M_1^t+8M_2^t}\xi$ |

**Table 8.1.** Deviations in coupling constants with the Higgs boson in MCHM4, MCHM5 and MCHM14. The formulae in the table are approximated for $\xi \ll 1$. The table is taken from [3].

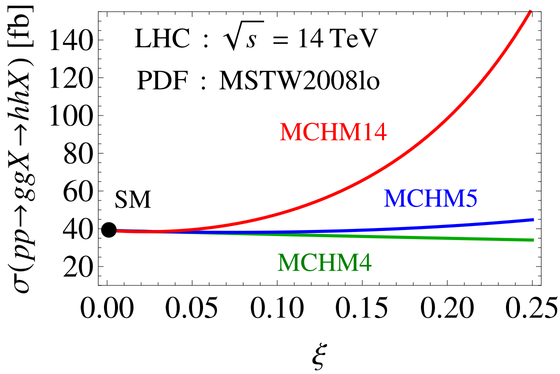


**Fig. 8.1.** The production cross section of $pp \to ggX \to hhX$ in MCHM4(green), MCHM5(blue) and MCHM14(red) at LHC with the collision energy of 14 TeV. The figure is taken from [3].

## 8.3 Numerical results for double Higgs production

First, we show the numerical results for the double Higgs boson production at LHC. The double Higgs boson production at LHC is dominated by the gluon fusion process, $pp \to ggX \to hhX$. In the MCHMs, the cross section is affected by deviations in the top Yukawa coupling constant and the triple Higgs boson coupling constant. In addition to these contributions, the dimension five interaction $hht\bar{t}$ enhances the cross section. As a result, the cross section of this process depends on the parameters $\kappa_t$, $\kappa_{hhh}$, and $c_{hhtt}$. In Fig. 8.1, the production cross section of $pp \to ggX \to hhX$ in each MCHM at the LHC with $\sqrt{s} = 14$ TeV is shown as a function of the compositeness parameter $\xi$. As shown there, the cross section is suppressed in MCHM4, and it is enhanced in MCHM5 and MCHM14.

Second, we consider the double Higgs production at an electron-positron collider. This process at the lepton collider is sensitive to the triple Higgs boson coupling $hhh$ and the contact interaction $hhVV$. Fig. 8.2 shows the $\sqrt{s}$ dependence of the production cross section of the process $e^+e^- \to hh\bar{\nu}\nu$ in MCHM4, MCHM5 and MCHM14 for fixed values of the compositeness parameter $\xi = 0.1$ and 0.2. The cross section of $e^+e^- \to hh\bar{\nu}\nu$ is dominated by $Z$-strahlung which is always suppressed by the scale factors in the MCHMs for $\sqrt{s} \lesssim 600$ GeV and by $W$-fusion which is enhanced as a result of unitarity non-cancellation for $\sqrt{s} \gtrsim 600$ GeV. Within the expected accuracy of measurements[5,6], such a specific behaviour

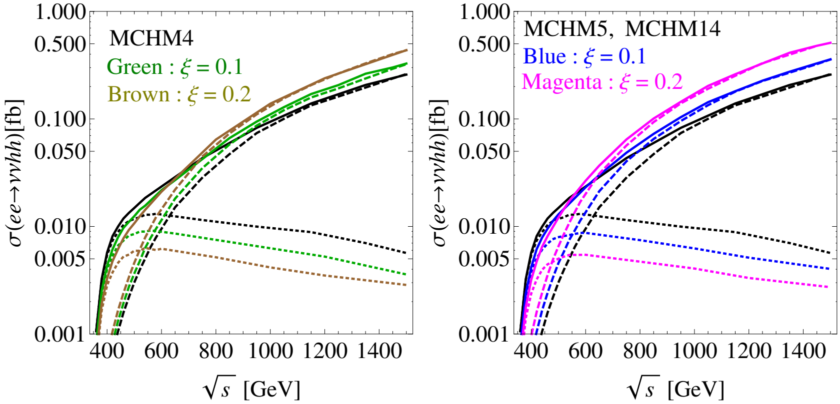


**Fig. 8.2.** The production cross sections for $e^+e^- \to \bar{\nu}\nu hh$ in MCHM4 (Left) and in MCHM5 and MCHM14 (Right). The solid curve is for the total cross section. The green(blue) and the brown (magenta) curves are for the case of $\xi = 0.1$ and $\xi = 0.2$, respectively. The dashed and dotted curves show the $W$-fusion and the $Z$-strahlung subprocesses, respectively, and the black curves show the SM prediction. The figures are taken from [3].

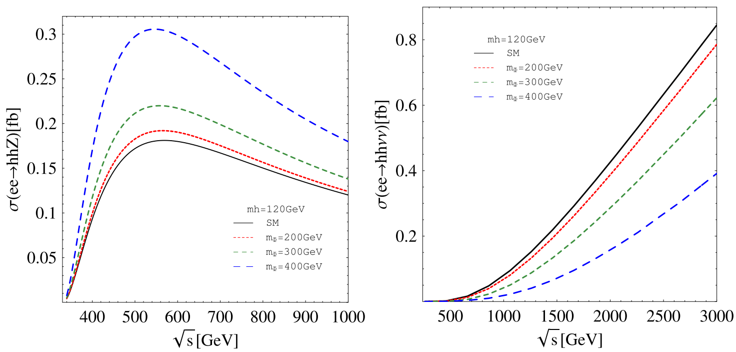


**Fig. 8.3.** Left: The cross section of $e^+e^- \to hhZ$ in the two Higgs doublet model. Right: The cross section of $e^+e^- \to hh\bar{\nu}\nu$ in the model. Here, the SM-like Higgs boson mass is fixed to be 120 GeV and the masses of extra Higgs bosons are taken to be degenerate as $m_\Phi \equiv m_H = m_A = m_{H^\pm}$. These figures are taken from Ref. [7].

might be observed by the $\sqrt{s}$ scan at the ILC and the CLIC in the cases with a significant size of $\xi$. This $\sqrt{s}$ dependence of the double Higgs boson production cross section in the MCHMs is different from that in other new physics models such as the two Higgs model with a significant deviation of the triple Higgs boson coupling from the SM predction[7] as shown in . Fig. 8.3. In the two Higgs doublet model, large enhancement of the triple Higgs boson coupling enhances the double Higgs boson production cross section via the $Z$-strahlung, while the cross section by $W$-fusion contribution is suppressed. This behavior is opposed to the case of MCHMs.

## 8.4 Summary

The scenario where the SM-like Higgs boson is identified with pNGBs is attractive new physics model from the view point of gauge hierarchy problem. In this talk, we review a phenomenological study in MCHMs. In particular, we focus on the double Higgs boson production both at LHC and at future lepton collider experiments. We show that MCHMs can be probed by using this process. Especially, the predicted production process at lepton collider $e^+e^- \to hh\nu\bar{\nu}$ shows a specific behavior so that we might be able to distinguish MCHMs from other new physics scenarios by this process unless the parameter $\xi$ is too small.

## Acknowledgements

It is partially supported by the Kogakuin University Grant for the project research "Phenomenological study of new physics models with extended Higgs sector".

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# Discussion Section

The discussion section is reserved for those open problems presented and discussed during the workshop, that might start new collaboration among participants or at least stimulate participants to start to think about possible solutions of particular open problems in a different way, or to invite new collaborators on the problems, or there was not enough time for discussions and will hopefully be discussed in the next Bled workshop.

Since the time between the workshop and the deadline for contributions for the proceedings is very short and includes for most of participants also their holidays, it is not so easy to prepare there presentations or besides their presentations at the workshop also the common contributions to the discussion section.

However, the discussions, even if not presented as a contribution to this section, influenced participants' contributions, published in the main section. Contributions in this section might not be yet pedagogically enough written, although they even might be innovative and correspondingly valuable indeed.

As it is happening every year also this year quite a lot of started discussions have not succeeded to appear in this proceedings. Organizers hope that they will be developed enough to appear among the next year talks, or will just stimulate the works of the participants.

There are seven contributions in this section this year.

The author of one contribution presents his own inovative model (which has been started by using the binary code to express the spins and charges of fermions, and correlated later the binary code with the Clifford algebra basis of the spin-charge-family theory), representing the elementary fermions as defects in the periodical tessalations of small charged domains.

The relations between the Clifford algebra and the Dirac matrices with the apperance of families in (3+1)-dimensional space, embedded into (5+1)-dimensional space, so that spin in the fifth and sixth dimensions represents the charge of fermions, are presented.

One contribution has started the generalization of the new way of the second quantized fermions in the Clifford space, presented in the talk section, trying to reformulate the cross products of the Hilbert space of indefinite numbers of fermions.

The contribution, reviewing the novel string field theory of authors, are pointing out that the possibility for objects to annihilate and create needs to be included.

In one contribution it is assumed that neutrinos are composition of Dirac and Majorana neutrinos, fitting correspondingly the parametrization of mass matrices to the experimental data.

There is the contribution studyng the possibility that the dark matter particles might decay and annihilate fast enough that the corresponding gamma rays should be observable, but yet they are not because of absorbtion.

One contribution considers clusters of primordial black holes, decoupled from the cosmological expansion and therefore heated as compared to the surrounding matter.

All discussion contributions are arranged alphabetically with respect to the authors' names.

# Diskusije

Ta razdelek je namenjen odprtim vprašanjem, o katerih smo med delavnico razpravljali in bodo morda privedli do novih sodelovanj med udeleženci, ali pa so pripravili udeležence, da razmislijo o možnih rešitvah odprtih vprašanj na drugačne načine, ali pa bodo k sodelovanju pritegnili katerega od udeležencev, ali pa ni bilo dovolj časa za diskusijo na določeno temo in je upati, da bo prišla na vrsto na naslednji blejski delavnici.

Ker je čas med delavnico in rokom za oddajo prispevkov zelo kratek, vmes pa so poletne počitnice, je zelo težko pripraviti prispevek in še težje poleg prispevka, v katerem vsak udeleženec predstavi lastno delo, pripraviti še prispevek k temu razdelku.

Tako se precejšen del diskusij ne bo pojavil v letošnjem zborniku. So pa gotovo vplivale na prispevek marsikaterega udeleženca. Nekateri prispevki še morda niso dovolj pedagoško napisani, so pa vseeno lahko inovativni in zato dragoceni.

Organizatorji upamo, da bodo te diskusije do prihodnje delavnice dozorele do oblike, da jih bo mogoče na njej predstaviti.

Letos je v tem razdelku sedem prisepvkov.

Avtor enega prispevka predstavi svoj inovativni model (začel ga je z uporabo binarne kode za zapis spinov in nabojev fermionov, zapis pa nadgradil s tem, da je povezal binarni zapis s Cliffordovo algebro v teoriji spinov-nabojev-družin), v katerem osnovne fermione predstavi kot defekte v periodičnem razcepu prostora (teselacijo) na majhne nabite podcelice.

Avtorji predstavijo zvezo med Cliffordovo algebro, s katero opišejo poleg spina in ročnosti tudi družine, in Diracovimi matrikami v $(3+1)$-razsežnem prostoru, ki ga vstavijo v $(5+1)$-razsežni prostor, tako da spin v peti in šesti dimenziji predstavlja naboj fermiona.

Avtorja želita v njuni novi formulaciji druge kvantizacije, ki pojasni Diracovo drugo kvantizacijo (v predavanjih v tem zborniku pojasnita ta novi predlog druge kvantizacije), posplošiti produkt Hilbertovih prostorov z nedoločenim številom fermionov.

Avtorja predstavita svojo novo teorijo polj s strunami ter namero, da vkljucita v to teorijo tudi anihilacijo in tvorbo objektov te teorije.

V prispevku, ki privzame, da nevtrine sestavljajo Diracovi in Majoranini nevtrini, avtor išče parametrizacijo, ki ustreza eksperimentalnim podatkom.

Avtorji prispevka obravnavajo možnost, da delci temne snovi razpadajo in se anihilirajo tako hitro, da bi morali opaziti nastale žarke $\gamma$, vendar jih zaradi absorbcije ne opazimo.

Prispevek obravnava zgručo prvotnih črnih lukenj, ki ni sklopljena s kozmološko širitvijo vesolja in se zato segreva glede na snov v okolici.

Prispevki v tej sekciji so, tako kot prispevki v glavnem delu, urejeni po abecednem redu priimkov avtorjev.

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# 9 Analysis of Programming Tools in Framework of Dark Matter Physics and Concept of New MC-generator ⋆

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**Abstract.** We analyse here some programming tools (MC-generators) from viewpoint of their application to the tasks of dark matter (DM) interpretation of cosmic rays puzzles. We shortly describe our tasks, where the main goal is the solution of the problem of suppression of gamma-rays induced by the products of DM decay or annihilation in Galaxy. We show that existing MC-generators do not fully satisfy our task, comparing them, and suggest our own one.

**Povzetek.** Avtorji domnevajo, da delci temne snovi razpadajo in se tudi anihilirajo dovolj pogosto, da bi pri tem nastale žarke gama morali opaziti. Študirajo procese, ki povzročajo absorpcijo žarkov gama. Analizirajo obstoječa programska orodja in predlagajo svoje ustreznejše orodje.

Keywords: dark matter physics, MC-generators, interaction Lagrangians

## 9.1 Introduction

The necessity of the usage of different MC-programs[1] appears in different areas. One of them is connected with dark matter (DM) processes. DM can give signal in cosmic rays (CR) due to their decay or annihilation. Positron anomaly [1,2] or possible excess of electrons and positrons [3] at high energy in CR is one of such subject.

DM physics is unknown, what requires a respective flexibility of calculations of the predicted signal in CR $e^+e^-$. Realization of this with the help of using some programming tools imposes definite requirements on them about which we will talk. We do not pretend to comprehensive review, we are reviewing it from point of view of our task, what can be useful for many adjacent ones too.

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[1] MC is decoded as Monte Carlo. Such programs are called as ME (Matrix Element) as well, implying the program tools able to simulate new (high energy) physics process.

The physical task itself comes from our previous works [4–14] studying compatibility of DM interpretation of CR $e^+e^-$ with cosmic gamma-ray data. The main problem is that, when we are trying to explain CR $e^+e^-$ anomalies we start to contradict to cosmic gamma-ray data even in the framework of, seeming, minimal model case from viewpoint of gamma-ray production. The latter is pure $e^+e^-$ decay or annihilation mode where gamma appears as (a) FSR (Final State Radiation) and (b) due to interaction of $e^+e^-$ with interstellar medium. Both contributions seem to be unavoidable. Nonetheless, even in this minimal case we got contradiction with gamma-ray data.

There are a few attempts to try to avoid this contradiction (we reviewed them in [5,12,11]), i.e. to suppress gamma coming from DM. It can relate to specifics of space distribution of DM like clumping or existence of dark disk component (supposing that a dominant halo DM component does not produce CR), or specifics in DM interaction. The latter includes both different decay/annihilation modes and Lagrangian of DM particles interaction with ordinary matter.

Specifics of DM physics may involve also opportunity of decay of DM particles onto two identical fermions like $X^{++} \to e^+e^+$. In such model it is supposed there exist two types of double charged DM particles, $X^{++}$ and $Y^{--}$. It is assumed that the last one is in form of electrically neutral bound state states with He, $X^{++}$ form bound state with $Y^{--}$ and decay [15–18]. In case of $X^{++} \to e^+e^+\gamma$ decay, we have factor two of suppression of FSR gamma per one $e^+$ (because they are two in one decay), and also extra suppression is expected due to identity itself of fermions in final state. The last reason takes place explicitly in classical case (dipole radiation of same charged particles is zero) and somehow partially in quantum case – due to so called single photon theorem [19].

All this accounts for necessity to have respective programming tool able to calculate the processes in the aforementioned tasks and, of course, not only. It does not cancel a desirability of analytical calculations. But the latter is often difficult to do and a crosscheck is necessary even when it is possible. It, in its turn, requires opportunity of step by step tracking calculations making with programming tools.

We demonstrate here the work of some such tools (Section 9.2). They does not provide identical and, therefore, reliable results at the absolutely same initially set parameters. It related to our tasks. We here come to conclusion of creation of MC (HEP) generator (Section 9.3,9.4) which would allow simple step by step checking of calculation procedure.

## 9.2 Programming tools analysis

As we told, it is impossible to build a model of dark matter in framework of dark halo or dark disk that would completely explain the positron anomaly in cosmic rays. Such attempts will lead to an excess of FSR arising from the decay/annihilation of a dark matter particle into two charged leptons or during the propagation in the interstellar medium.

This task requires to create a new physical models that go beyond the Standard Model (BSM). It is necessary to find the most suitable programming tools for such a task that would correspond the following minimum requirements:

1. the possibility to implement new physical models (BSM),
2. compute a matrix element and squared matrix element in analytical form,
3. the possibility of an explicit description of charge conjugation,
4. high enough precision of calculation.

To describe the decay or annihilation of DM particles, taking into account possible FSR, the different programming tools such as MadGraph [20], CompHEP [21], CalcHEP [22] and FormCalc [23] were considered.

Implementing BSM models in a generator such as MadGraph requires describing the model using the FeynRules [24] package. FeynRules is a package with Mathematica [25] source code that allows calculating the Feynman rules in momentum space for any physical model of quantum field theory.

One of the reasons for using this package is the possibility of describing charge conjugation for fermions, which is necessary in our models.

In FeynRules, we started with the following DM models: the simplest model of DM particle X decay on two opposite charged leptons and the model of double charged scalar particles X. In both models particle X is hypothetical long-lived scalar particle with a mass of about 1-3 TeV. Feynman rules for the Lagrangians presented below, which describes the decay of this particle, were tested:

$$\mathcal{L} = X\overline{\psi}(a + b\gamma^5)\psi + \overline{\psi}\gamma^\mu A_\mu \psi \tag{9.1}$$

$$\mathcal{L} = X\overline{\psi^C}(a + b\gamma^5)\psi + X^*\overline{\psi}(a - b\gamma^5)\psi^C - \overline{\psi}\gamma^\mu A_\mu \psi \tag{9.2}$$

where $a$ and $b$ are the unknown constant parameters.

At the output, sets of model files written in the Universal FeynRules Output (UFO) were obtained that can be used for calculations and modeling of various processes in the MC-generator MadGraph5aMC@NLO.

MadGraph is programming tool wich allows calculating cross-sections and squared matrix elements in numerical form.

Using the FeynRules model files, several decay modes of the DM particle X, namely, the processes $X \to e^+e^+$ and $X \to e^+e^+\gamma$, were simulated in this generator. MadGraph allows calculating cross-section, but it does not allow geting the squared matrix element in an analytical form, so this generator does not corresponds to all the previously set requirements.

The next two MC-generators that we used in our task are CompHEP and CalcHEP. These tools have attracted our attention since they have the ability to obtain a squared matrix elements. Obtaining the squared matrix elements in analytical form for each of the processes $X \to e^+e^\pm$, $X \to e^+e^\pm\gamma$, we get the opportunity to monitor the correctness of the results and compare them with those that were obtained manually.

To implement our models to CalcHEP, one can use the LanHEP [26] package. LanHEP has been designed as part of the MC-generator CalcHEP. This package,similar to the FeynRules package, is used to generate Feynman rules in a momentum representation based on a given Lagrangian.The output can be written in the form of CalcHEP's model files, which allows to start computing processes in a new physical model.

One of the alternatives to the MC-generators that we considered in framework of this task was FormCalc. FormCalc is the tool wich based on the FORM syntax and implemented as Mathematica package that allows one to calculate Feynman diagrams . Receiving input Feynman diagrams generated by the FeynArts (FeynArts [27] tool for generating Feynman diagrams), FormCalc is able to make calculations of the squared matrix element and write it out in Fortran code. The advantage of this program is that one can see some intermediate results, such as squared matrix element. However, FormCalc is a complex modular system of several packages and tools.

Figures 9.2 and 9.3 show approximate schemes for working with some MC-generators.

The main task at the first stage was the need to determine which programming tools is the most suitable for aforementioned task. An analysis of the above MC-generators was carried out, which consisted in comparing the results obtained from different MC-generators using the same model created using LanHEP. A positive result would be a complete

(within the errors) agreement between their results. We considered dependencies of the decay width of the DM particle on its mass (fig. 9.1). These graphs do not show the results obtained from the MadGraph MC-generator, since the decay width obtained using this tool is too large and could not be used in the general analysis. The reason for such deviations has not yet been found.

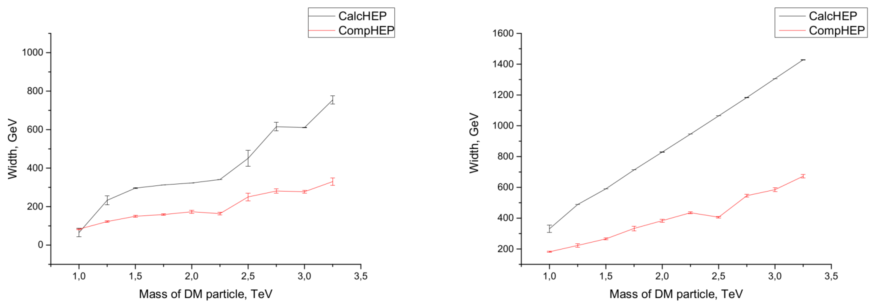


**Fig. 9.1.** Comparative analysis of MC-generators, using two processes as an example. Left: $X \rightarrow e^+e^-\gamma$ , Right: $X \rightarrow e^+e^+\gamma$
.

Figure 9.1 shows the results of the tests. As can be seen, the decay widths for the same model and masses of particle X differ. This deviation motivates us to look for additional verification tools.

It is almost impossible to determine the cause of such differences, since in the process of decay modeling it is impossible to obtain any intermediate results, such as, for example, matrix elements, etc.

The summary table (table 9.1) of the capabilities of some MC-generators was compiled, as applied, in particular, to BSM processes.

Summing up, we can conclude that none of the programming tools we have use are not fully suitable for our task.

| Options | CompHEP | CalcHEP | Madgraph | Pythia |
|---|---|---|---|---|
| Implementing of new models | + | + | + | − |
| Charge conjugation | + | + | + | − |
| Matrix element in analytical form | − | − | − | − |
| $\|M\|^2$ in analytical form | + | + | − | − |
| High precision | ±[2] | ± | ± | + |
| Performance[3] | ± | + | ± | ± |
| Have an implementation packages[4] | − | + | + | − |
| Hadronisation | − | − | − | + |

**Table 9.1.** Comparison of different MC-generators from viewpoint of calculation DM particle processes.

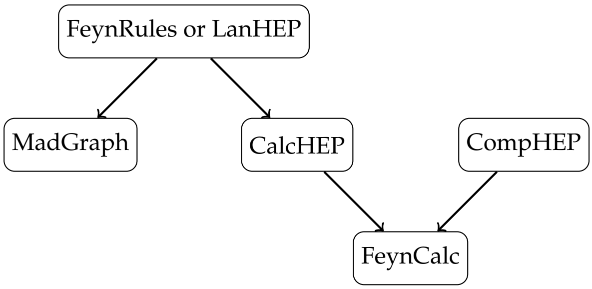


**Fig. 9.2.** Approximate schemes for working with some MC-generators

## 9.3 Idea of creating of new MC-generator

From analysis of existing MC-generators, given above, we come to conclusion that there is so far a necessity of creation of new one adjusted for our (of course, not only) tasks. The proposed new HEP generator allows calculating and displaying all intermediate results of calculations - i.e. analytical form of matrix element, the square of the matrix element in the form of traces of gamma matrices, the square of matrix element in form of kinematic variables and result of integrating the square of the matrix element of the given process over the phase volume.

Estimation of intermediate calculation results can be useful for validation of calculation processes and in the phenomenological areas of high energy physics to

[2] Hereinafter, the sign ± will mean that this tool does not fit exclusively to our task, but it copes well with other processes and models.

[3] Characterizes the speed of calculations

[4] New models can be loaded into CalcHEP and MadGraph with the help, for example, FeynRules and LanHEP packages, while in CompHEP one can add new models only by hand.

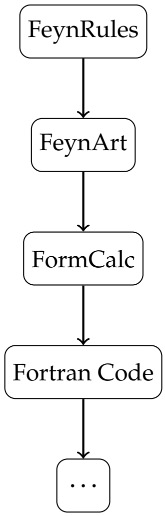


**Fig. 9.3.** Modular system of FormCalc using

understand the contribution of specific Lagrangian terms to the various distributions.

In specific of our work on dark matter interaction physics [4–7] we need to estimate why given components of Interaction Lagrangian lead to certain effects.

The developing generator is based on FORM symbolic manipulation system [28], which is designed to work with algebraic expressions and constructions. It reads text files containing definitions of mathematical expressions as well as statements which tell it how to manipulate these expressions. It is widely used in the theoretical particle physics community, but it is not restricted to applications in this specific field.

FORM ″doesn't know″ anything about the particle physics processes and calculations of amplitudes and cross sections. Everything that FORM makes - it searches in the string the substrings matching the pattern and replaces them with the developer-specified expressions. Then it leads similar terms and displays the result.

User can enter the expression of Lagrangian or the expression of partial term of a perturbation theory series. It is also necessary to explicitly indicate the types of fields used in the Lagrangian and ″in″ and ″out″ states. See Figure 9.4.

We want to note the monolithic architecture of the developing generator. That is all described above tasks are performed within one single program.

The matrix element calculation algorithm is based on the principle of secondary canonical quantization. That is if user enter the expression of lagrangian, program approximate the T-exponent by Teylor series, that give the perturbation theory series.

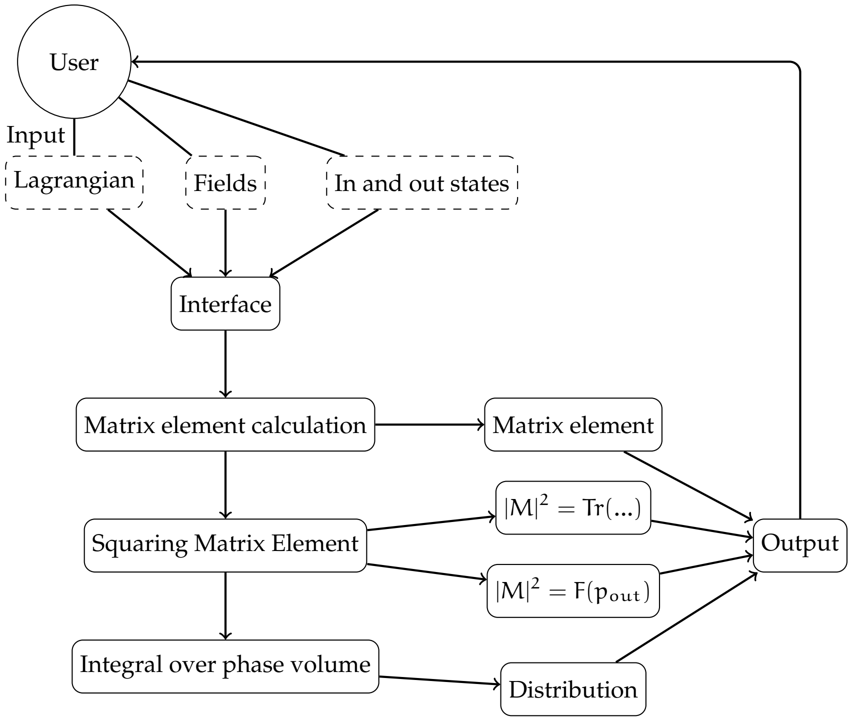


**Fig. 9.4.** General structure of the developing generator modules

$$e^{-iS} = 1 - iS + \frac{(-iS)^2}{2} + \frac{(-iS)^3}{3!} + ... \tag{9.3}$$

where $S \equiv \int d^4x \, \mathcal{L}$ - is the action of model.

And take interesting term of this one. After that generator takes the fields of considering model and performs the second quantization[5]:

$$\begin{gathered} \mathcal{L} \equiv \mathcal{L}(\phi, \partial_\mu \phi) \\ \phi \to \hat{\phi} \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (\hat{a}_p e^{-ipx} + \hat{a}_p^\dagger e^{ipx}) \end{gathered} \tag{9.4}$$

where $\hat{a}_p$ - is the lattice operator such that $[\hat{a}_p, \hat{a}_q^\dagger] = (2\pi)^3 \delta^{(3)}(p - q)$

FORM can perform specified instructions with given expressions taking into account the non-commutativity of variables.

Developing generator should include explicitly the permutation rules of the given non-commuting variables in the form of instructions which patterns should be replaced by other expressions.

That is the replacing of bosonic rising operators at each iteration schematically looks like:

$$... \cdot \hat{a}_p \hat{a}_q^\dagger \cdot ... \to ... \cdot \left((2\pi)^3 \delta^{(3)}(p - q) - \hat{a}_q^\dagger \hat{a}_p\right) \cdot ... \tag{9.5}$$

[5] This means that the symbols $\Phi$ are replaced by other text expressions corresponding to operators.

Then program takes the expression of matrix element in form of approximated T-exponent by the Teylor series with second quantization (see Eq.9.4):

$$\left\langle \mathrm{out}\middle|e^{-iS}\middle|\mathrm{in}\right\rangle = \left\langle \mathrm{out}\middle|\left(1 - iS + \frac{(-iS)^2}{2} + \frac{(-iS)^3}{3!} + ...\right)\middle|\mathrm{in}\right\rangle. \tag{9.6}$$

Here $|\mathrm{in}\rangle \equiv \hat{a}^\dagger_{q_1}...\hat{a}^\dagger_{q_k}|0\rangle$ and $\langle\mathrm{out}| \equiv \langle 0|\hat{a}_{p_1}...\hat{a}_{p_n}$ are the initial and final states of process which are specified by user and are expressed by specific character sets.

Then the program performs the normal ordering of rising operators according to the instructions indicated explicitly in the algorithm and described schematically (9.5) above.

One of features of the developing generator is the opportunity for the user to indicate perturbation theory order, as well as choose or enter only the interesting term of perturbation theory for consider only it's contribution.

After the matrix element of the process has been calculated - its analytical expression is displayed to the user on the screen (See Figure 9.4 - Matrix element calculation).

The part of the program described above has already been developed.

The next block of the algorithm in the Figure 9.4 (Squaring of the matrix element) takes an expression for the matrix element, which was calculated in the previous block of the diagram, and builds an expression for hermitian conjugate operator in the form of a specific string of characters.

Then the product $|M|^2 = M \cdot M^\dagger$ should be reduced to a trace of gamma matrices and displays to the user.

After substituting kinematic variables into the obtained expression and taking the trace, integration over the phase volume is performing to obtain the distribution.

## 9.4 Application of programming tools

We compare the results, computed by developing generator with the standard processes of particle physics and the specific processes of our work, previously calculated manually. The results are follows:

1)Two-particle decay of a neutral Dark Matter particle into an electron and a positron user enter the fields $X$, $\Psi$, $\bar{\Psi}$ and interaction lagrangian of the model

$$\mathcal{L} = X\bar{\Psi}(a + b\gamma^5)\Psi \tag{9.7}$$

Then he indicates the statistic of fields, that is $X$ - is the scalar field and $\Psi$ - is the spinor field.

This leads to:

$$M = FB(e, k_1) \cdot (a + b \cdot G(5)) \cdot FC(e, k_2) \cdot S(X, k_3) \tag{9.8}$$

that means:

$$M = \bar{u}(k_1)(a + b\gamma^5)\nu(k_2) \tag{9.9}$$

2) Two-parrticle decay of a double charged Dark Matter particle into two positrons.

Similarly:

$$\mathcal{L} = X\bar{\Psi}(a + b\gamma^5)\Psi^{(c)} + \text{H.C.} \tag{9.10}$$

with fixed initial and final states as $|in >\equiv |X>$ and $|fin >\equiv |e^+, e^+>$

$$\begin{aligned} M = &-FCT(e, k_1) \cdot iG(2) \cdot G(0) \cdot (a + b \cdot G(5)) \cdot FC(e, k_2) \cdot S(X, k_3) + \\ &+FCT(e, k_2) \cdot iG(2) \cdot G(0) \cdot (a + b \cdot G(5)) \cdot FC(e, k_1) \cdot S(X, k_3) \end{aligned} \tag{9.11}$$

that means:

$$M = -\nu^T(k_1)i\gamma^2\gamma^0(a + b\gamma^5)\nu(k_2) + \nu^T(k_2)i\gamma^2\gamma^0(a + b\gamma^5)\nu(k_1) \tag{9.12}$$

## 9.5 Conclusion

Here we considered capabilities of several MC-generators (CompHEP, CalcHEP, MadGraph with applications to some of them such packages as LanHEP, FeynRules and etc. and some modular tools like FormCalc). This was done in framework of our task concerning DM signal search in CR. More concretely, we considered decay of DM particles with different interaction Lagrangians. We see that the considered tools do not quite satisfy our requests. We need some single tool what would allow providing to show "step by step" results of calculations. We suggest it here on the base of code FORM.

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# 10 Tessellation Approach in Modeling Properties of Physical Vacuum and Fundamental Particles

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**Abstract.** The approach of representing fundamental particles by defects in the periodical tessellations built of small electrically-charged domains is discussed in this paper. We give reasons for its use, enumerate the assumptions underlying it, formulate the main tasks that arise with this approach and provide some of solutions for them that we found.

**Povzetek.** Avtor predstavi svoj model za opis lastnosti osnovnih gradnikov snovi – kvarkov, leptonov in njihovih antidelcev ter interakcij med njimi. Elementarne delce predstavi kot defekte v periodični teselaciji prostora, ki jih določajo majhna električno nabita območja. Pove od kod je črpal vzpodbudo za svoj model, našteje privzetke, na katerih je model zdrajen ter napovedi, ki jih model ponuja.

Keywords: tessellation, bit graph, particle, defect, triple-periodical, satori

## 10.1 Introduction

The *Tessellation approach* is the denotation for using some analogy between fundamental particles, on one hand, and structure defects in periodical spatial tessellations, on another hand, in calculation of particle properties and speculations about particle physics problems.

We do not know exactly, how deep this analogy is, and what causes such a correspondence, but we found this approach useful and productive, and also found it interesting to explore its limits, trying to extend them.

We formulated assumptions of the approach while developing several particle models based on bit graphs, aiming to get digital, more calculable by computers, representation of particles instead of usual quantum mechanical one [1].

There are approaches, that have some correspondences to our approach, among them are the Spin-charge-family theory [2], the Cellular automata interpretation [3], [4], and the ether hypothesis [5].

The bit graphs generalize the idea of *numbers as bit sequences* by allowing not just ordered, i.e. sequential, but also non-ordered and partially-ordered bit combinations. For instance, three bit organized in a closed loop appeared suitable to describe both the color charge of quarks or anti-quarks, and the color absence, characteristic of leptons and anti-leptons [1].

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Two three-bit loops was found enough to represent also gluons, weak bosons, and the electrical charge for all the particles. Adding two more bits to the graph, we get a model suitable to describe three fermion families, triplet- and singlet-states of bosons, Higgs scalar and the photon.

All these models provide the correct quantum numbers of the corresponding particles as combinations of their bit's values. The only thing one must assume is that the bit's values are not 0 and 1 but $+\frac{1}{6}$ or $-\frac{1}{6}$ and they have the physical sense of electric charge.

The weak points of our bit-graph models, including the most advanced one, was that they were completely **C**-symmetrical, and therefore they did not provide the representation of the handedness and the parity asymmetry. To overcome this obstacle, we modified the principal three-bit loop graph, assuming it *directed*, and, therefore, we get the whole model chiral and **CP**-symmetrical. The charge conjugation **C**, meaning exchange of all bit values from 0 to 1 ($+\frac{1}{6}$ to $-\frac{1}{6}$) or back, and **P**, meaning the reverse of all loops' directions, being applied together, turn the model back to the original state.

This trick helped, but the *bits* looked this time rather less like binary digits because they must somehow carry, in addition to the electric charge, some extra information about the direction.

According to our eight-bit model, there must be two different versions for all the bosons, one of them more, and another one less symmetrical, which we associated with the triplet and single state of them. The scalar Higgs boson H took, in this model, the place of the singlet Z. Because of **CP**-symmetry, the same thing also happened to the photon representation, predicting some new scalar chargeless particle taking place of the singlet, or longitudinal, photon.

Stacking, like children's blocks, several copies of our Higgs boson model graphs with each other, we found that an electrically and color-neutral filling of space with unlimited size is easily obtained in this way. We associate it with the vacuum condensate. It is chiral because its **CP**-symmetrical partner is another condensate, which is produced the same way by stacking with each other the copies of longitudinal photon model graphs.

We recognized that it can be very effective to consider this condensate as vacuum background, instead of empty free space. It looks like regular periodic directed bit graph, infinite or big enough, consisting of multiple copies of the background bit combination, either Higgs or longitudinal photon. Some of these copies can be easily replaced with other model graphs corresponding to any of known particles, so particles will be just defects in the regular structure, with one or more bits with inverted charge.

Since the background is chiral, the left- and right-handed configurations become completely different. As an example, the photon and Z boson, that were **CP**-partners, went far one from another. Heavy and short-living Z has 6 defect bits in respect to the background while the light-weight and stable photon has only two defected bits.

The Higgs boson manifests itself as a scalar neutral particle on the background of longitudinal photon condensate. On its own background it would be non-distinguished from it, and thus experimentally not observable, i.e. non-existent

- the same way as the longitudinal photon does not exist on the background of itself.

That was the first time we think about the space as filled with the regular structure so it can be treated as a start point of our tessellation approach. In contrast to the purely mathematical structure of the bit graph, filling of the space with regions of different charge is a picture that can be called physical. It can be explored to find out what laws can exist in this 'world' and under which of them it will be more similar to ours.

## 10.2 Assumptions

The assumptions we listed below constitute an essential part of the approach. Changing them, we usually get a model that significantly differs from the observables.

Generally, they are as follows:

- the idea of tessellation,
- the statement of electrical charge carried by domains in it, and
- grouping of charged domains into triplets and pairs.

### 10.2.1 The ground state is a domain tessellation

The principal assumption of the *tessellation approach* is to treat the vacuum not as an empty space, either with fluctuations or without them, but, instead, as a *dense filling of small regions*, or *domains*. The domains can be either similar or different from each other, and may be either separated or not separated by some kind of walls. These are details that can vary in particular models.

This filling, or tessellation, is assumed to be the ground state, so that all fluctuations, defects, geometric distortions should be considered against this background.

In principle, the tessellation can be assumed global, crystal-like, or local, similar to some fluid, and even finite, looking like gas of domain clusters. In the last case, though, it is not the tessellation, but something more close to the classical empty space with free distinct particles in it. The liquid tessellation, with just near order of domains, should have some secondary unordered walls separating these ordered regions from each other, that, on our opinion, contradicts to the observations. So we assume the long-ranged, up to the infinity, and, in the first approximation, strictly periodical crystal-like space filling as the basic object for our model. In fact, each defect is the local violation of the periodicity, and the vicinity of a defect also can be slightly distorted. Also, there can be waves of the distortion, but all this is considered as excitations of the ideally periodical ground state.

### 10.2.2 Domains are electrically charged

As the second assumption, we take the statement that the principal difference between domains, and probably the only one, is the difference in their *electric charge*. We assume it to be *either* $+1/6$ *or* $-1/6$ in units of proton charge e.

Fulfillment of this requirement is necessary to ensure that all the particles will have their electrical charges proportional to $\pm\frac{1}{3}$ only. In this case, the tessellation model gets compatible with the bit graph models we studied before, so we can use the bit values 1 and 0, converted to the charges $+1/6$ or $-1/6$, to represent the particle that we want to explore.

In fact, it is not mandatory for absolutely all domains to carry these charges: it is only required for those domains that can change their charges individually or along with another domains of the same charge.

In case a pair of domain can participate just in mutual charge exchanges, or in case of individual domains that can not change its charge at all, these domains could have any charge as long as they keep compensating each other.

However, this is a kind of complification, that we try to avoid. Our 8-bit graph model allows exchange between *any* pair of bits, so the tessellation, that is compatible to it, must have all the domains charged with either $+1/6$ or $-1/6$ only.

On our opinion, the scalar electric potential of these charges can play the role of Higgs field in explanation of particle masses, so we *do not assume an extra Higgs field* for this purpose. The electric field of domains is the only primary field assumed [7].

This hypothetical unification of both fields allows to estimate the domain radius:

$$r \approx \frac{\alpha}{6^2\eta} \approx \frac{1}{36 \cdot 137 \cdot 246\mathrm{GeV}} \approx 8,2 \cdot 10^{-7}\mathrm{GeV}^{-1} \approx 1.7 \cdot 10^{-20}\mathrm{cm}. \tag{10.1}$$

The whole picture of vacuum as a scalar field, non-zero almost everywhere (excepting walls), looks now close to the vacuum domain model [6] with the difference that domain sizes are not on cosmological but on sub-particle scale. This could, in our opinion, explain the paradox of absence of domain observations while they are predicted as a consequence of symmetry break in the electroweak theory.

### 10.2.3 Triplets and Pairs

It is well known that all the fundamental particles have their electrical charge values in the range from -1 to 1. All the multi-charged particles are considered composite, as bound states. In the tessellation approach we consider this limit as an evidence in favor of assumption that the *count of domains* that are able to possess simultaneous inversion in the same direction, is *exactly three*. We suppose them to reside in the tessellation in the close vicinity of each other, most likely being the *immediate neighbors*.

In other words, the tessellation consists of multiple positive- and negative-charged domain triplets, each carrying electric charge of $\pm\frac{1}{2}$, and one, two of three domains in a triplet can be defected, i.e. have the charge inverted.

This assumption immediately leads to the phenomena of the *isotopic symmetry*, because it stipulate existence of two variants for each defect configuration, depending on place where it occurs: either instead of positive triplet in the ground state tessellation, or instead of negative one. The difference of electrical charge between them is exactly 1, so defects in the positive triplets are down particles; the same defects in negative triples are up.

The relocation of the defected triplet from originally negative place to the positive one, is, in fact, its exchange with the positive triplet resided in its place. It causes, besides the transformation of the up particle into down one, the appearance of new positive-charged triple defect in the negative place. It corresponds to the weak boson, so all this exchange should be considered as an example of weak interaction, for instance: $u^{\frac{2}{3}} \to d^{-\frac{1}{3}} + W^{+}$. This defect can migrate, exchanging its place with triplets in negative places, or cause the relocation of some defected triplet from the positive place into negative one.

In addition to triplets, we assume the possibility of domain pairs. It is the artificial construction, serving as the simplest way to represent several different particles with the same charge. The exchange between domains in a pair affects neither color nor electric charge, but the result combination differs from the original.

## 10.3 Objectives of the tessellation approach

To be applied to problems in particle physics, the tessellation approach requires the concrete suitable tessellation. To calculate energies, including masses, it is necessary to figure out, what is the energy in this case. For the dynamic processes, including interactions, the way of defect migration also should be identified.

So the determining of the most optimal structure, obtaining the appropriate Hamiltonian and definition of dynamic may be considered as main objectives for the research.

Also it is possible that there are some physical systems, analogous to the tessellations, for instance foams and liquid-liquid mixtures, so the approach could be applied to them, and some observations and experiments with these systems can improve the knowledge of this subject.

### 10.3.1 Finding the optimal structure

There are a lot of mathematically possible different spatial fillings that, in principle, can be used in the tessellation approach. Each of them provides, as its defect combinations, the spectrum of possible fundamental particles. Some of them are better than others, i.e. their defect combinations looks more similar to the particles found in the real world. So there should be one or several tessellations that provide the best correspondence to experimental data. So, **Determining of the optimal structure** is the first and main task of the tessellation approach.

We examined five structures, in the following order:

- 1-dimensional probe tessellation of 8-bit 'V' bit graphs
- simple cubic grid (**NaCl** type),
- body-centered cubic grid (**CsCl**),
- Weaire-Phelan [8] structure, or A15 phase [9] (β-**W**, **Nb**$_3$**Sn**) [10], and

- 4-dimensional 'Satori' structure [11], built as alternation of two modified A15 grids.

All the structures are compatible with, but not limited by, our 8-bit model.

In all these cases we considered electrically-neutral grids containing equal quantities of positive- and negative-charged domains in their nodes.

In the simple cubic grid, to ensure both the neutrality, and also the **CP**-symmetry, we used as node's *charge* its *parity*, calculated as product of its row's, column's and layer's parities.

Since all subsequent grids can be produced from the (hyper-)cubic grid performing shifts of its rows, columns and/or layers, the parity is still defined for their nodes so we distribute the charge in the same way.

To obtain the domain structure from the grid, we use the Voronoi diagram [12] built for the nodes. In case of simple cubic grid, the Voronoi diagram is also simple cubic, dual to the original. In case of body-centered grid, the Voronoi diagram is the Kelvin structure, the tessellation of equal tetrakaidecahedra, each of them is truncated octahedron.

In both cases, the structure is not chiral, so both even and odd domains have identical shape and spatial orientation.

The key difference of the Weaire-Phelan structure in respect to simple and body-centered grids is that in it the domains of different parity have different orientation, being mirror reflections of each other. Moreover, there are two different kinds of domains: for each three tetrakaidekahedra of three different orientations, there is one dodecahedron. Each translation unit consist of two equilateral triangles built from tetrakaidecahedra and two dodecahedra of opposite parities. So it is obviously compatible with the 8-bit graph model, while the first two are not.

The last tessellation that is 4-dimensional, now it is constructed but not well-studied yet. We needed the four-dimensional structure in order to have any model of three-dimensional defect motion (see below). Like A15, from which it is derived, it has minimal wall pro cell ratio, but, in contrast to it, is built of the domains having the same shape.

### 10.3.2 Constructing the Hamiltonian

The electrical charge of particles, factorized into ones' complement bit representation, define most of the quantum numbers as bit combinations: weak charge, hyper-charge, baryon- and lepton-numbers, and matter type (matter-or-anti-matter bit). The unary triplet-bit-loop represents the color charge. So, it is easy to determine bit combinations and corresponding defects for the properties that influence on the electric charge.

It is more difficult to guess the possible combinations, that would represent the equal-charged particles of both handedness-es, different spin, members of three (or more) families, or possessing boson and fermion kind of statistic. For instance, they are up, charm, and top quarks, or $W^-$ boson, tau, muon, and electron. However, it can be done, following the symmetry of the tessellation structure.

But the problem of particle masses, which are very different, very special, and do not manifest any dependence on the particle's charge, on our opinion, can be solved in the tessellation approach just by applying some additional assumption about mass origin.

Since there is nothing in the model but spatially distributed electrical charge, the mass of particle, which appears as some difference in the distribution structure in respect to background one, should depend on this difference, that can be expressed analytically in geometric terms.

We start with choosing of the suitable definition for mass. The best one, on our opinion, is to treat as the particle's mass, the part of energy, associated with it regardless of its state of motion and of its interactions with other particles. It is preferable to the inertial mass definition, because it does not depend on motion, and to the gravitational one, since it does not require more than one particle.

It means that if we prepare the model containing one non-moving defect, corresponding to a particle, in the infinite periodic tessellation, and calculate the difference in energy between pure and defected models, we should get the particle's mass.

The tensor field of tessellation distortion, that might emerge around the defect, as we suppose, should be associated with the gravitational field of the particle. In this approach, the field of gravity is not created by mass nor by energy, but it is an essential part of the energy, and particularly, of mass. Following it, we should consider the total mass as split in two different parts: one of them is connected to the changes of not only sizes but also of the topology of domain walls, that is occurred in the place of defect; while another part is connected to the minor residual changes in shape of domains around it, that retain the tessellation topology, but can spread on rather bigger distances. Both parts are supposed to be able to exchange their energy and minimize it.

So, **obtaining the appropriate Hamiltonian** is the second task of the approach, essential for its application to mass and energy prediction. The energy function could depend on domains' and walls' volume, area, curvature, thickness, charge density and so on. To check it, we calculate the Hamiltonian for the sample pure background tessellation (that should be as large as possible, ideally infinite). After that we figure out how the appearance of the particular defect rearranges the tessellation components in-place and in the vicinity, and calculate the Hamiltonian again, this time for the defected tessellation. The difference we treat as defect energy, which should be equivalent to the particle mass in absence of interactions and movements (in the reference frame where the domain centers are motionless).

In addition to the mass calculation, the Hamiltonian can play another significant role. Both the initial assumption about existing of the domain tessellation, and choosing concrete structure for it, need some physical grounding for them, aside of their usefulness in explaining or predicting the particle and vacuum properties. We suppose that the energy depends on the structure shape so that it has the locally or globally minimum corresponding to the tessellation in the ground state.

Taking the Hamiltonians gradient as analogue of tension force, we can allow the model to relax under it, and do not care anymore about maintaining of correct form of domains.

In the most preferable case, we can omit the step of choosing the shape of tessellation, allowing the Hamiltonian minimization to self-assemble the tessellation.

This task looks rather real because, for instance, the tessellation A15 is an example of extremal case: it has minimal known wall area to given domain volume ratio among all 3-dimensional equal-volumed tessellations.

Nevertheless, the use of just such a Hamiltonian is not necessary: for simplicity, tessellation can be given imperatively, by the coordinates of points, or analytically, for example by a trigonometric or exponential function.

We have considered some simple rules of calculating energy, as follows:

- The simplest hypothesis is to estimate the energy as being proportional to the count of bits or domains that are inverted with respect to the ground state. Its advantage is that it can be applied to infinite or even to the finite bit graphs regardless of their structure.
  The results are mostly qualitative, and can only be considered valid for a few cases. For instance, the smallest but non-zero masses must correspond to the photon and neutrino because they are represented with just two inverted bits. The most heavy particle should be Higgs boson, built from eight defects. $Z$ corresponds to six defects while triplet-$W$ does with five ones. So the mass ratios should be $\frac{m_W}{m_H} = \frac{5}{8} = 0,625$, $\frac{m_Z}{m_H} = \frac{6}{8} = 0,75$, while experimental values are $0.643$ and $0.728$.
- Considering two kinds of bits, that reside in triplets and in pairs, as different, and treating solo changes of domain in pairs as having no influence on the mass, we could improve these results. This caused us to move from bit models to tessellations, where we can take in account the geometric properties.
- In the polyhedral approximation of A15 structure, constructed from domains of two parities (and, of course, two corresponding charges), there are three kinds of faces of different area, and they can separate domains of either equal or opposite charge.
  We supposed that the energy is proportional to area of the domain walls and it is different for two types of wall: for double-layered walls between opposite-charged domains, containing zero-charged film in their core, and for walls between domains of the same charges: these walls supposed to have another structure, without zero surface inside.
  The particle, as combinations of several defects, define the configuration of walls, that can be calculated manually, even without computer simulation, just by counting faces of particular type.
  For A15 model, this energy calculation leads to existence of massless, low-massive, and highly massive particles. The massless particles correspond to inversions in dodecahedra, that have six equal pentagonal faces of each type, and after recharging they have six equal faces of each type, again.
  Since the changes can be in both directions, and the difference between arithmetic mean of two face's area and the third face is very small, the particles containing combinations, compensating each other, are lite-weight. Others are massive.
  We could not reproduce all the known masses in this simple scheme, but slightly varying the tessellation geometry, we found some defect combinations, that simultaneously give correct quantum numbers and also correct masses, for the photon, neutrinos, electron, weak bosons and Higgs.

Gluon threads in mesons, supposed as 1-dimensional condensates of diagonal ($r\tilde{r}$, $g\tilde{g}$, $b\tilde{b}$) gluons, also appear massless excepting their ends. The solo gluons, not stacked in threads, have in this schema sufficient masses on $GeV$ scale, so the conception of threads is preferable. Quarks do not look like individual particles, but as indispensable ends of diagonal gluon thread or, for closed non-diagonal one, as sites where it changes its direction.
Some mass values, for example $105.65MeV$ for the muon, could not be represented this way unless we allow not just even but also odd count of changed faces, even though they always appear in pairs. This can mean that the second family should be considered in dynamics only, as oscillation or combination of two forms, having both even but different changed faces count, producing odd arithmetic mean.

So by now we have not suggested the Hamiltonian that we could call ultimate nor close to it. The task seems to be complex because it should allow to take in account the particle's motion, including relativistic case.

### 10.3.3 Dynamics, time and motion

To be able to represent dynamic effects we needed at least the tessellation that can get changed. However, we did not see that such an ability is present in any of the three-dimensional tessellations that we considered. Both the ground state, and the defects, manifest their tendency to be stable, motionless, especially under the Hamiltonian minimization. Nothing forces the defects to jump into another locations and also nothing causes them to keep jumping conserving their momentum or velocity.

**Cellular automaton as 4d tessellation** One thing we could do is to consider consequential 'snapshots' of the same tessellation, where the defects took different places, 'moving' in the same sense as 'move' the motionless frames on a film. By assuming some external, additional rules of the jumps we could get the working model that would be a kind of cellular automaton.

Geometrically, the cellular automaton build on the basis of three-dimensional Kelvin or A15 structure is the infinite four-dimensional tessellation with the dedicated direction, that is the direction of computation, orthogonal to the other three. Each 3-dimensional domain turns in it into the 4-dimensional cylinder or prism.

From the viewpoint of the tessellation approach, there is no reason to believe that the shape of prism or cylinder is the best shape for the domain in the tessellation, suitable for the modeling. Instead, we should get one step back and suggest some 4-dimensional tessellation that would be 'good' or may be 'the best' according to its abilities to reproduce the phenomena we want in our model.

**Cross-sections with 'moving' domains** On another hand, observing the 2-dimensional cross-sections[1] of the 3-dimensional A15 model, we found out that its

[1] In the trigonometrical approximation, with $\rho = \frac{1}{4}(\sin z(1-\cos x)(1+\cos y) + \sin x(1-\cos y)(1+\cos z) + \sin y(1-\cos z)(1+\cos x))$

sequential cross-sections, that can be taken continually, look like a cartoon film, showing perpetually moving two-dimensional domains, even in the pure non-defected tessellations. The character of movement could be described as kind of oscillation or rotation, but since the similar-charged domains are indistinguishable, when they meet, they can exchange, so the movement also can be treated as directed relocation of domains on any distance and in any direction with the limited velocity.

Any defect, occurred in this tessellation, in order to conserve its charge, must participate in its neighborhood's movement. Otherwise, it would overlay with other domain of the same charge, causing the double-charged domain, or mutually cancel the domain of opposite charge, forming the domain with reduced or zero charge. Both cases violate the principal assumption of the domain behavior, postulating their constant charge. So the charge conservation can be treated as the cellular automaton law, determining domain migration into the appropriate place on the each step.

**Hypothetical speculations about modeling movements and time** Each time when the defected domain meets two neighbors of the opposite charge, it must choose, which place to take. Manipulating with this choice, we can control the movement: if it happens predominantly in one direction, than the defect moves there; otherwise it moves randomly or oscillating, keeping close to the point of origin.

The small distortions of the domain's walls shape, caused by last choice made, can play role of the short-term memory, keeping some information about it, and make influence on the next upcoming choice. This possibility turns the process to be analog of Markov chain and allows keeping the movement direction, for instance, with the mechanism similar to the Bresenham's line algorithm.

We also supposed that the number of situations of making some choice of direction, can play role of the own time for the particle, that influences on the probability of the particle's decay. Propagating with high velocities, close to the limit, defected domains have less freedom in choosing direction, that can be treated by the low-velocity observer as the time dilation of the quickly propagating particle.

Unfortunately, the effect of 'moving' domains could not be used directly to represent the movement in the 3-dimensional model, because it reduces the dimension count by one, so in each temporal moment, i.e. cross-section, the model space is flat.

Combining the idea of cellular automaton, as a 4-dimensional tessellation, with the observations of movement-like behavior of domains in flat cross-sections, we supposed that there exists a 4-dimensional tessellation allowing cross-sections, which in turn are 3-dimensional tessellations, able to represent known set of particles, and the movement observed from within 3d sections is a certain process in 4d one, equivalent to sampling successive sections in some direction with strict conservation of charge for each domain in the section.

So the third task of the tessellation approach can be formulated as to **find the appropriate 4-dimensional tessellation**. It must offer the same possibilities as 3-dimensional ones, but, additionally, provide the way to represent momentum and, ideally, the law that causes domains to conserve it.

**4d tessellation 'Satori'** Since the most successive 3-dimensional model was the *optimal* space tessellation, we looked for the references to optimal tessellation in the 4-dimensional space, but did not find any. So we analyzed the way how the optimal tessellations in 2 and 3 dimensions are build, and found out that they are relaxed Voronoi diagrams of square or cubic point grids, with some nodes shifted on the unit half-size along the rows, columns or through layers.

We noted that the optimal 3d node grid is produced from two isomeric optimal 2d grids (in one of them each second row is shifted while in another one the points are shifted in each second column). Being placed in alternating adjacent layers, they offer possibility to perform additional shift of $\frac{1}{4}$ points along the straight lines orthogonal to the layers, so the ratio of shifted points raises from 0 in 1 dimension through $\frac{1}{2}$ in 2d up to $\frac{3}{4}$ in 3d, and the calculated value of the optimality criterion[2] was reduced, which meant compaction.

This procedure also produces two 3d-isomers, depending on selection of even-odd or odd-even order of 2d isomers used.

Following this way, we repeated the same operation once more, placing two alternating isomeric 3d grids in adjacent spaces. Doing so, we got all the remaining non-shifted $\frac{1}{4}$ points disposing on straight lines perpendicular to the spatial layers, so we could perform the ultimate shift along these lines.

Calculating the Voronoi diagram (using the qhull package [13]), we found out that it consists of all the regions having the same size and the same shape. They are 78-verticed polytopes, with 26 3d faces, two of which are distorted dodecahedra while the remaining 24 are nonahedra. They have 4 orthogonal orientations, that can be defined by the vector connecting centers of their dodecahedral 3d-faces. Polytopes of the same orientation stack together sharing dodecahedral 3d-faces along each of four orthogonal axis. Even and odd polytopes are alternating along the stack, being the mirror reflections of each other.

Calculating the optimality criterion, we found it[3] $\approx 4.9\%$ less than in 3d, so since all the points are yet shifted, it is impossible to get more compact tessellation with the same way. It means that, probably, this 4d tessellation that we called 'Satori' is the most compact one in all the Euclidean spaces.

[2] The optimality criterion we calculate as $c = \frac{D_{d-1}}{d \cdot \sqrt[d]{N D_d^{d-1}}}$ where $d$ is the space dimension count, $D_{d-1}$ is the hyper-area of walls in the sample of $N$ domains, and $D_d$ is the hyper-volume of the sample. It has the value of 1 for simple hyper-cubic grids in all dimensions. The optimal flat honeycomb has $c = \sqrt{\frac{3}{2\sqrt{3}}} \approx 0.93060$ while non-relaxed A15 has $c \approx 0.882825$.

[3] $c = \frac{1}{8}\left(1 + 7\sqrt{\frac{2}{3}}\right) \approx 0.83943$

Checking the cross-sections[4], we made sure that they keep the 'moving' behavior of domains, now in three dimensions. The section is to be made orthogonal to one of the axis . In contrast, when the section is performed orthogonal to the diagonal of the Cartesian reference frame, the 'movement' looses its stochastic character, keeping all domains in 4-beat oscillating near points close to their centers.

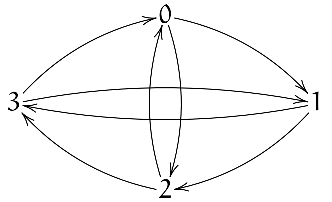
The new structure is made of equal domains so it is supposed to be stable under the relaxation with the tension applied with suitable Hamiltonian.

**4d Cylinder tessellation** With all its advantages, the Satori structure has at least two drawbacks that make us look for improvements. First, there is no more D-type domains that had equal count of neighbors of both parities, which allowed us to easily build models for massless particles using them. Now each domain shares two dodecahedral 3d-faces with two its neighbors, so even in mutual charge exchange between two neighbors the opposite 3d-faces would change their kind, that we usually treat as a sign of some mass connected with such a defect.

Second, the tessellation looks having the lack of causality from the viewpoint of observer inside 3d cross-section. Propagating in some direction, the process of cross-sectioning can meet regions, containing other defects, that for the 3d observer would be miracle artifacts, appearing from nowhere and violating the conservation laws.

We see that the possible solution for both problems listed above is the restriction in one of four dimensions *with only one translating unit*, turning the tessellation into the 4-dimensional cylinder, infinite in three dimensions but periodical in the fourth one.

In this case two domains of opposite parity lying along the *periodical* axis would share both dodecahedral 3d-faces, so they both will remain intact in the mutual charge exchanges. The process of cross-sectioning is limited now with only four domain layers, so it cannot meet anything that does not exists in these layers. That ensures the same reality for both 4d and 3d observers. The sectioning process degenerates to the directed oscillation or rotation between four 3d-spaces, schematically shown below:




in which states of domains in each space depend only on states of domains in two previous spaces, and also influence only on the state of domains in two subsequent spaces (rules for even and odd spaces are different due to their different structures).

[4] In the trigonometric approximation of the 'Satori' structure that we constructed having extremal points in the domain centers: $\rho = \frac{1}{4}(\sin x(\cos y - \cos z + \cos t - \cos y \cos z \cos t) + \sin y(\cos z - \cos x + \cos t - \cos z \cos x \cos t) + \sin z(\cos x - \cos y + \cos t - \cos x \cos y \cos t) - \sin t(\cos x + \cos y + \cos z + \cos x \cos y \cos z))$

## 10.4 Discussion

It is not obvious whether the tessellation approach is compatible with the known 'no-go' theorems. For instance, it should not be considered as deterministic because it is based on bit graphs, which are multivalent, producing multiple eigenvalues as result of the serialization, which corresponds to the quantum measurement. Also, it offers some combination of spatial and internal degrees of freedom so it is interesting to check against the Coleman-Mandula theorem.

## 10.5 Conclusion

The tessellation approach that we define and discuss in this paper allow us to formulate and solve problems of the particle modeling. Some of them have also the general mathematical meaning, for instance the problem of multi-dimensional filling optimality and measurement of information that tessellation holds.

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# 11 Mass Matrix Parametrization for Pseudo-Dirac Neutrinos

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**Abstract.** An overview of pseudo-Dirac neutrino framework is given starting from general spinor phenomenology. The framework is then tested by simulation of oscillations for T2K experiment parameters. Two possible derivations [7] and [8] of oscillation parameters are indicated to have the same result.

**Povzetek.** Avtor poda pregled modela psevdo Diracovega nevtrina v okviru splošne fenomenologije spinorjev. Model preizkusi s simulacijo oscilacij za parametre poskusa TK2. Pokaže, da izpeljavi [7] in [8] privedeta do enakega rezultata.

Keywords: neutrino oscillations, sterile neutrinos, pseudo-Dirac neutrinos, neutrino oscillation experiments

## 11.1 Introduction

Massive neutrinos directly indicate presence of physics beyond the Standard model (BSM). Precise measurements of neutrino oscillations provide the possibility to probe various BSM theories.

Since the absolute values of neutrino masses are currently beyond direct measurements various experiments are focused on the standard neutrino model ($\nu$SM) oscillation parameters – square mass differences $\Delta m^2$ and $\delta$-phase.

Some experiments however reported the existence of anomalies in experimental data. These anomalies can find explanation in theories with additional neutrino interactions, most notably the sterile neutrinos.

Recently a number of short-baseline reactor experiments declared an observation of sterile neutrinos with the significance of $3\sigma$. However the observations are not entirely compatible to each other. The matter is under investigation in the ongoing STEREO, PROSPECT, SoLid and Neutrino-4 experiments. Experimental evidences suggesting sterile neutrino with mass $\sim 1$ eV can be explained in the simplest way in 3+1 neutrino model.

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Standard unitary 3+1 data fit suffers from strong tension between MINOS and MINOS+ bound on $\overset{(-)}{\nu}_\mu$ disappearance [2] and LSND&MiniBooNE $\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e$ appearance [3,3,4]. There are two ways to approach this problem.

First possibility is to consider 3+1 non-unitary mixing scenario [5]. It can be used to explain short-baseline disappearance experiments however the anomalies observed in LSND and MiniBooNE experiments [6] remain unexplained.

Second possibility is addressing to more than one sterile neutrino. 3+2 scenario can be studied in general framework of 3 active and 3 sterile neutrino. Here we are probing the pseudo-Dirac scenario with 3 active and 3 sterile neutrinos.

In Section 11.2 we will describe how pseudo-Dirac neutrinos naturally arise when the neutrino is a composition of Dirac and Majorana spinors.

In Section 11.3 we will show that pseudo-Dirac neutrinos can be effectively described by three parameters. Then the mass matrix can be effectively diagonalized which we show using two different approaches. Then we will plot the oscillation probability for pseudo-Dirac scenario against pure Dirac neutrinos for the setup of T2K experiment.

In Section 11.4 we will discuss what can be further done to address the problem of streile neutrinos and neutrino mass generation.

## 11.2 General spinor formalism

Lagrangian mass term for two spinors $\chi$ and $\eta$ has the form

$$\mathcal{L}_{mass} = \frac{1}{2} \left(\chi\ \eta\right) \mathbf{M} \begin{pmatrix} \chi \\ \eta \end{pmatrix} \tag{11.1}$$

where mass is given by $\mathbf{M} = \begin{pmatrix} A & M \\ M & B \end{pmatrix}$ and $M, A, B$ are 2x2 matrices.

For the most general free field case we can write down "Weyl-Majorana-Dirac equation"

$$\begin{aligned} i\sigma_\mu \partial^\mu \psi_L - \eta_{D,R} m_{D,R} \psi_R - \eta_L m_L (i\sigma_2) \psi_L^* = 0 \\ i\bar{\sigma}_\mu \partial^\mu \psi_R - \eta_{D,L} m_{D,L} \psi_L - \eta_R m_R (i\sigma_2) \psi_R^* = 0 \end{aligned} \tag{11.2}$$

with non-negative mass terms $m$ and phase terms $\eta = e^{i\varphi}$ from unitary group $U(1)$. Defining $\tilde{m} = \eta m$ and $\psi_R = \begin{pmatrix} \psi_1 + i\psi_2 \\ \psi_3 + i\psi_4 \end{pmatrix}$, $\psi_L = \begin{pmatrix} \psi_5 + i\psi_6 \\ \psi_7 + i\psi_8 \end{pmatrix}$ this equation can be transformed into the form [1]:

$$\Box \Phi + \hat{M}^2 \Phi = 0 \tag{11.3}$$

where $\Phi = (\psi_1 .. \psi_8)^T$

Now let us illustrate only the simple case $m_{D,L} = m_{D,R} = m_D$. For this case general spinor mass matrix is positive semi-definite Hermitian matrix of the form

$$\hat{M}^2 = \begin{pmatrix} M_R & 0 & 0 & A \\ 0 & M_R & -A & 0 \\ 0 & -B & M_L & 0 \\ B & 0 & 0 & M_L \end{pmatrix} \tag{11.4}$$

where $M_R = \begin{pmatrix} \nu_1 + m_R^2 & -\nu_2 \\ \nu_2 & \nu_1 + m_R^2 \end{pmatrix}$, $M_L = \begin{pmatrix} \nu_1 + m_L^2 & -\nu_2 \\ \nu_2 & \nu_1 + m_L^2 \end{pmatrix}$,

$B = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & -\mu_1 \end{pmatrix}$, $A = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$ and $\tilde{m}_D \tilde{m}_L + \tilde{m}_D^* \tilde{m}_R = k \geqslant 0$ and moreover $\tilde{m}_D^* \tilde{m}_L + \tilde{m}_D \tilde{m}_R = \mu_1 + i\mu_2$ $\tilde{m}_D^2 = \nu_1 + i\nu_2$. This matrix has four doubly degenerate eigenvalues. Considering real and positive $m_R$ and $m_D$ and complex $m_L$ we are down to just two eigenvalues.

Now consider $\chi$ and $\eta$ in 11.1 to be the left- and right-handed neutrino fields $\nu_L$ and $\nu_R$. We can work with two Majorana neutrinos if we stipulate $\nu_R = \nu_L'^C$. Then $\mathbf{M} = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}$ There are three commonly known special cases for the values of the elements of this matrix:

- First case is $m_L = m_R$. In this scenario we have a pair of eigenvalues $m_D \pm m_L$ and mixing angle between $\nu_L$ and $\nu_R$ is given by $tan 2\theta = \frac{2m_D}{m_R - m_L} = \frac{\pi}{4}$. No active-sterile oscillations are realized in this case.
- Second case is $m_L = m_R = 0$. In this scenario we have a pure Dirac neutrino.
- Last case is $m_L, m_R \ll m_D$. This scenario is referred to as pseudo-Dirac case.

In general, neutrino can have Majorana and Dirac parts

$$\mathcal{L}_{mass}^{D+M} = \mathcal{L}_{mass}^{D} + \mathcal{L}_{mass}^{L} + \mathcal{L}_{mass}^{R} \tag{11.5}$$

and Dirac neutrino can be represented as two Majorana neutrinos. Left-handed neutrinos are concerned active while right-handed are sterile i.e. they are singlets under $SU(2)_L \times U(1)_Y$.

For the Pseudo-Dirac neutrino the symmetry of mass matrix is not the symmetry of the weak interaction. It is easy to obtain Pseudo-Dirac neutrino decomposition

$$\begin{aligned} \psi_{\pm L} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \eta_1 \pm i\eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} (N_{1L} \pm iN_{2L}) \to \frac{1}{\sqrt{2}} (N_{1L} \pm e^{i\varphi} N_{2L}) \\ \psi_{\pm R} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -i\sigma^2 (\eta_1^* \pm i\eta_2^*) \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (N_{1L}^C \pm iN_{2L}^C) \to \frac{1}{\sqrt{2}} (N_{1L}^C \pm e^{i\varphi} N_{2L}^C) \end{aligned} \tag{11.6}$$

for a pair of almost degenerate mass Majorana neutrino with opposite CP sign and lepton number not being conserved in higher order weak interaction.

Because of the small value of mass matrix distortions the mixing angle between two Majorana neutrinos is $\sim \frac{\pi}{4}$.

## 11.3 Modeling

### 11.3.1 Mass matrix diagonalization

For chirality preserving processes it is suffice to diagonalize $M^\dagger M$. We will now consider two possibilities – $M^2$ and $M$ diagonalization and show that in the leading order they provide the same result for pseudo-Dirac neutrinos.

In general, 6x6 mass matrix diagonalization gives 15 mixing angles, multiple violating CP phases and 6 eigenvalues. Under Pseudo-Dirac assumption this can be approximated by ordinary 3x3 Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [7].

$$M^\dagger M \simeq \begin{pmatrix} m_D^\dagger m_D & m_L^* m_D^T + m_D^\dagger m_R^* \\ m_D^* m_L + m_R m_D & m_D^* m_D^T \end{pmatrix} \tag{11.7}$$

consider bi-unitary transformation $U_R^\dagger m_D U_L = diag(m_1, m_2, m_3) = m$ then $V = \begin{pmatrix} U_L & 0 \\ 0 & U_R^* \end{pmatrix}$ and

$$V^\dagger(M^\dagger M)V = \begin{pmatrix} m^2 & U_L^\dagger m_L^\dagger U_L^* m + m U_R^\dagger m_R^* U_R^* \\ m U_L^T m_L U_L + U_R^T m_R U_R m & m^2 \end{pmatrix} \tag{11.8}$$

If we completely ignore off-diagonal parts then it is just Dirac scenario with doubly-degenerate eigenvalues. Otherwise in the first order approximation each pair takes the form $\begin{pmatrix} m_i^2 & \epsilon_i^* m_i \\ \epsilon_i m_i & m_i^2 \end{pmatrix}$

Now we obtain 6 mass eigenstates $\nu_{iS} = \frac{1}{\sqrt{2}}(\nu_{iL} + e^{i\varphi_i}\nu_{iR})$ $\nu_{iA} = \frac{1}{i\sqrt{2}}(\nu_{iL} - e^{i\varphi_i}\nu_{iR})$ such that $e^{i\varphi_i} = \frac{\epsilon_i}{|\epsilon_i|}$ for decomposition 11.6 and mass eigenvalues given by $m_{iS,A}^2 = m_i^2 \pm \epsilon_i m_i$.

Another method for diagonalization $M$ itself is completely removing left-handed Majorana spinor part of the Dirac one – mass matrix takes the form $M = \begin{pmatrix} \mathbf{0} & m_D^{'} \\ m_D^{'} & M_s \end{pmatrix}$ In [8] it is shown that the appropriate diagonalizing transformation is given in form

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} U^\dagger & 1 \\ U & 1 \end{pmatrix} \begin{pmatrix} 1 & \delta \\ -\delta^\dagger & 1 \end{pmatrix} \tag{11.9}$$

where U diagonalizes $m_D^{'}$ and $\delta = U(\epsilon/2 + \varepsilon)$, $\varepsilon^T = -\varepsilon$ and $M_s = 2\epsilon m_D - \varepsilon m_D + m_D\varepsilon$. This produces

$$M = V^\dagger m V$$

where

$$m = \begin{pmatrix} m_D(1+\epsilon) & 0 \\ 0 & -m_D(1-\epsilon) \end{pmatrix}$$

Now $m^2$ in the leading order have the eigenvalues $m_i^2 \pm \epsilon_i^{'} m_i$ which are the same as in the previous case.

### 11.3.2 Probing the pseudo-Dirac scenario

With these eigenvalues we can write down the oscillation probability in terms of ordinary PMNS matrix. Assume that mass eigenvalues splitting for pseudo-Dirac neutrino is given by $m_{iS,A}^2 = m_i^2 \pm \epsilon_i m_i$. Using the results from [7] it is easy to model $\nu_\mu \to \nu_e$ oscillation probability which is

$$P(\nu_\alpha \to \nu_\beta) = \frac{1}{4}\left|\sum_{j=1}^{3} U_{\beta j}\left(e^{i\frac{m_{jS}^2}{2E}t} + e^{i\frac{m_{jA}^2}{2E}t}\right)U_{\alpha j}^*\right|^2 \qquad (11.10)$$

To illustrate potentially observable differences between Dirac and pseudo-Dirac scenario we will simulate oscillations for T2K experiment parameters:

- $L = 295$ km and $E \leq 2$ GeV.
- $\delta = -\frac{\pi}{2}$ and $\sin^2\theta_{12} = 0.307$ $\sin^2\theta_{23} = 0.5$ $\sin^2\theta_{13} = 0.218$.
- $\Delta m_{12}^2 = 7.53 \cdot 10^{-5}\text{eV}^2$ $\Delta m_{23}^2 = 2.44 \cdot 10^{-3}\text{eV}^2$.
- normal mass hierarchy.

This allows us to probe the impact of small Majorana additives. Please also note that energy spectrum now depends on the absolute mass of neutrino because of the splitting. First we will model the situation where $\epsilon_i = 0.1$, Fig. 11.1.

Please note that neutrino beam in T2K experiment has energy distribution with maximum at $0.6$ GeV and almost all neutrinos have energy in the interval $0.5 \div 1$ GeV. So we cannot make any assumptions considering pseudo-Dirac neutrinos using only T2K data.

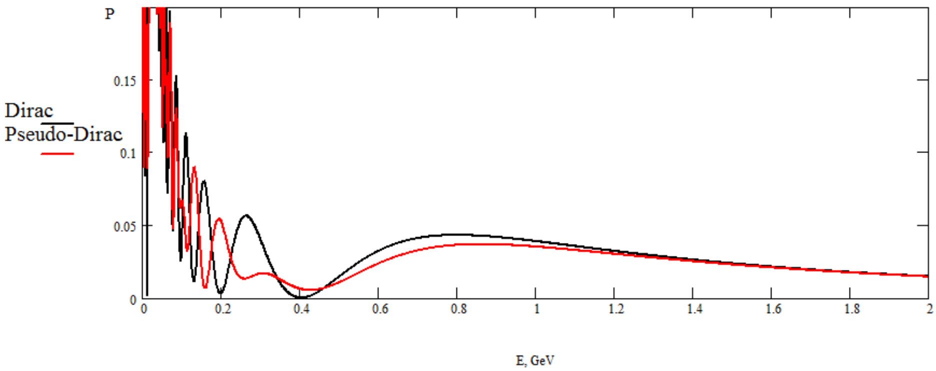


**Fig. 11.1.** Pseudo-Dirac neutrino $\nu_\mu \to \nu_e$ oscillation probability compared to pure Dirac scenario for T2K experiment parameters and naive assumptions for pseudo-Dirac mass eigenvalues.

Let us illustrate the difference in energy spectrum for more realistic $\epsilon_i$ parameters. In Fig. 11.2 we have taken $m_1 = 0.01$ eV, $\epsilon_1 = 2.6 \cdot 10^{-3}$,$\epsilon_2 = 4.0 \cdot 10^{-3}$ and $\epsilon_3 = 5.0 \cdot 10^{-3}$ proportional to mass squares differences.

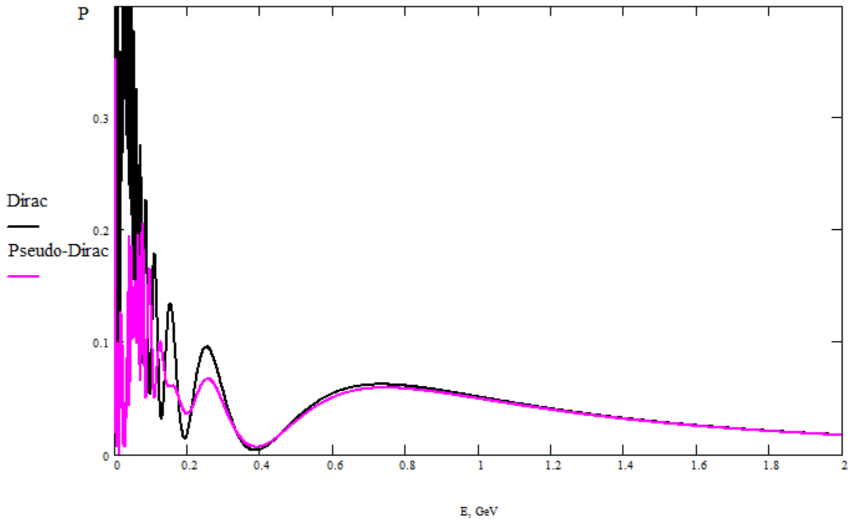


**Fig. 11.2.** Pseudo-Dirac neutrino $\nu_\mu \to \nu_e$ oscillation probability compared to pure Dirac scenario for T2K experiment parameters for more realistic mass splittings.

## 11.4 Discussion and Conclusion

Now we are in the situation where combined experimental data from atmospheric, reactor and accelerator neutrino experiments is in good agreement with 3 active neutrino model for the first three oscillation peaks. Upcoming experiments can provide more experimental data thus clarifying the situation.

Long-baseline experiments can provide precise values of $\nu$SM oscillation parameters and provide enough data to determine the neutrino mass hierarchy.

Short-baseline experiments can either improve their statistics and cancel out all anomalies or successfully approve that the $\nu$SM needs expansion.

Using precise $\beta$-decay and K-capture measurements it would be arguably possible to measure neutrino masses directly or at least put a constraints on them.

$\beta\beta$ and $0\nu\beta\beta$ observations as well as atmospheric, solar, galactic and extra-galactic neutrino experiments are important for probing different neutrino mass generation mechanisms.

It is also important to consider theoretical models for processes in early Universe – the constraints from these models are generally less strict than from direct observations but still helpful either for a cross-checking or for limiting the potential of exotic mass generation and mixing models.

Here we presented the derivation of pseudo-Dirac neutrino from general spinor formalism.

For the parameters of T2K experiment the probability of $\nu_\mu \to \nu_e$ oscillation was modeled. The current setup of the experiment however is not sensitive to differences in Dirac and pseudo-Dirac oscillations.

It was shown that in the leading order approximation PD neutrino can be effectively described by three $\epsilon$ parameters of mass splitting – it is valid for $M^2$ and M diagonalization.

There are questions arising naturally in the context of neutrino mass generation mechanism.

First question is whether it is suffice to consider pseudo-Dirac neutrino to fit observations or general framework is needed? This question will be addressed by the future observations.

Second question is about the compatibility of particular mass generation mechanism with pseudo-Dirac scenario in particular and it's rigidity to possible observational data as a whole. Which mechanisms are the best candidates, Yukawa coupling or multiple scalar fields (like in Zee model) or maybe even geometric models of mass generation?

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# 12 Relations Between Clifford Algebra and Dirac Matrices *

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**Abstract.** In the *spin-charge-family* theory [2–7] there are $\forall n \in \mathbb{N}$, $2^d$ Clifford operators, forming the vector space. Space can have for given $n \in \mathbb{N}$ dimension $d = 2(2n+1)$ or $4n$. Half of them are Clifford odd operators with the properties of fermion creation and annihilation operators for $2^{\frac{d}{2}-1}$ family members of $2^{\frac{d}{2}-1}$ families, fulfilling for each momentum $p_k$ the anticommutation relations for the second quantized fermions [8]. Families in Clifford space are reachable by $\tilde{S}^{ab} = \frac{1}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, $a \neq b$ and family members by $S^{ab} = \frac{i}{2}\gamma^a\gamma^b$, $a \neq b$. In this paper the basis in $d = (3+1)$ Clifford space is discussed, chosen in a way that the matrix representation of $\gamma^a$ and of generators of the Lorentz transformations in internal space, $S^{ab}$, coincide for each family quantum number, determined by $\tilde{S}^{ab}$, with Dirac matrices. The appearance of charges in Clifford space is discussed by embedding $d = (3+1)$ space into $d = (5+1)$-dimensional space.

**Povzetek.** V teoriji *spina-naboja-družin* [2–7] je v d dimenzionalnem prostoru $2^d$ Cliffordovih operatorjev, ki določajo vektorski prostor. Teorija izbere $d \geq (13+1)$. Če uredimo vektorski prostor tako, da so vektorji lastni vektorji Cartanove podalgebre Lorentzove grupe, izpolnjujejo lihi Cliffordovi vektorji $2^{\frac{d}{2}-1}$ družin s po $2^{\frac{d}{2}-1}$ člani vse Diracove pogoje za fermione v drugi kvantizaciji. Družinske člane določajo generatorji Lorentzove grupe $\tilde{S}^{ab}$ ($= \frac{1}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, $a \neq b$), družine pa $S^{ab} = \frac{i}{2}\gamma^a\gamma^b$, $a \neq b$.

V tem prispevku predstavijo avtorji bazo v $d = (3+1)$ razsežnem Cliffordovem prostoru ter matrično upodobitev za operatorje $\gamma^a$, $S^{ab}$, $\tilde{S}^{ab}$, $\tilde{\gamma}^a$ ter $\tilde{S}^{ab}$. $d = (3+1)$ razsežni Cliffordov prostor vgradijo v prostor $d = (5+1)$ ter komentirajo pojav naboja fermionov v $d = (3+1)$ .

## 12.1 Introduction

In the Grassmann graded algebra of anticommuting coordinates $\theta^a$ there are in d-dimensional space $2^d$ vectors, which define, together with the corresponding derivatives $\frac{\partial}{\partial\theta_a}$, two kinds of the Clifford algebra objects: $\gamma^a$ and $\tilde{\gamma}^a$ [2,6–8], both with the anticommutation properties of the Dirac $\gamma^a$ matrices, while the

* Talk presented by N.S. Mankoč Borštnik

anticommutators among $\gamma^a$ and $\tilde{\gamma}^b$ are equal to zero.

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ , \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 ,\\
(\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a , \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a ,\\
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) , \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a) ,\\
\{S^{ab}, \tilde{S}^{ab}\}_+ &= 0 ,\\
(a, b) &= (0, 1, 2, 3, 5, \cdots, d) . \qquad (12.1)
\end{aligned}$$

The two Clifford algebras, $\gamma^a$'s and $\tilde{\gamma}^a$'s, are obviously completely independent and form two independent spaces, each with $2^d$ vectors [9].

Sacrificing the space of $\tilde{\gamma}^a$'s by defining

$$\tilde{\gamma}^a B(\gamma^a) = (-)^B\, i\, B\gamma^a , \qquad (12.2)$$

with $(-)^B = -1$, if B is an odd product of $\gamma^a$'s, otherwise $(-)^B = 1$ [7], we end up with vector space of $2^d$ degrees of freedom, defined by $\gamma^a$'s only.

A general vector can correspondingly be written as

$$\mathbf{B} = a_0 + \sum_{k=1}^{d} a_{a_1 a_2 \dots a_k}\, \gamma^{a_1}\gamma^{a_2}\dots\gamma^{a_k}|\psi_o> , a_i < a_{i+1} , k = 1, \dots, d \quad (12.3)$$

where $|\psi_o>$ is the vacuum state.

We arrange these vectors as products of nilpotents and projectors

$$\begin{aligned}
\stackrel{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) , \quad (\stackrel{ab}{(k)})^2 = 0 .\\
\stackrel{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) , \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]} , \qquad (12.4)
\end{aligned}$$

where $k^2 = \eta^{aa}\eta^{bb}$. Their Hermitian conjugated values follow from Eq. (12.1).

$$\stackrel{ab}{(k)}{}^\dagger = \eta^{aa} \stackrel{ab}{(-k)}, \quad \stackrel{ab}{[k]}{}^\dagger = \stackrel{ab}{[k]} . \qquad (12.5)$$

Vectors in Clifford space are chosen to be eigenstates of the Cartan subalgebra, Eq. (12.6), of the generators of the Lorentz transformations $S^{ab}$ in the internal space of $\gamma^a$'s.

$$\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d} ,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d} , \qquad (12.6)
\end{aligned}$$

with the eigenvalues $S^{ab} \stackrel{ab}{(k)} = \frac{1}{2}k \stackrel{ab}{(k)}$, $S^{ab} \stackrel{ab}{[k]} = \frac{1}{2}k \stackrel{ab}{[k]}$. All the relations of Eq. (12.1) remain unchanged after the assumption of Eq. (12.3), while each irreducible representation of the Lorentz algebra $S^{ab}$ receives the additional quantum number $f$, defined by $\tilde{S}^{ab}$.

$$\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)} , \qquad \tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)} ,\\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]} , \qquad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]} . \qquad (12.7)
\end{aligned}$$

Eq. (12.7) demonstrates that the eigenvalues of $S^{ab}$ on nilpotents and projectors generated by $\gamma^a$'s differ from the eigenvalues of $\tilde{S}^{ab}$.

States, which are products of projectors and nilpotents, have well defined handedness of both kinds, $\Gamma^{(d)}$ and $\tilde{\Gamma}^{(d)}$.

$$\begin{aligned}\Gamma^{(d)} :&= (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\gamma^a)\,, \quad \text{if} \quad d = 2n\,,\\ \tilde{\Gamma}^{(d)} :&= (i)^{d/2} \prod_a (\sqrt{\eta^{aa}}\tilde{\gamma}^a)\,, \quad \text{if} \quad d = 2n\,.\end{aligned} \tag{12.8}$$

The *spin-charge-family* theory [2–7] of N.S. Mankoč Borštnik uses products of nilpotents, $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)$, and projectors, $\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$, to define $2^d$ vectors in this space of the Clifford graded algebra [3–5]. In this theory $S^{ab}$ determine in $d = (3+1)$ space, which is a part of $d = (13+1)$-dimensional space, spins and charges of quarks and leptons, while $\tilde{S}^{ab}$ determine families of quarks and leptons.

It is interesting to notice ([9,8] and references therein): *Vectors, which are superposition of odd products of nilpotents and projectors, anticommute fulfilling the anticommutation relations postulated by Dirac [1] for second quantized fermions, explaining correspondingly Dirac's postulate* [9,8].

In Sect. 12.2 the properties of products of nilpotents and projectors are discussed, arranged in eigenvectors of the Cartan subalgebra, defining the internal vector space of fermions in d-dimensional space when $d = (3+1)$-dimensional space is embedded into $d = (5+1)$-dimensional space, so that the spin in $d = (5, 6)$ determines the charge of fermions in $d = (3+1)$.

In Sect. 12.2.3 the matrix representation of vectors are presented.

## 12.2 Properties of vectors in Clifford space

In Refs. [9,8] the fact that the Clifford vectors, spanned by products of an odd number of $\gamma^a$'s, fulfill the anticommutation relations postulated by Dirac for the second quantized fermions, explains these Dirac's anticommutation relations. Let us see on the case that $d = (5+1)$ how this happens.

Let us denote vectors in $d = (5+1)$, presented in Table 12.1 as products of three nilpotents or projectors or both, by $\hat{b}^{f\dagger}_m$, $m = (ch, s)$, the member quantum number $m$ includes the charge, $ch$ and the spin $s$. The corresponding Hermitian conjugated partner is denoted by $(\hat{b}^{f\dagger}_m)^\dagger = \hat{b}^f_m$.

The first member $m = (\frac{1}{2}, \frac{1}{2})$ of the first family $a$, which is the product of three nilpotents, is correspondingly denoted by $\hat{b}^{a\dagger}_{(\frac{1}{2},\frac{1}{2})} = \overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{(+)}$. All the rest vectors of the family $f = a$ follow by the application of $S^{ab}$. The families $f = (b, c, d)$ follow from $f = a$ by the application of $\tilde{S}^{ab}$. The Hermitian conjugated partners follow by the application of Eq. (12.1).

Table 12.1, taken from Table IV of Ref. [8], represents four families of Clifford odd vectors and their Hermitian conjugated partners. All the families have the same quantum numbers $m$ of the corresponding members, $(S^{03}, S^{12}, S^{56})$, each family carries its own family quantum number $f$.

| f(amily) m | (ch, s) | $\hat{b}^{f\dagger}_{m}$ | $\hat{b}^{f}_{m}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{3+1}$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ |
|---:|---:|---:|---:|---:|---:|---:|---:|---:|---:|---:|
| a 1 | $(\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{(+i)}\stackrel{12}{(+)}\,\vert\,\stackrel{56}{(+)}$ | $(-)\stackrel{56}{(-)}\,\vert(-)\stackrel{12}{(-)}\stackrel{03}{(-i)}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| a 2 | $(\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{[-i]}\stackrel{12}{[-]}\,\vert\,\stackrel{56}{(+)}$ | $(-)\stackrel{56}{(-)}\,\vert\,\stackrel{12}{[-]}\stackrel{03}{[-i]}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| a 3 | $(-\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{[-i]}\stackrel{12}{(+)}\,\vert\,\stackrel{56}{[-]}$ | $\stackrel{56}{[-]}\,\vert(-)\stackrel{12}{(-)}\stackrel{03}{[-i]}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| a 4 | $(-\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{(+i)}\stackrel{12}{[-]}\,\vert\,\stackrel{56}{[-]}$ | $\stackrel{56}{[-]}\,\vert\,\stackrel{12}{[-]}\stackrel{03}{(-i)}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| b 1 | $(\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{[+i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(+)}$ | $(-)\stackrel{56}{(-)}\,\vert\,\stackrel{12}{[+]}\stackrel{03}{[+i]}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| b 2 | $(\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{(-i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(+)}$ | $(-)\stackrel{56}{(-)}\,\vert(-)\stackrel{12}{(+)}\stackrel{03}{(+i)}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| b 3 | $(-\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{(-i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[-]}$ | $\stackrel{56}{[-]}\,\vert\,\stackrel{12}{[+]}\stackrel{03}{(+i)}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| b 4 | $(-\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{[+i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[-]}$ | $\stackrel{56}{[-]}\,\vert(-)\stackrel{12}{(+)}\stackrel{03}{[+i]}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ |
| c 1 | $(\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{[+i]}\stackrel{12}{(+)}\,\vert\,\stackrel{56}{[+]}$ | $\stackrel{56}{[+]}\,\vert(-)\stackrel{12}{(-)}\stackrel{03}{[+i]}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| c 2 | $(\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{(-i)}\stackrel{12}{[-]}\,\vert\,\stackrel{56}{[+]}$ | $\stackrel{56}{[+]}\,\vert\,\stackrel{12}{[-]}\stackrel{03}{(+i)}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| c 3 | $(-\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{(-i)}\stackrel{12}{(+)}\,\vert\,\stackrel{56}{(-)}$ | $(-)\stackrel{56}{(+)}\,\vert(-)\stackrel{12}{(-)}\stackrel{03}{(+i)}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| c 4 | $(-\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{[+i]}\stackrel{12}{[-]}\,\vert\,\stackrel{56}{(-)}$ | $(-)\stackrel{56}{(+)}\,\vert\,\stackrel{12}{[-]}\stackrel{03}{[+i]}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| d 1 | $(\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{[+]}$ | $\stackrel{56}{[+]}\,\vert\,\stackrel{12}{[+]}\stackrel{03}{(-i)}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| d 2 | $(\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{[-i]}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{[+]}$ | $\stackrel{56}{[+]}\,\vert(-)\stackrel{12}{(+)}\stackrel{03}{[-]}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| d 3 | $(-\frac{1}{2},\frac{1}{2})$ | $\stackrel{03}{[-i]}\stackrel{12}{[+]}\,\vert\,\stackrel{56}{(-)}$ | $(-)\stackrel{56}{(+)}\,\vert\,\stackrel{12}{[+]}\stackrel{03}{[-i]}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| d 4 | $(-\frac{1}{2},-\frac{1}{2})$ | $\stackrel{03}{(+i)}\stackrel{12}{(-)}\,\vert\,\stackrel{56}{(-)}$ | $(-)\stackrel{56}{(+)}\,\vert(-)\stackrel{12}{(+)}\stackrel{03}{(-i)}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

**Table 12.1.** The basic creation operators, which are sums of odd products of $\gamma^a$'s — $\hat{b}^{f\dagger}_{m}$, $m = (ch, s)$, $ch$ represents the spin in $d = (5,6)$, manifesting in $d = (3+1)$ as the charge, and $s$ represents the spin in d=(1,2), according to the choice of the Cartan subalgebra, Eq. (12.6) — and their annihilation partners — $\hat{b}^{f}_{m}$ — are presented for the $d = (5+1)$-dimensional case. The basic creation operators are the products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra generators, ($S^{03}$, $S^{12}$, $S^{56}$) and ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), presented in Eq. (12.6). The Clifford odd parts of creation operators, belonging to $d = (3+1)$ space, are marked.

Half of vectors, the eigenvectors of the Cartan subalgebra, Eq. (12.6), which are products of nilpotents and projectors, are odd products of $\gamma^a$'s and half of them are even products of $\gamma^a$'s. On Table 12.1 only Clifford odd vectors are presented.

Let us make a choice of the vacuum state [6–9]. (In the case of a general even $d$ the normalization factor is $\frac{1}{\sqrt{2^{\frac{d}{2}-1}}}$, since the vacuum states, generated by projectors only, follows from the starting products of $\frac{d}{2}$ projectors, let say $\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\,\stackrel{56}{[-]}\;\stackrel{d-1\,d}{[-]}$), by changing all possible pairs of [−]...[−], with [−i] included, to [+]...[+], leading therefore to $2^{\frac{d}{2}-1}$ summands.

$$|\psi_o> = (\frac{1}{\sqrt{2}})^2\,(\stackrel{03}{[-i]}\stackrel{12}{[-]}\,|\,\stackrel{56}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\,|\,\stackrel{56}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\,|\,\stackrel{56}{[+]} + \stackrel{03}{[-i]}\stackrel{12}{[+]}\,|\,\stackrel{56}{[+]})\,|1> \;. \tag{12.9}$$

It then follows that

$$
\begin{aligned}
&\hat{b}^{f\dagger}_{m}\,|\psi_o>=|\psi^f_m>\,,\\
&\hat{b}^{f}_{m}\,|\psi_o>=0\,|\psi_o>\,,\\
&\{\hat{b}^{f\dagger}_{m},\hat{b}^{f'}_{m'}\}_{+}=\delta^{ff'}\delta_{mm'}\,|\psi_o>\,,\\
&\{\hat{b}^{f\dagger}_{m},\hat{b}^{f'\dagger}_{m'}\}_{+}=0\,|\psi_o>\,,\\
&\{\hat{b}^{f}_{m},\hat{b}^{f'}_{m'}\}_{+}=0\,|\psi_o>\,,\\
&\forall\, m\,\text{and}\,\forall\, f\,.
\end{aligned}
\tag{12.10}
$$

Eq. (12.10) represents all the requirements for the second quantized fermions.

### 12.2.1 Action

The action for a free massless fermion is needed and the corresponding equations of motion to take into account the ordinary space as well.

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$
\mathcal{A}=\int d^dx\,\frac{1}{2}\,(\psi^{\dagger}\gamma^0\,\gamma^a p_a\psi)+\text{h.c.}\,,
\tag{12.11}
$$

$p_a = i\,\frac{\partial}{\partial x^a}$, leading to the equations of motion

$$
\gamma^a p_a|\psi>=0\,,
\tag{12.12}
$$

which fulfill also the Klein-Gordon equation

$$
\gamma^a p_a\gamma^b p_b|\psi>=p^a p_a|\psi>=0\,,
\tag{12.13}
$$

for each of the basic vectors $|\psi^m_f>=\hat{b}^{f\dagger}_{m}\,|\psi_o>$. ($\gamma^0$ appears in the action to take care of the Lorentz invariance of the action.)

Solutions of equtions of motion, Eq. (12.12), for a free massless fermions with momentum $p^a=(p^0,p^1,p^2,p^3,0,0)$ and a particular charge $\pm\frac{1}{2}$, are superposition of vectors with spin $\frac{1}{2}$ and $-\frac{1}{2}$, multiplied by the plane wave $e^{-ip_a x^a}$. Coefficients in superposition depend on the momentum $p^a$.

### 12.2.2 Creation and annihilation operators in $d=(3+1)$ space embedded in $d=(5+1)$

The creation and annihilation operators of Table 12.1 are all of an odd Clifford character (they are superposition of odd products of $\gamma^a$'s). The rest of $2^4$ creation operators of an even Clifford character can be found in Refs. [9,8].

Taking into account Eq. (12.1) one recognizes that $\gamma^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$
\begin{aligned}
&\gamma^a\stackrel{ab}{(k)}=\eta^{aa}\stackrel{ab}{[-k]},\ \gamma^b\stackrel{ab}{(k)}=-ik\stackrel{ab}{[-k]},\ \gamma^a\stackrel{ab}{[k]}=\stackrel{ab}{(-k)},\ \gamma^b\stackrel{ab}{[k]}=-ik\eta^{aa}\stackrel{ab}{(-k)}\,,\\
&\tilde{\gamma}^a\stackrel{ab}{(k)}=-i\eta^{aa}\stackrel{ab}{[k]},\ \tilde{\gamma}^b\stackrel{ab}{(k)}=-k\stackrel{ab}{[k]},\ \tilde{\gamma}^a\stackrel{ab}{[k]}=\ i\stackrel{ab}{(k)},\ \tilde{\gamma}^b\stackrel{ab}{[k]}=-k\eta^{aa}\stackrel{ab}{(k)}
\end{aligned}
\tag{12.14}
$$

With the knowledge presented in Eq. (12.14) it is not difficult to reproduce Table 12.2, representing vectors that belong to $d = (3+1)$ space. Vectors carry no charge and have either an odd or an even Clifford character. Multiplying these vectors by the appropriate charge (that is by either the nilpotent— if the $d = (3+1)$ part has an even Clifford character — or the projector — if the $d = (3+1)$ part has an odd Clifford character — both must be the eigenfunction of $S^{56}$) we end up with the Clifford odd vectors from Table 12.1.

The properties of vectors of Table 12.2 are analyzed in details in order that the correspondence with the Dirac $\gamma$ matrices in $d = (3+1)$ space would be easy to recognize. Superposition of vectors with the spin $\pm\frac{1}{2}$ (either Clifford even or odd) solve the equations of motion, Eq. (12.12), for free massless fermions.

As seen in Table 12.2 $\gamma^a$ as well as $\tilde{\gamma}^a$ change the handedness of states. $S^{ab}$, which do not belong to Cartan subalgebra, generate all the states of one representation of particular handedness, Eq. (12.8), and particular family quantum number. $\tilde{S}^{ab}$, which do not belong to Cartan subalgebra, transform a family member of one family into the same family member of another family, $\tilde{\gamma}^a$ change the family quantum number as well as the handedness $\tilde{\Gamma}^{(3+1)}$, Eq. (12.8).

Dirac matrices $\gamma^a$ and $S^{ab}$ do not distinguish among the families: Corresponding family members of any family have the same properties with respect to $S^{ab}$ and $\gamma^a$, manifesting for $d = (3+1)$ space four times twice $2\times 2$ by diagonal matrices, which are, up to a phase, identical. The operators $\gamma^a$ and $S^{ab}$ are correspondingly four times $4\times 4$ matrices.

One finds that half of vectors of Table 12.2 are Hermitian conjugated to each other. In the Clifford odd part of Table 12.2 one finds that $\hat{b}^{a\dagger}_{m=(3,4)}$ ($\stackrel{03}{[-i]}\stackrel{12}{(+)}$, $\stackrel{03}{(+i)}\stackrel{12}{[-]}$) have as the Hermitian conjugated partners $\hat{b}^{(d,c)}_{m=2}$ ($\stackrel{03}{[-i]}\stackrel{12}{(-)}$, $\stackrel{03}{(-i)}\stackrel{12}{[-]}$), respectively. And $\hat{b}^{b\dagger}_{m=(3,4)}$ ($\stackrel{03}{(-i)}\stackrel{12}{[+]}$, $\stackrel{03}{[+i]}\stackrel{12}{(-)}$) have as the Hermitian conjugated partners $\hat{b}^{(d,c)}_{m=1}$ ($\stackrel{03}{(+i)}\stackrel{12}{[+]}$, $\stackrel{03}{[+i]}\stackrel{12}{(+)}$), respectively.

The vacuum state for the $d = (3+1)$ case is correspondingly:
$(\frac{1}{\sqrt{2}})^2\,(\stackrel{03}{[-i]}\stackrel{12}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]} + \stackrel{03}{[+i]}\stackrel{12}{[-]} + \stackrel{03}{[-i]}\stackrel{12}{[+]})$.

Embedding $\hat{b}^{b\dagger}_{m=3}$ ($=\stackrel{03}{[-i]}\,\stackrel{12}{(+)}$) into odd part of Table 12.1 the creation operator extends into $\stackrel{03}{[-i]}\,\stackrel{12}{(+)}\,\stackrel{12}{[-]}$, manifesting in $d = (3+1)$ the charge $-\frac{1}{2}$.

### 12.2.3 $\gamma^a$ matrices in $d = (3+1)$

There are $2^4 = 16$ basic states in $d = (3+1)$, presented in Table 12.2. They all can be found as well as a part of states in Table 12.1 with either nilpotent or projector, expressing the charge, added. We make a choice of products of nilpotents and projectors, which are eigenstates of the Cartan subalgebra operators, Eq. (12.6), as presented in Eqs. (12.7).

The family members of a family are reachable by either $S^{ab}$ or by $\gamma^a$, and represent twice two vectors of definite handedness $\Gamma^{(d)}$ in $d = (3+1)$. Different families are reachable by either $\tilde{S}^{ab}$ or by $\tilde{\gamma}^a$. Each state carries correspondingly

| | $\psi^f_m$ | $\gamma_0\,\psi^f_m$ | $\gamma_1\,\psi^f_m$ | $\gamma_2\,\psi^f_m$ | $\gamma_3\,\psi^f_m$ | $\tilde{\gamma}_0\,\psi^f_m$ | $\tilde{\gamma}_1\,\psi^f_m$ | $\tilde{\gamma}_2\,\psi^f_m$ | $\tilde{\gamma}_3\,\psi^f_m$ | $S^{03}$ | $S^{12}$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\Gamma^{3+1}$ | $\tilde{\Gamma}^{3+1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\psi^a_1$ | $(+i)(+)$ | $\psi^a_3$ | $\psi^a_4$ | $i\psi^a_4$ | $\psi^a_3$ | $-i\psi^c_1$ | $-i\psi^d_1$ | $\psi^d_1$ | $-i\psi^c_1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| $\psi^a_2$ | $[-i][-]$ | $\psi^a_4$ | $\psi^a_3$ | $-i\psi^a_3$ | $-\psi^a_4$ | $i\psi^c_2$ | $i\psi^d_2$ | $-\psi^d_2$ | $i\psi^c_2$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| $\psi^a_3$ | $[-i](+)$ | $\psi^a_1$ | $-\psi^a_2$ | $-i\psi^a_2$ | $-\psi^a_1$ | $i\psi^c_3$ | $i\psi^d_3$ | $-\psi^d_3$ | $i\psi^c_3$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $-1$ | 1 |
| $\psi^a_4$ | $(+i)[-]$ | $\psi^a_2$ | $-\psi^a_1$ | $i\psi^a_1$ | $\psi^a_2$ | $-i\psi^c_4$ | $-i\psi^d_4$ | $\psi^d_4$ | $-i\psi^c_4$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $-1$ | 1 |
| $\psi^b_1$ | $[+i][+]$ | $\psi^b_3$ | $\psi^b_4$ | $i\psi^b_4$ | $\psi^b_3$ | $-i\psi^d_1$ | $i\psi^c_1$ | $\psi^c_1$ | $i\psi^d_1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | 1 | 1 |
| $\psi^b_2$ | $(-i)(-)$ | $\psi^b_4$ | $\psi^b_3$ | $-i\psi^b_3$ | $-\psi^b_4$ | $i\psi^d_2$ | $-i\psi^c_2$ | $-\psi^c_2$ | $-i\psi^d_2$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | 1 | 1 |
| $\psi^b_3$ | $(-i)[+]$ | $\psi^b_1$ | $-\psi^b_2$ | $-i\psi^b_2$ | $-\psi^b_1$ | $i\psi^d_3$ | $-i\psi^c_3$ | $-\psi^c_3$ | $-i\psi^d_3$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-1$ | 1 |
| $\psi^b_4$ | $[+i](-)$ | $\psi^b_2$ | $-\psi^b_1$ | $i\psi^b_1$ | $\psi^b_2$ | $-i\psi^d_4$ | $i\psi^c_4$ | $\psi^c_4$ | $i\psi^d_4$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-1$ | 1 |
| $\psi^c_1$ | $[+i](+)$ | $\psi^c_3$ | $-\psi^c_4$ | $-i\psi^c_4$ | $\psi^c_3$ | $i\psi^a_1$ | $i\psi^b_1$ | $-\psi^b_1$ | $-i\psi^a_1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | 1 | $-1$ |
| $\psi^c_2$ | $(-i)[-]$ | $\psi^c_4$ | $-\psi^c_3$ | $i\psi^c_3$ | $-\psi^c_4$ | $-i\psi^a_2$ | $-i\psi^b_2$ | $\psi^b_2$ | $i\psi^a_2$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | 1 | $-1$ |
| $\psi^c_3$ | $(-i)(+)$ | $\psi^c_1$ | $\psi^c_2$ | $i\psi^c_2$ | $-\psi^c_1$ | $-i\psi^a_3$ | $-i\psi^b_3$ | $\psi^b_3$ | $i\psi^a_3$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-1$ | $-1$ |
| $\psi^c_4$ | $[+i][-]$ | $\psi^c_2$ | $\psi^c_1$ | $-i\psi^c_1$ | $\psi^c_2$ | $i\psi^a_4$ | $i\psi^b_4$ | $-\psi^b_4$ | $-i\psi^a_4$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $-1$ | $-1$ |
| $\psi^d_1$ | $(+i)[+]$ | $\psi^d_3$ | $-\psi^d_4$ | $-i\psi^d_4$ | $\psi^d_3$ | $i\psi^b_1$ | $-i\psi^a_1$ | $-\psi^a_1$ | $i\psi^b_1$ | $\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | 1 | $-1$ |
| $\psi^d_2$ | $[-i](-)$ | $\psi^d_4$ | $-\psi^d_3$ | $i\psi^d_3$ | $-\psi^d_4$ | $-i\psi^b_2$ | $i\psi^a_2$ | $\psi^a_2$ | $-i\psi^b_2$ | $-\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | 1 | $-1$ |
| $\psi^d_3$ | $[-i][+]$ | $\psi^d_1$ | $\psi^d_2$ | $i\psi^d_2$ | $-\psi^d_1$ | $-i\psi^b_3$ | $i\psi^a_3$ | $\psi^a_3$ | $-i\psi^b_3$ | $-\frac{i}{2}$ | $\frac{1}{2}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1$ |
| $\psi^d_4$ | $(+i)(-)$ | $\psi^d_2$ | $\psi^d_1$ | $-i\psi^d_1$ | $\psi^d_2$ | $i\psi^b_4$ | $-i\psi^a_4$ | $-\psi^a_4$ | $i\psi^b_4$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $\frac{i}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1$ |

**Table 12.2.** In this table $2^d = 16$ vectors, describing internal space of fermions in $d = (3+1)$, are presented. Each vector carries the family member quantum number $f = (a, b, c, d)$ — determined by $S^{03}$ and $S^{12}$, Eqs. (12.7) — and the family quantum number $m$ — determined by $\tilde{S}^{03}$ and $\tilde{S}^{12}$, Eq. (12.6). Vectors $\psi^f_m$ are obtained by applying $\hat{b}^{f\dagger}_m$ on the vacuum state. Vectors — that is the family members of any family — split into even (they are sums of products of even number of $\gamma^a$'s) and odd (they are sums of products of odd number of $\gamma^a$'s). If these vectors are embedded into the vectors of $d = (5+1)$ (by gaining the appropriate nilpotent or projector), they gain charges. The Clifford odd parts of vectors are marked, entering into Table 12.1.

quantum numbers of the two kinds of the Cartan subalgebra. In Table 12.2 also $\Gamma^{(3+1)}$ $(= -4iS^{03}S^{12})$ and $\tilde{\Gamma}^{(3+1)}$ $(= -4i\tilde{S}^{03}\tilde{S}^{12})$ are presented.

When once the basic states are chosen and Table 12.2 is made it is not difficult to find the matrix representations for the operators ($\gamma^a$, $S^{ab}$, $\tilde{\gamma}^a$, $\tilde{S}^{ab}$, $\Gamma^{(3+1)}$, $\tilde{\Gamma}^{(3+1)}$). They are obviously $16 \times 16$ matrices with a $4 \times 4$ diagonal or off diagonal or partly diagonal and partly off diagonal substructure.

Let us define, to simplify the notation, the unit $4 \times 4$ submatrix and the submatrix with all the matrix elements equal to zero as follows

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12.15}$$

We also use $(2 \times 2)$ Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.16}$$

It is easy to find the matrix representations for $\gamma^0$, $\gamma^1$, $\gamma^2$ and $\gamma^3$ from Table 12.2

$$\gamma^0 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.17}$$

$$\gamma^1 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.18}$$

$$\gamma^2 = \begin{pmatrix} \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} & 0 & 0 & 0 \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.19}$$

$$\gamma^3 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} \end{pmatrix}, \tag{12.20}$$

manifesting the $4 \times 4$ substructure along the diagonal of $16 \times 16$ matrices.

The representations of the $\tilde{\gamma}^a$ do not appear in the Dirac case. They manifest the off diagonal structure as follows

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix} \tag{12.21}$$

$$\tilde{\gamma}^1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.22}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.23}$$

$$\tilde{\gamma}^3 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}. \tag{12.24}$$

Matrices $S^{ab}$ have again along the diagonal the $4\times 4$ substructure, as expected, manifesting the repetition of the Dirac $4 \times 4$ matrices, up to a phase, since the Dirac $S^{ab}$ do not distinguish among families.

$$S^{01} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^1 & 0 \\ 0 & \frac{i}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^1 & 0 \\ 0 & -\frac{i}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}, \tag{12.25}$$

$$S^{02} = \begin{pmatrix} \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^2 & 0 \\ 0 & -\frac{i}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{i}{2}\sigma^2 & 0 \\ 0 & \frac{i}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{12.26}$$

$$S^{03} = \begin{pmatrix} \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{i}{2}\sigma^3 & 0 \\ 0 & -\frac{i}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{12.27}$$

$$S^{12} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^3 & 0 \\ 0 & \frac{1}{2}\sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{12.28}$$

$$S^{13} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^2 & 0 \\ 0 & -\frac{1}{2}\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{12.29}$$

$$S^{23} = \begin{pmatrix} \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\frac{1}{2}\sigma^1 & 0 \\ 0 & -\frac{1}{2}\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \frac{1}{2}\sigma^1 & 0 \\ 0 & \frac{1}{2}\sigma^1 \end{smallmatrix} \end{pmatrix}. \tag{12.30}$$

$$\Gamma^{3+1} = -4iS^{03}S^{12} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{12.31}$$

The operators $\tilde{S}^{ab}$ have again off diagonal $4 \times 4$ substructure, except $\tilde{S}^{03}$ and $\tilde{S}^{12}$, which are diagonal.

$$\tilde{S}^{01} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.32}$$

$$\tilde{S}^{02} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.33}$$

$$\tilde{S}^{03} = \begin{pmatrix} \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \end{pmatrix}, \tag{12.34}$$

$$\tilde{S}^{12} = \begin{pmatrix} \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \end{pmatrix}, \tag{12.35}$$

$$\tilde{S}^{13} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{i}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{i}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \frac{i}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{12.36}$$

$$\tilde{S}^{23} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{1}{2}\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{12.37}$$

$$\tilde{\Gamma}^{3+1} = -4i\tilde{S}^{03}\tilde{S}^{12} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} . \tag{12.38}$$

## 12.3 Conclusions

We present in this contribution the matrix representations of operators applying on the basis, defined by the creation and annihilation operators in $d$-dimensional Clifford space — $d = 2(2n+1)$, or $4n$, $n$ is a positive integer. We make a choice of $d = (3+1)$ and $d = (5+1)$.

Creation and annihilation operators, which define the vector space, are in our case products of nilpotents and projectors (applying on the vacuum state, Eq. (12.9)), which are eigenvectors of the Cartan subalgebra, Eq. (12.6), of the Lorentz algebra of $S^{ab}$, as well as of the corresponding Cartan subalgebra, Eq. (12.6), of the Lorentz algebra of $\tilde{S}^{ab}$. Creation and annihilation operators are Hermitian conjugated to each other. We make a choice of the creation operators by choosing the vacuum state, Eq. (12.9), to be the sum of the Clifford odd (they are superposition of an odd number of $\gamma^a$'s) annihilation operators multiplying their Hermitian conjugated partners from the left hand side.

$S^{ab}$ generate $2^{\frac{d}{2}-1}$ family members of a particular family of an odd Clifford character, $\tilde{S}^{ab}$ generate the corresponding $2^{\frac{d}{2}-1}$ families. The Hermitian conjugation determines their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ partners (which are reachable also by $\gamma^a\tilde{\gamma}^a$). The Clifford even representations follow from the odd $2^{d-1}$ vectors by the application of $\gamma^a$'s or $\tilde{\gamma}^a$'s. There are correspondingly $2^d$ vectors in $d$-dimensional space $(d = 2(2n+1), 4n)$.

The Clifford even operators keep the Clifford character unchanged. $\gamma^a$'s and $\tilde{\gamma}^a$'s change the Clifford character of vectors — from odd to even or opposite.

Embedding $SO(3+1)$ into $SO(d)$, $d > (3+1)$, $d$ even, spins in $d \geq (5+1)$ manifest in $d = (3+1)$ as charges.

One can check that the creation operators of an odd Clifford character and their Hermitian conjugated partners, applied on the vacuum state, Eq.(12.9), fulfill the anticomutation relations for the second quantized fermions, Eq. (12.10), postulated by Dirac, what explains the Dirac's second quantization postulates.

One can also observe the appearance of families, used in the *spin-charge-family* theory for the explanation of families of quarks and leptons [3–5].

In this contribution the matrix representations for operators ($\gamma^a$'s, $S^{ab}$'s, $\tilde{\gamma}^a$'s, $\tilde{S}^{ab}$'s) are presented for the basis in which creation operators are eigenstates of the Cartan subalgebras of both kinds, Eq. (12.7). It is discussed how do Clifford odd and even products of nilloptents and projectors in $(3+1)$ become a part of creation and annihilation operators of an odd Clifford character in $d = (5+1)$, manifesting the spin in $a = (5,6)$ as the charge in $d = (5+1)$.

There are $2^4 = 16$ basic vectors in $d = (3+1)$ and correspondingly all the matrices have dimension $16 \times 16$, which are for the operators, determined by $\gamma^a$'s, by diagonal and for the operators, determined by $\tilde{\gamma}^a$'s, off diagonal. We keep the

Clifford odd and the Clifford even vectors as the basic vectors. We treat in the Clifford odd part the creation and annihilation operators as they would all define the vector space, to point out, that if space of $d = (3+1)$ is embedded into $d \geq 6$ , all the parts, even and odd contribute to the enlarged vector space as factors.

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# 13 Second Quantization as Cross Product

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**Abstract.** In the contributions [4,5] of this proceedings the new way of the second quantization of fermions is proposed, inspired by the fact that the Clifford and Grassmann algebra by themselves offer basis in internal space, presented as creation operators on the corresponding vacuum state, which together with their Hermitian conjugated annihilation partners fulfill all the requirements for the second quantized fermions, provided that the part of the basis in the ordinary space is orthogonal. In the Hilbert space of indefinite number of fermions it is assumed that each fermion has to distinguish from all the others either in ordinary or in internal space or in both spaces. The purpose of this contribution is to generalize this last requirement for either fermions or bosons.

**Povzetek.** V prispevkih [4,5] tega zbornika predstavita avtorja nov način druge kvantizacije fermionov. Cliffordov in Grassmanov prostor ponudita namreč bazo v notranjem prostoru fermionov, ki jo določajo kreacijski operatorji na vakuumskem stanju, ti pa skupaj s Hermitsko andjungiranimi operatorji (annihilacijskimi operatorji) izpolnjujejo vse Diracove zahteve za fermione v drugi kvantizaciji pod pogojem, da je baza v prostoru gibalnih količin ortogonalna. V Hilbertovem prostoru nedoločenega števila fermionov mora vsakemu fermionu ustrezati drugačen notranji prostor ali drugačna gibalna količina, V drugi kvantizaciji je Hilbertov prostor direkten produkt neskončne množice Hilbertovih prostorov za izbrano vrednost gibalne količine. Namen tega prispevka je posplošiti ta drugi del zahteve tako za fermione kot za bozone.

Keywords: second quantization, bosons, fermions, cross product

## 13.1 Introduction

We present in this contribution the possibility to make a new step in the new way of the second quantization of fermions, presented in the contributions [4,5] of this proceedings, for indefinite number of fermions and bosons.

It is the purpose of the present discussion to seek to use such a formulation of second quantized theories to generalize them to possibly quite new types of second quantization like theories. This is inspired from the type of theory put forward by one of us as being unification theory of spin, charges and families [1–5]

* H.B. Nielsen presented the talk.

**Second quantization as Cross product**

It is rather trivial and welknown that a second quantized (free) theory of bosons has a second quantized Hilbert space, that can be written as a Cartesian cross product over an (infinite) set of (smaller) Hilbert spaces, each of which is attached for example to the momentum, and tells how many particles have just this momentum.

**Simplest case: A scalar without internal degrees of freedom**

If we think of a charged scalar - like $\pi^+$ - it may be natural to even include in our "momentum" also the sign of the energy and use that as the 'factorisation **p**arameter" $\mathbf{p}$. We like to do it as abstract and general as possible, so we now use the letter $\mathbf{p}$ and you can think of it as "(factorization) **p**arameter" or as momentum as you like.

*In the* $\pi^+$ *case we take the "factorization parameter"* $\mathbf{p}$ *to be:*

$$\mathbf{p} = (\vec{p}, sign(E)). \tag{13.1}$$

**The general form as factorized space:**

The Hilbert space for the second quantized boson system can always be written like

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}. \tag{13.2}$$

In the $\pi^+$ example, where $\mathbf{p} = (\vec{p}, sign(E))$, the Hilbert space $\mathcal{H}_{\mathbf{p}}$ is actually that of harmonic oscillator for which the number operator counts the number of $\pi^+$ particles with just the $\mathbf{p}$-specification $\mathbf{p}$.

(Here we stepped too fast over the Dirac sea for bosons problem, but that is not so crucial just now; just think of antiparticles instead, when formally $sign(E) < 0$.)

**Dream of generalization(s)**

In the formulation as the Cartesian product

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}} \tag{13.3}$$

one could dream about making a new - and perhaps interesting theory - by replacing the Hilbert spaces that are factors in the Cartesian product such as $\mathcal{H}_{\mathbf{p}}$ by some Hilbert spaces with a different structure, e.g. different dimensionality.

E.g. Could we decide that all these harmonic oscillators could only be excited up to their 7th level, after that it would not be possible to put more $\pi^+$ in with a given $\mathbf{p}$ ?

We could of course postulate such a "theory" but it would be rather strange physically. *A postulate of only up to 7 particles per* $\mathbf{p}$ *would violate locality*

In a big universe particles with the same momentum are so far from each other that one cannot from locality feel if there are more or less than 7 particles in the same momentum eigenstate.

*If we use* $\tilde{\mathbf{x}}$ *instead of* $\mathbf{p}$ *then locality would be automatic.*

If one thinks of a discretized (d-1)-space, i.e. really a (perhaps a bit irregular) lattice, and take the state of the universe to be described by the a state in the Hilbert space $\mathcal{H}$, then factorization of the type

$$\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}} \tag{13.4}$$

i.e. where we as "factorization parameter" use the spatial position $\vec{x}$ - the lattice point, if discretized - this Cartesian product would be automatically suited for locality, one should just only provide it with local interaction, but could for the structure and operators acting on the single factors $\mathcal{H}_{\vec{x}}$ be very free since everything would be o.k..

### Usual second quantization for the Norma's *spin-charge-family* theory

Once one has decided on the inner degrees of freedom, the statistics – fermion or boson – and of dimension of space time and thus of the dimension of the momentum vectors, one would than think that *there is only one way to second quantize.*

This way will then turn out in the boson case to indeed be of the form that the full second quantized Hilbert space $\mathcal{H}$ takes the product form, and thus *be written in the product way*.

However, if one starts by a product form and has not gotten it via the standard procedure, then we would feel a priori unsafe if this would be a physically meaningful way or not.

It probably depends strongly on the details.

### A couple of trivialities on component numbers

**i.** A Dirac (rather Weyl) massless spinor in an even number d of space time dimensions has $2^{\frac{d}{2}-1}$ components.

**ii.** In Norma's *spin-charge-family* theory ([3] and the references therein) there is not only the usual Dirac spin index with $2^{\frac{d}{2}-1}$ components, but a quite analogous family index again with the $2^{\frac{d}{2}-1}$ components. So in this model the number of components could be marked by two Dirac indices, or instead using another but equivalent formalism with projection and nilpotent "operators". But in any case of these two formalisms the number of components for a full fermion particle is the square of the number for an ordinary Dirac construction. The number of components is therefore $2^{d-2}$. One can learn in Ref. [4,5] in this proceedings that:

**a.** Only operators of an odd character can offer the second quantization fermions.

**b.** The operators of an odd character split into two parts, Hermitian conjugated to each other.

**iii.** If we ignore momentum and look at one single momentum only, then the number of different states one could produce by having for this single momentum various possible numbers with the $2^{d-2}$ different components filled or unfilled would be $2^{2^{d-2}}$. Let us add that the rest of possibilities belong to either the Hermitian conjugated partners or have the evenness Character and do not fulfil the anticommutation relations for fermions (and probably even not for bosons. In any case the number is much much more than the number of components.

### Standard second quantization procedure in factor language

Before telling this standard procedure of quantizing fermions by the factorization into the Cartesian product of "subHilbert" spaces, we have to admit that one *cannot* do that without some essential modification, which we though postpone to discuss below in the section called "The problem of fermions".

However, we are for the moment interested in reaching to the point, where we can see the problems when one attempts to make a new way of second quantizing by postulating some algebraic structure for the operators acting on the "subHilbert" spaces $\mathcal{H}_{\mathbf{p}}$ going into the Cartesian product. For this problem presumably the statistics being fermion or boson statistics may however not matter so much, so our postponing is not so crucial for that.

**iv.** Let us first look for a fixed momentum **p** and calculate which states are needed to describe the possibilities for filling with the allowed number of particles (up to one for fermions, and up to infinity for bosons) all the internal states.

**v.** Then we construct the Hilbert space $\mathcal{H}_{\mathbf{p}}$,of which is just the number of different ways of filling particles into the different combinations of internal states.

**vi.** Then finally you can take the Cartesian product and get the genuine Hilbert space for the full second quantized theory.

### Standard way dim($\mathcal{H}_{\mathbf{p}}$) = $2^{2^{d-2}}$ for Norma's theory.

Since there are $(2^{\frac{d}{2}-2})^2$ component combinations, namely say $2^{\frac{d}{2}-2}$ genuine Dirac components, and $2^{\frac{d}{2}-2}$ family index values, there for assumed fermion-statistic $2^{2^{d-2}}$ possibilities for filling or not filling these $2^{d-2}$ difference internal states.

Thus the Hilbert-space for only one momentum should have the dimension

$$\dim(\mathcal{H}_{\mathbf{p}}) = 2^{2^{d-2}} \,. \tag{13.5}$$

(*Notice that this space $\mathcal{H}_{\mathbf{p}}$ thus has a much bigger dimension than the space of single particle internal states, which has only dimension* = $2^{d-2}$ . )

**We ignored at first equations of motion.**

We have to modify the above simplified proposal by:
**vii.** Notice that using the momentum energy relation

$$E^2 - \vec{p}^2 = 0 \tag{13.6}$$

we have for each (d-1)-momentum $\vec{p}$ **two** values for the energy $E$ of the particle, so that we should let, as already mentioned, as a possibility

$$\mathbf{p} = (\vec{p}, E)\,, \tag{13.7}$$

meaning a doubling of the space of momenta to be used.

**viii.** Let us take into account that the (free) equation of motion (the Dirac equation, the Weyl equation indeed) for a choice of energy $E = \pm\sqrt{\vec{p}^2}$ only allow a subspace of the internal space of states for the (single) particle,

$$(\not{p})\psi = 0\,. \tag{13.8}$$

**Standard second quantization as product over** $(\vec{p}, sign(E))$**.**

Letting an index $_{emr}$ denote that we have restricted the single particle sates to the states obeying the equations of motion (emr = "equation of motion restricted") we write the true standard second quantized Hilbert space

$$\mathcal{H}_{emr} = \bigotimes_{(\vec{p},sign(E))} \mathcal{H}_{(\vec{p},sign(E)),emr}, \tag{13.9}$$

where now $\mathcal{H}_{(\vec{p},sign(E)),emr}$ is constructed from space of single particle internal states obeying the Dirac equation and having $E = sign(E)\sqrt{\vec{p}^2}$, which because of the restriction by the equation of motion has only half the dimensionality of $2^{d/2-1}$ in the simple Dirac case or half of $2^{d-2}$ in the case with families. So

$$\dim(\mathcal{H}_{(\vec{p},sign(E)),emr}) = 2^{2^{d-1}/2} = 2^{2^{d-2}}. \tag{13.10}$$

## 13.2 The problem of fermions

**Yet a problem for Cartesian product form for fermions.**

For just constructing the Hilbert space we could claim that this Cartesian product procedure is o.k. even for fermions, but for the *creation and annihilation operators or the field operators for fermions there is a problem more:*

If we take a true Cartesian product and let it be understood that the creation and annihilation operators for a state with $(\vec{p}, sign(E)) = \mathbf{p}$ *alone shall act on the Cartesian product factor* $\mathcal{H}_{\mathbf{p}}$*, then we cannot make such fermion creation or annihilation operators for different* $\mathbf{p}$ *anticommute!* Operators acting alone on different Cartesian product factors will namely always commute.

**Suggested trick to solve the anticommutation problem:**

*Use operators* $(-1)^{F_{\mathbf{p}}}$, where $F_{\mathbf{p}}$ is the fermion number for the fermions in the Cartesian factor $\mathcal{H}_{\mathbf{p}}$.

That is to say to construct the "true creation or annihilation operators" – $b^\dagger(i;\mathbf{p})$ or $b(i;\mathbf{p})$ – for the $\mathbf{p}$ Cartesian factor we modify the truly "local ones", $c^\dagger(i;\mathbf{p})$ and $c(i;\mathbf{p})$ defined so as to *only act on the Cartesian factor* $\mathcal{H}_{\mathbf{p}}$, not touching the other factors, by multiplying it with a lot of factors of the form $(-1)^{F_{\mathbf{p}'}}$.

Associate in fact to each essentially momentum $\mathbf{p}$ a subset of this kind of essential momenta $\mathbf{B}(\mathbf{p})$ and define

$$b^\dagger(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} (-1)^{F_{\mathbf{p}'}}\right] c^\dagger(i;\mathbf{p}) \tag{13.11}$$

$$b(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} (-1)^{F_{\mathbf{p}'}}\right] c(i;\mathbf{p}) \tag{13.12}$$

## 13.3 Dream of Algebra

Although we for fermions must introduce the modification from $c^\dagger(i;\mathbf{p})$ to $b^\dagger(i;\mathbf{p})$ in order to achieve the anticommutativity of the annihilation operators $b(i;\mathbf{p})$, when we build up the Hilbert space construction from a Cartesian product, we might dream of using this Cartesian product idea to make a generalization of the algebra for the operators acting on one of these Hilbert spaces $\mathcal{H}_{\mathbf{p}}$ (we could call them factor-Hilbert spaces) from which the Cartesian product is made up to a more general algebra, say $\mathbf{F}$. That is to say we imagine an algebra $\mathbf{F}$ consisting of operators acting on the Hilbert space $\mathcal{H}_{\mathbf{p}}$.

We can easily think of e.g. a couple operators/elements $f, g \in \mathbf{F}$, which e.g. anticommute $\{f, g\}_+ = 0$. Of course we shall then have such algebra elements for every factor-Hilbert-space $\mathcal{H}_{\mathbf{p}}$, and correspondingly we should of course distinguish analogous algebra elements related to different factor-Hilbert-spaces or equivalently different $\mathbf{p}$ as we decided to enumerate these factor-Hilbert-spaces. That is to say we should write $f(\mathbf{p})$ for the operator of a given structure in $\mathbf{F}$ when it acts on $\mathcal{H}_{\mathbf{p}}$.

But now if we do not even make the modification of inserting the $(-1)^{F_{\mathbf{p}'}}$-factors when in the ordering we had to have $\mathbf{p}$ and $\mathbf{p}'$ were in a certain relative order - say $\mathbf{p}' < \mathbf{p}$ - then of course any $f(\mathbf{p})$ and any $g(\mathbf{p})$ at one $\mathbf{p}$ will commute with any $f(\mathbf{p}')$ and any $g(\mathbf{p}')$ at **another** "momentum" $\mathbf{p}' \neq \mathbf{p}$, independent of how $f$ and $g$ for the same $\mathbf{p}$ may happen to commute or anticommute.

In other words **we cannot prevent the commutation due to independent factor-Hilbert-spaces for the operators**, what ever we take the local algebra to be, i.e. it does not modify this commutation to let the operators say anticommute locally, it does not help even if say $\{f(\mathbf{p}), g(\mathbf{p})\}_+ = 0$ to prevent $[f(\mathbf{p}), g(\mathbf{p}')] = 0$ for $\mathbf{p} \neq \mathbf{p}'$.

### 13.3.1 Even with $(-1)^{F_\mathbf{p}}$-factors

Even if we improve our purely Cartesian product construction with the $(-1)^{F_\mathbf{p}}$-factors as above, it will not bring us to get the commutation or anticommutation to progress from the "local" to the inter **p** commutator or anticommutator so easily. If we indeed include the type of factor (from (13.11,13.12) ) being the product over the factors $(-1)^{F_{\mathbf{p}'}}$ for all $\mathbf{p}'$ which are say "smaller" in the ordering than the **p** considered, then we will achieve that we get anticommutation all operators $g(\mathbf{p})$ say at **p** with all the ones at another place $\mathbf{p}'$ provided both operators carry a fermion number in the sense that they shift the value of the fermion number $F_\mathbf{p}$ for their factor Hilbert space by their action. So if e.g. two operators are fermionic in this quantum number $F$ sense and even if they commuted when at the same site, they will anticommute when they are at different sites. If oppositely they anticommute locally they will again anticommute when at different sites(= different **p**'s).

The conclusion from the remarks just above should be:

Using the starting point of the Cartesian product and only modifying by the extra factor of the type from equations (13.11,13.12) the commutation versus anticommutation of operators associated with **different p**-values depend alone on:

**a.** the fermion number of the operators,

**b.** from whether one introduce transformation (13.11, 13.12) above at all or not. But it does not depend on on how the algebra elements considered may commute or not in the "local algebra" i.e. for the **same p**-value.

### 13.3.2 More generally:

The above proposed method for making fermion-fields on the basis of a Cartesian product by means of an ordering of all the **p**-values is really not very attractive. In fact such an ordering does not match well with the topological structure of a momentum space or a position space except for the spatial dimension being $d_{spatial} = 1$. In higher dimensions you rather have to use the axiom of choice to even see that there exists such an ordering. We also need such a construction if we would like to make fermionization, and then this only by axiom of choice found ordering would not seem attractive at all either.

So attempting to generalize this method of constructing fermion fields from a Cartesian product is highly called for.

Now if there is in the theory some sort of gauge freedom one might not require quite as strict the properties of the extra factors introduced to convert the a priori commuting fields appearing from operators acting on different factor-Hilbert-spaces from (13.11.13.12). If one allows more freedom in the construction then one might optimistically hope to construct such factors to convert the boson-commuting operators into fermion ones to have some continuity and thus compatibility with the topology of a higher dimensional space(than just dimension =1).

We here at first write down the type of transformation to be made to construct fermions from commuting fields in a general way. Then one may investigate how much one needs to require about the multiplying factors $U(\mathbf{p},\mathbf{p}')$ converting the bosons to fermions so to speak.

Unfortunately we have not come far in developing these conditions, but just the thought of looking at it more generally might turn out useful:

$$
\begin{aligned}
b^{\dagger}(i;\mathbf{p}) &= \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')^{\dagger}\right] c^{\dagger}(i;\mathbf{p})\\
b(i;\mathbf{p}) &= \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')\right] c(i;\mathbf{p})
\end{aligned}
$$

**Not even crudely local $b^{\dagger}(\vec{x})$ unless the modification by $U(\vec{x},\vec{x}')$ inessential.**

So there should preferably be a "gauge" transformation which could be the effect of the modification $U(\vec{x},\vec{x}')$ or "jump over correction"-replacement.

Natural that the $U(\vec{x},\vec{x}')$ depends on the direction from $\vec{x}$ to $\vec{x}'$, and thus is a function of a point on thee sphere $S^{d-2}$.

Also the 'gauge"like modifications must lie in a group $G$. So need map $S^{d-2} \to G$.

### 13.3.3 Anyons

To exercise constructing other statistics than bosons from the Cartesian product one would of course like to exercise with two spatial dimensions because this is the first case after the one spatial dimension case in which there are essentially no problem and fermionization is already well done. But now just 2 spatial dimensions is the interesting case in which also Leinaas Myhrheim or anyon statistics is possible[6].

With the suspicion of the gauge symmetries being important in allowing a more developed choice of the conversion factors $U(\mathbf{p},\mathbf{p}')$ a first exercise might be to even construct a system of anyons or first just a pair by electromagnetic ingredients.

## Acknowledgement

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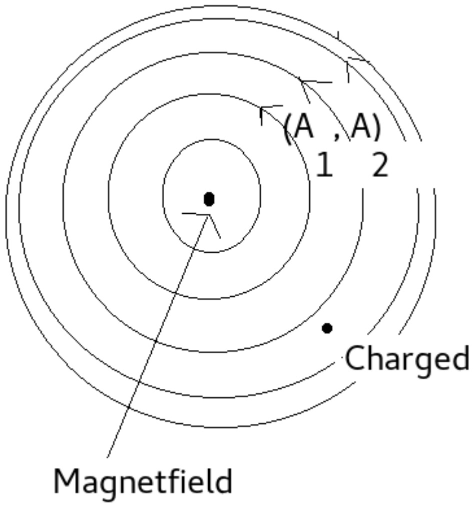


**Fig. 13.1.** Anyons as electric magnetic made.

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# 14 Novel String Field Theory and Remaining Problems ⋆

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**Abstract.** We review our Novel String Field theory, and then it is pointed out very important remaining problem.

**Povzetek.** Avtorja na kratko predstavita njuno Novo teorijo polj s strunami in obravnavata ključen problem te teorije.

Keywords: string field theory

## 14.1 Introduction

It has been shown that Super string Theories loop corrections constructed in the theories do not lead to ultraviolet divergences, contrary to the conventional field theories. Nevertheless they have so good physical properties, e.g. revealing Regge trajectories. Thus they are considered serious candidates for the theory of Nature.

From the point of view of the Novel String Field theory advocated by H.B. Nielsen and M. Ninomiya[1], they can considered the strings composite from an infinite number of what are called "objects" – to some extend similar to C. Thorn's string bits[2]. But they deviate by the fact that the objects correspond to a description of the right and left variables on the string $\tau - \sigma$ and $\tau + \sigma$ respectively, while C. Thorn rather discretizes the $\sigma$ variable.

We figure out that in the field theory typical diagrams of 2 particles $\rightarrow$ 2 particles scattering processes in perturbation as Fig 14.1.

If we took the particles to be closed strings the usual string theory formulation would lead to the corresponding string pictures in Fig 14.2. But it happens that if we consider the particles in Fig. 14.1 open strings, then in our formalism with cyclically ordered chains of objects, the second line could actually also represent the topological structure of the developments in our formalism (for open strings then).

In the string theory the diagrams of the first few diagram of 2 closed strings $\rightarrow$ 2 closed strings processes are given by Fig2 above.

⋆ Talk presented by H.B.N. at 22nd Bled workshop July 6. – 14., 2019, Bled, Slovenia

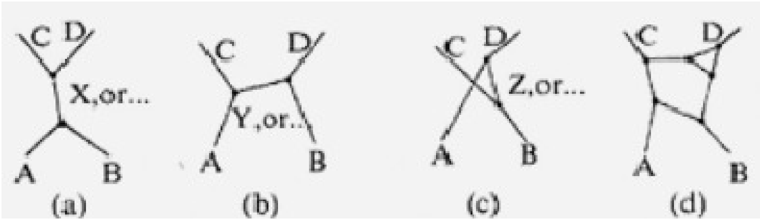


**Fig. 14.1.** Feynman diagrams for scatterings; Incoming particles are denoted as A and B, while intermediate ones are denoted as X, Y, Z $\cdots$ respectively.

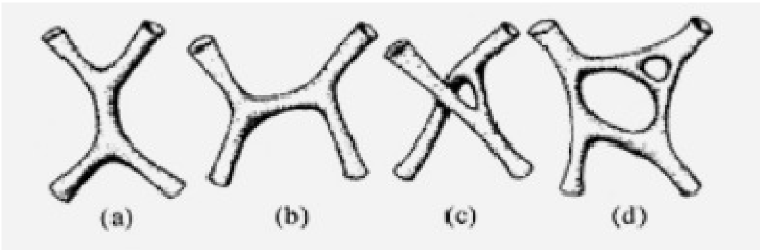


**Fig. 14.2.** The topological structure of the developments in our formalism (for open strings).

String theory actually avoids the ultraviolet divergences even in the loop corrections such as (d), corresponding to the loop corrections (c) and (d). In quantum field theory one has such divergences. However, in the string theories the loop corrections, e.g. (d′) falls off exponentially with a squared of the external momenta expression.

## 14.2 Analogy with parton model.

One can consider the string as composed objects of infinitely many constituents such as partons [3]. Thus they have Bjorken's [4] variable $x = 0$.

In deep inelastic scattering one often uses the concept that a hadron (e.g. proton) is composed of partons as a bound state; see e.g. Fig 14.3.

When Bjorken $x$ is non-zero one can obtain for sufficiently high collision energy large transverse momenta — jets — for scattering of constituent partons with $x \neq 0$.

Such scattering could again cause ultraviolet divergences, so to realize our analogy of getting rid of ultraviolet divergences for the bound, we should usu bound states with all Bjorken variables $x = 0$.

## 14.3 Some characteristic features of the novel string field theory

Our novel SFT [1] is a kind of string-bit theory similar to that of C.Thorn [2]; but we use the right moving and left moving fields $X_R$ and $X_L$ respectively. And that each of them are functions of the variables $\tau - \sigma$ and $\tau + \sigma$, contrary to those of Thorn who uses the genuine string parameter $\sigma$.

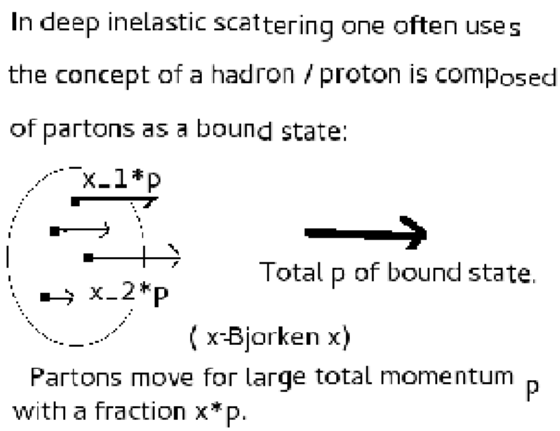


**Fig. 14.3.** The constituents i = 1, 2,.. carry the longitudinal momentum x_i*p where the longitudinal momentum of the total bound state is p.

Thus our constituents objects are associated rather with wave packets running along the string back and forth.

It turns out that our constituents equal to objects do not change at all. Thus scattering is exchange of objects rather than interpreted as collisions of the objects.

Other aspects of our SFT [1] is the following:

At first straight and resting string, you may produce a wave-packet in just one direction until it reflects at the end, and run back (see Fig 14.4 ).

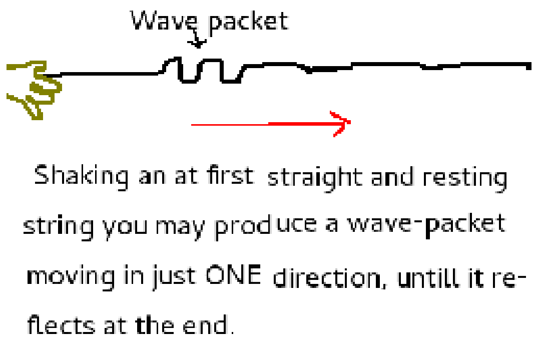


**Fig. 14.4.** Producing a wave packet.

The whole way around in a period would correspond to a run both forth and back and thus have the topology of $S^1$: see Fig 14.5.

## 14.4 One of the great points in our novel SFT: Objects do not change

We now stress that one of the great points of our novel SFT is that it corresponds to the fact that the wave packets run along the string without any change, we

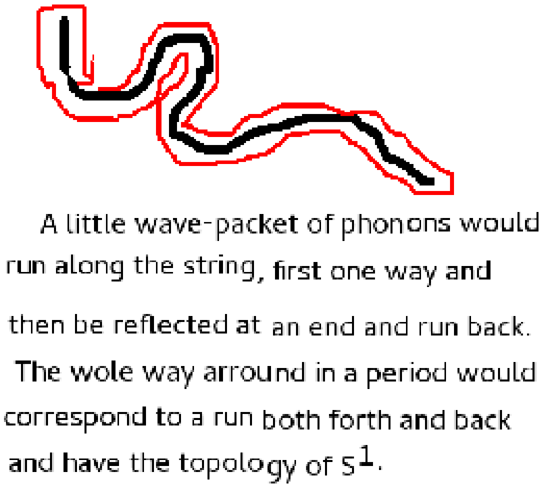


**Fig. 14.5.** The topology of $S^1$.

arrange that our objects - which describe these wave structures as moving along - do not change in time at all.

Thus our description of several string theory (= a string field theory has no development in the object formulation.The string theory is so to speak solved in terms of objects.

This is the great hallmark of our novel SFT: Nothing moves.

All scattering (etc.) is fake.

To form the cyclically ordered chains of objects corresponding to moving forth and backward along the open string we need a cyclic ordering of a series of objects. We could describe that by a successor function f that is mapping one object to the next one in the chain:

$$f(\mathrm{object}_1) = \mathrm{object}_1 + 2(\mathrm{mod} N) \tag{14.1}$$

(we only consider, due to a technical detail, with an even number in the cyclic chain series)

## 14.5 Conclusion and future outlook

We have constructed a String Field theory called “Novel String Field theory” by using objects. The strings are in our theory considered as bound states of several objects.

In our theory we can derive the Veneziano amplitude with recourse to exchange objects between incoming strings.

However we did not introduce the possibility for objects to annihilate and create will be our subject to be investigated in our String Field theory.

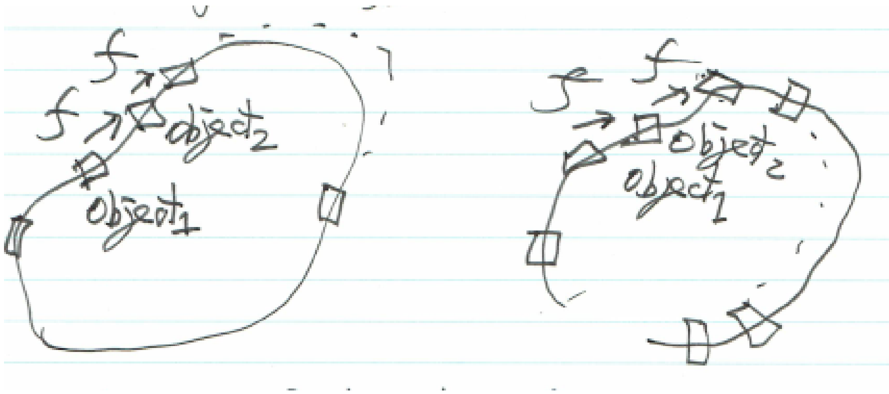


**Fig. 14.6.** Cyclicaly ordered chains($\rightarrow$ indicates f map).

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# 15 Local Temperature Distribution in the Vicinity of Gravitationally Bound Objects in the Expanding Universe ⋆

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**Abstract.** We consider a cluster of Primordial Black Holes which is decoupled from the cosmological expansion (Hubble flow) and this region is heated as compared to the surrounding matter. The increased temperature inside the region can be explained by several mechanisms of Primordial Black Holes formation. We study the temperature dynamics of the heated region of Primordial Black Holes cluster.

**Povzetek.** Avtorja obravnavata gručo prvotnih črnih lukenj, ki ni sklopljena s kozmološko širitvijo (Hubblovim tokom) v območju segretem glede na snov, ki območje obdaja. Povišano temperaturo lahko pojasnita z več mehanizmi nastanka prvotnih črnih lukenj. Obravnavata gibanje temperature segretega območja gruče prvotnih črnih lukenj.

## Introduction

The idea of the Primordial Black Holes (PBH) formation was predicted five decades ago [1]. Although they have not yet been identified in observations but some astrophysical effects can be attributed to PBH: supermassive black holes in early quasars. Therefore till now, PBH give information about processes in the Early Universe only in the form of restrictions on the primordial perturbations [2] and on physical conditions at different epochs. It is important now to describe and develop in detail models of PBH formation and their possible effects in cosmology and astrophysics.

There are several models of PBH formation. PBH can be formed during the collapses of adiabatic (curvature) density perturbations in relativistic fluid [3]. They could be formed as well at the early dust-like stages [4] and rather effectively on stages of dominance of dissipative superheavy metastable particles owing to a rapid evolution of star-like objects that such particles form [5]. There is also an exciting model of PBH formation from the baryon charge fluctuations [6]. Another set of models uses the mechanism of domain walls formation and evolution with the subsequent collapse [7]. Quantum fluctuations of a scalar field near a potential

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maximum or saddle point during inflation lead to the formation of closed domain walls [12]. After the inflation is finished, the walls could collapse into black holes in the final state. There is a substantial amount of the inflationary models containing a potential of appropriate shape. The most known examples are the natural inflation [8] and the hybrid inflation [9] (and its supergravity realization [10]). The landscape string theory provides us with a wide class of the potentials with saddle points, see review [11] and references within. Heating of the surrounding matter is the inherent property of the domain wall mechanism of PBH cluster formation. While collapsing the domain wall partially transfers its kinetic energy to the ambient matter. It would allow to distinguish different models by observations.

## 15.1 The first Chapman–Enskog approximation

According to the discussion above, PBH are gathering into the clusters with heated media inside them. It is assumed that after decoupling from the cosmological expansion the temperature of gas inside the cluster and its density is higher than that around the cluster. These factors can ignite a new chain of nuclear reactions changing chemical composition of the matter in given region. We are going to study the rate of temperature spreading into surrounding space and the temperature distribution within the cluster. The temperature dynamics is described by the appropriate equations in the framework of the Chapman–Enskog procedure.

The Chapman–Enskog method [13] makes it possible to obtain a solution to the transport equation and it can be applied to the relativistic transport equation in general case. The applicability condition of this method: macroscopic wavelengths should be significantly greater than the mean free path. This excludes the propagation velocity that is faster than the thermal velocity of particles [14]. Using this method, we can find linear laws for flows, thermodynamic forces and expressions for transfer coefficients based on the solution of linearized transfer equation. After that we apply this linear laws to continuity, energy and motion equations. This leads to the relativistic Navier–Stokes equations which form a closed system for hydrodynamic variables. In the first approximation various irreversible flows are linearly related to non-uniformities present in the system. In this case the relativistic generalization of the Fourier–law for the heat flux and the linear expression for the viscous pressure tensor has the form ($c = \hbar = k_B = 1$)

$$I_q^\mu = \lambda\Big(\nabla^\mu T - \frac{T}{hn}\nabla^\mu p\Big) \tag{15.1}$$

$$\Pi^{\mu\nu} = 2\eta\overline{\overset{\circ}{\nabla^\mu} u^\mu} + \eta_\nu \Delta^{\mu\nu}\nabla_\sigma u^\sigma \tag{15.2}$$

$\lambda$ – the heat conductivity, $\eta$ – the shear viscosity, $\eta_\nu$ – the volume viscosity, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ , $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and this operator acts as a projector: $\Delta^{\mu\nu}u_\nu = 0$. They are designed to select two hydrodynamic four–velocity expressions proposed by Eckart and Landau–Lifshitz. We will use the definition of Eckart [15] which relates the hydrodynamic four–velocity directly to the particle four–flow $N^\mu$

$$u^\mu = \frac{N^\mu}{\sqrt{N^\nu N_\nu}}. \tag{15.3}$$

The relativistic equation of motion and equation of energy are given by [16]

$$hnDu^{\mu} = \nabla^{\mu}p - \Delta^{\mu}_{\nu}\nabla_{\sigma}\Pi^{\nu\sigma} + (hn)^{-1}\Pi^{\mu\nu}\nabla_{\nu}p - \\ -\left(\Delta^{\mu}_{\nu}DI^{\nu}_{q} + I^{\mu}_{q}\nabla_{\nu}u^{\nu} + I^{\nu}_{q}\nabla_{\nu}u^{\mu}\right) \tag{15.4}$$

$$nDe = -p\nabla_{\mu}u^{\mu} + \Pi^{\mu\nu}\nabla_{\nu}u_{\mu} - \nabla_{\mu}I^{\mu}_{q} + 2I^{\mu}_{q}Du_{\mu}. \tag{15.5}$$

After linearization, the energy equation is reduced to

$$\frac{DT}{T} = \frac{1}{c_{\nu}}\left[\nabla_{\mu}u^{\mu} - \frac{\lambda}{p}\left(\nabla^{2}T - \frac{T}{hn}\nabla^{2}p\right)\right] \tag{15.6}$$

where we have taken into account the linear laws (15.1) and (15.2), $\nabla^2$ stands for $\nabla^2 = \nabla^{\mu}\nabla_{\mu}$ and $D = u^{\mu}\partial_{\mu}$.

If the hydrodynamic four–velocity is constant and $p = nT$ (we will see it in the next section) the energy equations reduce to the relativistic heat-conduction equation:

$$nc_{\nu}DT = -\lambda\left(\nabla^{2}T - \frac{T}{hn}\nabla^{2}p\right). \tag{15.7}$$

## 15.2 The thermodynamic values

The equilibrium distribution function with no external fields takes the form of the Juttner distribution function

$$f(p) = \frac{1}{(2\pi)^3}\exp\left(\frac{\mu - p^{\mu}u_{\mu}}{T}\right). \tag{15.8}$$

It allows to calculate the particle four–flow in equilibrium

$$N^{\mu} = \frac{1}{(2\pi)^3}\int\frac{d^3p}{p^0}p^{\mu}\exp\left(\frac{\mu - p^{\mu}u_{\mu}}{T}\right). \tag{15.9}$$

The Juttner distribution function outlines one direction in space–time. As a result, it must be proportional to four–velocity, where the proportionally factor of this relation is the particle density

$$n = \frac{1}{(2\pi)^3}\int\frac{d^3p}{p^0}p^{\mu}u_{\mu}\exp\left(\frac{\mu - p^{\nu}u_{\nu}}{T}\right). \tag{15.10}$$

The integral is a scalar and it can be calculated at selected $u^{\mu} = (1, 0, 0, 0)$. This result can be expressed in the modified Bessel function of the second kind

$$n = \frac{4\pi m^2 T}{(2\pi)^3}K_2\left(\frac{m}{T}\right)\exp\left(\frac{\mu}{T}\right). \tag{15.11}$$

We can obtain the equilibrium pressure following the same reasoning to calculate the energy–momentum tensor in equilibrium:

$$p = -\frac{1}{3}T^{\mu\nu}\Delta_{\mu\nu} = -\frac{1}{3}\int \frac{d^3p}{p^0} p^\mu p^\nu \Delta_{\mu\nu} f(p) = \tag{15.12}$$
$$= \frac{4\pi m^2 T^2}{(2\pi)^3} K_2\left(\frac{m}{T}\right) \exp\left(\frac{\mu}{T}\right) = nT\,.$$

Hence, if we identify T with the temperature of the system the standard scheme of thermodynamics could be clearly seen.

Using recurrence relation for the modified Bessel function of the second kind and taking into account particle density the expression has the form

$$e = m\,\frac{K_3(m/T)}{K_2(m/T)} - T\,. \tag{15.13}$$

Considering the result of (15.12) for pressure we can find the enthalpy per particle

$$h = e + pn^{-1} = m\,\frac{K_3(m/T)}{K_2(m/T)} - T + T = m\,\frac{K_3(m/T)}{K_2(m/T)}\,. \tag{15.14}$$

The heat capacity per particle at constant volume by definition

$$c_v = \partial e/\partial T\,. \tag{15.15}$$

We can get asymptotic behaviour of these values for large arguments of the modified Bessel function of the second kind (which corresponds to the case of low temperatures) and for small arguments (which corresponds to the case of massless particles). For small values of temperature we have the asymptotic ratio for large arguments ($w = m/T$):

$$K_n(w) \simeq \frac{1}{e^w}\sqrt{\frac{\pi}{2w}}\left[1 + \frac{4n^2-1}{8w} + \frac{(4n^2-1)(4n^2-9)}{2!(8w)^2} + \ldots\right]. \tag{15.16}$$

It allows to obtain the enthalpy per particle:

$$h = m + \frac{5}{2}T + \frac{15}{8}\frac{T^2}{m} + \ldots \tag{15.17}$$

and to derive the caloric equation of state of relativistic perfect gas (15.13) and the heat capacity per particle at constant volume (15.15):

$$e = m + \frac{3}{2}T + \frac{15}{8}\frac{T^2}{m} + \ldots \tag{15.18}$$

$$c_v = \frac{\partial e}{\partial T} = \frac{3}{2} + \frac{15}{4}\frac{T}{m} + \ldots \tag{15.19}$$

Massless particles are essential in relativistic kinetic theory. For this purpose we should expand our formulas in this special case. The results can be obtained by taking the limit in $m \to 0$ with the asymptotic relation for the modified Bessel function of the second kind:

$$\lim_{w \to 0} w^n K_n(w) = 2^{n-1}(n-1)! \tag{15.20}$$

$$e = 3T\,, \quad h = 4T\,, \quad c_v = 3\,. \tag{15.21}$$

In this case, with the caloric equation of state of relativistic gas and $p = nT$ we can obtain well-known expression for pressure for massless particles: $p = en/3$.

We can find Fourier differential equation of the heat conduction in the non–relativistic case. Following expression (15.17) in the case of low temperatures ($T \ll m$) and considering the ratio $p = nT$ we obtain

$$nc_v DT = -\lambda\Big(\nabla^2 T - \frac{T}{hn}\nabla^2 p\Big) \approx -\lambda\nabla^2 T. \tag{15.22}$$

In this case, the heat–conduction equation allows an infinite propagation velocity. Although this feature is already present in the non–relativistic theory in the relativistic theory it becomes a paradox: the thermal disturbances can not propagate faster than the speed of light. This paradox is easily resolved in the framework of the Chapman–Enskog procedure. In fact the restriction inherent in the Chapman–Enskog method (the macroscopic wave lengths has to be much greater than the mean free path) prevents the existence of propagation velocities faster than the thermal velocity of particles.

## 15.3 Thermal equilibrium

We should check the applicability of our results by estimating to what extent the electron–proton–photon plasma is close to kinetic equilibrium before and during recombination. All our previous calculations were made under the assumption that the distribution functions have equilibrium form and all components have the same temperature equal to the photon temperature. To make sure that the temperature of electron–proton component coincides with the photon temperature we have to study the following effect. The effective temperature of photons would decrease in time slower than that of electrons and protons. Thus we have to check that energy transfer from photons to electrons and protons is sufficiently fast.

Electrons get energy from photons via Compton scattering process that occurs with Thomson cross section. The time between two subsequent collisions of a given electron with photons is

$$\tau = \frac{1}{n_\gamma \sigma_T} \tag{15.23}$$

here $\sigma_T$ – the Compton cross section and $n_\gamma$ – the number density of photons. For energy transfer the time $\tau_E$ in which an electron obtains kinetic energy of the same order of magnitude as temperature due to the Compton scattering should be found. We note that the typical energy transfer in a collision of a slow electron with a low energy photon is actually suppressed for estimation of this time. The estimation for number of scattering events needed to heat up a moving electron is given by [17]

$$N \sim \left(\frac{T}{\Delta E}\right)^2 \sim \frac{m_e}{T}. \tag{15.24}$$

We can obtain the time of electron heating [17]

$$\tau_E(T) \sim N\tau(T) \sim \frac{m_e}{n_\gamma(T)\sigma_T T}. \tag{15.25}$$

At the moment of recombination $\tau_E(T_{rec}) \simeq 6$ yrs. It is much smaller than the Hubble time and energy transfer from photons to electrons is efficient. Thus electrons and protons have the photon temperature.

What about the heating of protons? Doing the same procedure (with $m_p$ substituted for $m_e$ in (25)) we obtain that process of direct interaction of proton with photons is irrelevant. Since the Thomson cross section is proportional to $m_e^{-2}$ the time for protons is larger by a factor $(m_p/m_e)^3$ and this time is larger than the Hubble time. Energy transfer to protons occurs due to elastic scattering of electrons off protons. The energy transfer time [17] is

$$\tau_E(T) \sim \frac{m_e m_p}{16\pi n_e(T)\alpha^2 \ln(6Tr_D/\alpha)}\left(\frac{3T}{m_e}\right)^{3/2} \tag{15.26}$$

here $r_D = \sqrt{\dfrac{T}{4\pi n_e \alpha}}$ and during recombination $\tau_E(T_{rec}) \sim 10^4$ s and this time is very small compared to the Hubble time at recombination. The estimation done for protons is valid for electrons as well with $m_e$ substituted for $m_p$ and numeric factor. This means that electrons and protons have equilibrium distribution functions with temperature equal to photon temperature.

## 15.4 Transport coefficients

The divergence of the collision integrals is the main difficulty encountered when applying the transport equation to plasma. The many particle correlations which provide the Debye shielding are not included in the transport equation due to the long range nature of electromagnetic interaction. In the Standard Model of the Universe Compton scattering between photons and electrons was the dominant mechanism for energy and momentum transfer in the radiation–dominated era (RD–stage). It seems worthy to present a quantitative description of the non–equilibrium processes that can be expected in a hot photon gas coupled to plasma by Compton scattering.

In case of low temperatures we have the following expression for heat conductivity [16]

$$\lambda = \frac{4}{5}\frac{x_\gamma}{x_e}\frac{1}{\sigma_T} \tag{15.27}$$

here $x_i$ – the fraction of particles, $\sigma_T$ – the Compton cross section and the ratio of electron and photon number densities through baryon–to–photon ratio with electric neutrality of the Universe

$$\eta_B = \frac{n_B}{n_\gamma} = 0.6\,10^{-9}\,. \tag{15.28}$$

## 15.5 Dependence of the equation on the rate of expansion of space

We should set the form of operators included in the equation (15.7). If the matter of the surrounding space is stationary as a whole then the four–velocity takes the form $u_\mu = (1, 0, 0, 0)$ hence $D = u^\mu \partial_\mu = \partial_t$.

We need to make the following substitution: $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu \longrightarrow \Delta^{\mu\nu}\mathcal{D}_\nu$ in order to take into account the expansion of space. For a scalar field covariant differentiation is simply partial differentiation: $\mathcal{D}_\mu = \partial_\mu$, for a covariant vector we have: $\mathcal{D}_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha$ and for a contravariant vector: $\mathcal{D}_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\alpha\mu} A^\alpha$. The Christoffel symbols of the second kind: $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\big)/2$.

Thus our operator $\nabla^2$ is explicitly dependent on the metric

$$\nabla^2 = \nabla^\mu \nabla_\mu = \nabla^\mu g_{\mu\nu} \nabla^\nu = \Delta^{\mu\nu}\mathcal{D}_\nu g_{\mu\nu} \Delta^{\nu k}\partial_k\,. \tag{15.29}$$

The rate of temperature spreading into the surrounding space will be calculated with respect to the Friedmann–Lemaître–Robertson–Walker metric. The metric tensor in this case has the form

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2(t), -a^2(t)r^2, -a^2(t)r^2\sin^2\theta)\,. \tag{15.30}$$

The scale factor $a(t)$ can be found from Friedmann equations

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho\,, \qquad \frac{d\rho}{da} = -3\xi\frac{\rho}{a} \tag{15.31}$$

here $\xi = 4/3\ (1)$ for RD–stage (MD–stage).

Finally we get the following dependence for scale factor

$$a(t) = \left[1 + \frac{3\xi}{2}\sqrt{\frac{8\pi G\rho_0}{3}}(t - t_0)\right]^{2/3\xi} \tag{15.32}$$

obtained under the conditions $a(t_0) = 1$, $\rho(t_0) = 0.53 \cdot 10^{-5}\,\mathrm{GeV/cm^3}$, $t_0 \simeq 14 \cdot 10^9$ yrs– the age of Universe.

## 15.6 Final statement of the problem and result of calculation

We consider spherical symmetry for simplicity. The heat–conduction equation (15.7) with expression (15.12) for pressure and in case of stationary matter takes the form

$$\frac{nc_v}{\lambda}\frac{\partial}{\partial t}\Big(T(r,t)\Big) = -\nabla^2 T(r,t) + \frac{T(r,t)}{hn}\nabla^2 n T(r,t) \tag{15.33}$$

with boundary conditions

$$\begin{cases} \left.\dfrac{\partial}{\partial r}T(r,t)\right|_{r=0} = 0\,, \\ T(r,t)|_{r=\infty} = \dfrac{T_{out}}{a(t)} \end{cases} \tag{15.34}$$

here the dependence for scale factor $a(t)$ is taken from (15.32).

The initial condition is

$$T(r,0) = T_{in}\exp(-r^2/r_0^2) + T_{out} \tag{15.35}$$

here $T_{in}$ and $T_{out}$ – temperatures of matter inside cluster and the surrounding space respectively, $r_0$– temperature distribution parameter.

In general the obtained expressions can also be used in calculations at the RD–stage (stage of radiation dominance) and the MD–stage (stage of the matter domination). For this purpose the expression for the scale factor (15.32) should be taken at different $\xi$ and with modified heat conductivity. Presumably the cluster of primordial black holes virializes at the end of the RD–stage. It makes sense to estimate its cooling before the end of this stage. We need to choose specific values of the following parameters:

· temperature distribution parameter $r_0 = 1$ pc;
· temperature inside the area $T_{in} = 100$ keV;
· temperature of the surrounding space $T_{out} = 1$ keV;
· dependence a(t) in boundary condition is selected for RD–stage;
· for enthalpy and heat capacity we should select forms in case of low temperatures (15.17) and (15.19) accordingly.

Using numerical simulation in MAPLE by the BackwardEuler method with the interval of spatial points on a discrete grid 1/60 we have Fig.15.1. As can be seen from the figure, the gravitationally bound region almost completely retains temperature which was obtained during the formation at the RD–stage. The next step is to determine what happens with this heated region at the MD–stage.

## 15.7 Estimation for MD–stage

We will be interested in the internal temperature of the gravitationally bound region during the MD–stage. At the end of the RD–stage we have a region with higher temperature. It is possible to ignite a new chain of nuclear reactions changing chemical composition of the matter in given region. The temperature inside the cluster can be calculated in Minkowski space and we can find the dependence of the thermal conductivity on temperature in the non–relativistic case. The thermal diffusivity by definition is given by [1]

$$\chi = \frac{\lambda}{n_e c_v} = 3.16\,\frac{T_e \tau_e}{m_e c_v} = \frac{3.16}{2\sqrt{2\pi}\,\sqrt{m_e}\,q_e^4 Z n_e(T)\ln(6Tr_D/\alpha)} T_e^{5/2} \tag{15.36}$$

[1] Here the values are expressed in the CGS system and the temperature in eV

The thermal diffusivity in $\left[\frac{\mathrm{pc}^2}{\mathrm{year}}\right]$ is

$$\chi(T) \simeq \frac{2.3 \cdot 10^{-14}}{\sqrt{T_e(\mathrm{eV})}}. \tag{15.37}$$

The calculated value allows to retain the increased temperature inside the cluster until the recombination starts. The heat is conserved within a region starting from the moment of its formation. Thus, there are significant prerequisites for anomalies in the chemical composition of this region which makes sense to consider in future.

## Conclusion

We investigated the temperature dynamics of the heated region around the primordial black holes cluster. For this purpose the relativistic heat–conduction equation (without convective terms) was considered taking into account the expansion of space in the framework of the Chapman–Enskog relativistic procedure. The numerical solution was found with the corresponding initial and boundary conditions. According to our calculations, the gravitationally bound region almost completely retains temperature which was obtained during the formation. At the MD–stage the increased temperature inside the cluster is conserved until then recombination will start. Thus, there are significant prerequisites for anomalies in the chemical composition of this region. In prospect, we are going to study possible anomalies in the chemical content of the region with comparison to the observed data.


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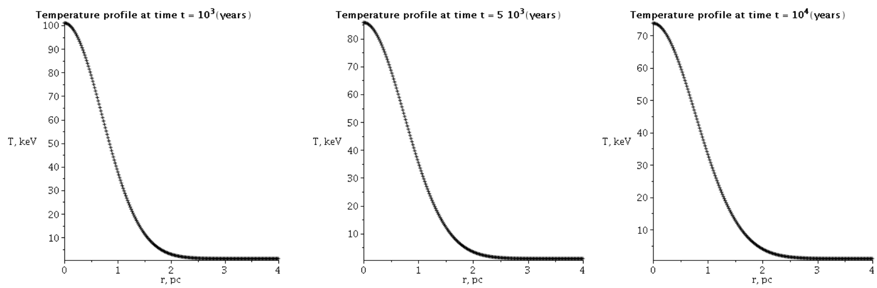


**Fig. 15.1.** Numerical calculation results of local temperature distribution at the stage of radiation dominance.

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# Virtual Institute of Astroparticle Physics Presentation

# 16 The Platform of Virtual Institute of Astroparticle Physics for Studies of BSM Physics and Cosmology

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**Abstract.** Being a unique multi-functional complex of science and education online, Virtual Institute of Astroparticle Physics (VIA) operates on website **http://viavca.in2p3.fr/site.html**. It supports presentation online for the most interesting theoretical and experimental results, participation online in conferences and meetings, various forms of collaborative scientific work as well as programs of education at distance, combining online videoconferences with extensive library of records of previous meetings and Discussions on Forum. Since 2014 VIA online lectures combined with individual work on Forum acquired the form of Open Online Courses. Aimed to individual work with students the Course is not Massive, but the account for the number of visits to VIA site converts VIA in a specific tool for MOOC activity. VIA sessions are now a traditional part of Bled Workshops' programme. At XXII Bled Workshop they involved not only remote presentations but also online streaming of most of the talks and discussions, supporting world-wide propagation of the main ideas, presented at this meeting. Special VIA sessions were dedicated at the XXII Bled Workshop to scientific debuts of students.

**Povzetek.** Virtual Institute of Astroparticle Physics (VIA, **http://viavca.in2p3.fr/site.html**), ponuja direktne predstavitve najbolj zanimivih in aktualnih teoretičnih spoznanj ter eksperimentalnih rezultatov, odprtih diskusij na konferencah, delavnicah, videokonferencah in drugih srečanjih, ponuja tudi izobraževanje preko spleta. Na svoji spletni strani hrani zapis vseh predavanj, diskusij in drugih dogodkov. Po letu 2014 so postala predavanja z odrtimi diskusijami na daljavo tudi izobraževalni tečaji, ki vključujejo tudi individualno mentorstvo za posamezne študente (MOOC). Seje VIA so postale tradicionalen del programa vsakoletnih Blejskih delavnic z nalovom "Kako preseči oba standardna modela, elektrošibkega in barvnega ter kozmološkega". Ponujajo ne le živ (sproten) prenos predavanj in diskusij, ampak omogočajo raziskovalcem iz vseh laboratorijev po svetu, ki jih teme delavnice zanimajo, komentarje, vprašanja. 22. delavnica je ponudila podoktorskim študentom na posebni seji prve predstavitve raziskovalnih spoznanj.

Keywords: astroparticle physics, physics beyond the Standard model, e-learning, e-science, MOOC

## 16.1 Introduction

Studies in astroparticle physics link astrophysics, cosmology, particle and nuclear physics and involve hundreds of scientific groups linked by regional networks (like ASPERA/ApPEC [1,2]) and national centers. The exciting progress in these studies will have impact on the knowledge on the structure of microworld and Universe in their fundamental relationship and on the basic, still unknown, physical laws of Nature (see e.g. [3,4] for review). The progress of precision cosmology and experimental probes of the new physics at the LHC and in nonaccelerator experiments, as well as the extension of various indirect studies of physics beyond the Standard model involve with necessity their nontrivial links. Virtual Institute of Astroparticle Physics (VIA) [5] was organized with the aim to play the role of an unifying and coordinating platform for such studies.

Starting from the January of 2008 the activity of the Institute takes place on its website [6] in a form of regular weekly videoconferences with VIA lectures, covering all the theoretical and experimental activities in astroparticle physics and related topics. The library of records of these lectures, talks and their presentations was accomplished by multi-lingual Forum. Since 2008 there were **207 VIA online lectures**, VIA has supported distant presentations of **132 speakers at 27 Conferences** and provided transmission of talks at **74 APC Colloquiums**.

In 2008 VIA complex was effectively used for the first time for participation at distance in XI Bled Workshop [7]. Since then VIA videoconferences became a natural part of Bled Workshops' programs, opening the virtual room of discussions to the world-wide audience. Its progress was presented in [8–17].

Here the current state-of-art of VIA complex, integrated since 2009 in the structure of APC Laboratory, is presented in order to clarify the way in which discussion of open questions beyond the standard models of both partcile physics and cosmology were presented at the XXII Bled Workshop with the of VIA facility to the world-wide audience. Active involvement of young scientists in VIA sessions and discussions and VIA streaming of virtually all the talks were specific new features of VIA activity at XXII Bled Workshop.

## 16.2 VIA structure and activity

### 16.2.1 VIA activity

The structure of the VIA complex is illustrated by the Fig. 16.1. The home page, presented on this figure, contains the information on the coming and records of the latest VIA events. The upper line of menu includes links to directories (from left to right): with general information on VIA (About VIA); entrance to VIA virtual rooms (Rooms); the library of records and presentations (Previous), which contains records of VIA Lectures (Previous → Lectures), records of online transmissions of Conferences (Previous → Conferences), APC Colloquiums (Previous → APC Colloquiums), APC Seminars (Previous → APC Seminars) and Events (Previous → Events); Calender of the past and future VIA events (All events) and VIA Forum (Forum). In the upper right angle there are links to Google search engine

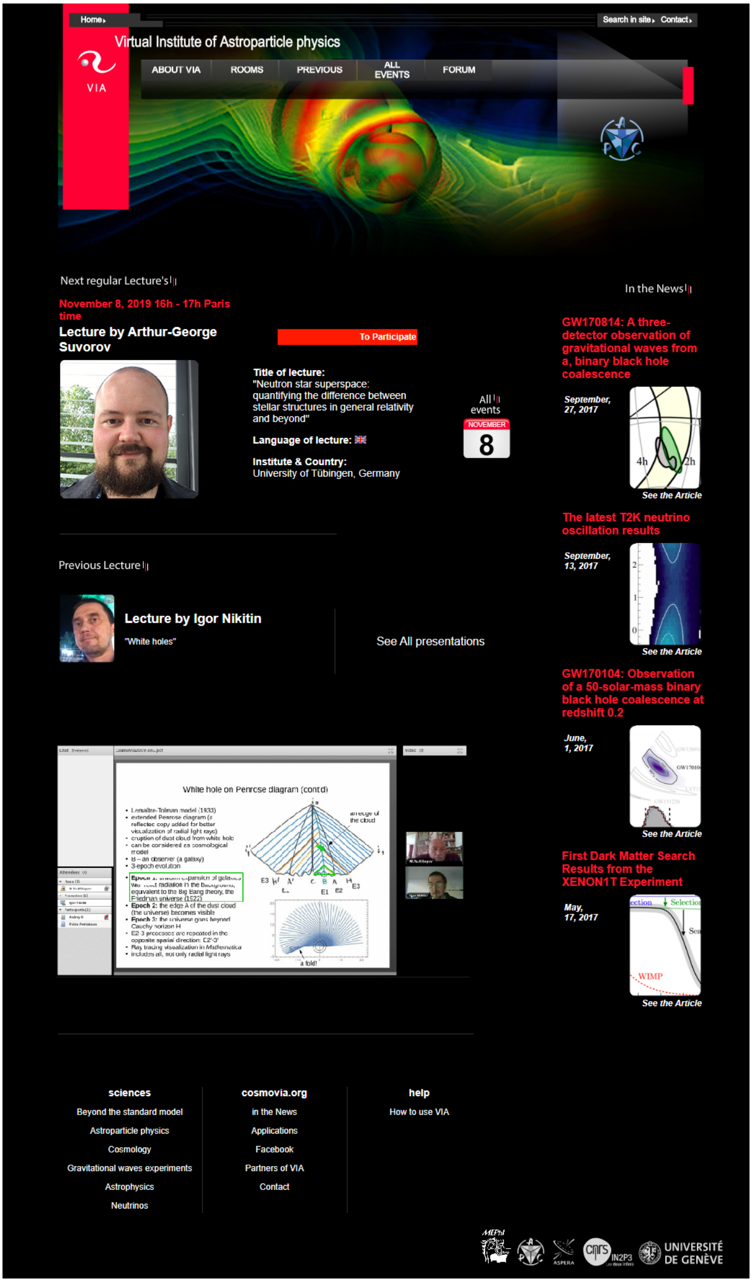


**Fig. 16.1.** The home page of VIA site

(Search in site) and to contact information (Contacts). The announcement of the next VIA lecture and VIA online transmission of APC Colloquium occupy the main part of the homepage with the record of the most recent VIA events below. In the announced time of the event (VIA lecture or transmitted APC Colloquium) it is sufficient to click on "to participate" on the announcement and to Enter as Guest (printing your name) in the corresponding Virtual room. The Calender shows the program of future VIA lectures and events. The right column on the VIA homepage lists the announcements of the regularly up-dated hot news of Astroparticle physics and related areas.

In 2010 special COSMOVIA tours were undertaken in Switzerland (Geneva), Belgium (Brussels, Liege) and Italy (Turin, Pisa, Bari, Lecce) in order to test stability of VIA online transmissions from different parts of Europe. Positive results of these tests have proved the stability of VIA system and stimulated this practice at XIII Bled Workshop. The records of the videoconferences at the XIII Bled Workshop are available on VIA site [18].

Since 2011 VIA facility was used for the tasks of the Paris Center of Cosmological Physics (PCCP), chaired by G. Smoot, for the public programme "The two infinities" conveyed by J.L.Robert and for effective support a participation at distance at meetings of the Double Chooz collaboration. In the latter case, the experimentalists, being at shift, took part in the collaboration meeting in such a virtual way.

The simplicity of VIA facility for ordinary users was demonstrated at XIV Bled Workshop in 2011. Videoconferences at this Workshop had no special technical support except for WiFi Internet connection and ordinary laptops with their internal webcams and microphones. This test has proved the ability to use VIA facility at any place with at least decent Internet connection. Of course the quality of records is not as good in this case as with the use of special equipment, but still it is sufficient to support fruitful scientific discussion as can be illustrated by the record of VIA presentation "New physics and its experimental probes" given by John Ellis from his office in CERN (see the records in [19]).

In 2012 VIA facility, regularly used for programs of VIA lectures and transmission of APC Colloquiums, has extended its applications to support M.Khlopov's talk at distance at Astrophysics seminar in Moscow, videoconference in PCCP, participation at distance in APC-Hamburg-Oxford network meeting as well as to provide online transmissions from the lectures at Science Festival 2012 in University Paris7. VIA communication has effectively resolved the problem of referee's attendance at the defence of PhD thesis by Mariana Vargas in APC. The referees made their reports and participated in discussion in the regime of VIA videoconference. In 2012 VIA facility was first used for online transmissions from the Science Festival in the University Paris 7. This tradition was continued in 2013, when the transmissions of meetings at Journées nationales du Développement Logiciel (JDEV2013) at Ecole Politechnique (Paris) were organized [21].

In 2013 VIA lecture by Prof. Martin Pohl was one of the first places at which the first hand information on the first results of AMS02 experiment was presented [20].

In 2014 the 100th anniversary of one of the foundators of Cosmoparticle physics, Ya. B. Zeldovich, was celebrated. With the use of VIA M.Khlopov could contribute the programme of the ”Subatomic particles, Nucleons, Atoms, Universe: Processes and Structure International conference in honor of Ya. B. Zeldovich 100th Anniversary” (Minsk, Belarus) by his talk ”Cosmoparticle physics: the Universe as a laboratory of elementary particles” [22] and the programme of ”Conference YaB-100, dedicated to 100 Anniversary of Yakov Borisovich Zeldovich” (Moscow, Russia) by his talk ”Cosmology and particle physics” [23].

In 2015 VIA facility supported the talk at distance at All Moscow Astrophysical seminar ”Cosmoparticle physics of dark matter and structures in the Universe” by Maxim Yu. Khlopov and the work of the Section ”Dark matter” of the International Conference on Particle Physics and Astrophysics (Moscow, 5-10 October 2015). Though the conference room was situated in Milan Hotel in Moscow all the presentations at this Section were given at distance (by Rita Bernabei from Rome, Italy; by Juan Jose Gomez-Cadenas, Paterna, University of Valencia, Spain and by Dmitri Semikoz, Martin Bucher and Maxim Khlopov from Paris) and its work was chaired by M.Khlopov from Paris [28]. In the end of 2015 M. Khlopov gave his distant talk ”Dark atoms of dark matter” at the Conference ”Progress of Russian Astronomy in 2015”, held in Sternberg Astronomical Institute of Moscow State University.

In 2016 distant online talks at St. Petersburg Workshop ”Dark Ages and White Nights (Spectroscopy of the CMB)” by Khatri Rishi (TIFR, India) ”The information hidden in the CMB spectral distortions in Planck data and beyond”, E. Kholupenko (Ioffe Institute, Russia) ”On recombination dynamics of hydrogen and helium”, Jens Chluba (Jodrell Bank Centre for Astrophysics, UK) ”Primordial recombination lines of hydrogen and helium”, M. Yu. Khlopov (APC and MEPHI, France and Russia)”Nonstandard cosmological scenarios” and P. de Bernardis (La Sapiensa University, Italy) ”Balloon techniques for CMB spectrum research” were given with the use of VIA system [29]. At the defense of PhD thesis by F. Gregis VIA facility made possible for his referee in California not only to attend at distance at the presentation of the thesis but also to take part in its successive jury evaluation.

Since 2018 VIA facility is used for collaborative work on studies of various forms of dark matter in the framework of the project of Russian Science Foundation based on Southern Federal University (Rostov on Don). In September 2018 VIA supported online transmission of **17 presentations** at the Commemoration day for Patrick Fleury, held in APC [30].

The discussion of questions that were put forward in the interactive VIA events is continued and extended on VIA Forum. Presently activated in English,French and Russian with trivial extension to other languages, the Forum represents a first step on the way to multi-lingual character of VIA complex and its activity. Discussions in English on Forum are arranged along the following directions: beyond the standard model, astroparticle physics, cosmology, gravitational wave experiments, astrophysics, neutrinos. After each VIA lecture its pdf presentation together with link to its record and information on the discussion during it are put in the corresponding post, which offers a platform to continue discussion in replies to this post.

### 16.2.2 VIA e-learning, OOC and MOOC

One of the interesting forms of VIA activity is the educational work at distance. For the last eleven years M.Khlopov's course "Introduction to cosmoparticle physics" is given in the form of VIA videoconferences and the records of these lectures and their ppt presentations are put in the corresponding directory of the Forum [24]. Having attended the VIA course of lectures in order to be admitted to exam students should put on Forum a post with their small thesis. In this thesis students are proposed to chose some BSM model and to study the cosmological scenario based on this chosen model. The list of possible topics for such thesis is proposed to students, but they are also invited to chose themselves any topic of their own on possible links between cosmology and particle physics. Professor's comments and proposed corrections are put in a Post reply so that students should continuously present on Forum improved versions of work until it is accepted as admission for student to pass exam. The record of videoconference with the oral exam is also put in the corresponding directory of Forum. Such procedure provides completely transparent way of evaluation of students' knowledge at distance.

In 2018 the test has started for possible application of VIA facility to remote supervision of student's scientific practice. The formulation of task and discussion of progress on work are recorded and put in the corresponding directory on Forum together with the versions of student's report on the work progress.

Since 2014 the second semester of the course on Cosmoparticle physics is given in English and converted in an Open Online Course. It was aimed to develop VIA system as a possible accomplishment for Massive Online Open Courses (MOOC) activity [25]. In 2016 not only students from Moscow, but also from France and Sri Lanka attended this course. In 2017 students from Moscow were accompanied by participants from France, Italy, Sri Lanka and India [26]. The students pretending to evaluation of their knowledge must write their small thesis, present it and, being admitted to exam, pass it in English. The restricted number of online connections to videoconferences with VIA lectures is compensated by the wide-world access to their records on VIA Forum and in the context of MOOC VIA Forum and videoconferencing system can be used for individual online work with advanced participants. Indeed Google Analytics shows that since 2008 VIA site was visited by more than **242 thousand** visitors from **153** countries, covering all the continents by its geography (Fig. 16.2). According to this statistics more than half of these visitors continued to enter VIA site after the first visit. Still the form of individual educational work makes VIA facility most appropriate for PhD courses and it is planned to be involved in the International PhD program on Fundamental Physics, which can be started on the basis of Russian-French collaborative agreement. In 2017 the test for the ability of VIA to support fully distant education and evaluation of students (as well as for work on PhD thesis and its distant defense) was undertaken. Steve Branchu from France, who attended the Open Online Course and presented on Forum his small thesis has passed exam at distance. The whole procedure, starting from a stochastic choice of number of examination ticket, answers to ticket questions, discussion by professors in the absence of student and announcement of result of exam to him was recorded and put on VIA Forum [27].


**Fig. 16.2.** Geography of VIA site visits according to Google Analytics

In 2019 in addition to individual supervisory work with students the regular scientific and creative VIA seminar is in operation aimed to discuss the progress and strategy of students scientific workin the field of cosmoparticle physics.

### 16.2.3 Organisation of VIA events and meetings

First tests of VIA system, described in [5,7–9], involved various systems of videoconferencing. They included skype, VRVS, EVO, WEBEX, marratech and adobe Connect. In the result of these tests the adobe Connect system was chosen and properly acquired. Its advantages are: relatively easy use for participants, a possibility to make presentation in a video contact between presenter and audience, a possibility to make high quality records, to use a whiteboard tools for discussions, the option to open desktop and to work online with texts in any format.

Initially the amount of connections to the virtual room at VIA lectures and discussions usually didn't exceed 20. However, the sensational character of the exciting news on superluminal propagation of neutrinos acquired the number of participants, exceeding this allowed upper limit at the talk "OPERA versus Maxwell and Einstein" given by John Ellis from CERN. The complete record of this talk and is available on VIA website [31]. For the first time the problem of necessity in extension of this limit was put forward and it was resolved by creation of a virtual "infinity room", which can host any reasonable amount of participants. Starting from 2013 this room became the only main virtual VIA room, but for specific events, like Collaboration meetings or transmissions from science festivals, special virtual rooms can be created. This solution strongly reduces the price of the licence for the use of the adobeConnect videoconferencing, retaining a possibility for creation of new rooms with the only limit to one administrating Host for all of them.

The ppt or pdf file of presentation is uploaded in the system in advance and then demonstrated in the central window. Video images of presenter and participants appear in the right window, while in the lower left window the list of all the attendees is given. To protect the quality of sound and record, the participants are required to switch out their microphones during presentation and to use the upper left Chat window for immediate comments and urgent questions. The Chat window can be also used by participants, having no microphone, for questions and comments during Discussion. The interactive form of VIA lectures provides oral discussion, comments and questions during the lecture. Participant should use in this case a "raise hand" option, so that presenter gets signal to switch out his microphone and let the participant to speak. In the end of presentation the central window can be used for a whiteboard utility as well as the whole structure of windows can be changed, e.g. by making full screen the window with the images of participants of discussion.

Regular activity of VIA as a part of APC includes online transmissions of all the APC Colloquiums and of some topical APC Seminars, which may be of interest for a wide audience. Online transmissions are arranged in the manner, most convenient for presenters, prepared to give their talk in the conference room in a normal way, projecting slides from their laptop on the screen. Having uploaded in advance these slides in the VIA system, VIA operator, sitting in the conference room, changes them following presenter, directing simultaneously webcam on the presenter and the audience. If the advanced uploading is not possible, VIA streaing is used - external webcam and microphone are directed to presenter and screen and support online streaming.

## 16.3 VIA Sessions at XXII Bled Workshop

VIA sessions of XXII Bled Workshop continued the tradition coming back to the first experience at XI Bled Workshop [7] and developed at XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX and XXI Bled Workshops [8–17]. They became a regular part of the Bled Workshop's program.

In the course of XXII Bled Workshop, the list of open questions was stipulated, which was proposed for wide discussion with the use of VIA facility. The list of these questions was put on VIA Forum (see [32]) and all the participants of VIA sessions were invited to address them during VIA discussions. During the XXII Bled Workshop the announcement of VIA sessions was put on VIA home page, giving an open access to the videoconferences at VIA sessions. Though the experience of previous Workshops principally confirmed a possibility to provide effective interactive online VIA videoconferences even in the absence of any special equipment and qualified personnel at place, VIA Sessions were directed at I Workshop by M.Khlopov at place. Only laptop with microphone and webcam together with WiFi Internet connection was proved to support not only attendance, but also VIA presentations and discussions.

Starting from the Openening of the Workshop VIA streaming of most of the talks was arranged for distant participents. This new form of VIA transmission that avoids the necessity upload presentations in advance made possible to convert VIA sessions with a very limited set of talks to online streaming of practically all the conference accompanied by its record in the VIA library [33].

In the framework of the program of XXII Bled Workshop, E. Kiritsis, gave his talk "Emergent gravity (from hidden sector)" (Fig. 16.4), from Paris (see records in [33]).

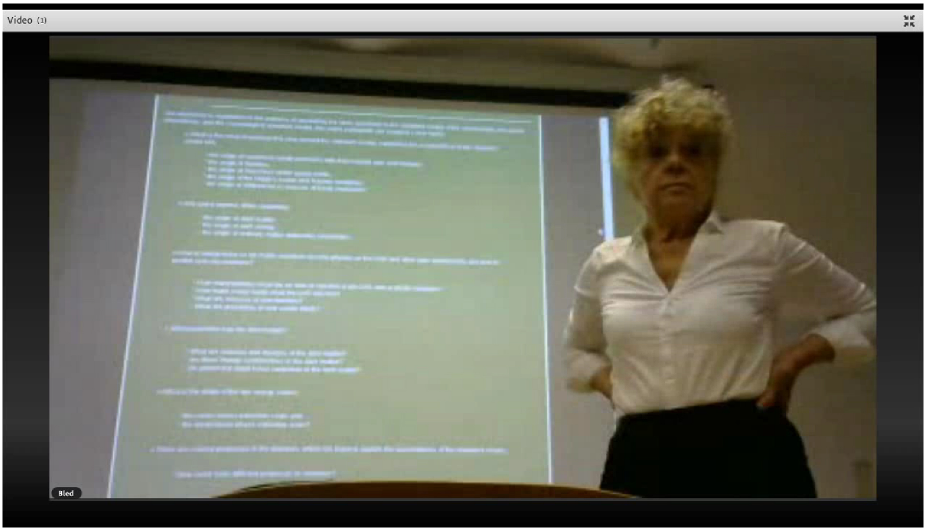


**Fig. 16.3.** VIA streaming of Opening of XXII Bled Workshop by Norma Mankoc- Borstnik

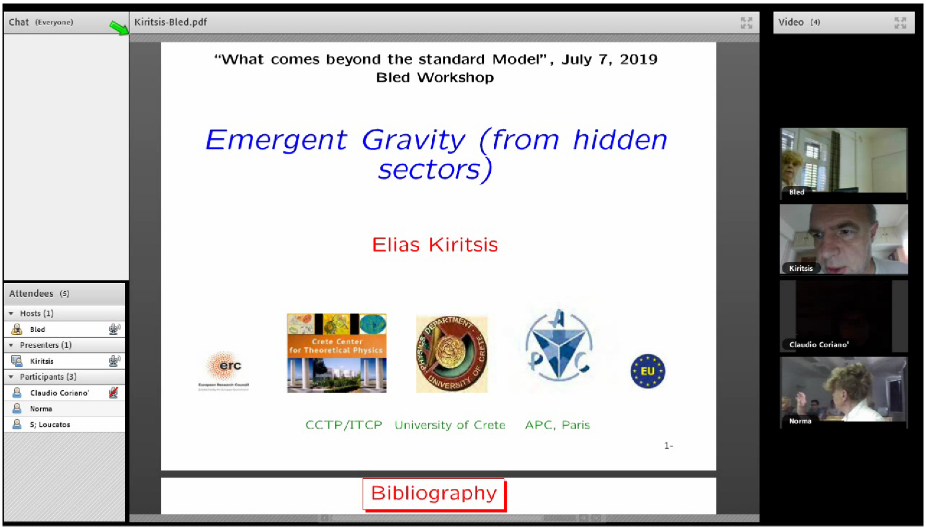


**Fig. 16.4.** VIA talk ”Emergent gravity (from hidden sector)” by E. Kiritsis from Paris at XXII Bled Workshop

The talks "Conspiracy of BSM Physics and BSM Cosmology" by Maxim Yu. Khlopov (Fig. 16.5) "Experimental consequences of spin-charge family theory" by Norma Mankoc-Borstnik (Fig. 16.6), as well as virtually all other talks were transmitted from Bled in the regime of streaming, inviting distant participants to join the discussion and extending the creative atmosphere of these discussions to the world-wide audience.

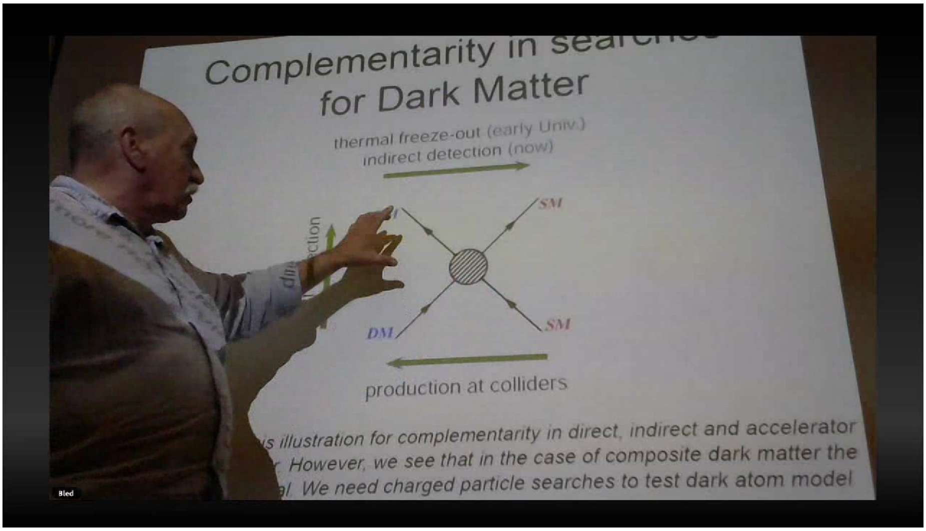


**Fig. 16.5.** VIA talk by Maxim Yu. Khlopov "Conspiracy of BSM Physics and BSM Cosmology" at XXII Bled Workshop

Two special VIA sessions provided remote presentation of students' scientific debuts in BSM physics and cosmology as it was the talk by Valery Nikulin (Fig. 16.7) who could not attend the Workshop, but could manage to present his interesting results with the use of VIA facility. The records of all these lectures and discussions can be found in VIA library [33].

## 16.4 Conclusions

The Scientific-Educational complex of Virtual Institute of Astroparticle physics provides regular communication between different groups and scientists, working in different scientific fields and parts of the world, the first-hand information on the newest scientific results, as well as support for various educational programs at distance. This activity would easily allow finding mutual interest and organizing task forces for different scientific topics of astroparticle physics and related topics. It can help in the elaboration of strategy of experimental particle, nuclear, astrophysical and cosmological studies as well as in proper analysis of experimental data. It can provide young talented people from all over the world to get the highest level education, come in direct interactive contact with the world known

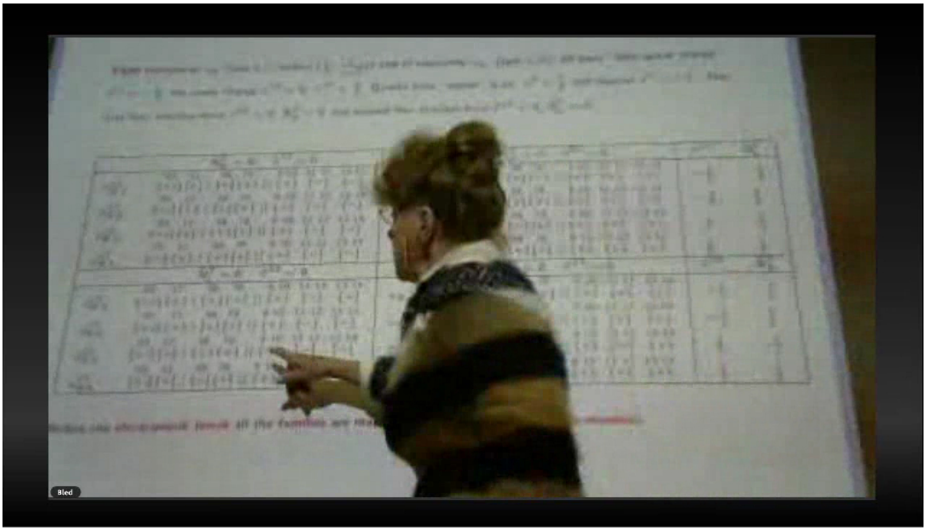

**Fig. 16.6.** VIA talk ”Dark matter, Matter-antimatter and spin-charge-family theory” by Norma Mankoc-Borstnik at XXII Bled Workshop

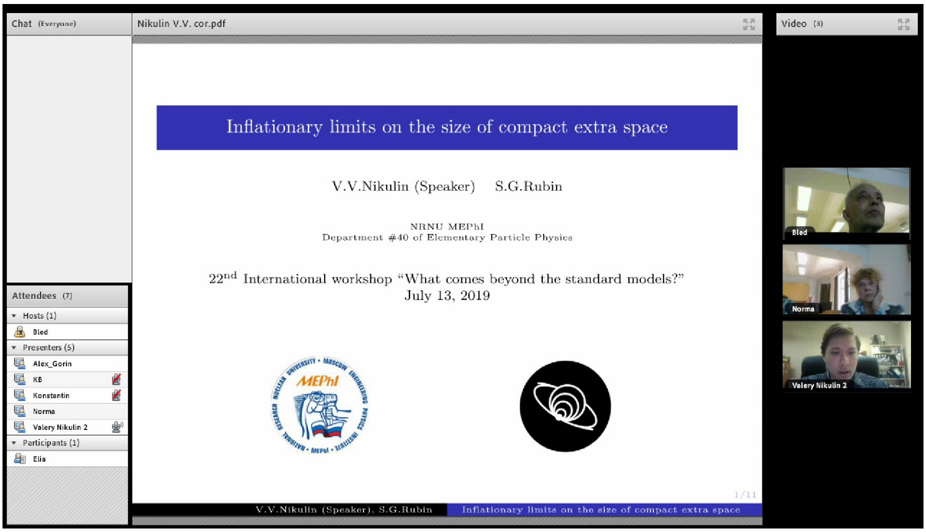


**Fig. 16.7.** VIA talk ”Inflationary limits on the size of compact extra space” by Valery Nikulin at XXII Bled Workshop

scientists and to find their place in the fundamental research. These educational aspects of VIA activity is now being evolved in a specific tool for International PhD programme for Fundamental physics. Involvement of young scientists in creative discussions was an important step of VIA activity at XXII Bled Workshop. VIA applications can go far beyond the particular tasks of astroparticle physics and give rise to an interactive system of mass media communications.

VIA sessions became a natural part of a program of Bled Workshops, maintaining the platform of discussions of physics beyond the Standard Model for distant participants from all the world. This discussion can continue in posts and post replies on VIA Forum. The experience of VIA applications at Bled Workshops plays important role in the development of VIA facility as an effective tool of e-science and e-learning.

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