Second Quantization as Cross Product

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Second quantization as Cross product

It is rather trivial and welknown that a second quantized (free) theory has a second quantized Hilbert space that can be written as a Cartesian cross product over an (infinite) set of (smaller) Hilbert spaces each of which is attached for example to the momentum, and tells how many partcles have just this momentum.
It is the purpose of the present discussion to seek to use such a formulation of second quantized theories to generalize them possible quite new types of second quantization like theories.

Simplest case: A scalar without internal d.o.f.

If we think of a charged scalar - like π^+ - it may be natural to even include in our "momentum" also the sign of the energy and use that as the 'factorisation **p**arameter" p. We like to do so abstract and general as possible, so we now use the letter p and you can think of it as "(factorization) **p**arameter" or as momentum as you like.

In the π^+ case we take the "factorization parameter" p to be:

$$\mathbf{p} = (\vec{p}, sign(E)). \tag{1}$$

The general form as factorized space:

The hilbert space for the second quantized system can always be written like

$$\mathcal{H} \quad = \quad \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}. \tag{2}$$

In the π^+ example, where $\mathbf{p} = (\vec{p}, sign(E))$, the hilbert space $\mathcal{H}_{\mathbf{p}}$ is actually that of harmonic oscillator for which the number operator counts the number of π^+ particles with just the **p**-specification **p**. (Here we stepped too fast over the dirac sea for bosons problem, but that is not so crucial just now; just think of antiparticles instead, when fromally $sign(E) < 0$.)

Dream of Generalization(s)

In the formulation as the Cartesian product

$$\mathcal{H} = \bigotimes_{\mathbf{p}} \mathcal{H}_{\mathbf{p}} \tag{3}$$

one could dream about making a new - and perhaps interesting theory - by replacing the Hilbert spaces that are factors in the Cartesian product such as $\mathcal{H}_{<bfp}$ by some Hilbert spaces with a different structure, e.g. different dimensionality.
E.g. Could we decide that all these harmonic oscillators could only be excited up to their 7th level, after that it would not be possible to put more π^+ in with a given **p** ?
We could of course postulate such a "theory" but it would be rather strange physically.

A posulate of only up to 7 particles per p would violate locallity

In a big universe particles with the same momentum are so far from each other than one cannot from locallity feel if there are more or less than 7 particles in the same momentum eigenstate. So such a generalization would violate locallity.

If we used $\vec{x}$ for p then locality would be automatic.

If one thinks of a discretized (d-1)-space, i.e. really a (perhaps a bit irregular) lattice, and take the state of the universe to be described by the a state in the Hilbert space $\mathcal{H}$, then factorization of the type

$$\mathcal{H} = \bigotimes_{\vec{x}} \mathcal{H}_{\vec{x}} \tag{4}$$

i.e. where we as "factorization parameter" used the spatial position $\vec{x}$ - the lattice point, if discretized - this Cartesian product would be automatically suited for locality, one should just only provide it with local interaction, but could for the structure and operators acting on the single factors $\mathcal{H}_{\vec{x}}$ be very free since everything would be o.k..

Usual second quantization in e.g. Normas model.

Once you have decided on the inner degrees of freedom, the statistics - fermion or boson - and of course dimension of space time and thus the dimension of the momentum vectors, I would a priori have said that now **there is only one way to second quantize.**
This way will then turn out to indeed be of the form that the full second quantized Hilbert space $\mathcal{H}$ takes the product form, and thus CAN be written that way.
However, if you start by a product form and have not gotten it via the standard proceedure, then I feel a priori very unsafe it will be a physically meaningfull model or not.
It probably depends strongly on the details.

A couple of trivialities on component numbers:

- A dirac spinor in an even number d of space time dimensions has $2^{d/2}$ components.
- In Normas model there is not only the usual Dirac spin index but a quite analogous family index. So in this model the number of components could be marked by two Dirac-indices, or instead using anoother but equivalent formalism with projection and nilpoten "operators". But in any case of these two formalisms the number of components for a full fermion particle is the square of the number for an ordinary Dirac construction. (then one may make reductions by Weyl reductions, but let us ignore that now.) The number of components is 2^d.

Trival numbers continued:

- If we ignored momentum and looked at one single momentum only, then the number of different states one could produce by having for this single momentum various possible numbers with the 2^d different components filled or unfilled would be 2^{2^d}. (I.e. much much more than the number of components)

Standard second Quantization Proceedure in Factor Language.

How one may get the $\mathcal{H}_{\mathbf{p}}$ spaces to be used for a second quantized Hilbert space by the Cartesian product formula:

- You first look for a fixed momentum $\mathbf{p}$ an calculate which states are needed to describe the possibilities for filling with the allowed number of particles (up to one for fermions, and up to infinity for bosons) all the internal states.
- Then construct the Hilbert space $\mathcal{H}_{\mathbf{p}}$ the dimension of which is just the number of different ways of filling particles into the different combinations of internal states.
- The finally you can take the Cartesian product and get the genuine Hilbert space for the full second quantized theory.

Standard Way $\dim(\mathcal{H}_{\mathbf{p}}) = 2^{2^d}$ for Normas Theory.

Since there are 2^d component combinations, namely say $2^{d/2}$ genuine Dirac components, and $2^{d/2}$ family index values, there for assumed fermion-statistic 2^{2^d} posibilities for filling or not filling these 2^d difference internal states.

Thus the Hilbert-space for only one momentum should have the dimension

$$dim(\mathcal{H}_{\mathbf{p}}) = 2^{2^d}. \tag{5}$$

(**Notice that this space $\mathcal{H}_{\mathbf{p}}$ thus has a much bigger dimension than the space of single particle internal states, which has only dimension $= 2^d$.**)

We ignored at first equations of motion.

We have to modify the above simplified proposal by:

- Notice that using the momentum energy relation

$$E^2 - \vec{p}^2 = m \tag{6}$$

  we have for each (d-1)-momentum $\vec{p}$ **two** values for the energy E of the particle, so that we should let - as already mentioned as a possibility - let

$$\mathbf{p} = (\vec{p}, E), \tag{7}$$

  meaning a doubling of the space of momenta to be used.
- Secondly take into account that the (free) equation of motion (=Dirac equation) for a choice of energy $E = \pm\sqrt{\vec{p}^2 + m^2}$ only allow a subspace of the internal space of states for the (single) particle,

$$(\not{p} - m)\psi = 0. \tag{8}$$

Standard Second Quantization as product over $(\vec{p}, sign(E))$.

Letting an index $_{emr}$ denote that we have restricted the single particle sates to the states obeying the equations of motion (emr = "equation of motion restricted") we write the true standard second quantized Hilbert space

$$\mathcal{H}_{emr} = \bigotimes_{(\vec{p}, sign(E))} \mathcal{H}_{(\vec{p}, sign(E)), emr}, \tag{9}$$

where now $\mathcal{H}_{(\vec{p}, sign(E)), emr}$ is constructed from space of single particle internal states obeying the Dirac equantion and having $E = sign(E)\sqrt{\vec{p}^2 + m^2}$, which because of the restriction by the equation of motion has only half the dimensionallity of $2^{d/2}$ in the simple Diarac case or half of 2^d in the case with families. So

$$dim(\mathcal{H}_{(\vec{p}, sign(E)), emr}) = 2^{2^d/2} = 2^{2^{d-1}}. \tag{10}$$

Yet a problem for Cartesian Product form for Fermions.

For just constructing the Hilbert space we could claim that this Cartesian product proceedure is o.k. even for fermions, but for the **creation and annihilation operators or the field operators for fermions there is a problem more:**
If we take a true Cartesian product and let it be understood that the creation and annihilation operators for a state with $(\vec{p}, sign(E)) = \mathbf{p}$ alone shall act on the Cartesian product factor $\mathcal{H}_{\mathbf{p}}$, **then we cannot make such fermion creation or annihilation operators for different p anticommute !** Operators acting alone on different Cartesian product factors will namely always commute.

Suggeted Trick to Solve the Anticommutation Probelm:

Use operators $(-1)^{F_{\mathbf{p}}}$, where $F_{\mathbf{p}}$ is the fermion number for the fermions in the Cartesian factor $\mathcal{H}_{\mathbf{p}}$.
That is to say to construct the "true creation or annihilation operators $b^{\dagger}(i;\mathbf{p})$ or $b(i;\mathbf{p})$" for the $\mathbf{p}$ Cartesian factor we modify the truly "local ones", $c^{\dagger}(i;\mathbf{p})$ and $c(i;\mathbf{p})$ defined so as to **only act on the Cartesian factor** $\mathcal{H}_{\mathbf{p}}$, not touching the other factors, by multiplying it with a lot of factors of the form $(-1)^{F_{\mathbf{p}'}}$.
Associate in fact to each essentially momentum $\mathbf{p}$ a subset of this kind of essential momenta $\mathbf{B}(\mathbf{p})$ and define

$$b^{\dagger}(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} (-1)^{F_{\mathbf{p}'}}\right] c^{\dagger}(i;\mathbf{p})$$

More generally:

$$b^\dagger(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')^\dagger\right] c^\dagger(i;\mathbf{p})$$

$$b(i;\mathbf{p}) = \left[\prod_{\mathbf{p}'\in\mathbf{B}(\mathbf{p})} U(\mathbf{p},\mathbf{p}')\right] c(i;\mathbf{p})$$

Not even crudely local $b^\dagger(\vec{x})$ unless the modification by $U(\vec{x},\vec{x}')$ inessential.

So there should preferably be a "gauge" transformation which could be the effect of the modification $U(\vec{x},\vec{x}')$ or "jump over correction"-replacement.
Natural that the $U(\vec{x},\vec{x}')$ depends on the direction from $\vec{x}$ to $\vec{x}'$, and thus is a function of a point on thee sphere S^{d-2}.
Also the 'gauge" like modifications must lie in a group G. So need map $S^{d-2} \rightarrow G$.

Anions as electric magnetic maded