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What Comes Beyond the Standard Models

Bled, July 4–12, 2020

[Virtual Workshop] [July 6.–10. 2020]

Volume 1: Invited Talks

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Preface

The series of annual workshops on "What Comes Beyond the Standard Models?" started in 1998 with the idea of Norma and Holger for organizing a real workshop, in which participants would spend most of the time in discussions, confronting different approaches and ideas. Workshops have taken place in the picturesque town of Bled by the lake of the same name, surrounded by beautiful mountains and offering pleasant walks and mountaineering. This year 2020 we still had a workshop in July, but without personal conversations all day and late at night, even between very relaxing walks and mountaineering due to COVID-19 pandemic. We have, however, a very long tradition of videoconferences (cosmovia), enabling discussions and explanations with laboratories all over the world. This enables us to have this year the total virtual workshop, resembling Bled workshops as much as possible.

In our very open minded, friendly, cooperative, long, tough and demanding discussions several physicists and even some mathematicians have contributed.

Most of topics presented and discussed in our Bled workshops concern the proposals how to explain physics beyond the so far accepted and experimentally confirmed both standard models — in elementary particle physics and cosmology — in order to understand the origin of assumptions of both standard models and be consequently able to propose new theories, models and to make predictions for future experiments.

Although most of participants are theoretical physicists, many of them with their own suggestions how to make the next step beyond the accepted models and theories, experts from experimental laboratories were and are very appreciated, helping a lot to understand what do measurements really tell and which kinds of predictions can best be tested.

The (long) presentations (with breaks and continuations over several days), followed by very detailed discussions, have been extremely useful, at least for the organizers. We hope and believe, however, that this is the case also for most of participants, including students. Many a time, namely, talks turned into very pedagogical presentations in order to clarify the assumptions and the detailed steps, analyzing the ideas, statements, proofs of statements and possible predictions, confronting participants' proposals with the proposals in the literature or with proposals of the other participants, so that all possible weak points of the proposals, those from the literature as well as our own, showed up very clearly. The ideas therefore seem to develop in these years considerably faster than they would without our workshops.

This year neither the cosmological nor the particle physics experiments offered much new, as also has not happened in the last two years, which would offer new insight into the elementary particles and fields and also into cosmological events, although a lot of work and effort have been put in, and although there are some indications for the existence of the fourth family to the observed three, due to the fact that the existence of the fourth family might explain the existing experimental data better, what is mentioned in this proceedings, as we did in the last year proceedings. Also the newest analyses of the data from LHC and other experiments has not changed the situation much. Of particular interest is the observed gravitational waves signal triggered by black holes of around 150 solar masses. These measurements are of the central interest of many a contribution in this proceedings.

However, there are more and more cosmological evidences, which require the new step beyond the standard model of the elementary fermion and boson fields. Understanding the universe through the cosmological theories and theories of the elementary fermion and boson fields, have, namely, so far never been so dependent on common knowledge and experiments in both fields.

We are keeping expecting that new cosmological experiments and new experiments in laboratories together will help to resolve the open questions in both fields.

On both fields there appear proposals which should explain assumptions of these models. Most of them offer small steps beyond the existing models. The competition, who will have right, is open.

The new data might answer the question, whether laws of nature are elegant (as predicted by the spin-charge-family theory and also — up to the families — other Kaluza-Klein-like theories and the string theories) or "she is just using gauge groups when needed" (what many models assume, also some presented in this proceedings). Can the higgs scalars and the Yukawa couplings be guessed by small steps from the standard model case, or they originate in gravity in higher dimensions as also the vector and scalar gauge fields do?

Is there only gravity as the interacting field, which manifests in the low energy regime all the vector gauge fields as well as the scalar fields, those observed so far and those predicted by the spin-charge-family theory, with the scalar colour triplets included ? Should correspondingly gravity be a quantized field like all the vector and the scalar gauge fields — possibly resulting from gravity — are?

Is masslessness of all the bosons and fermions, with scalar bosons included, essential, while masses appear at low energy region due to interactions and breaks of symmetries? Do the observed fermion charges indeed origin from spins of fermions in higher dimensions? What is then the dimension of space-time? Infinite, or it emerges from zero? One of contributions discusses also this problem. Does "nature use" odd Clifford algebra to describe fermions, what leads to anticommutation relations for second quantized fermions, explaining the Dirac's postulates, making already the creation operators for single fermion state anticommuting?

What "forces" fermions to appear in families? How many families do we have and what is their relation to the observed ones? What are reasons for breaking symmetries — discrete, global and local? Is The Lorentz invariant really violated? Does

the symmetry between fermions and antifermions manifest also in the presence of gravity?
Do the baryons of the stable family, decoupled from the observed ones, and predicted by the spin-charge-family theory (or can follow from heterotic string model), contribute to the dark matter? Do new stable quarks constitute neutral particles like neutrons, or form negatively charged particles, bound with primordial helium in dark atoms? How close are the additional new fermions, added to quarks and leptons of the standard model "by hand", to the stable fifth family of the spin-charge-family theory? Are also the charged "nucleons" of OHe's atoms explainable with the stable nucleons of the fifth family? Is the dark matter explainable within the standard model? Or does the dark matter manifest in dark stars, which are a kind of black holes?
What are indeed the black holes? If they ought to be created in the primordial time during the inflation (early matter stages or phase transitions), what kind of fermions and antifermions should contribute to the creation of black holes, massless (that is before the electroweak transition) or massive? What did cause the inflation? If there are singularities inside a black hole what is the status of fermions and fields inside the black hole? Do they make phase transitions into massless state within the black hole, loosing identity they have in $d = (3 + 1)$? Do we really understand black holes inside the the horizon?
We discussed these and many other open topics during Bled workshop 2020. Like it is the new idea of theory of strings, represented by particle objects, which do not develop in time.
The DAMA/LIBRA experiment convinced us again that the group in Gran Sasso do measure the dark matter particles scattering on the nuclei of their measuring apparatus. It is expected that sooner or latter other laboratories will confirm the DAMA/LIBRA results. This has not yet happened and our discussions clarified the reasons for that.
Although cosmovia served the discussions all the time (and we are very glad that we did have in spite of pandemia the 23^{rd} workshop), it was not like previous workshops. Discussions were fiery and sharp, at least during some talks. But this was not our Bled workshop. Effective discussions require the personal presence of the debaters, as well as of the rest of participants, which interrupt the presentations with questions all the time. As students need personal discussions with a good teacher, Internet discussions can never replace the real one.
Let us point out that we still succeeded to discuss the open problems on present understanding of the elementary particle physics and cosmology in the fully online regime, trying to save the most important feature of Bled Workshops - their free streaming discussion resulting in the comprehensive view on the discussed phenomena and ideas.
And let us add that due to the on line presentations we have students participants, who otherwise would not be able to attend the Bled conference, the travel expenses are too high for them. Their presentations are published in the second part of the proceedings, together with the invited talks, which came at the very last moment.
The organizers strongly hope that next year the covid-19 will be defeated, this is the hope for the whole world, for the young generation in particular and for all

of us, with the Bled workshop 2021 included. Let us meet at Bled! (This year's experience made us to think on more practical videoconferencing tools, like Zoom to facilitate extension of our discussions online.)

Since, as every year, also this year there has been not enough time to mature the discussions into the written contributions, only two months, authors can not really polish their contributions. Organizers hope that this is well compensated with fresh contents.

Questions and answers as well as lectures enabled by M.Yu. Khlopov via Virtual Institute of Astroparticle Physics (viavca.in2p3.fr/site.html) of APC have in ample discussions helped to resolve many dilemmas. Google Analytics, showing more than 242 thousand visits to this site from 154 countries, indicates world wide interest to the problems of physics beyond the Standard models, discussed at Bled Workshop. At XXIII Bled Workshop VIA streaming made possible to webcast practically all the talks.

The reader can find the records of all the talks delivered by cosmovia since Bled 2009 on viavca.in2p3.fr/site.html in Previous - Conferences.

Most of the talks can be found on the workshop homepage
http://bsm.fmf.uni-lj.si/.

Having a poet among us, we kindly asked Astri to contribute a poem for our proceedings. It is our pleasure that she did listen us and send two poems. We publish both, in each volume one.

Let us conclude this preface by thanking cordially and warmly to all the participants, present through the teleconferences at the Bled workshop, for their excellent presentations and also, in spite of all, for really fruitful discussions.

Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,
(the Organizing comittee)

Norma Mankoč Borštnik, Holger Bech Nielsen, Dragan Lukman,
(the Editors)

Ljubljana, December 2020

Predgovor (Preface in Slovenian Language)

Vsakoletne delavnice z naslovom „Kako preseči oba standardna modela, kozmološkega in elektrošibkega" ("What Comes Beyond the Standard Models?") sta postavila leta 1998 Norma in Holger z namenom, da bi udeleženci v izčrpnih diskusijah kritično soočali različne ideje in teorije. Delavnice domujejo v Plemljevi hiši na Bledu ob slikovitem jezeru, kjer prijetni sprehodi in pohodi na čudovite gore, ki kipijo nad mestom, ponujajo priložnosti in vzpodbudo za diskusije.
Tudi to leto je bila delavnica v juliju, vendar nam je tokrat covid-19 onemogočil srečanje v Plemljevi hiši. Tudi diskutirali nismo med hojo okoli jezera ali med hribolazenjem. Vendar nam je dolgoletna iskušnja s "cosmovio" — videopovezavami z laboratoriji po svetu — omogočila, da je tudi letos stekla Blejska delavnica, tokrat prek interneta.
K našim zelo odprtim, prijateljskim, dolgim in zahtevnim diskusijam, polnim iskrivega sodelovanja, je prispevalo veliko fizikov in celo nekaj matematikov. V večini predavanj in razprav so udeleleženci poskusili razumeti in pojasniti predpostavke obeh standadnih modelov, elektrošibkega in barvnega v fiziki osnovnih delcev ter kozmološkega, predpostavke in napovedi obeh modelov pa vskladiti z meritvami in opazovanji, da bi poiskali model, ki preseže oba standardna modela, kar bi omogočilo zanesljivejše napovedi za nove poskuse.
Čeprav je večina udeležencev teoretičnih fizikov, mnogi z lastnimi idejami kako narediti naslednji korak onkraj sprejetih modelov in teorij, so še posebej dobrodošli predstavniki eksperimentalnih laboratorijev, ki nam pomagajo v odprtih diskusijah razjasniti resnično sporočilo meritev in nam pomagajo razumeti kakšne napovedi so potrebne, da jih lahko s poskusi dovolj zanesljivo preverijo.
Organizatorji moramo priznati, da smo se na blejskih delavnicah v (dolgih) predstavitvah (z odmori in nadaljevanji preko več dni), ki so jim sledile zelo podrobne diskusije, naučili veliko, morda več kot večina udeležencev. Upamo in verjamemo, da so veliko odnesli tudi študentje in večina udeležencev. Velikokrat so se predavanja spremenila v zelo pedagoške predstavitve, ki so pojasnile predpostavke in podrobne korake, soočile predstavljene predloge s predlogi v literaturi ali s predlogi ostalih udeležencev ter jasno pokazale, kje utegnejo tičati šibke točke predlogov. Zdi se, da so se ideje v teh letih razvijale bistveno hitreje, zahvaljujoč prav tem delavnicam.
Tako kot v preteklih dveh letih tudi to leto niso eksperimenti v kozmologiji in fiziki osnovih fermionskih in bozonskih polj ponudili rezultatov, ki bi omogočili nov vpogled v fiziko osnovnih delcev in polj, čeprav je bilo vanje vloženega veliko truda in četudi razberemo iz eksperimentov, da četrta družina k že izmerjenim trem mora biti, saj lahko s štirimi družinami lažje pojasnimo izmerjene podatke, kar je omenjeno tudi v tem zborniku.

Tudi zadnje analize rezultatov merjenj na LHC in drugih merilnikih niso pripomogle k boljšemu razumevanju naravnih zakonov v fiziki osnovnih delcev in kozmologiji. Posebno pozornost so vzbudile meritve gravitacijskih valov, ki so jih povzročile črne luknje z masami okoli 150 sončnih mas. Prav te meritve poskušajo razložiti nekateri prispevki v letošnjem zborniku.
Vse več je tudi kozmoloških meritev, za katere se zdi, da jih standardni model osnovnih fermionski in bozonskih polj ne more pojasniti. Še nikoli doslej niso bili predlogi za kozmološke teorije in iskanje nove teorije v fiziki osnovnih polj tako zelo soodvisne od poizkusov in razumevanja predpostavk na obeh področjih.
Pričakujemo, da bodo kozmološka merjenja in meritve v laboratorijih pomagala razrešiti odprta vprašanja na obeh področjih. Na obeh področjih je predlogov za novo teorijo čedalje več, vendar velika večina teh predlogov ponuja majhna odstopanja od standardnih modelov. Tekma, kdo ima prav, je odprta.
Nove meritve bodo morda kmalu ponudile odgovor na vprašanje, ali so naravni zakoni elegantni (kot napoveduje teorija spina-naboja-družin in tudi druge teorije Kaluze in Kleina, vendar brez družin in ne tako "udarno") ali pa "narava uporabi grupe, ki in ko jih ravno potrebuje" (kar predlaga velika večina modelov, tudi nekateri v tem zborniku). Ali je smotrno pojav Higgsovega skalarnega polja in Yukawinih sklopitev dodati k standardnemu modelu osnovnih delcev kot dodatno polje, ki ga zahtevajo poskusi, ali pa je v resnici skalarnih polj več, njihov izvor pa je gravitacijsko polje v razsešnostih $d > (3+1)$?
So vsa osnovna fermionska in bozonska polj, tudi skalarna, brezmasna in je njihova masa, ki jo merimo pri nizkih energijah, posledica sil in zlomitve simetrij? Ali izvirajo naboji fermionov, ki jih izmerimo pri nizkih energijah, v spinih, ki jih ti fermioni nosijo v $d > (3+1)$? Kaj tedaj prostor in čas v resnici pomenita? Sta neskončna, ali pa se rodita iz nič ? Ali "narava uporabi" liho Clifordovo algebro za opis fermionov, kar zagotovi antikomutacijske relacije med kreacijskimimi in anihilacijskimi operatorji že med enofermionskimi stanji, kar pojasni Diracove postulate za fermione v drugi kvantizaciji? Kaj "prisili" fermione, da se pojavijo v družinah? Koliko je družin kvarkov in leptonov in kako so povezani, če sploh, z izmerjenimi tremi družinami? Kaj povzroči zlomitev simetrij, diskretnih, globalnih, lokalnih? Ali je Lorentzova simetrija zlomljena in če je, pod kakšnimi pogoji se zlomi? Ali je simetrija med fermioni in antifermioni v gravitacijskem polju zlomljena?
Kaj so gradniki temne snovi? Ali so barioni družin, ki niso sklopljene z izmerjenimi družinami kvarkov in leptonov in jih napove teorija spina-nabojev-družin, del temne snovi v vesolju? Ali se lahko novi fermioni, ki jih dodajo k kvarkom in leptonom standardnega modela osnovnih fermionskih in bozonskih polj, dajo pojasniti s stabilnimi barioni, ki jih napove teorija spina-nabojev-družin? So tudi temna jedra atoma O-He-lija člani stabilne družine? Se da temna snov pojasniti s skupki kvarkov in leptonov standardnega modela? Ali pa k temni snovi prispevajo temne zvezde, ki imajo lastnosti črnih lukenj?
Kaj pa so v resnici črne luknje? Če so nastajale ob inflaciji, kakšni fermioni in antifermioni so sodelovali pri nastanku črnih lukenj, z maso nič (to je pred elektrošibkim faznim prehodom) ali z neničelnimi masami? Kaj je povzročilo inflacijo? Če ima črna luknja singularnost, kako se spremenijo lastnosti fermionov in an-

tifermionov znotraj črne luknje? Ali izgubijo lastnosti, ki so jih imeli v $d = (3 + 1)$-razsežnem prostoru? Ali razumemo, kaj se dogaja v črni luknji znotraj horizonta? Ta in še marsikatera druga vprašanja smo načeli v času Blejske delavnice 2020. Denimo kot to, da v novi teoriji strun, ki jo sestavljajo točkasti delci, čas sploh ne nastopa.
Meritve DAMA/LIBRA v Gran Sassu so nas znova prepričale, da so delci, ki se sipljejo na atomskih jedrih merilcev in ki skozi leto periodično spreminjajo svojo intenzivnost, delci temne snovi. Pričakujemo, da bo laboratorijem po svetu, ki poskušajo potrditi njihove meritve, prej ali slej to tudi uspelo. Vprašanja in odgovori so pomagali razumeti, zakaj nobenemu doslej potrditev še ni uspela.
Četudi je cosmovia poskrbela, da so diskusije tekle ves čas, tako kot je bilo na vseh delavnicah doslej, blejskih diskusij v živo diskusije po internetu niso mogle nadomestiti. Diskusije so bile ognjevite in ostre, vsaj pri nekaterih predavanjih, vendar potrebujejo učinkovite diskusije osebno prisotnost diskutantov in poslušalcev, ki z vprašanji poskrbijo, da je debata razumljiva vsem. Tudi študentom internet ne more nadomestiti dobrega učitelja.
Poudariti je potrebno, da nam je kjub temu uspela dokaj plodna diskusija o tem, kako dobro razumemo danes obe področji, fiziko osnovnih delcev in polj ter dinamiko našega vesolja. In dodajmo, da je delavnica preko interneta omogočila šudentom aktivno in plodno sodelovanje, ki bi se ga v živo zaradi stroškov potovanja ne mogli udeležiti.
Šiudentski prispevki so zbrani v drugem zborniku Blejske delavnice, skupaj s prispevki vabljenih predavateljev, katerih prispevke smo prejeli zadnji trenutek.
Organizatorji upamo, da bo naslednje leto virus premagan, naše upanje velja za ves svet, za mlado generacijo pa še posebej, pa tudi za Blejsko delavnico 2021, da bo stekla v živo na Bledu.
Ker je vsako leto le malo casa od delavnice do zaključka redakcije, manj kot dva meseca, avtorji ne morejo dovolj skrbno pripravti svojih prispevkov, vendar upamo, da to nadomesti svezina prispevkov.
Bralec najde zapise vseh predavanj, objavljenih preko "cosmovia" od leta 2009, na viavca.in2p3.fr/site.html v povezavi Previous - Conferences. Večino predavanj najde bralec na spletni strani delavnice na http://bsm.fmf.uni-lj.si/.
Prosili smo Astri, da nam pošlje kako od svojih pesmi. Prijazno nam je ugodila in poslala dve. Objavljamo obe, v vsakem zborniku po eno.
Naj zaključimo ta predgovor s prisrčno in toplo zahvalo vsem udeležencem, prisotnim preko videokonference, za njihova predavanja in še posebno za zelo plodne diskusije in kljub vsemu odlično vzdušje.

Norma Mankoč Borštnik, Holger Bech Nielsen, Maxim Y. Khlopov,
(Organizacijski odbor)

Norma Mankoč Borštnik, Holger Bech Nielsen, Dragan Lukman,
(uredniki)

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Talk Section

All talk contributions are arranged alphabetically with respect to the authors' names.

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1 The DAMA Project: Achievements, Implications and Perspectives

R. Bernabei[1], P. Belli[1], A. Bussolotti[1], V. Caracciolo[1], R. Cerulli[1], N. Ferrari[1], V. Merlo[1], F. Cappella[2], A. d'Angelo[2], A. Incicchitti[2], A. Mattei[2], A. Di Marco[3], F. Montecchia[4], C.J. Dai[5], X.H. Ma[5], X.D. Sheng[5], Z.P. Ye[6]

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Abstract. Experimental observations and theoretical arguments at galactic and larger scales pointed out that a large fraction of the Universe is composed of Dark Matter (DM) particles. This has motivated the pioneer DAMA experimental efforts to investigate the presence of such particles in the galactic halo by exploiting a model-independent signature and highly radio-pure apparata in deep underground. In this paper the long-standing DM model-independent annual modulation effect measured by DAMA with various experimental set-ups made of highly radiopure NaI(Tl) is shortly summarized. Efforts to further improve the performance of the experiment at very low energy are mentioned.

Povzetek. Eksperimentalna opažanja in teoretične analize opažanj kažejo, da je temne snovi mnogo več kot običajne. Experiment DAMA je bil postavljen, da z direktno meritvijo delcev temne snovi, ki bo neodvisna od teoretičnih modelov, meri interakcijo med temi delci in delci v merilnikih. Experiment DAMA že vrsto let ugotavlja prisotnost delcev temne snovi v naši galaksiji na način, ki je neodvisen od predpostavk in napovedi modelov. Detektorji iz zelo čistih materialov, ki so postavljeni globoko v podzemlju, merijo letno modulacijo signala. Članek na kratko povzame rezultate, izmerjene z zelo (radio) čistim detektorji NaI(Tl). Avtorji na kratko poročajo o nameravanih izbolšanju eksperimenta za območje zelo nizkih energij.

Keywords: Dark Matter, Candidates, Rare Events, Annual Modulation, Low-background Scintillators, NaI(Tl).

1.1 Introduction

The constant progress in the last century and the numerous astronomical and cosmological observations in the last decades have collected a lot of information about our Galaxy and the Universe itself. It was argued that much of it must be in form of relic particles from the early stages of the formation of the Universe; this opened up a field for the detection of such particles.

In particular, the Dark Matter (DM) direct detection approach is the most direct method to investigate the presence of DM particles in our galactic halo, picking up rare events directly induced on suitable Earth detectors, settled deep underground. This investigation is difficult and delicate since many questions are still open on the topic, such as:

- Considering how rich in particles the luminous matter is, although its extremely modest density in the Universe, can it be argued that the particle DM is multicomponent?
- Which is the nature of the DM particle(s) and of its(their) interactions?
- Which is the proper description of the dark galactic halo?
- What about the interplay among Nuclear Physics, Particle Physics and Astrophysics/Cosmology that heavily enters in the choice of related physical parameters and scenarios in corollary model dependent analyses?

These quests and many others on related experimental and theoretical arguments make the efforts very challenging.

The DAMA project has been working as an observatory to investigate various kinds of rare processes, in particular by developing and using low radioactive scintillators. It is operative deep underground in the Gran Sasso National Laboratory (LNGS) of the INFN. Among the many experiments carried on, dedicated R&D, developments and highly radio-pure apparata have been set and empowered to explore the presence of DM particles in the galactic halo by exploiting mainly the DM model-independent annual modulation signature (see Ref. [1] and Refs. therein). In particular, the development of Ultra Low Background (ULB) NaI(Tl) target–detectors ensures sensitivity to a wide range of DM candidates, masses, interaction types and astrophysical scenarios.

1.2 DAMA/LIBRA and the Dark Matter annual modulation

The expected DM particles differential counting rate depends on the Earth's velocity in the galactic frame:

$$v_E(t) = v_\odot + v_\oplus \cos\gamma \cos\omega(t - t_0), \tag{1.1}$$

where the Sun velocity with respect to the galactic halo is $v_\odot \simeq v_0 + 12$ km/s, with v_0 local velocity), and $v_\oplus \simeq 30$ km/s is the Earth's orbital velocity around the Sun on a plane with inclination $\gamma = 60^o$ with respect to the galactic one. Moreover, $\omega = 2\pi/T$ with $T = 1$ year and roughly $t_0 \simeq$ June 2^{nd} (when the Earth's speed in the galactic halo is at maximum). Thus, the expected counting rate averaged in a given energy interval can be conveniently worked out through a first order Taylor expansion:

$$\mathcal{S}(t) = \mathcal{S}_0 + \mathcal{S}_m \cos\omega(t - t_0), \tag{1.2}$$

with the contribution from the highest order terms being less than 0.1%. The $\mathcal{S}_m$ and $\mathcal{S}_0$ are the modulation amplitude and the un-modulated part of the expected differential counting rate, respectively.

Therefore, in the DAMA experiments the experimental observable is the modulation amplitude, $\mathcal{S}_m$, as a function of the energy, and the identification of the constant part of the signal, $\mathcal{S}_0$, is not required to point out the presence of a signal in the exploited model-independent annual modulation approach. This has several advantages; in particular, the only background of interest is the one able to mimic the signature, i.e. able to account for the whole observed modulation amplitude and to simultaneously satisfy all its many specific peculiarities (see e.g. Ref. [2]). No background of this sort has been found, see Refs. [2–13].

The model-independent evidence for the presence of DM particles in the galactic halo has been investigated on the basis of the exploited DM annual modulation signature by the first six annual cycles of DAMA/LIBRA–phase2 [2,4–7] after the previous DAMA/LIBRA–phase1 [3,8–15] and the former DAMA/NaI [16,17] experiments. The cumulative Confidence Level (C.L.) is increased from the previous 9.3 σ (data from 14 independent annual cycles for an exposure of 1.33 ton $\times$ yr) to 12.9 σ (data from 20 independent annual cycles for an exposure of 2.46 ton $\times$ yr).

The modulation amplitudes, S_m, for the whole data sets: DAMA/NaI, DAMA/ LIBRA–phase1 and the first 6 annual cycles of DAMA/ LIBRA–phase2 (total exposure 2.46 ton$\times$yr) are plotted in Fig. 1.1; the data below 2 keV refer only to the first 6 annual cycles of DAMA/LIBRA–phase2 exposure (1.13 ton$\times$yr). It can be inferred that positive signal is present in the (1–6) keV energy interval, while S_m values compatible with zero are present just above [2]. Dedicated data analyses descriptions are given e.g. in Refs. [1–19]. See also Fig. 1.2 in the following.

In order to continuously monitor the running conditions, several pieces of information are acquired with the production data and quantitatively analysed; information on technical aspects of DAMA/LIBRA has been given in Ref. [8,14], where the sources of possible residual radioactivity have also been analysed. In particular, all the time behaviours of the running parameters, acquired with the production data, have been investigated. Table 1.1 shows the modulation amplitudes obtained for each annual cycle of DAMA/LIBRA–phase1 and phase2, when fitting the time behaviours of the values of the main parameters including a cosine modulation with the same phase and period as for DM particles. As can be seen, all the measured amplitudes are well compatible with zero and the stability of the measurements conditions is better than 1%.

Careful investigations on absence of any systematics or side reaction able to account for the measured modulation amplitude and to simultaneously satisfy all the requirements of the signature have been quantitatively carried out (see e.g. Refs. [3], and references therein); some of them will be mentioned in the following. The cases of muons, neutrons and neutrinos have also been carefully investigated [12,13]. In particular, no modulation

	DAMA/LIBRA–phase1						
	1	2	3	4	5	6	7
Temperature (°C)	$-(0.0001 \pm 0.0061)$	(0.0026 ± 0.0086)	(0.001 ± 0.015)	(0.0004 ± 0.0047)	(0.0001 ± 0.0036)	(0.0007 ± 0.0059)	(0.0000 ± 0.0054)
Flux N_2 (l/h)	(0.13 ± 0.22)	(0.10 ± 0.25)	$-(0.07 \pm 0.18)$	$-(0.05 \pm 0.24)$	$-(0.01 \pm 0.21)$	$-(0.01 \pm 0.15)$	$-(0.00 \pm 0.14)$
Pressure (mbar)	(0.015 ± 0.030)	$-(0.013 \pm 0.025)$	(0.022 ± 0.027)	(0.0018 ± 0.0074)	$-(0.08 \pm 0.12) \times 10^{-2}$	$(0.07 \pm 0.13) \times 10^{-2}$	$-(0.26 \pm 0.55) \times 10^{-2}$
Radon (Bq/m^3)	$-(0.029 \pm 0.029)$	$-(0.030 \pm 0.027)$	(0.015 ± 0.029)	$-(0.052 \pm 0.039)$	(0.021 ± 0.037)	$-(0.028 \pm 0.036)$	(0.012 ± 0.047)
Hardware rate (Hz)	$-(0.20 \pm 0.18) \times 10^{-2}$	$(0.09 \pm 0.17) \times 10^{-2}$	$-(0.03 \pm 0.20) \times 10^{-2}$	$(0.15 \pm 0.15) \times 10^{-2}$	$(0.03 \pm 0.14) \times 10^{-2}$	$(0.08 \pm 0.11) \times 10^{-2}$	$(0.06 \pm 0.10) \times 10^{-2}$
	DAMA/LIBRA–phase2						
	1	2	3	4	5	6	
Temperature (°C)	(0.0012 ± 0.0051)	$-(0.0002 \pm 0.0049)$	$-(0.0003 \pm 0.0031)$	(0.0009 ± 0.0050)	(0.0018 ± 0.0036)	$-(0.0006 \pm 0.0035)$	
Flux N_2 (l/h)	$-(0.15 \pm 0.18)$	$-(0.02 \pm 0.22)$	$-(0.02 \pm 0.12)$	$-(0.02 \pm 0.14)$	$-(0.01 \pm 0.10)$	$-(0.01 \pm 0.16)$	
Pressure (mbar)	$(1.1 \pm 0.9) \times 10^{-3}$	$(0.2 \pm 1.1) \times 10^{-3}$	$(2.4 \pm 5.4) \times 10^{-3}$	$(0.6 \pm 6.2) \times 10^{-3}$	$(1.5 \pm 6.3) \times 10^{-3}$	$(7.2 \pm 8.6) \times 10^{-3}$	
Radon (Bq/m^3)	(0.015 ± 0.034)	$-(0.002 \pm 0.050)$	$-(0.009 \pm 0.028)$	$-(0.044 \pm 0.050)$	(0.082 ± 0.086)	(0.06 ± 0.11)	
Hardware rate (Hz)	$-(0.12 \pm 0.16) \times 10^{-2}$	$(0.00 \pm 0.12) \times 10^{-2}$	$-(0.14 \pm 0.22) \times 10^{-2}$	$-(0.05 \pm 0.22) \times 10^{-2}$	$-(0.06 \pm 0.16) \times 10^{-2}$	$-(0.08 \pm 0.17) \times 10^{-2}$	

Table 1.1. Modulation amplitudes (1 σ error) obtained – for each annual cycle of DAMA/LIBRA–phase1 and phase2 – by fitting the time behaviours of main running parameters including a possible annual modulation with phase and period as for DM particles. These running parameters, acquired with the production data, are: i) the operating temperature of the detectors; ii) the HP nitrogen flux in the inner Cu box housing the detectors; iii) the pressure of the HP nitrogen atmosphere of that inner Cu box; iv) the environmental radon in the inner part of the barrack from which the detectors are however excluded by other two sealing systems (see Ref. [3–5,8,9,12,13,15] for details); v) the hardware rate above single photoelectron threshold. All the measured amplitudes are compatible with zero.

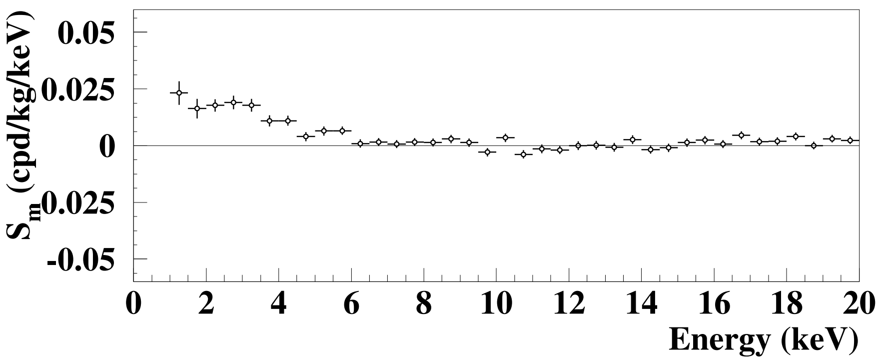


Fig. 1.1. Modulation amplitudes, S_m, for the whole data sets: DAMA/NaI, DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2 (total exposure 2.46 ton×yr) above 2 keV; below 2 keV only the DAMA/LIBRA–phase2 exposure (1.13 ton × yr) is available and used. The energy bin ΔE is 0.5 keV. A clear modulation is present in the lowest energy region, while S_m values compatible with zero are present just above [1].

has been found in any possible source of systematics or side reactions; thus, upper limits (90% C.L.) on the possible contributions to the DAMA/LIBRA measured modulation amplitude are summarized in Table 1.2. In particular, they cannot account for the measured modulation both because quantitatively not relevant and unable to mimic the observed effect.

1.2.1 Any effect from long–term decay in DAMA?

The adopted cautious procedure in the investigation of the DM particles annual modulation signature, as discussed several times in the DAMA papers [2–7,9–11], is that the data taking of each annual cycle starts from autumn (when $\cos\omega\,(t-t_0)\simeq 0$) towards summer (maximum expected). In such a way, during the annual cycle the expected minimum (December) of the DM signal occurs before of the maximum (June). Thus, any possible decay of long–term–living isotopes cannot simulate the observed positive signal. On the contrary, assuming in the analysis a constant background within each annual cycle, a possible decay of long–term–living isotopes may only lead to an underestimate of the observed annual modulation signal, depending on the radio–purity of the set-up as mentioned already e.g. in Ref. [20], pag. 573[1].

Despite this clear argument, recently Ref. [21] claims that the DAMA annual modulation result may be biased by a slow variation in the rate, possibly due to either some indefinite background or signal; even that the total rate at low energy in DAMA/LIBRA may have an odd behavior, increasing with time (see Fig. 2 of Ref. [21]). At first, this odd time behaviour of the counting rate was already excluded by the DAMA/LIBRA published results. In particular, the contaminants of the DAMA set-ups are reported in several papers; for example in Refs. [1,3,8]

[1] We also recall that the data collection of the first analysed annual cycle occurred more than one and half year after the installation of the detectors underground.

Source	Main comment (see also Ref. [8])	Cautious upper limit (90%C.L.)
Radon	Sealed Cu Box in HP nitrogen atmosphere, 3-level of sealing	$< 2.5 \times 10^{-6}$ cpd/kg/keV
Temperature	Air conditioning + huge heat capacity	$< 10^{-4}$ cpd/kg/keV
Noise	Efficient rejection	$< 10^{-4}$ cpd/kg/keV
Energy scale	Routine + intrinsic calibrations	$< 1 - 2 \times 10^{-4}$ cpd/kg/keV
Efficiencies	Regularly measured	$< 10^{-4}$ cpd/kg/keV
Background	No modulation above 6 keV; no modulation in the (1 – 6) keV *multiple–hit* events; this limit includes all possible sources of background	$< 10^{-4}$ cpd/kg/keV
Side reactions	From muon flux variation measured by MACRO	$< 3 \times 10^{-5}$ cpd/kg/keV
In addition: no effect can mimic the signature		

Table 1.2. Summary of the results obtained by investigating possible sources of systematics or side reactions in the DAMA/LIBRA annual cycles. None able to give a modulation amplitude different from zero has been found; thus cautious upper limits (90% C.L.) on possible contribution to the measured modulation amplitude have been calculated and are shown here (see e.g. Ref. [1] and references therein).

and references therein (see also above); none of them increases with time. Moreover, the stability with time of the running parameters are shown e.g. in Refs. [1–7,9–11] (also see Table 1.1). Thus, the assumptions in the paper of Ref. [21] are untenable and the conclusions are flawed.

In addition, to quantitatively show the possible amount of long–term decaying isotopes in DAMA/LIBRA, the following cases have been analyzed [1]:

- We recalculate the (2–6) keV *single–hit* residual rates of Fig. 25 in Ref. [1] (reference case in Fig. 1.2–*Top*), by considering a possible time behaviour given by the signal searched for and by different straight lines, one for each annual cycle, simulating a time–varying background (hereafter, B hypothesis). The residuals, once subtracting the so-obtained background, are reported in Fig. 1.2–*Bottom*. The (2–6) keV *single–hit* residual rates have been fitted with the function: $A \cos \omega(t - t_0)$, considering a period $T = \frac{2\pi}{\omega} = 1$ yr and a phase $t_0 = 152.5$ day (June 2^{nd}) as expected from the DM annual modulation signature. The obtained modulation amplitude in case of B hypothesis is $A = (0.0093 \pm 0.0008)$ cpd/kg/keV, to be compared with $A = (0.0095 \pm 0.0008)$ cpd/kg/keV (also see Ref. [1]) for the reference case in Fig. 1.2–*Top*. The χ^2/dof

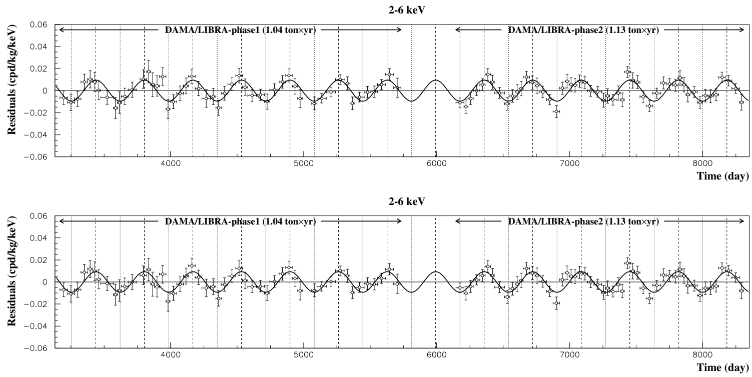


Fig. 1.2. Experimental residual rate of the (2–6) keV *single–hit* scintillation events measured by DAMA/LIBRA–phase1 and by the first 6 annual cycles of DAMA/LIBRA–phase2, calculated according to the prescriptions of Sect. 5.1 in Ref. [1] (*Top*), reference case, and according to B hypothesis, see text (*Bottom*). The superimposed curve is the cosinusoidal functional form $A \cos \omega(t - t_0)$ with a period $T = \frac{2\pi}{\omega} = 1$ yr, a phase $t_0 = 152.5$ day (June 2^{nd}) and modulation amplitude, A, equal to the central value obtained by best fit on the data points of DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2. The two fitted modulation amplitudes are well compatible, as described in the text.

are rather good in either case: 60.4/75 and 71.8/101. A χ^2–difference ($\Delta\chi^2$) test have been applied to investigate the two nested cases. The $\Delta\chi^2 = 11.4$ is a χ^2 variable with 26 degrees of freedom and, consequently, the B hypothesis is not favoured with respect to the reference case by the data at 90% C.L.: $P\left(\Delta\chi^2 < 11.4 \mid dof = 26\right) = 5.9 \times 10^{-3}$.

- In addition, the (2–6) keV *single–hit* residuals have also been fitted by keeping free the period and the phase in the procedure. The period and the phase are well compatible with expectations for a DM annual modulation signal; they are for the B hypothesis: $T = (0.9985 \pm 0.0009)$ yr and $t_0 = (143 \pm 5)$ days; the modulation amplitude is $A = (0.0094 \pm 0.0008)$ cpd/kg/keV. These values can be compared with those of Table 5 in Ref. [1], showing that the effect of long–term time–varying background – if any – has a negligible role in the given results.
- A possible long–term time–varying background would also induce a (either positive or negative) fake modulation amplitudes (Σ) on the tail of the $\mathcal{S}_m$ distribution above the energy region where the signal has been observed. Taking as reference the (6–14) keV energy interval, the averaged modulation amplitudes are: $\langle \mathcal{S}_m \rangle_{(6-14)} = (0.00028 \pm 0.00075)$ cpd/kg/keV, and $\langle \mathcal{S}_m \rangle_{(6-14)} = (0.0006 \pm 0.0006)$ cpd/kg/keV for DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2, respectively [2–7, 9–11]. They are both compatible with zero, as also previously reported in Refs. [2–7, 9–11]. Thus, applying the Feldman and Cousins procedure [22], one can obtain an upper limit on the absolute value of Σ at 90% C.L.: $\mid \Sigma \mid < 1.5 \times 10^{-3}$ cpd/kg/keV, and $\mid \Sigma \mid < 1.6 \times 10^{-3}$ cpd/kg/keV for DAMA/LIBRA–phase1 and DAMA/LIBRA–phase2, respectively.

Thus, taking into account that the observed annual modulation amplitude at low energy is order of 10^{-2} cpd/kg/keV, any possible effect of long–term time–varying background – if any – is negligible.

- The maximum likelihood analysis has been repeated including in the background model a linear term decreasing with time [1]. The obtained $\mathcal{S}_m$ are shown in Fig. 1.3 in comparison with those already obtained considering a constant background for each annual cycle. It clearly shows that the systematic error on the determination of the $\mathcal{S}_m$ previously reported is marginal.

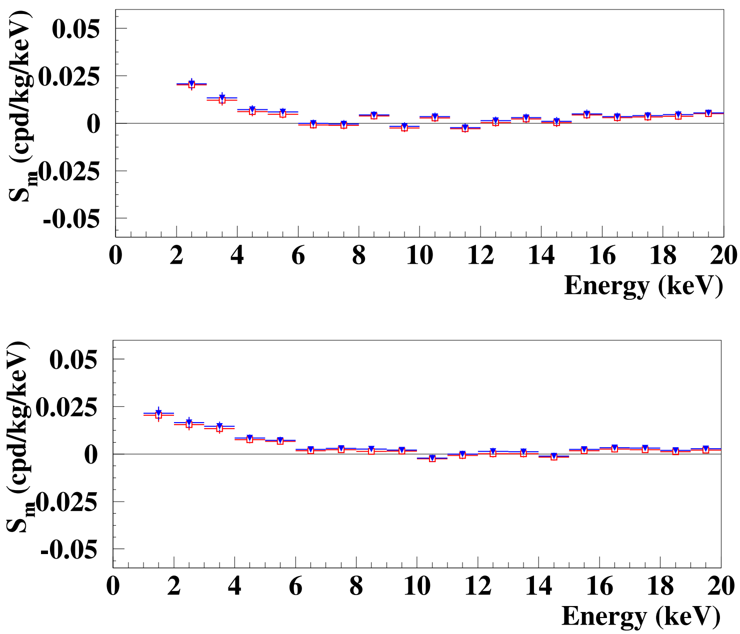


Fig. 1.3. Modulation amplitudes, $\mathcal{S}_m$, calculated in the analyses of Ref. [1] (red data points) and including in the background model a linear term decreasing with time (blue data points). Here the energy bin is 1 keV. *Top:* DAMA/LIBRA–phase1 and *Bottom:* DAMA/LIBRA–phase2.

In conclusion, we have shown few simple examples demonstrating that the possible effect of long–term time–varying background in DAMA/LIBRA is negligible and the *reference* analyses, that assume a constant background within each annual cycle as reported in Ref. [1], can be safely adopted. Similar conclusions have also been reported in Ref. [23].

1.2.2 A prior for the corollary model dependent analyses

Implications of the model-independent DAMA result on several of the many possible scenarios, already investigated with lower exposure and higher software energy threshold in the past, have also been updated by including data of DAMA/LIBRA–phase2 as reported in [24–26]. In fact, we have discussed in several papers the effects of the existing experimental and theoretical uncertainties existing in model dependent interpretations of the DAMA model-independent DM results, and comparisons.

Here we shortly remind the study of the measured low energy spectrum which offers a useful prior for those kinds of corollary analyses. Few structures in such spectrum allow the identification of contaminants present in traces in the detectors. As an example, the low energy spectrum of *single–hit* scintillation events for one detector in DAMA/LIBRA–phase2 is reported in Fig. 1.4, where – as also always in the paper – the correction for efficiencies is already applied.

There are represented the measured contributions of: (i) the internal cosmogenic ^{129}I: (947 ± 20) μBq/kg; (ii) the internal ^{210}Pb: (26 ± 3) μBq/kg, which is in a rather–good equilibrium with ^{226}Ra in the ^{238}U chain; (iii) the electron capture of ^{40}K (producing the 3.2 keV peak, binding energy of shell K in ^{40}Ar): 14.2 ppb of ^{nat}K, corresponding to 450 μBq/kg of ^{40}K in this detector. The broader structure around 12–15 keV can be ascribed to ^{210}Pb either on the PTFE, wrapping the bare crystal, and/or on the Cu housing, at level of 1.20 cpd/kg. The continuum due to high energy γ/β contributions is also reported. Below 5 keV a sharp decreasing curve represents the derived upper limit on $\mathcal{S}_0$.

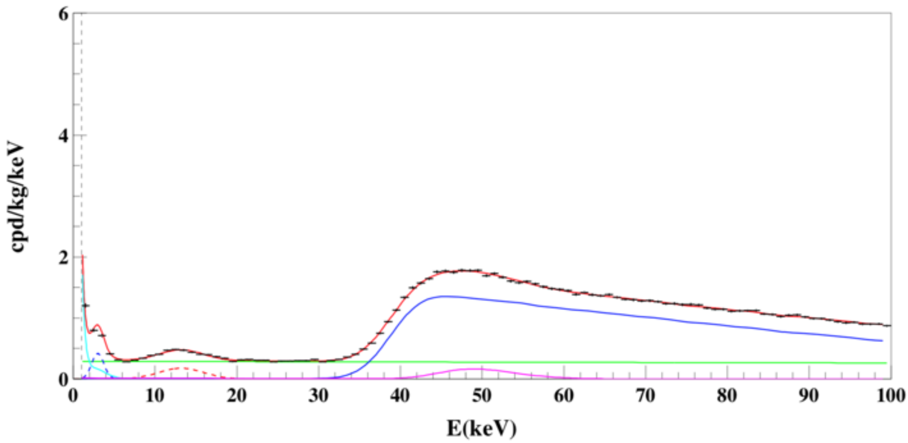


Fig. 1.4. Example of the energy spectrum of the *single–hit* scintillation events collected by one DAMA/LIBRA–phase2 detector in one annual cycle. The software energy threshold of the experiment is 1 keV. The identified components of the background are reported: internal ^{129}I (full blue curve on-line), internal ^{40}K (dashed blue curve on-line), ^{210}Pb (internal: solid pink curve; external: dashed pink curve; on line), continuum due to high–energy γ/β contributions (light green line on-line). Finally, the cyan (on-line) curve at low energy represents the upper limit on $\mathcal{S}_0$. The red (on-line) line is the sum of the previously mentioned contributions.

The cumulative (over all the detectors and DAMA/LIBRA–phase2 annual cycles) low–energy distribution of the *single–hit* scintillation events (that is each detector has all the others as veto) is reported in Fig. 1.5. Superimposed there is the model of the ^{40}K structure and the continuum. As can be seen, there is excess at low energy (online blue line in Fig. 1.5), which can be considered as an upper limit on $\mathcal{S}_0$ in corollary model dependent analyses.

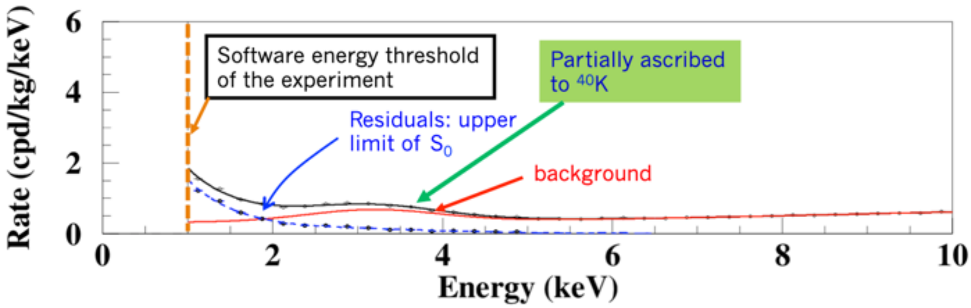


Fig. 1.5. Cumulative low–energy distribution of the *single–hit* scintillation events (that is each detector has all the others as veto), as measured by DAMA/LIBRA–phase2 in an exposure of 1.13 ton × yr.

In particular, in DAMA/LIBRA–phase2 $\mathcal{S}_0 \leq 0.80$ cpd/kg/keV in the (1–2) keV energy interval, $\mathcal{S}_0 \leq 0.24$ cpd/kg/keV in the (2–3) keV energy interval, and $\mathcal{S}_0 \leq 0.12$ cpd/kg/keV in the (3–4) keV energy interval, are obtained. These upper limits have to be properly taken into account as prior when corollary model–dependent analyses for a specific DM candidate and scenario through maximum likelihood procedure is pursued. The accounting of these priors assures more suitable determinations of allowed parameters space in this kind of analyses we published in literature.

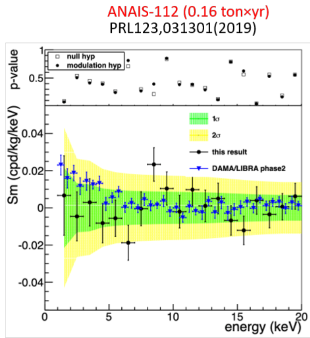
1.3 Some arguments on comparisons in the DM direct detection field

Let us shed light on few of the several arguments needed to correctly depict the present situation in the DM direct detection field.

- It is important to remind that the number of DM particles species, their nature, their interaction types, and the related astrophysical, nuclear and particle physics aspects (which also play a relevant role in model-dependent results and comparisons) are actually unknown at present level of knowledge. Indeed, a lot of different DM scenarios have been proposed. Furthermore, in several cases, complementary sensitivities are intrinsic in the use of different target-detectors or approaches. Sometimes it is even not possible to completely depict the more-than-2-dimensional volumes allowed/excluded at given C.L. under the considered theoretical assumptions and adopted experimental parameters. Therefore, considering both the large theoretical and experimental uncertainties, space for compatibility in the DM field can exist in various scenarios even assuming the sensitivity to $\mathcal{S}_0$ claimed by some other experiments. In addition, the exclusion plots have no general meaning, but are linked to the assumed model and to the adopted experimental parameters, and – in every case – they lose validity in presence of systematics. We already discussed related arguments in previous papers.

- As to the recently published results by COSINE–100 (from part of its detectors) and by ANAIS, they presently have no impact on the observed DAMA annual modulation result. See Fig. 1.6, which summarizes the situation. When DM candidates inducing nuclear recoils are considered, one has to introduce the quenching factors (q.f.). In addition, the q.f.'s have large variations among different detectors since they are property of the specific detector and not a general property of the material. See discussion e.g. in Ref. [24]. In details the q.f. values for nuclear recoils in COSINE–100 are lower than those in DAMA detectors; this is consistent with the q.f. for α's higher in DAMA than in COSINE–100. In conclusion, in case of DM candidates inducing nuclear recoils, experiments with lower q.f.'s values have different and lower sensitivity. As a matter of fact different q.f.'s are actually expected e.g. because of the different adopted procedures for the detectors' production.

Energy interval	Experiment	Exposure ton x yr	Rate (cpd/kg/keV)	Amplitude (cpd/kg/keV)
(2,6) keV	DAMA/LIBRA (ph1 + ph2)	2.17	0.8	0.0095 ± 0.0008
	COSINE-100	0.098	3.0	0.0083 ± 0.0068
	ANAIS-112	0.16	3.2	- 0.0044 ± 0.0058
(1,6) keV	DAMA/LIBRA-phase2	1.13	0.7	0.0105 ± 0.0011
	ANAIS-112	0.16	3.6	- 0.0015 ± 0.0063



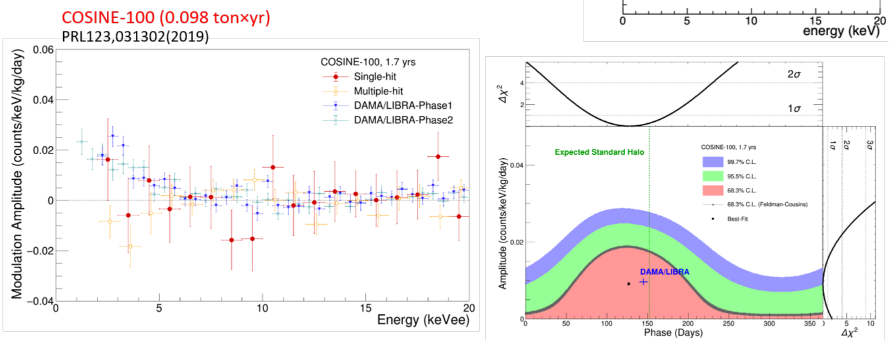


Fig. 1.6. Some aspects of comparison among DAMA/LIBRA annual modulation results and COSINE–100 and ANAIS (compared there by those authors with a partial exposure of DAMA). As evident they had no sufficient sensitivity to investigate the DAMA observed signal. Here both keV and keVee means keV electron equivalent; other problems arise when recoiling energy is addressed; see text.

- COSINE–100 has also used a different approach trying to extract the $\mathcal{S}_0$ observable – instead of $\mathcal{S}_m$ – from its large counting rate. In fact, the overall energy spectrum has been fitted with some assumed background components to try predicting with high precision the background at keV region. As we mentioned above, this is a dangerous and uncertain procedure which affects the result. Here, e.g., the existence of ^{129}I was omitted leading to an overestimate of the ^{210}Pb contents to fit with high precision the background at keV energy, with a evident systematic error model dependent that instead the investigation of $\mathcal{S}_m$ overcome at all. This has been somehow amended more recently, but exclusion limits were not corrected. In every case this further demonstrates how the "precise" determination of background at keV level through such method is untenable. Anyhow, even assuming their model dependent subtraction – and their given error estimates – as they claim in their paper, one can derive that (Data - model) = $-(0.105 \pm 0.276)$ cpd/kg/keV, corresponding an upper limit on $\mathcal{S}_0$: $\mathcal{S}_0 < 0.36$ cpd/kg/keV 90% C.L. in the (2–6) keV energy region; thus, no contradiction exists with the limit on $\mathcal{S}_0$ arising from DAMA/LIBRA (see Sect. 1.2.2). Moreover, as mentioned above, the poorer quenching factors in COSINE–100 with respect to the DAMA ones implies that the (2–6) keV interval corresponds to different recoil energies between the two experiments. On the other hand, in general, an experiment with much larger counting rate, much lower exposure, etc. cannot intrinsically be more sensitive than one with lower counting rate, much larger exposure, etc.
- Finally, as regards the DM indirect searches, that study the annihilation products of DM particles in galactic halo, no quantitative comparison can be directly performed with results obtained in direct investigations; it strongly depends on assumptions and on the considered model framework. In fact, the counting rate in direct search is proportional to the direct detection cross-sections, while the flux of secondary particles is connected also to the annihilation cross-section. In principle, these cross-sections can be correlated, but only when a specific model is adopted and by non-directly proportional relations.

In conclusion, there is no direct model independent contradiction with any available experiment so far, and DAMA results have deeply verified in very different conditions over many years. Its detectors have well different features than those recently developed because e.g. of the different adopted growing procedures, starting materials, purification methods and protocols procedures, of the long underground storage and of the exploited handling/running protocols. Finally, it should be recalled that positive hints have been published both by direct and indirect approaches. In both cases compatibility with DAMA results are possible in various scenarios as shown also in literature at some extent.

1.4 Efforts towards the further lowering of the software energy threshold

One of the possibilities to improve the DAMA/LIBRA performance at low energy foresees a change of all the PMTs with new high Q.E. metal PMTs of increased radio-purity, equipped with miniaturized low background new concept preamplifiers mounted on the same socket holding the components of the miniaturized HV divider, and few related improvement of the electronic chain. The aim of this upgrade is the improvement of the experimental sensitivity through a lower software energy threshold with large acceptance efficiency. In particular, the experimental sensitivity to the DM annual modulation signature is connected to the product: $\epsilon \times \Delta E \times M \times T \times (\alpha - \beta^2)$, where ϵ is the overall efficiency, ΔE is the energy region where the DM annual modulation is present, M is the exposed mass, T is the running time, and $(\alpha - \beta^2)$ shows how the data are collected along each annual cycle; it should approach 0.5 for a full year of data taking, crucial for a reliable investigation on DM annual modulation signature.

An increase of the sensitivity can allow to explore more extensively the DM annual modulation signature and to improve the measurement of the modulation parameters such as the phase, which brings important information. On the other hand, the goal of the DAMA project was not only to point out model-independent evidence for the presence of DM particles in the galactic halo, but also to investigate the nature of such particles and related astrophysical, nuclear and particle physics scenario. In particular, the main aims of an experiment with lower energy threshold are:

- to explore the DM annual modulation signature at lower software energy threshold with high overall efficiency, also offering the possibility to more effectively disentangle among some of the different proposed scenarios;
- to investigate possible presence of streams in the Galaxy (as we already did for Sagittarius in the past) also in the light of the recent GAIA results;
- to investigate possible presence of caustics or of effects of gravitational focusing of the Sun; and also to investigate the nuclear quantities entering in model dependent corollary analyses;
- to investigate with increased sensitivity the diurnal modulation, and other possible diurnal effects due e..g. to Earth shadow and channeling (refer to DAMA literature for detailed discussions);
- to investigate rare processes other than Dark Matter by analyzing either other parts of the energy spectrum or specific features of processes searched for, as previously done and published with the DAMA/NaI and DAMA/LIBRA–phase1 data (from keV up to tens MeV; see DAMA literature).

Therefore, the DAMA collaboration has been working towards this direction. In particular, the aim of the R&D is to improve the signal/noise ratio near software energy threshold, in order to disentangle the noise events (time decay of order of tens ns) and the scintillation pulses (time decay of order of 240 ns) with high overall efficiency also below 1 keV [8,14,15]. Thus, new high Q.E. and low radioactivity PMTs have been developed by the Hamamatsu co. on the basis of our own specifications. The main ones are: 1) Q.E. at λ= 420 nm 30%- 40%; 2) dark current <100 cps; 3) 3″ window diameter; 4) particular size and shape designed to fit the already-existing honeycomb shaped copper shield around PMTs, and to minimize the amount of material; 5) multiplication factor $> 10^6$; 6) peak/valley ratio >2.5; 7) PMTs radio-purity at level of few mBq/PMT.

Moreover, new voltage divider-preamplifier systems mounted on the same Pyralux board have been developed. New preamplifiers for the new metal PMTs have been developed to realize a single device with high signal/noise ratio, where the voltage divider and the preamplifier are integrated on the same board. The preamplifier is based on the operational amplifier LMH6624 by Texas Instruments working at ±5V, with an input bias current of -15 μA at 300 K and a bandwidth of 1.5 GHz.

The preamplifier and the voltage divider are printed on the first and third layer of a Pyralux board [27]; between these two layers a second ground layer is placed. The board is directly mounted in the back of the PMT inside the copper honeycomb structure of the shield. In such a way, the preamplifiers are as close as possible to the input source – that is the anode of the PMT – instead of those used so far in DAMA/LIBRA–phase2 which are allocated outside the internal part of the set-up's shield. The new configuration is now possible thanks to the improved radio-purity both of the new metallic PMTs and of the new preamplifiers. Thus summarizing, a better signal/noise factor can be obtained with respect to the case of an external and more distant preamplifier and the overall radio-purity is improved as well.

Measurements with and without the preamplifier have been performed to characterize the metallic PMTs, their response and the voltage divider integrated with the preamplifier:

- Spectral response of the PMT, expressed by the Q.E., as a function of λ;
- Single Photon Pulse Height Distribution of the metallic PMTs obtained by irradiating the device with single photon pulse;
- Dark Pulse Height Distribution of the PMTs;
- Dark Current.

The produced boards, allocating the voltage divider and the preamplifier, have been tested from the radioactivity point of view. Three boards equipped with all the components (total mass of 14.1 g) were measured for 6.87 day in a Germanium detector of the STELLA facility of the LNGS. The measured residual activity of ^{232}Th, ^{235}U, ^{40}K, ^{137}Cs and ^{60}Co is much lower than that of the single PMT. The only concern is for ^{226}Ra, which stays at level of 13 mBq per piece. This is a typical feature for such devices: the Pyralux support has generally activity about hundred times lower for ^{226}Ra and about 30 times lower for ^{228}Ra and ^{228}Th. Thus, the small residual activity is mainly due to the electronic components. Considering that the system voltage divider integrated with the preamplifier is placed behind the PMT, and shielded in part by the honeycomb Cu structure, its role in the total background is negligible.

At present four of the DAMA/LIBRA detectors are already equipped and they are in data taking using the configuration with metal R11065-20 MOD PMTs and the developed voltage divider-preamplifier system. In the light of these developments and goals, also other alternative and cheaper configurations are under study to further lower the software energy threshold of the detectors.

1.5 Conclusions

The DAMA has been a pioneer project in the direct detection of DM, obtaining the first model-independent evidence for the presence of a particle component of the DM in the galactic halo on the basis of the exploited DM annual modulation signature.

Three independent experimental set-ups, and their upgrades have confirmed the presence of a peculiar annual modulation of the *single–hit* events in the energy region (1–6) keV, that meets all the many requirements of the DM annual modulation signature; the cumulative exposure, considering them all together is 2.46 tons $\times$ yr (over 20 independent annual cycles).

In particular:

- the *single–hit* events show a clear modulation in accordance with the cosine function, as expected for a signal induced by DM particles;
- the measured period is (0.999 ± 0.001) yr, well compatible with a period of 1 year, as expected for a signal of DM;
- the measured phase: (145 ± 5) d, is compatible with about 152.5, which is the expected value for a DM signal;
- modulation is present only in the low energy range (1–6) keV and not in other higher energy regions, consistently as required for a DM signal;
- modulation is present only in the *single–hit* events, while it is absent in *multiple–hit* events, as expected for a DM signal;
- the measured modulation amplitude using a NaI(Tl) target for the *single–hit* scintillation events in the energy range (2–6) keV is: (0.0103 ± 0.0008) cpd/kg/keV (12.9 σ C.L.).

No systematic or side processes is able to account for the observed signal are available. Corollary investigations on the nature of the DM particle(s) in given scenarios have been performed by corollary model-dependent analyses. Various models and parameters (experimental and theoretical) are possible and many hypotheses have to be considered [24]. In particular, the model-independent evidence obtained by DAMA is compatible with a wide set of astrophysical, nuclear and particle physics scenarios for high and low mass candidates that induce nuclear recoil and/or electromagnetic radiation, as shown extensively in literature.

The experiment is collecting data; moreover, R&Ds have been funded and developed to further lower the software energy threshold with high acceptance efficiency – among others – to further efficiently disentangle among at least some of the many possible DM candidates and scenarios.

Finally, for completeness, we note that also all the other DAMA low background set-ups are running and related developments are in progress.

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2 On the Reactions Involving Neutrinos and Hidden Mass Particles in Hypercolor Model

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Abstract. We present and discuss some basic elements of the Standard Model hypercolor extension. Appearance of a set of hyperquarks bound states is resulted from σ−model using; due to specific symmetries of this minimal extension, there arise stable hypermesons and hyperbaryons which are interpreted as the Dark Matter candidates. Knowing estimations of their masses from analysis of Dark Matter annihilation kinetics, some processes of high energy cosmic rays scattering off these particles are analyzed for the search of Dark Matter manifestations.

Povzetek. Avtorja obravnavata σ−model z dodano grupo hiper kvarkov. V tem modelu poiščeta stabilna vezana stanja hyper mesonov in hyper barionov. Interpretirata jih kot gradnike temne snovi. Iz izmerjenih lastnosti temne snovi ocenita mase teh hiper delcev. Obravnavata morebitne opazljive efekte sipanja kozmičnih žarkov visokih energij na teh delcih.

Keywords: vector hypercolor, Dark Matter, cosmic rays
PACS: 12.60 - i, 96.50.S-,95.35.+d.

2.1 Introduction

The presence in the Universe of so-called hidden mass, which manifests itself in the formation of the observed structure of galaxies and their clusters, is confirmed mainly by a quantitative analysis of the gravitational interaction of stellar systems with these invisible neutral stable objects for which neither dynamics nor evolution in time is exactly known. Astrophysical confirmations of distributed hidden mass influence on the star clusters dynamics, the alleged effects of Dark Matter (DM) particles induced by their presence in the massive objects composition, an increased density of DM near active galactic nuclei (AGN), a possible DM effect on the composition and parameters of propagation of high-energy cosmic ray fluxes in the Universe — search for answers on these and some other questions of high energy physics, astrophysics and cosmology are the primary tasks of fundamental physics. Elucidation of the mysterious nature of Dark Matter is complicated by

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the fact that terrestrial experimental physics at colliders, as it has become clear by now, cannot detect any traces of DM particles. The impossibility to identify processes with large missed energies and momenta which are characteristic of the DM production at the colliders, i.e. in an active experiments, is accompanied by the absence of signals of the DM particles interactions with nuclei and nucleons in passive experiments, in underground laboratories. Another type of passive experiments gathering astrophysical data by space telescopes play very important role, recording the specific spectra of cosmic photons, leptons, and baryons in the vicinity of the Earth.

In this situation, any astrophysical data capable of shedding light on the hidden mass nature are valuable. In particular, these are indirect signals about possible DM effects, which can be interpreted consistently. For an unambiguous interpretation within the framework of a certain paradigm about the origin and dynamics of the DM particles, it is of particular importance to study the correlations between characteristics of various astrophysical phenomena and to consider also DM interactions with different types of particles and astrophysical objects at various space-time scales and in a wide energy ranges. Investigation of all aspects of DM physics within the framework of multi-messenger approach becomes key for establishing the SM extension type. Because of lack of new information about possible hidden mass carriers, we should examine various reasonable ideas allowing to move beyond the SM. In this way, we come to consideration of new objects with some new dynamics, and we can realize a quantitative analysis of known or expected physical effects interpreting as the DM manifestations. Here, we present some results on high-energy cosmic rays interaction with the DM candidates arising in hyper-color SM extension. In the following Section 2 we present in brief some detail of minimal vectorial hyper-color model; then, Section 3 contains discussion of cosmic rays scattering off DM objects in H-color scenario. There are some new preliminary results of high-energy cosmic proton interaction with the hyper-pions, ones of the DM components, in the Section 4. We also add Conclusion and Discussion of this scenario in the end.

2.2 Basics of hypercolor extension of Standard Model

Hyper-color approach modifies the SM by extending it with additional heavy fermions charged under some new gauge group [1–9]. In fact, these new fields, hyperquarks, are similar to techniquarks, however, in this case vectorial interaction of H-quark currents with the gauge bosons can be provided by some transformation of initial fermion doublets. In this way, some problems of Technicolor can be eliminated. It is the vector-like interaction is the reason why the model is in accordance with strong electro-weak constraints. Certainly, the H-quarks are confined with new strong interaction and, remembering the Technicolor ideas, these models with extra heavy H-quarks can result to the scenarios with composite Higgs doublets (see e.g. [10]) or a small mixing between fundamental standard Higgs bosons and composite hadron-like states of new strong sector. In this way, we come to partially composite Higgs boson. Due to accidental symmetries in these models, there occur neutral stable states which can be interpreted as DM candidates.

Among the simplest realizations of the scenario described, there are models with two or three vector-like H-flavors confined by strong H-color force $Sp(2n_F)$, $n_F \geqslant 1$. The models with H-color group SU(2) (see Refs. [8, 9] and references therein) can be considered as particular cases as a consequence of isomorphism $SU(2) = Sp(2)$. The global symmetry group of the strong sector with symplectic H-color group is larger than for the special unitary case—it is the group $SU(2n_F)$ broken spontaneously to $Sp(2n_F)$, with n_F being a number of H-flavors. Here we consider the case when the elementary Higgs doublet is preserved in the set of Lagrangian field operators. Then, the scalar doublet mixes with H-hadrons, and we get the physical Higgs partially composite. Note also that the same coset $SU(2n_F)/Sp(2n_F)$ can be used to construct composite two Higgs doublet model [10] or little Higgs models [11–15].

In fact, the model has the symmetry $G = G_{SM} \times Sp(2n_f)$ with $n_f \geqslant 1$, here G_{SM} and $Sp(2n_F)$ are the gauge SM group and a symplectic hypercolor group respectively. In its field content, the model introduces a doublet and a singlet of heavy vector-like H-quarks charged under H-color group. In the most general form, renormalizable and invariant under G Lagrangian can be written as

$$L = L_{SM} - \frac{1}{4} H^{\mu\nu}_{\underline{a}} H^{\underline{a}}_{\mu\nu} + i\bar{Q}DQ - m_Q \bar{Q}Q + i\bar{S}DS - m_S \bar{S}S + \delta L_Y, \tag{2.1}$$

$$D^{\mu}Q = \left[\partial^{\mu} + \frac{i}{2} g_1 Y_Q B^{\mu} - \frac{i}{2} g_2 W^{\mu}_{a} \tau_a - \frac{i}{2} g_{\tilde{c}} H^{\mu}_{\underline{a}} \lambda_{\underline{a}}\right] Q, \tag{2.2}$$

$$D^{\mu}S = \left[\partial^{\mu} + i g_1 Y_S B^{\mu} - \frac{i}{2} g_{\tilde{c}} H^{\mu}_{\underline{a}} \lambda_{\underline{a}}\right] S, \tag{2.3}$$

where $H^{\mu}_{\underline{a}}$, $\underline{a} = 1 \ldots n_F(2n_F + 1)$ are hypergluon fields and $H^{\mu\nu}_{\underline{a}}$ are their strength tensors; τ_a are the Pauli matrices; $\lambda_{\underline{a}}$, $\underline{a} = 1 \ldots n_F(2n_F + 1)$ are $Sp(2n_F)$ generators satisfying the relation

$$\lambda^{T}_{\underline{a}} \omega + \omega \lambda_{\underline{a}} = 0, \tag{2.4}$$

where T stands for the transition operation, ω is an antisymmetric $2n_F \times 2n_F$ matrix, $\omega^T \omega = 1$. All underscored indices correspond to representations of the H-color group $Sp(2n_F)$. In the Lagrangian (2.1), the contact Yukawa couplings δL_Y of the H-quarks and the SM Higgs doublet H are permitted by the symmetry G if the hypercharges Y_Q and Y_S satisfy an additional linear relation:

$$\delta L_Y = y_L \left(\bar{Q}_L H\right) S_R + y_R \left(\bar{Q}_R \varepsilon \bar{H}\right) S_L + \text{h.c.} \quad \text{for } \frac{Y_Q}{2} - Y_S = +\frac{1}{2}; \tag{2.5}$$

$$\delta L_Y = y_L \left(\bar{Q}_L \varepsilon \bar{H}\right) S_R + y_R \left(\bar{Q}_R H\right) S_L + \text{h.c.} \quad \text{for } \frac{Y_Q}{2} - Y_S = -\frac{1}{2}. \tag{2.6}$$

Indeed, the hypercolor part of the H-quark Lagrangian (2.1) can be rewritten in terms of a left-handed sextet as follows:

$$\delta L_{Hq,\,kin} = i\bar{P}_L D P_L, \qquad P_L = \begin{pmatrix} Q_L \\ \epsilon\omega {Q_R}^C \\ S_L \\ -\omega {S_R}^C \end{pmatrix}, \tag{2.7}$$

$$D^\mu P_L = \left[\partial^\mu - \frac{i}{2} g_{\tilde{c}} H^\mu_{\underline{a}} \lambda_{\underline{a}}\right] P_L, \tag{2.8}$$

where $\epsilon = i\tau_2$, the operation C denotes the charge conjugation. In the absence of the electroweak interactions, the H-quark Lagrangian is invariant under an extension of the chiral symmetry—a global SU(6) symmetry [16, 17]. The set of SU(6) subgroups is the following:

- the chiral symmetry $SU(3)_L \times SU(3)_R$,
- SU(4) subgroup corresponding to the two-flavor model without singlet H-quark S,
- two-flavor chiral group $SU(2)_L \times SU(2)_R$, which is a subgroup of both former subgroups.

The global symmetry of the model is broken both explicitly and dynamically:

- explicitly—by the electroweak and Yukawa interactions and the H-quark masses;
- dynamically—by H-quark condensate [18, 19]:

$$\langle \bar{Q}Q + \bar{S}S \rangle = \frac{1}{2} \langle \bar{P}_L M_0 P_R + \bar{P}_R M_0^\dagger P_L \rangle, \qquad P_R = \omega {P_L}^C, \tag{2.9}$$

$$M_0 = \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}. \tag{2.10}$$

Note, condensate (2.9) is invariant under $Sp(6) \subset SU(6)$ transformations U that satisfy a condition

$$U^T M_0 + M_0 U = 0, \tag{2.11}$$

so the global SU(6) symmetry is dynamically broken to Sp(6) subgroup. Further, H-quarks mass terms break the symmetry to $Sp(4) \times Sp(2)$:

$$\delta L_{Hq} = -\frac{1}{2} \bar{P}_L M_0' P_R + \text{h.c.}, \; M_0' = -{M_0'}^T = \begin{pmatrix} 0 & m_Q \varepsilon & 0 \\ m_Q \varepsilon & 0 & 0 \\ 0 & 0 & m_S \varepsilon \end{pmatrix}. \tag{2.12}$$

The model under consideration is free of gauge anomalies and is in agreement with the electroweak precision constraints, since the H-quarks are vector-like, i.e. their electroweak interactions are chirally symmetric.

The case of two-flavor model (without the singlet H-quark) is completely analogous to the three-flavor model but is simpler than the latter one— global

SU(4) symmetry is broken dynamically to its Sp(4) subgroup by the condensate of doublet H-quarks; corresponding Lagrangian of the model is presented in detail in [8,9]. To operate with interacted constituent H-quarks and their bound states, there were used a linear σ-model; the model Lagrangian consists of kinetic terms for the constituent fermions and the lightest (pseudo)scalar composite states, Yukawa terms for the interactions of the (pseudo)scalars with the fermions, and a potential of (pseudo)scalar self-interactions U_{scalars} [8,9] (remind that we consider here fundamental Higgs doublet of the SM).

It also postulated that the constituent H-quarks interact with the gauge bosons as the fundamental H-quarks. Then, we have transformation laws for the covariant derivative for the scalar field M. The complete set of covariant derivatives which are involved into the model Lagrangian is as follows:

$$D_\mu H = \left[\partial_\mu + \frac{i}{2} g_1 B_\mu - \frac{i}{2} g_2 W_\mu^a\right] H, \tag{2.13}$$

$$D^\mu P_L = \left[\partial^\mu + i g_1 B^\mu \left(Y_Q \Sigma_Q + Y_S \Sigma_S\right) - \frac{i}{2} g_2 W_a^\mu \Sigma_W^a\right] P_L,$$

$$\begin{aligned} D_\mu M &= \partial_\mu M + i Y_Q g_1 B_\mu (\Sigma_Q M + M \Sigma_Q^T) \\ &+ i Y_S g_1 B_\mu (\Sigma_S M + M \Sigma_S^T) - \frac{i}{2} g_2 W_\mu^a (\Sigma_W^a M + M \Sigma_W^{aT}). \end{aligned} \tag{2.14}$$

For detail see [8,9], matrices Σ_Q, Σ_S, Σ_W^a, $a = 1, 2, 3$ also are presented there.

Setting the H-quarks hypercharges to zero, the model Lagrangian is invariant under an additional symmetry—hyper G-parity [20,21]:

$$Q^{\tilde{G}} = \varepsilon \omega Q^C, \qquad S^{\tilde{G}} = \omega S^C. \tag{2.15}$$

This transformation does not involve H-gluons and SM fields, so the lightest $\tilde{G}$-odd H-hadron becomes stable. It happens to be the neutral H-pion π^0. Besides, the numbers of doublet quarks are conserved in the minimal SU(4) model, because of global U(1) symmetry group of the Lagrangian, so we get the neutral singlet H-baryon B stable. Note also that the $\tilde{G}$-parity is induced by a discrete symmetry, and not with a continuous transformation of the H-pion states. So, higher order corrections cannot destabilize neutral weakly interacting H-pion, which is the lightest state in the pseudoscalar triplet. But charged H-pion states should decay by several channels producing charged leptons and neutral H-pion.

In any case, we can interpret both stable objects in H-color model as two-components of the DM (multi-component structure od DM was analyzed in a number of papers [22–26]). Importantly, to be in a correspondence with precision SM data the angle of mixing between $\tilde{\sigma}-$ meson and Higgs boson should be small $\sin\theta \ll 0.1$, then Peskin-Tackeuchi parameters agree with experimental restrictions [8].

2.3 Cosmic lepton scattering off Dark Matter

Now, in strong and EW channels the width of the charged H-pion decay [8] can be found as

$$\Gamma(\tilde{\pi}^{\pm} \to \tilde{\pi}^0 l^{\pm} \nu_l) = 6 \cdot 10^{-17}\,\text{GeV}, \quad \tau_{\pi} = 1.1 \cdot 10^{-8}\,\text{sec}, \quad c\tau_{\pi} \approx 330\,\text{cm};$$
$$\Gamma(\tilde{\pi}^{\pm} \to \tilde{\pi}^0 \pi^{\pm}) = 3 \cdot 10^{-15}\,\text{GeV}, \quad \tau_l = 2.2 \cdot 10^{-10}\,\text{sec}, \quad c\tau_l \approx 6.6\,\text{cm}. \qquad (2.16)$$

The DM candidates, $\tilde{\pi}^0$ and B^0, have equal tree level masses, but the mass splitting $\Delta M_{B\tilde{\pi}} = m_{B^0} - m_{\tilde{\pi}^0}$ in one loop depends on a renormalization point as a consequence of couplings of these pNG states with different H-quark currents. We also assume that not-pNG H-hadrons (vector H-mesons, etc.) manifest itself at much more larger energies. It results from the smallness of the scale of explicit SU(4) symmetry breaking comparing with the scale of dynamical symmetry breaking.

Obviously, the search for two-component DM signals is possible in the (approximately) known range of DM candidates masses. Calculating cross sections of DM components annihilation in all channels, we can analyze kinetics of the DM freezing out. Assuming both of mass splittings are small in comparison with the mass, coupled system of five Boltzmann kinetic equations for all stable components (with an account of co-annihilation reactions) is numerically solved.

As it is shown in detail in [27], there are a set of regions in a plane of H-pion and H-sigma masses (see Fig.1), where it is possible to fix the DM relic density in agreement with the modern data. Because only H-pions interact with vector

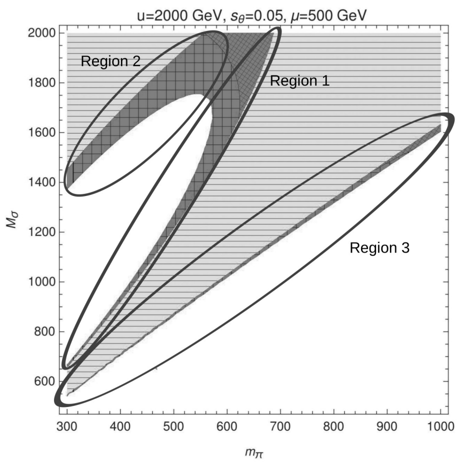


Fig. 2.1. Phase diagram in terms of $M_{\tilde{\sigma}}$ and $m_{\tilde{\pi}}$ which is resulted from numerical solution of the kinetic equations system.

bosons, there are no regions where this component dominates in the DM density. The stable B^0-baryons interact with matter only via H-quark and H-pion loops and scalar exchange channels. It is a specific feature of SU(4) vector-like model with two stable pNG states. From numerical tree level analysis there are three allowable regions of parameters (masses):

Region 1: here $M_{\tilde{\sigma}} > 2m_{\tilde{\pi}^0}$ and $u \geq M_{\tilde{\sigma}}$; at small mixing, $s_\theta \ll 1$, and large mass of H-pions we get a reasonable value of the relic density and a significant H-pion fraction;

Region 2: here again $M_{\tilde{\sigma}} > 2m_{\tilde{\pi}^0}$ and $u \geq M_{\tilde{\sigma}}$ but $m_{\tilde{\pi}} \approx 300 - 600$ GeV; H-pion fraction is small here, approximately, $(10 - 15)\%$;

Region 3: $M_{\tilde{\sigma}} < 2m_{\tilde{\pi}}$ — this region is possible for all values of parameters, but decay $\tilde{\sigma} \to \tilde{\pi}\tilde{\pi}$ is prohibited. Here, H-pion fraction can be up to 40% for $m_{\tilde{\pi}^0} \sim 1$ TeV and small mixing between scalars.

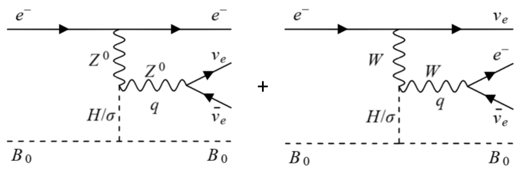
So, possible values of H-pion mass should vary approximately in the range $(600–1200)$ GeV in agreement with recent astrophysics data. Because some hopeful results from colliders are absent, we consider indirect searches of DM manifestations in astrophysical data [28–32]. Now, we come to study of high-energy cosmic rays quasi-elastic scattering off the DM [27, 33–36]. Most simple reaction in H-color scenario is cosmic ray electrons scattering off H-pion component via weak boson in the process $e\tilde{\pi}^0 \to \nu_e\tilde{\pi}^-$, then charged $\tilde{\pi}^-$ will decay as it was indicated above. In the narrow-width approximation we get for the cross section: $\sigma(e\tilde{\pi}^0 \to \nu_e\tilde{\pi}^0 l\nu_l') \approx \sigma((e\tilde{\pi}^0 \to \nu_e\tilde{\pi}^-) \cdot Br(\tilde{\pi}^- \to \tilde{\pi}^0 l\nu_l')$, branchings of charged hyperpion decay channels are: $Br(\tilde{\pi}^- \to \tilde{\pi}^0 e\nu_e') \approx 0.01$ and also $Br(\tilde{\pi}^- \to \tilde{\pi}^0\pi^-) \approx 0.99$. Considering final charged hyperpion $\tilde{\pi}^-$ near its mass shell, standard light charged pion produces neutrino $e\nu_e$ and $\mu\nu_\mu$ with following probabilities: $\approx 1.2 \cdot 10^{-6}$ and ≈ 0.999, correspondingly.

Then, an energetic cosmic electron produces electronic neutrino and soft secondary $e'\nu_e'$ or $\mu\nu_\mu$ arise from charged H-pion decays. Now, there are final states with $Br(\tilde{\pi}^0\nu_e\mu'\nu_\mu') \approx 0.99$ and $Br(\tilde{\pi}^0\nu_e e'\nu_e') \approx 10^{-2}$. These results are justified in the framework of the factorization approach [37].

We get that initial electron with energies in the range $E_e = (100 - 1000)$ GeV interacts with cross section decreases from $O(10)$ up to $O(0.1)$ nb and there is a maximum at small angles between electron and the neutrino emitted [27]). In this approximation, energy of the neutrino produced is proportional to incident electron energy and depends on the mass of the Dark Matter particle very weakly. The neutrino flux is calculated by integrating of spectrum, dN/dE_ν, this flux depends on H-pion mass very weakly. In the interval $(50 - 350)$ GeV it decreases most steeply, and then, down to energies ~ 1 TeV the fall is more smoother.

We also estimate number of neutrino landings on the IceCube surface and (even with an amplification the DM density near the Galaxy center for the symmetric Einasto profile), we get very small number of neutrino events per year: $N_\nu = (6 - 7)$. Indeed, this number can be increased for cosmic rays energy in multi-TeV region. However, the electron flux is only small part of the cosmic rays total flux especially at energies $\geq 10^2$ TeV. As a result, we predict a very small fluxes of secondary neutrinos and, consequently, small probability to detect such events at IceCube [27, 36]. Cosmic rays scattering off DM clusters of very high density [38, 39] can result in amplifying secondary neutrino flux [40, 41].

It seems that there is a chance to introduce the B^0 interaction through H-pion and/or H-quark loops, however for the scattering channels these loops are exactly zero [27]. Thus, we need to consider of more complex tree diagrams, in particular, tree diagrams with the exchange of Higgs boson and its partner, $\tilde{\sigma}-$meson in t-channel give dominant non-zero contribution to the process $e^- B \to \nu_e W^- B$ (see Fig.2.a). Virtual W^-—boson eventually decays to $l\bar{\nu}_l$ or into light ordinary mesons. Of course, there is similar scattering reaction with the scalar states exchange, $e^- \tilde{\pi}^0 \to \nu_e W^- \tilde{\pi}^0$, whose amplitude is half as it is seen from the model Lagrangian. We have found, these diagrams give dominant tree level part of cosmic particles scattering cross section, and we do not take into account small contributions from H-quark loops, hhZ and other multi-scalar vertices [36]. To calculate total cross



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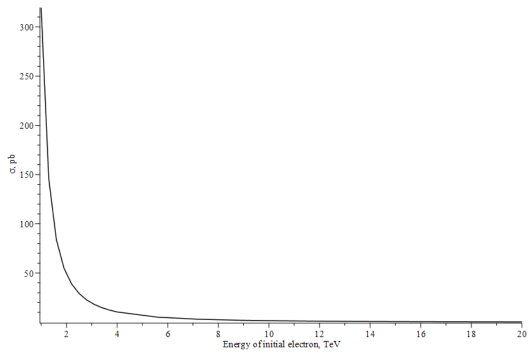


b)

Fig. 2.2. Quasi-elastic cosmic lepton scattering with energies $(1 - 20)$ TeV off H-baryon Dark Matter component: a) necessary Feynman diagrams; b) total cross section for $m_B = 1200$ GeV.

section of the process with final state $B^0 e^- \nu\bar{\nu}$ or $\tilde{\pi}^0 e^- \nu\bar{\nu}$, it was used factorization method [37] considering independently amplitudes squared of subprocesses with intermediate W and Z-bosons and then estimating the (negative) interference of these contributions. The approach allows to estimate with reasonable accuracy (no worse than $\sim 10\%$ due to approximate estimation of the interference) cross section of an "averaged" process where final electron and neutrinos are produced by different vertices, $W \to l\nu_l$ and $Z \to \nu_l \bar{\nu}_l$.

So, we get a reasonable evaluation of total cross section (Fig.2b) and can estimate also the possibility to detect at IceCube the neutrino signal producing by the process of electron scattering off the DM. In this calculation, we restrict ourselves those phase space regions which do not include an acceleration of initial DM particle. In other words, final DM components are slow and nearly all energy of the incident lepton is distributed between three final massless leptons (electron and pair of neutrinos). Of course, for neutrino scattered via W-vertex cross section is the same. The secondary neutrino fluxes calculated are very small in comparison with expected neutrino fluxes from AGN which can be $\sim 10^5$ $cm^{-2}s^{-1}sr^{-1}$. Atmospheric neutrino fluxes with neutrino energies $\leq$ 2TeV are also much larger [42–44]: $\sim 10^{-10} - 10^{-9}$ $cm^{-2}sr^{-1}s^{-1}$. Namely, we get the secondary neutrino flux resulted from the cosmic electrons scattering is $\sim 10^{-19} - 10^{-22}$ $cm^{-2}sr^{-1}s^{-1}$ [36].

Interestingly to note a specific scattering process when high-energy intergalactic neutrino interact with the DM via neutral Z-vertex as $\nu_l + DM \to \bar{\nu}_l + Z^* + DM \to \bar{\nu}_l + \nu_k\bar{\nu}_k + DM$. Then three secondary neutrinos are produced and can be accompanied with the accelerated DM particle. This process can be informative especially because both of these neutral objects can be messengers from regions of high DM density — regions near AGN or from possible DM inhomogeneities of some other nature —and early epoch of the Universe [45]. Work on analysis of such reactions is in progress.

Thus, there are some points which are important for study of the cosmic rays scattering off the DM. Independently of the SM extension, processes with scalar exchanges results in a strong dependence of the cross section on the DM particle mass giving dominant part of the total ctoss section. In H-color scenario, increasing of the DM of 10 % provides the cross section grow up to 50 %. The opening of channels with scalar exchanges allows to consider an additional ways to produce secondary high-energy leptons and photons by ultra high-energy cosmic rays (UHECR) scattering off the DM.

2.4 High energy protons and the Dark Matter particles acceleration

Remind that expected number of neutrino events is too small to be measured in experiments at modern neutrino observatories. The weakness of the signal is also resulted from effective bremsstrahlung of electrons and the smallness of electron fraction in cosmic rays, ~ 1 %. Therefore, they are not so good probe for the DM structure; only if there are sharply non-homogeneous spatial distribution of hidden mass, the signal of production of energetic neutrino by cosmic electrons can be detected. It is an important reason to study inelastic scattering of cosmic protons, because they are more energetic and have a much larger flux.

The possibility to accelerate light DM particles due to scattering of high-energy cosmic rays off the DM was recently supposed and numerically analyzed in a numerous papers [46–52]. In [53] kinetic energy of DM particle which was initially at rest and then has been accelerated by high-energy cosmic ray particle, was calculated assuming the scattering reaction is elastic and isotropic. This simplified analysis of the DM acceleration was used for the light DM candidates. It seems, however, in the approximation of elastic reaction in the CMS we can use the same simple formula from [53] to estimate possibility of boosting for heavy DM objects. Namely, we have

$$T_{DM} = \frac{1}{2}(T_{CR}^2 + 2mT_{CR})(T_{CR} + M/2)^{-1} \cdot (1 + \cos\theta), \tag{2.17}$$

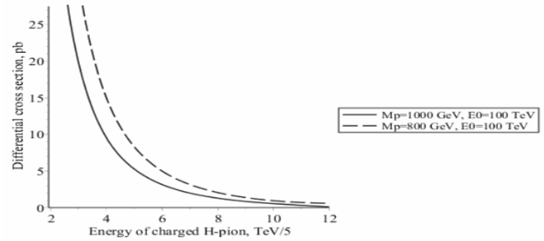
where m and M - masses of cosmic ray and DM particles, correspondingly, T_{DM} and T_{CR} - kinetic energies of DM particle after the scattering and projectile (cosmic ray particle), θ - angle of scattering in CMS (here we consider $M >> m$). Further, for $T_{CR} >> M$ we get an estimation $T_{DM} \sim T_{CR}$. So, heavy DM objects with masses ~ 1 TeV, as it takes place in H-color scenario, can be effectively accelerated by cosmic ray protons of high energies $\sim 10^2$ TeV. Cosmic rays with such energies can be generated near AGN, in particular. In other words, fast protons from blazar's jets can interact with heavy DM particles from halo having the largest density near AGN. In the framework of the H-color scenario, a significant part of protons energy can be transferred due to charged current to heavy H-pion and to both DM component in the diagrams involving scalar exchange as it takes place for leptons scattering off H-baryon component (see Fig.2a).

As it is seen from Fig.3a,b, if an initial proton with energy in the range $50-200$ TeV interacts with neutral H- pion which is nearly at rest, cross section of the scattering process where final charged H-pion is produced with energies $(40-50)$ TeV is $\sim (10-15)$ pb. Certainly, secondary charged H-pion predominantly decays as $\tilde{\pi}^{\pm} \to \tilde{\pi}^{o}\pi^{\pm}$ with the width $\Gamma \approx 3\cdot 10^{-15}$ GeV. So, besides neutral H-pion there appear secondary muon and muonic anti-neutrino in the final state.

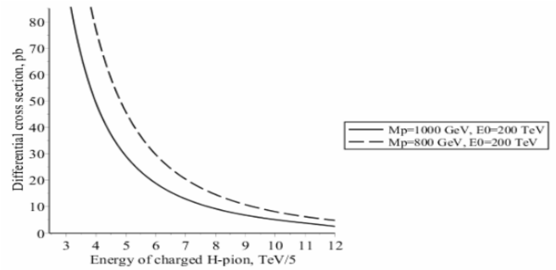
Thus, in the deep inelastic reaction the main charged component of UHECR i.e. protons, can transform in part into flux of high energy neutrino and leptons accompanied with accelerated DM particle. From our estimations it follows that $\sim (10-25)\%$ of the proton energy is transferred to heavy neutral component of the DM with cross section $\approx (10-100)$ pb. When the UHECR scatter off B-component, total cross section is of the same order but there can appear additional neutrinos generated by decay of intermediate vector bosons.

It should be noted, to get an estimation of these cross sections we have used a very simple model of quark distribution function in the proton: $q(x) \approx A\cdot x^{\alpha}\cdot(1-x)^{\beta}$. In other words, we used very simplified an analytical expression for pdf's (see also Refs. [54, 55]) because at these high energies we do not know an exact form of pdf's and only try to get some reasonable evaluation of the cross section. We assume, an error in these estimations cannot be more than in one order. Indeed, cross section of the scattering with the DM particle acceleration is small, but in these rare events heavy DM particles can accelerate significantly and pass away from the DM halo. Then, they move like neutrino but slower, and keep nearly constant direction due to weak interactions with the matter. So, this rare process when the charge component of cosmic rays can be ruined in the deep inelastic reaction and as a result neutral DM particle moves like a neutrino towards the Earth. Remind that above considered high-energy electron scattering off the DM can also accelerate the DM but in quasi-elastic process high energy neutrino are generated with more probability.

Thus, from this brief description of some processes of scattering of high-energy cosmic ray particles off the DM we can conclude that these reactions can enrich the cosmic rays composition with boosted heavy neutral DM particles [56]. At energies of these projectiles $\sim (10-100)$ TeV cross sections of their interactions with nucleons and nuclei, $\sim (10^{-34}-10^{-37})$ cm^2, are compared with cross sections of neutrino-nucleons scattering. In this deep-inelastic process nucleons or nuclei



a)



b)

Fig. 2.3. Differential cross section for cosmic ray proton scattering off H-pion DM component: a) initial proton energy 100 TeV; b) initial proton energy 200 TeV.

are transformed a multiparticle final states consisting of charged leptons, photons and neutrino. Additional neutrinos are generated by the charged H-pion decay and in the processes with resonant decay of Z-boson. So, the accelerated neutral DM components can produce rare events - specific types of extended air showers (EAS), which can be separated in the atmosphere from other types of showers [57,58].

It is known that as usually cosmic rays generate a shower of secondary particles which are mainly muons, electrons and photons. They go to ground detectors and can be fixed as measured signals registering also due to fluorescence and Cherenkov light, and radio emission generated by charged component, electrons, in atmosphere of the Earth. It seems, such type of shower is similar to neutrino induced shower and its initial point also should be deeply in atmosphere, however, the neutral DM particle can not disappear from the EAS composition and will interact with the ground detector producing some radiation from secondary electrons or from excited nuclei in the detector. The DM showers, as they generated by intergalactic DM objects which were accelerated by UHECR or AGN jets from halo of other galaxies, or DM particles boosted from halo of our Galaxy by intergalactic UHECR do not have to be mostly inclined or nearly horizontal. It is supposed, these accelerated DM components and EAS produced by them should be distributed more or less isotropic. May be, the EAS axis can be connected with direction to some blazar, as it was found for some very high-energy neutrino events at IceCube.

So, we can conclude that EAS from heavy DM particles are distinguished from EAS generated by protons or neutrino because in the former event the shower contains in his composition neutral stable object up to the final moment when this fast DM particle scattered on nucleon in the detector (see also [59] and references therein). In contrary, in composition of EAS which was induced by neutrino or protons (or light nuclei), there is no any heavy stable particles, only leptons, photons and neutrino are detected as final states. Note also that interaction of DM component with nucleons in detector should have specific signature: the scattering in charged current channel is accompanied with creation and following decay of charged H-pion, so, the event can be seen due to charged lepton bremsstrahlung. We hope, observing and measurement of characteristics new types of EAS containing heavy neutral stable particle will be possible at modern complex LHAASO [60], in other words, the DM candidates can manifest itself in a specific types of EAS.

2.5 Conclusions and Discussion

It is known, hadrons, leptons, photons with energies $E \geq 10^7$ TeV cannot reach the Earth because they interact with γ-bkg and loss the energy. High-energy photons with $E \geq 10$ TeV also practically cannot reach the Earth due to interactions with γ-bkg of various wave lengths — electron-positron pairs creation decreases photon energies below 10 TeV. But intergalactic high energy neutrino can move to the Earth being generated, for example, in blazars jets. Neutrinos being produced in decays of high-energy hadrons and in reactions of scattering and conserve their energy on the way from remote sources – at cosmological distances in 10-100 Mps or more. Certainly, neutrinos can be also produced by supermassive X-particles decay and in virtual Z-boson resonant transition to neutrino or hadronic pairs or from resonant generation of lepton + neutrino pair or hadronic pairs from virtual W-boson. And when neutrinos reach the Earth's atmosphere, they can produce Extended Atmospheric Showers (EAS) with high portion of neutrino energy despite of small interaction cross section (which, however, increases with energy).

However, as we see, there is a possibility to accelerate (heavy) neutral DM particles which also can move from the distant sources, as the neutrino. EAS generated by neutrino can be successfully discriminated from other types of events (from EAS induced by fast protons, for instance) because they are produced at large depth in atmosphere and are mostly strongly inclined or they are even nearly horizontal. It is an important "fingerprint" for the EAS detection at modern complex , LHAASO. We assume, heavy accelerated DM particle also would produce EAS more deeply than ordinary cosmic rays, mimicking, in fact, neutrino event but with different secondary particles spectrum and total energy release in the process. So, if the DM particles entrance into atmosphere with sufficiently high energy, they can produce specific EAS similar to neutrino-induced ones only in some aspects.

There are known a number of neutrino events with energies up to $E \approx 10^7$ TeV which were registered at IceCube. The source of such super high-energy neutrinos is still unknown, and maybe resonance at $q^2 = M_Z^2$ can contribute to high-energy neutrino creation when high-energy proton interact with the DM particles. In this process some part of proton energy transferred to multiple secondary decaying mesons and neutral accelerated DM. High-energy intergalactic neutrino scattered by DM in halo can transfer its energy to secondary leptons and accelerated neutral DM object. This event can be detected as correlated EAS produced by neutral objects with energies up to $\sim 10^3$ TeV. In this range of energies the atmospheric bkg should be small. In fact, cross section of neutrino-DM interaction is $\sim 10^2$ pb, it is much lower than cross section for annihilation of high-energy neutrino with relic neutrino which is ≈ 10 nb but such type events, in principle, can be registered at IceCube and LHAASO as correlated EAS.

Note, cosmic ray proton scattering off the DM can give rise to increasing of positrons number — they are products of eventually decay of (positively) charged secondary hyperpions. Energy spectrum of these secondary positrons are determined by energies of cosmic rays primaries , masses of the DM candidates, type of the scattering reaction and kinematics of charged H-pion decay. This process does not considered in detail yet.

If it wold be found some increasing of secondary particles (neutrino and/or leptons) flux and the number of detected events from some fixed direction, it can follow from the UHECR scattering off the DM clumps. In other words, to study the DM space distribution, an analysis of EAS induced by (different) neutral objects, their correlation together with measurements of secondary neutrino spectrum and the number of events, can be used.

Moreover, there are some other features of the hypercolor two-component DM scenario, namely, at high energies inelastic reactions with the exciting of higher states of the pNG unstable H-hadrons can occur. Arising and decays of these excited states can be manifested as heavy H-hadron jets that eventually decay to neutral stable DM particles accompanied with photons, leptons and decaying standard light mesons. To study these processes we should know (or suppose) the mass spectrum of unstable H-hadrons, their possible decay channels and widths. In any case, the DM two-component structure can be seen studying of correlations in the set of quantitative and qualitative results in vector and scalar UHECR scattering channels.

Acknowledgements

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3 New Trends in BSM Physics and Cosmology

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Abstract. Physics beyond the Standard Model (BSM) of elementary particles is considered as the physical basis for the now Standard model of the Universe, involving inflation, baryosynthesis and dark matter /energy. BSM physics of these necessary elements of the modern cosmology inevitably leads to cosmological predictions beyond the Standard model of Cosmology. We outline some new trends in the relationship between BSM physics and cosmology in the context of multimessenger cosmological probes for new physics, underlying the modern theory of the structure and evolution of the Universe.

Povzetek. Fiziko onkraj Standardnega Modela elementarnih delcev imajo za fizikalno osnovo sedaj Standardnega Modela Vesolja, ki vkljucje inflacijo, sintezo barionov in temne snovi/energije. Fizika, potrebna za opis teh bistvenih elementov moderne kozmologije sega onkraj Standardnega modela in neizogibno vodi do napovedi izven okvira standardnega kozmološkega modela. Avtor predstavi nekatere nove povezave med fiziko in kozmologijo onkraj obeh standardnih modelov porojene s kozmološkimi opazovanji z različnimi fizikalnimi nosilci informacij in oriše sodobno teorijo zgradbe in razvoja Vesolja.

Keywords: cosmology, particle physics, cosmoparticle physics, inflation, baryosynthesis, dark matter, primordial black holes, antimatter, decaying particles, stable particles, dark atoms

3.1 Introduction

Inflationary models with baryosynthesis and dark matter/energy, underlying the now standard cosmology, are based on physics Beyond the Standard Model (BSM) of elementary particles (see [1–8] for review and references). The choice of this physics involves specific model dependent predictions, which can make possible their effective observational and experimental test.

There are two principally different basic theoretical approaches to description of BSM physics:

- Extension of the Standard model SM by additional sector G involving new particles and/or fields, $SM \otimes$ G.

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- Embedding of SM in more general framework C, $C \supset SM$. It inevitably involves additional particles and fields, completing the SM content to the general framework C.

The first approach adds new symmetries and particles without an ambition to explain everything and doesn't pretend to provide a complete physical basis for the modern cosmological paradigm.

The second approach, pretending to be a overwhelming theory of everything, is to provide such basis and offer candidates for dark matter together with mechanisms of inflation and baryosynthesis. It makes necessary to develop methods to reveal model dependent signatures, specifying the particular ways to probe the predicted BSM physics and the experimental and observational evidence for the proposed theoretical framework. In particular, it is the challenge for the approach of [9] to provide not only a comprehensive basis for the modern cosmology, but also to reveal the specific features of its deviations from the now Standard cosmological paradigms

The paradox of the modern situation, outlined in [10], is that the data of precision cosmology favor now standard cosmological scenario and can be considered as the evidence of BSM physics, proved by our existence, while the laboratory and collider experimental probes for this physics only tighten the data around the prediction of the Standard model.

To specify the model of BSM physics, its additional model dependent signatures are needed. They involve effects, reflecting the fundamental structure and symmetry breaking pattern of the BSM model [7, 8, 10] and can provide multi-messenger cosmological probes for new physics [11].

Taking into account the recently published extensive review [12], here we give only brief general review of probes for BSM physics with special emphasis on cosmological messengers of new physics, involving new stable and meta-stable particles, multi-component dark matter, composite dark matter and dark atoms, primordial black holes and primordial nonlinear structures. The latter includes antimatter stars in the baryon asymmetrical Universe as a profound signature of strongly nonhomogeneous baryosynthesis in inflationary Universe. Some probes of this kind look exotic and highly improbable, but the evidence for their existence would provide a very refined selection of the proper BSM approach and model.

3.2 Messengers of BSM physics

In general, new physics is characterized by energy scale V and appears with full strength at the energies $E \geq V$. This scale V determines the mass of new particles and at these energies they can be copiously produced, as well as their exchange is not suppressed.

At smaller energies $E < V$ new particles can be produced in virtual states and their effects are suppressed by some power of (E/V). It defines the way to probe super-high energy new physics at laboratory energies $E \ll V$ - by rare processes, whose exotic features provide their distinction om the background of the SM physics events. The set of these rare processes is rather small and basically involves processes with baryon or lepton number nonconservation

On the other hand, cosmology, predicting the stages of early Universe with very high energy density, can be considered as a natural laboratory of new physics with high energy scale V, like Supergravity with subPlanckean scale [13]. The corresponding processes take place at very early stages of cosmological evolution and their signatures require some messengers, which retain information on these processes and provide confrontation with the astrophysical data on the phenomena, taking place at much later stage of cosmological evolution.

Such approach implies sufficiently long-living particles and objects, surviving sufficiently long time after their creation. From the view point of particle theory such particles and objects reflect the fundamental symmetry of BSM model and mechanisms of its symmetry breaking, making them cosmological messengers of the fundamental symmetry of microworld. Here we briefly discuss some forms of cosmological messengers of new physics.

3.2.1 Dark matter messengers of BSM physics

Nonbaryonic dark matter, dominating in the matter content of the modern Universe, is associated with the new stable form of the nonrelativistic matter. It should be nonluminous and must decouple from plasma and radiation before the beginning of the matter dominated stage. The first condition follows from the "darkness" of this form of matter. The second comes from the condition that dark matter provides effective development of gravitational instability in the beginning of matter dominated stage before recombination of hydrogen (see e.g. [7, 8, 10] for reviews and references). The simplest theoretical possibility to satisfy these conditions is to assume the existence of stable neutral Weakly Interacting Massive particles (WIMP).

The attractive feature of the WIMP dark matter candidates was their miraculous property to explain the observed dark matter density by primordial gas of stable particles with mass of the order of several hundred GeV with annihilation cross section of the order of the ordinary weak interaction. These conditions naturally lead to the predicted abundance corresponding to the measured density of dark matter.

Strong theoretical support for WIMPs came from predictions of stable lightest supersymmetric (SUSY) neutral particles with mass and annihilation cross section, corresponding to the desired WIMP parameter range. The advantage of supersymmetry with SUSY scale within 1 TeV was its principle possibility to solve the problems of Standard model related with divergence of Higgs boson mass and origin of the scale of the electroweak symmetry breaking. The expected discovery of supersymmetric partners of ordinary quarks, leptons and gauge bosons with the mass in the range 100 GeV-1 TeV, was the challenge for experimental search at the LHC.

However, the results of the direct WIMP search in underground experiments are controversial, as well as there is no positive results of SUSY particle searches at the LHC in the indicated mass range, It stimulates the substantial extension of the list of possible dark matter particle candidates.

Stability of dark matter implies stability of its constituents, which involves new stable or very long-living particles, predicted by BSM models. It assumes extension of the symmetry of the Standard model, which leads to new conserved charges, corresponding the the new additional symmetry. The lightest particle, which possess new charge is stable, if the charge is strictly conserved.

There are several strongly motivated extensions of the Standard model, predicting various types of dark matter candidates (see [12] for review and references):

- Sterile neutrinos, having no ordinary weak interaction and involved in the see-saw mechanism of neutrino mass generation;
- axion, a pseudo Nambu-Goldstone boson related with the Peccei-Quinn solution of the problem of strong CP violation in QCD;
- mirror or shadow matter, restoring equivalence of left- and right- handed coordinate systems. Being in the same space-time with the ordinary matter they have gravitational interaction and can also interact with matter due to strongly suppressed kinetic mixing of neutral bosons, like mixing of ordinary and mirror photons.
- gravitino, SUSY partner of graviton in Supergravity. By construction gravitino has super-weak semi-gravitational interaction. At very high sub-Planckean SUSY energy scale it can be also superheavy

These extensions of the Standard model lead to non-WIMP dark matter candidates. Sterile neutrinos, mirror or shadow particles or gravitino are superWIMPs with superweak interaction with matter, while axions have a very small mass, but still play the role of Cold Dark Matter. The list of these candidates can be extended by neutral stable particles originated by any extension of the group of the SM symmetry SU(3) x SU(2) x U(1) by any additional strict symmetry group G. In particular, new stable colored objects that possess the corresponding new conserved charge can form Strongly Interacting Massive Particles (SIMP) (see .

3.2.2 Multicomponent dark matter

The motivation for existence of various dark matter particle candidates comes from different solutions for the internal problems of SM. It makes possible their co-existence and can lead to multicomponent dark matter scenarios.

In such scenarios dark matter can represent mixture of primordial particles with different properties, like mixture of Hot and Cold Dark matter. Another possibility is co-existence of absolutely stable and metastable particles. The latter can lead to observable effects of deviations from the Standard cosmological scenario.

To be of cosmological significance metastable particles with the mass m must be sufficiently long living. Their lifetime τ should be much larger than m_{Pl}/m^2. Then they retain in the Big Bang Universe at $T \ll m$ and their presence can lead to observable signatures.

3.2.3 Cosmoarcheology of new physics

The set of astrophysical data puts constraints on any new forms of matter present in the Universe at various periods of cosmological evolution. The very fact of their presence means that they contribute to the total energy density and such contribution is restricted by the measurements of the modern total density, or by effects of their presence in the period of Big Bang Nucleosynthesis or Large Scale Structure formation.

Metastable particle with lifetime τ exceeding the age of the Universe t_U should contribute the modern dark matter density as decaying dark matter component. If leptons, quarks, gluons or photons are among the decay products, their contributions in the cosmic ray fluxes can provide constraints on the lifetime, branching ratios and abundance of metastable particles.

Metastable particles with lifetime $\tau < t_U$ cannot be considered as the candidates for the modern dark matter, but their presence in the period of structure formation can lead to unstable dark matter (UDM) scenarios, which are severely constrained by the condition of the effective growth of density fluctuations, which can be strongly suppressed after decays, if UDM dominates in the period of large scale structure formation.

The sensitivity of astrophysical data to the presence and decays of metastable particles strongly depends on the contribution of the decaying particles into the total density and on the possibility of decay products to influence the observable features of the CMB spectrum, light element abundance or cosmic neutrino, gamma ray or cosmic ray fluxes. This sensitivity strongly increases, if decay products influence observable features of subdominant component, which is baryonic matter at the radiation dominated stage and radiation at the matter dominated stage. Such sensitive probes assume specific decay channels and are strongly model dependent. Contribution to the total density of the Universe at various periods of cosmological evolution avoids such specific model dependence, but on this reason is much less sensitive to the presence of new particles in the Universe (see [11, 12] for recent review and references).

3.2.4 Composite dark matter

Problem of stable charged particles BSM models try to avoid predictions of stable electrically charged particles. Positively charged stable particles should bind with electrons and form anomalous isotopes of chemical elements. The constraints on the presence of such anomalous isotopes in the terrestrial matter put severe constraint on their abundance. Only superheavy subPlanckean Charged Massive particles (CHAMP) can avoid these constraints due to very small number density and rapid diffusion to the center of Earth, strongly reducing their abundance in the sea and terrestrial layers near the surface.

Similar to baryonic matter, charged stable particles may be hidden in neutral atomic states and play the role of dark matter. The only condition is to avoid overproduction of anomalous isotopes in this case. The main problem is that in the expanding Universe recombination of electrically charged particles is never complete and freezing out of free charged particles is inevitable. Free +1 charge particles form anomalous hydrogen, severely constrained by the experimental data. Free -1 charged particles E^- form +1 charged ion (EHe) with primordial helium nuclei, as soon as they are produced in the Big Bang Nucleosynthesis. Similar problems arise for all positively charged stable particles and negatively charged particles with charge $-2n-1$. It leaves only $-2n$ charged particles as possible constituents of dark atoms.

Multicharged stable particles Multicharged particles may be composite or elementary. The example of composite -2 charged particles give models with new stable U-type quark. They predict existence of stable Δ^{--} - like state ($\bar{U}\bar{U}\bar{U}$). It's positively charged antiparticle (UUU) can bind with electrons in anomalous helium and special mechanisms are needed to suppress their abundance. Such mechanisms may naturally appear, if the ($\bar{U}\bar{U}\bar{U}$) excess over (UUU) is generated similar to baryon excess in baryon asymmetrical Universe.

The balance between excess of new particles and baryon asymmetry can be established by sphaleron transitions, if new particles possess electroweak SU(2) charges. Such balance with proper (negative) sign of the excessive new particles takes place in Walking Technicolor (WTC) models, predicting technibaryons composed of techniquarks and elementary technileptons. The absolute values of electric charges of technibaryons and technileptons are free parameter of the model. The only condition for the charge assignment is the cancellation of anomalies that fixes the relationship between the charges of technileptons and technibaryons, while the absolute value of these charges depends on the free parameter of this model. New stable charged techniparticles may be technibaryons, if technibaryon charge is conserved, technileptons, if technilepton charge is conserved, or both, if the both charges are conserved. In the latter case two-component techniparticle dark matter scenario is possible. Both technibaryons and technileptons look like elementary leptons at energies below WTC confinement.

Dark atoms of dark matter Independent of the mechanism of baryon excess generation, sphaleron transitions establish equilibrium between baryon excess and excess of charged techniparticles. Choice of reasonable parameters of the model provides excess of even negatively charged stable techniparticles, which provides their explanation of the observed dark matter density for the masses of the order of 1 TeV (see [11, 12] for recent review).

After Big Bang Nucleosynthesis these excessive $-2n$ charged techniparticles bind with n helium nuclei in dark atoms. O^{--} with charge -2 form OHe atoms - Bohr like systems with O^{--} leptonic core and strongly interacting helium shell. The Bohr radius in OHe atom is equal to the size of He. The lack of usual approximations of atomic physics (small size of nuclear interacting nucleus relative to Bohr orbit and electronic shell with electroweak interaction, supporting perturbation methods of calculations) makes proper quantum mechanical treatment of OHe interaction with matter a very complicated and still unresolved problem. The first steps towards self-consistent numerical simulations of OHe interaction with nuclei, taking into account both Coulomb and nuclear forces acting between OHe components and nucleus, are discussed in [15].

Multimessenger probes for dark atoms Cosmological scenario of dark atom evolution leads to Warmer than Cold dark matter scenario of structure formation.

Owing to low number density of nuclei OHe gas decouples from plasma and radiation before the beginning of matter dominated stage and supports growth of density fluctuations with spectrum with slightly suppressed short wave part as compared with the standard Cold dark matter scenario [11,12].

In spite of its strong interaction with matter ($\sigma \approx 2 10^{-25}$ cm^2), only sufficiently dense matter objects of the size R with density

$$\rho > \frac{1}{\sigma R m_p},$$

where m_p is the mass of proton, are opaque for OHe, while the average matter density makes the Galaxy transparent for it. OHe gas in the Galaxy is collisionless, but in the region of the Galaxy center, where OHe density is higher rare OHe collisions can lead to OHe excitations. De-excitation of OHe, excited in collisions, by emission of electron-positron pairs can provide explanation for the excess of positron annihilation line radiation from the galactic bulge, observed by INTEGRAL. Such explanation implies the mass of O^{--} in the narrow window near 1.25 TeV, challenging the search of such stable double charged particles at the LHC.

Due to strong interaction with matter cosmic OHe is slowed down in the terrestrial matter and cannot be detected in underground experiments by effects of nuclear recoil, used for direct WIMP searches. However, annual modulation in low energy binding of OHe with intermediate mass nuclei, like sodium, can explain the positive results of DAMA/NaI and DAMA/LIBRA experiments with their puzzling contradictions with negative results of direct WIMP searches.

Created after helium production in the Big Bang Nucleosynthesis OHe can catalyze pregalactic production of heavier nuclei, like carbon or oxygen. Captured by stars OHe can play interesting but still unexplored role in stellar evolution. Liberated in stellar interiors and accelerated at Supernova explosions multiple charged dark atom constituents can form high energy flux of exotic multiple charged leptonic component that can lead to specific type of atmospheric showers in LHAASO experiment [11].

3.3 Tracers of very early Universe

Together with baryon asymmetry or primordial gas of dark matter particles physics of very early Universe can provide many other model dependent observable tracers. Second order phase transitions can lead to formation of topological defects like monopoles, strings, walls or many other types of stable or unstable topological defects. Strong first order phase transitions can be the source of gravitational wave background. These processes can lead to appearance of inhomogeneities in homogeneous and isotropic Universe.

One of the profound signature of strong inhomogeneity of very early Universe is formation of primordial black holes. Their spectrum contains information on the mechanisms of their formation, reflecting the fundamental structure of the particle theory at very high energy scale [7,16].

3.3.1 Primordial Black Holes as the messenger of new physics

To form black hole in the expanding Universe, one should stop its expansion within the cosmological horizon [17]. It corresponds to nonhomogeneity $\delta = \delta\rho/\rho \sim 1$ in the nearly homogeneous and isotropic Universe with dispersion of small density fluctuations

$$\left\langle \delta^2 \right\rangle = \delta_o^2 \ll 1 \tag{3.1}$$

. Probability for such a high amplitude fluctuation depends on the equation of state $p = \gamma\epsilon$ (where p is pressure, ϵ is energy density and $\gamma = 0$ for matter dominance (MD) and $\gamma = 1/3$ for radiation dominated (RD) stage) and is given by [18]

$$W_{PBH} \propto \exp\left(-\frac{\gamma^2}{2\left\langle \delta^2 \right\rangle}\right).$$

This probability is exponentially suppressed for small amplitude density fluctuations at the RD stage. At MD stage there is no exponential suppression. It makes primordial black holes a sensitive indicator of early MD stages [19,20].

Physics of early MD stages Early MD stage may be a consequence of existence of a supermassive metastable particle, dominating in the Universe before decay [16,19,20]. If such particles with mass m are created in the Big Bang Universe with frozen out relative abundance $\nu = n_m/n_r$, where n_m and n_r are number densities of considered particles and relativistic species, respectively, at the temperature $T < T_o = \nu m$, corresponding to the period $t > t_o = m_{Pl}/m^2$ such particles start to dominate in the Universe until their decay at $t = \tau$, where τ is the particle lifetime.

Growth of density fluctuations at the MD stage leads to formation of gravitationally bound systems, separated from cosmological expansion. Evolution of these systems can lead to formation of black holes, retaining in the Universe at $t > \tau$, when particles, dominating in the Universe, decay.

The minimal estimation of the probability of PBH formation is determined by direct collapse into black hole of specially homogeneous and isotropic configurations, after they separate from the general expansion. This probability is given by [11,19]

$$W_{PBH} \propto \delta_o^{13/2}.$$

If configuration is specially homogeneous and isotropic it contracts within its gravitational radius as soon as it separates from cosmological expansion at $t_1 \approx t_0\delta_o^{-3/2}$. This mechanism leads to a flat spectrum of PBH masses ranging from $M_{min} = m_{Pl}^2 t_o$ to the maximal mas, determined by the condition that the configuration can separate from expansion and collapse in black hole before particles decay at $t = \tau$.

However, most of configurations don't contract directly into black . They form gravitationally bound systems, whose evolution strongly depends on the nature of particles, dominating at the MD stage.

If gas of massive particles is collisionless within configuration, gravitationally bound system of point like masses collapses into black hole due to evaporation of energetic particles in binary gravitational collisions at the timescale $t_{evbin} =$

$t_1 N/\ln N$ [21] or due collective effects at the timescale $t_{evcol} = t_1 N^{2/3}$ [22], where $N \gg 1$ is the number of particles in the gravitationally bound system [11,12,16].

If gas of massive particles is dissipational, its evolution to black holes takes place at much smaller timescale, comparable with t_1. In particular, if magnetic monopole abundance is not suppressed by inflation and magnetic monopoles dominate in the Universe before their abundance is suppressed by monopole-antimonopole annihilation in gravitationally bound systems formed at the stage of their dominance, collapse into black holes turns out to be more rapid, than annihilation in these systems and magnetic monopole overproduction would convert into overproduction of PBHs [16,23].

Inflation can end by sufficiently long MD stage of massive scalar field dominance, which can also result in PBH formation [24].

PBH formation in first order phase transitions If inflation ends by first order phase transition or the symmetry breaking phase transition is a strong first order, the process of bubble nucleation can lead to black hole production in bubble wall collisions [25]. In the course of transition bubbles of true vacuum, expanding in the false vacuum, collide and in the collision area the energy of bubble walls converts into a false vacuum bag, which separates from walls and pending on its mass either collapses in black hole [26] or converts in oscillon [27].

Bubble collisions become effective, when the bubble nucleation rate becomes equal to the rate of expansion, H, and the mass of forming black holes is determined by the energy of the false vacuum within a region with typical size of $1/H$.

3.3.2 Primordial nonlinear structures

Primordial objects created in the very early Universe seem to be constrained by the small size of cosmological horizon. However, inflation can provide large scale correlations in the space distribution of these objects, giving rise to the large scale primordial structures.

Archioles - large scale correlations of energy density of the axion-like fields In the axion-like models a complex scalar field $\Psi = \psi \exp(i\theta)$ acquires after spontaneous symmetry breaking of global U(1) symmetry vacuum expectation value $\langle\psi\rangle = f$, leaving continuous degeneracy of vacua with arbitrary values of the phase θ. This continuous degeneracy is broken by explicit symmetry breaking term

$$V_{eb} = \Lambda^4(1 - \cos\theta). \tag{3.2}$$

This term is negligible, if $f \gg \Lambda$. In the axion models it doesn't exist at high temperature and appears due to instanton effects in the period of QCD phase transition. Then at $T \sim \Lambda$ takes place the second phase transition, in which continuous degeneracy of vacua is broken by the term Eq.(3.2) and the vacua have discrete degeneracy, corresponding to $\theta_{vac} = 0, 2\pi, 4\pi....$ The value of phase $\theta - \theta_{vac}$

acquires the meaning of the amplitude of axion field, which determines the energy density of the axion field oscillations.

If first phase transition takes place after reheating, the continuous degeneracy of phase leads to singularities, having the geometric place of lines - axion strings.

After the second phase transition vacua with different values of θ_{vac} are separated by domain walls, surrounded by strings. This vacuum defect structure is unstable and rapidly decays, but the distribution of axion energy density follows the initial structure of walls-surrounded-by-strings. Since 80% of axion string length corresponds to infinite strings, this structure provides large scale correlation in the distribution of axion energy density (see [7] for review and references).

Clusters of massive PBHs If the first phase transition takes place at the inflationary stage, the now observed part of the Universe acquires at the corresponding e-folding $N_i = 60$ unique value of phase θ_i.

However, at successive steps of inflation with smaller e foldings $N < N_i$ the value of phase experiences fluctuations

$$\delta\theta \sim \frac{H_i}{2\pi f},$$

where H_i is the Hubble constant at the inflationary stage. Therefore, if $\theta_i < \pi$ at $N = N_i$ in some smaller regions fluctuations of θ can lead to values $\theta > \pi$. At successive stages of inflation with smaller N fluctuations can lead in some smaller regions to the value of $\theta < \pi$. This process continues until the end of inflation.

In the result, at successive second phase transition, which takes place after reheating at $T \sim \Lambda \ll f$, the regions with $\theta < \pi$ and $\theta > \pi$ should be separated by closed domain walls. The process described above leads to a system of closed walls. Collapse of closed walls results in formation of black holes, which are not distributed stochastically but appear in clusters, in which black holes of smaller mass are created around the locally most massive black hole [29].

This mechanism leads to formation of clusters of PBHs with masses, determined by the fundamental parameters of the model f and Λ, which can have stellar and superstellar values. The minimal mass is determined by the condition that the width of domain wall ($\sim f/\lambda^2$) doesn't exceed the size of the gravitational radius of the wall. It gives [28]

$$M_{min} = f(\frac{m_{Pl}}{\Lambda})^2. \tag{3.3}$$

The principally maximal mass of such PBHs is determined by the condition that the wall does not dominate locally before it enters the cosmological horizon. Otherwise, local wall dominance leads to a superluminal $a \propto t^2$ expansion for the corresponding region, separating it from the other part of the universe. This condition corresponds to the mass [16]

$$M_{max} = \frac{m_{Pl}}{f} m_{Pl}(\frac{m_{Pl}}{\Lambda})^2. \tag{3.4}$$

Formation of PBHs in the collapse of closed walls is accompanied by the primordial gravitational wave (GW) background. Its spectrum is peaked at

$$\nu_0 = 3 \times 10^{11} (\Lambda/f)\,\mathrm{Hz}$$

and the energy density can be estimated as [16] $\Omega_{GW} \approx 10^{-4}(f/m_{Pl})$. At $f \sim 10^{14}$ GeV this primordial gravitational wave background can reach $\Omega_{GW} \approx 10^{-9}$. For the physically reasonable values of $1 < \Lambda < 10^8$ GeV the maximum of the spectrum corresponds to

$$3 \times 10^{-3} < \nu_0 < 3 \times 10^5 \text{ Hz}. \tag{3.5}$$

This range may be within the reach of LIGO-VIRGO and future LISA detection of gravitational waves and this prediction may be of interest for interpretation of the recent results of the NANOGrav Collaboration [30].

The primordial origin of the observed massive and supermassive black holes [31] may find additional support in the recent detection by LIGO and VIRGO collaborations of gravitational wave signal from a binary black hole merging with total mass $150M_{odot}$ [32], corresponding to the gap in the predicted BH masses from first massive stars, can evidence for primordial origin of massive BHs [33].

3.3.3 Antimatter stars as probes for nonhomogeneous baryosynthesis

Any mechanism of baryosynthesis can under some conditions predict nonhomogeneous distribution of the baryon excess. In the extreme case, nonhomogeneity can lead not only to the spatial change of baryon asymmetry, but can also change its sign, so that antimatter excess can appear in some regions of baryon asymmetrical Universe [34–38].

Sufficiently large antimatter domains, corresponding to the mass, exceeding $10^3 M_{odot}$ can survive in the matter surrounding and form antimatter globular cluster in our Galaxy. Owing to its situation in the galactic halo, where the gas density is low, and the absence of significant amount of matter gas within the cluster, γ radiation from this cluster, can come dominantly from the surfaces of antimatter stars. It makes such object rather faint gamma source. Antimatter, lost by the cluster annihilates with the matter gas and is the source of gamma background. It puts upper limit on the mass of cluster around $10^5 M_{odot}$.

Antimatter supernova explosions can accelerate antinuclei and generate heavy antinuclear component of cosmic rays. Since the estimated flux of secondary cosmic antihelium, originated from cosmic ray interaction with matter, is far beyond the sensitivity of AMS02 experiment, detection of antihelium in this experiment would be a very strong evidence for its primordial nature and for existence of antimatter stars in our Galaxy [39]. It may provide distinction of this mechanisms from other predictions of possible forms of antimatter in our Galaxy [40].

The first claims on the detection of antihelium events in the AMS02 experiment can hardly find explanation by natural astrophysical sources [41] and, if confirmed in the future data analysis, may strongly evidence for the existence of antimatter stars in our Galaxy. The development of numerical simulation of production and propagation of antinuclei from antimatter globular cluster in our Galaxy is discussed in [42].

3.4 Conclusion

Recent hints to the deviations from the Standard cosmological paradigm in the interpretation of discrepancies in measurement of the Hubble constant [43], in the probes for hypothetical time variation of electron to proton mass ratio from the period of recombination to the modern time [44], in the indications to some problems of simple CDM model in the observational data on the structure and evolution of galaxies [45,46], as well as puzzles of the results of direct and indirect dark matter searches, of the origin of massive BH binaries, whose merging is detected in gravitational wave experiments, of the possible detection of antihelium in cosmic rays may reflect some specific features of BSM physics, on which now standard cosmology is based.

Multimessenger cosmology of new physics deals with hypothetical phenomena, which are not predicted with necessity in the framework of the now standard cosmological paradigm. Being model dependent cosmological consequences of BSM physics, their signatures can specify the underlying particle models and provide their effective selection. Positive results of the searches for such signatures would lead to nonstandard deviations from the modern cosmological standards, specifying true cosmological scenario and fundamental structure of the microworld, on which it is based.

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4 Squeezing the Parameter Space for Lorentz Violation in the Neutrino Sector by Additional Decay Channels

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Abstract. The hypothesis of Lorentz violation in the neutrino sector has intrigued scientists for the last two to three decades. A number of theoretical arguments support the emergence of such violations first and foremost for neutrinos, which constitute the "most elusive" and "least interacting" particles known to mankind. It is of obvious interest to place stringent bounds on the Lorentz-violating parameters in the neutrino sector. In the past, the most stringent bounds have been placed by calculating the probability of neutrino decay into a lepton pair, a process made kinematically feasible by Lorentz violation in the neutrino sector, above a certain threshold. However, even more stringent bounds can be placed on the Lorentz-violating parameters if one takes into account, additionally, the possibility of neutrino splitting, i.e., of neutrino decay into a neutrino of lower energy, accompanied by "neutrino-pair Cerenkov radiation". This process has negligible threshold and can be used to improve the bounds on Lorentz-violating parameters in the neutrino sector. Finally, we take the opportunity to discuss the relation of Lorentz and gauge symmetry breaking, with a special emphasis on the theoretical models employed in our calculations.

Povzetek. Domneva o morebitni kršitvi Lorentzove invariance pri nevtrinih vznemirja znanstvenike v zadnjih dveh do treh desetletjih. Vrsta teorijskih argumentov podpira pojav takih kršitev, predvsem za nevtrine, ki so "najbolj izmuzljivi" delci z "najmanj interakcijami", kar jih poznamo. Zato je pomembno omejiti parametre, ki dopuščajo kršitve Lorentzove invariance. Doslej so najbolj ostre omejitve kršitve Lorentzove invariance ponudili izračuni verjetnosti razpada nevtrina v leptonski par, pri katerem bi nad določenim pragom lahko prišlo do kršitve. Avtor ocenjuje omejitev verjetnosti za Lorentzovo invarianco pri procesu, ko nevtrino izgubi del svoje energije s čerenkovim sevanjem nevtrinskega para, ki se pri tem rodi. Prag za tak dogodek je zelo nizek. Na koncu avtor obravnava še zvezo med kršitvijo Lorentzove invariance in umeritveno simetrijo s posebnim poudarkom na teoretičnih modelih, ki jih uporabi.

Keywords: Lorentz Violation, Neutrinos, Gauge Invariance, Mass Mixing, IceCube Detector; Physics beyond the Standard Models

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4.1 Introduction

Neutrinos are the most elusive particles within the Standard Model of Elementary Interactions. Speculation about their tachyonic nature started with Ref. [1], and has led to the development of a few interesting scenarios [2]. Within the Lorentz-violating scenarios [3–11], one typically assumes a dispersion relation of the form $E = \sqrt{\vec{p}^2 v^2 + m^2}$ with $v > 1$. (In this article, we use physical units with $\hbar = \epsilon_0 = c = 1$.) Formalizing the Lorentz-violating ideas, the Lorentz–Violating Extension of the Standard Model (SME) was developed with a strong inspiration coming from neutrino physics [12]. Kinematically, decay among neutrino mass eigenstates is allowed due to their mass differences, while decay rates for "ordinary" neutrinos within the Standard Model formalism (for both Dirac as well as Majorana) exceed the lifetime of Universe by orders of magnitude. Lorentz-violating neutrinos undergo stronger decay and energy loss mechanisms than "ordinary" neutrinos because of their dispersion relation $E \approx |\vec{p}|\, v$ (at high energy), which makes a number of decay channels (without GIM suppression, see Refs. [13, 14]) kinematically possible.

There are a number of phenomena which inspire us to concentrate on the neutrino sector for Lorentz violation. The early arrival of neutrinos from the 1987 supernova still inspires (some) physicists. Specifically, under the Mont Blanc, in the early morning hours of February 23, 1987, a shower of neutrinos of interstellar origin arrived about 6 hours earlier then the visible light from the Siderius Nuntius SN1987A supernova. This event has been recorded in Ref. [15], and it was asserted that such an event could happen by accident once in about 1000 yrs. Direct measurements of neutrino velocities have given results that are consistent with the speed of light within experimental uncertainty, but with the experimental result being a littler larger than the speed of light. For example, the MINOS experiment [16] has measured superluminal neutrino propagation velocities which differ from the speed of light by a relative factor of $(5.1 \pm 2.9) \times 10^{-5}$ at an energy of about $E_\nu \approx 3\,\mathrm{GeV}$, compatible with an earlier FERMILAB experiment [17]. Furthermore, neutrinos cannot be used to transmit information (at least not easily) because of their small interaction cross sections. Superluminality of neutrinos would thus not necessarily lead to violation of causality at a macroscopic level, as demonstrated in Appendix A.2 of Ref. [18]. Similar arguments have been made in Ref. [19], where it was shown that problems with microcausality, in Lorentz-violating theories, are alleviated for small Lorentz-violating parameters and in so-called *concordant frames* where the boost velocities are not too large. For neutrinos, corresponding problems are further alleviated by the fact that their interaction cross sections are small; hence, it becomes very hard to transport information superluminally even if the dispersion relation indicates such effects (see also Appendix A.2 of Ref. [18]).

4.2 Threshold Considerations

We refer to the lepton-pair Cerenkov radiation process (LPCR) in Fig. 4.1(a) and the neutrino-pair Cerenkov radiation process (NPCR) depicted in Fig. 4.1(b). In order to make neutrino decay kinematically possible, it is necessary to fulfill certain

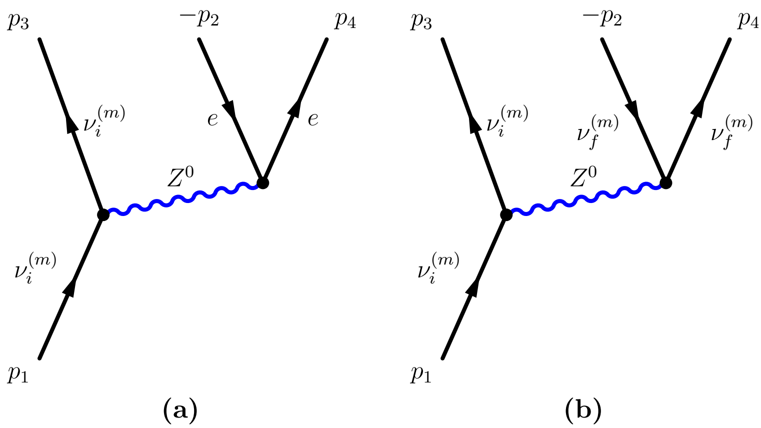

Fig. 4.1. In the lepton-pair Cerenkov radiation process (a), an oncoming Lorentz-violating initial neutrino mass eigenstate $\nu_i^{(m)}$ decays, under emission of a virtual Z^0 boson, into an electron-positron pair. The sum of the outgoing pair momenta is $p_2 + p_4$; one observes the inverted direction of the fermionic antiparticle line. The arrow of time is from bottom to top. The (blue) bosonic line carries the four-momentum q. Diagram (b) describes the neutrino-pair Cerenkov radiation process, with a final neutrino mass eigenstate $\nu_f^{(m)}$.

threshold conditions. Let us denote the outgoing fermions in the generic decay processes depicted in Fig. 4.1 by

$$\nu \to \nu + f + \bar{f} \,, \tag{4.1}$$

with a pair of a massive fermion f and its antiparticle $\bar{f}$ being emitted in the process.

Energy-momentum conservation implies that (in the notation of Fig. 4.1)

$$(p_1 - p_3)^2 = q^2 = (p_2 + p_4)^2 \,. \tag{4.2}$$

Let us first consider the case of a massive outgoing pair 2+4, with rest mass m_f, and vanishing Lorentz-violating parameter. Threshold is reached for collinear emission geometry. The incoming four-momentum is $p_1 = (E_1, \vec{p}_1)$, with $E_1 = p_1 \, v_i$, while $p_3 = (0, \vec{0})$, so that all transfer four-momentum q goes into the pair. For collinear geometry, one has $p_2^\mu = p_4^\mu = (E_f, \vec{p}_f)$, where $E_f = \sqrt{\vec{p}_f^{\,2} + m^2}$. Under these assumptions, $p_2 + p_4 = (2\sqrt{\vec{p}_f^{\,2} + m^2}, 2\vec{p}_f)$, so that $(p_2 + p_4)^2 = 4m_f^2$. The threshold condition becomes

$$p_1^2(v_i^2 - 1) \geq 4m_f^2 \,, \qquad p_1 \approx E_1 \geq \frac{2m_f}{\sqrt{v_i^2 - 1}} = \frac{2m_f}{\sqrt{\delta_i}} \,. \tag{4.3}$$

Here,

$$v_i = \sqrt{1 + \delta_i} \,. \tag{4.4}$$

The threshold condition $E_{th} = 2m_f/\sqrt{\delta_i}$ has been used extensively in Refs. [20–22]. The formula (4.3) implies that the threshold for NPCR is lower by at a least six orders of magnitude as compared to LPCR.

The kinematic considerations are very different in the high-energy regime, when *both* incoming the decaying particle as well as the outgoing particles are Lorentz violating. Masses can be neglected. In this case, one has at threshold $p_2 = p_4 = (E_f, \vec{p}_f)$, where $E_f = |\vec{p}_f|\, v_f$, so that at threshold

$$p_1^2(v_i^2 - 1) \geq 4p_f^2(v_f^2 - 1)\,, \tag{4.5}$$

Due to equipartition of the energy among outgoing pair at threshold, one has $p_f \approx E_f = E_1/2 \approx p_1/2$. In this case, the threshold condition reduces to

$$(v_i^2 - 1) \geq (v_f^2 - 1)\,, \qquad \delta_i > \delta_f\,. \tag{4.6}$$

Here, $v_f = \sqrt{1 + \delta_f}$. For $\delta_i = \delta_f$, no phase space is available in order to accommodate for the decay. This consideration explains why all results communicated in Ref. [22] display a factor $\delta_i - \delta_f$; decay takes place from "faster" to "slower" mass eigenstates.

4.3 Outline of the Calculation

The understanding of decay processes involving Lorentz-violating has been advanced through Refs. [20–22]. Let us briefly recall elements of the derivation given in Ref. [22]. One particular question concerns the question of how to express the decay rate for an (initially) flavor-eigenstate neutrino (the electroweak Lagrangian is flavor-diagonal) in terms of mass eigenstates. We have, in the same obvious notation as used in Ref. [22],

$$\nu_k^{(f)} = \sum U_{k\ell}\, \nu_\ell^{(m)}\,, \tag{4.7}$$

with the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix $U_{k\ell}$. The interaction interaction $\mathcal{L}_W$ with the Z^0 boson in the flavor basis is

$$\mathcal{L}_W = -\frac{g_w}{4\,\cos\theta_W} \sum_{k,\ell,\ell'} U^+_{\ell k}\, U_{k\ell'}\, \overline{\nu}_\ell^{(m)}\, \gamma^\mu (1 - \gamma^5)\, \nu_{\ell'}^{(m)}\, Z_\mu\,. \tag{4.8}$$

Here, g_w is the weak coupling constant, and θ_W is the Weinberg angle. A unitary transformation leads to

$$\mathcal{L} = -\frac{g_w}{4\,\cos\theta_W} \sum_{k,\ell,\ell'} U^+_{\ell k}\, U_{k\ell'}\, \overline{\nu}_\ell^{(m)}\, \gamma^\mu (1 - \gamma^5)\, \nu_{\ell'}^{(m)}\, Z_\mu\,. \tag{4.9}$$

The interaction with the Z^0 boson in the mass eigenstate basis therefore reads as follows,

$$\mathcal{L} = -\frac{g_w}{4\,\cos\theta_W} \sum_{\ell} \overline{\nu}_\ell^{(m)}\, \gamma^\mu (1 - \gamma^5)\, \nu_\ell^{(m)}\, Z_\mu\,. \tag{4.10}$$

In order to model the free Lorentz-violating neutrino Lagrangian, one introduces a metric with tilde:

$$\mathcal{L} = \sum_\ell i\, \overline{\nu}_\ell^{(m)}\, \gamma^\mu\, (1 - \gamma^5)\, \tilde{g}_{\mu\nu}(\nu_\ell)\, \partial^\nu \nu_\ell^{(m)}\,. \tag{4.11}$$

Here,

$$\tilde{g}_{\mu\nu}(\nu_\ell) = \mathrm{diag}(1, -\nu_\ell, -\nu_\ell, -\nu_\ell)\,. \tag{4.12}$$

The dispersion relation

$$E_\ell = |\vec{p}|\,\nu_\ell \tag{4.13}$$

follows as the massless limit of $E_\ell = \sqrt{(|\vec{p}|\,\nu_\ell)^2 + m_\ell^2}$. For neutrinos, we know that the m_ℓ terms are different. So, there is reason to assume that the $\delta_\ell = \sqrt{\nu_\ell^2 - 1}$ terms are also different among mass (flavor) eigenstates, if they are nonvanishing.

One defines a parameter ν_{int} for the unified description of LPCR and NPCR; the effective four-fermion Lagrangian for the process reads as

$$\mathcal{L}_{\mathrm{int}} = f_e\,\frac{G_F}{2\sqrt{2}}\,\overline{\nu}_i^{(m)}\,\gamma^\lambda\,(1-\gamma^5)\,\nu_i^{(m)}\tilde{g}_{\lambda\sigma}(\nu_{\mathrm{int}})\;\bar{\psi}_f\,\gamma^\sigma\,(c_V - c_A\,\gamma^5)\,\psi_f\,. \tag{4.14}$$

Cohen and Glashow [20] set $\nu_{\mathrm{int}} = 1$. (In Ref. [22], on a number of occasions, the parameter used in Ref. [20] had been inadvertently indicated as $\nu_{\mathrm{int}} = 0$, which is not the case. We take the opportunity to point out that of course, the parameter $\nu_{\mathrm{int}} = 1$ implies that $\delta_{\mathrm{int}} = 0$, which was the intended statement in Ref. [20].) Bezrukov and Lee [21] use the parameters $\nu_{\mathrm{int}} = 1$ ("model I") and $\nu_{\mathrm{int}} = \nu_i$ ("model II"). In Ref. [22], the parameter ν_{int} is kept as a variable. As explained in detail in Ref. [23], "gauge invariance" (with respect to a restricted subgroup of the electroweak sector) can be restored if one uses the value $\nu_{\mathrm{int}} = \nu_i\,\nu_f$. Both Cohen and Glashow [20], as well as Bezrukov and Lee [21], assume that $\delta_f = 0$ for LPCR. The parameter f_e characterizes the process:

$$f_e = \begin{cases} 1, & \psi_f = \nu_f^{(m)} \\ 2, & \psi_f = e \end{cases}\,. \tag{4.15}$$

Approximately, one has

$$(c_V, c_A) = \begin{cases} (1,1) & \psi_f = \nu_f^{(m)} \\ (0, -\frac{1}{2}), & \psi_f = e \end{cases}\,. \tag{4.16}$$

The matrix element

$$\mathcal{M} = f_e \frac{G_F}{2\sqrt{2}}\left[\bar{u}_i(p_3)\gamma^\lambda(1-\gamma^5)u_i(p_1)\right]\tilde{g}_{\lambda\sigma}(\nu_{\mathrm{int}})\left[\bar{u}_f(p_4)(c_V\gamma^\sigma - c_A\gamma^\sigma\gamma^5)v_f(p_2)\right]\,. \tag{4.17}$$

Key to the calculation is the fact that one can split the phase space of the three-particle outgoing phase space

$$\begin{aligned}\Gamma &= \frac{1}{2E_1}\int d\phi_3(p_2, p_3, p_4; p_1)\frac{1}{n_s}\sum_{\text{spins}}|\mathcal{M}|^2 \\ &= \frac{1}{2E_1}\int_{M^2_{\min}}^{M^2_{\max}}\frac{dM^2}{2\pi}d\phi_2(p_3, p_{24}; p_1)\,d\phi_2(p_2, p_4; p_{24})\frac{1}{n_s}\sum_{\text{spins}}|\mathcal{M}|^2\,. \end{aligned} \tag{4.18}$$

with appropriate limits for $M^2_{\rm min}$ and $M^2_{\rm max}$ being given as follows,

$$M^2_{\rm min} = \delta_f (|\vec{p}_2| + |\vec{p}_4|)^2 \,, \qquad M^2_{\rm max} = \delta_i (|\vec{p}_1| - |\vec{p}_3|)^2 \,. \tag{4.19}$$

The following splitting relation for the phase space is crucial to a simplification of the integrations [see Ref. [24] and Eq. (43) of Ref. [22]],

$$\begin{aligned} d\phi_3(p_2, p_3, p_4; p_1) = & \\ \int \frac{dM^2}{2\pi} & \underbrace{\frac{d^4 p_3}{(2\pi)^3} \delta_+(p_3^2 - \delta_i k_3^2) \frac{d^4 p_{24}}{(2\pi)^3} \delta_+(p_{24}^2 - M^2)(2\pi)^4 \delta^{(4)}(p_1 - p_3 - p_{24})}_{= d\phi_2(p_3, p_{24}; p_1)} \\ & \times \underbrace{\frac{d^4 p_2}{(2\pi)^3} \delta_+(p_2^2 - \delta_f k_2^2) \frac{d^4 p_4}{(2\pi)^3} \delta_+(p_4^2 - \delta_f k_4^2)(2\pi)^4 \delta^{(4)}(p_{24} - p_2 - p_4)}_{= d\phi_2(p_2, p_4; p_{24})} \\ & = \int \frac{dM^2}{2\pi} d\phi_2(p_3, p_{24}; p_1) d\phi_2(p_2, p_4; p_{24}) \,. \end{aligned} \tag{4.20}$$

The general result for the decay rate, unifying both processes depicted in Fig. 4.1, reads as follows,

$$\begin{aligned} \Gamma_{\nu_i \to \nu_i \psi_f \bar{\psi}_f} = & \frac{G_F^2 k_1^5}{192 \pi^3} f_e^2 \frac{c_V^2 + c_A^2}{420 n_s} (\delta_i - \delta_f) \Big[(60 - 43\sigma_i)(\delta_i - \delta_f)^2 \\ & + 2(50 - 32\sigma_i - 25\sigma_f + 7\sigma_i \sigma_f)(\delta_i - \delta_f)\delta_f \\ & + 7(4 - 3\sigma_i - 3\sigma_f + 2\sigma_i\sigma_f)\delta_f^2 + 7\delta_{\rm int}^2 \Big] \,. \end{aligned} \tag{4.21}$$

This result vanishes for $\delta_i = \delta_f$, per the discussion in Sec. 4.2. Cohen and Glashow [20] have $n_s = 2$ active spin states for the (initial) neutrino, while Bezrukov and Lee [21] calculate with $n_s = 1$. The σ parameters depend on the way in which spin polarization sums are carried out,

$$\sigma_i = \begin{cases} 0, & \text{CG spin sum for } \nu_i \\ 1, & \text{BL spin sum for } \nu_i \end{cases} , \quad \sigma_f = \begin{cases} 0, & \text{CG spin sum for } \psi_f \\ 1, & \text{BL spin sum for } \psi_f \end{cases} . \tag{4.22}$$

In Ref. [20], the Cohen–Glashow (CG) spin sum ("polarization sum") is taken as follows,

$$\sum_s \nu_{\ell,s} \otimes \bar{\nu}_{\ell,s} = p^\mu g_{\mu\nu} \gamma^\nu \,. \tag{4.23}$$

In Ref. [21], the Bezrukov–Lee (BL) spin sum is based on a somewhat more advanced treatment of the eigenspinors of superluminal neutrino mass eigenstates and reads as

$$\sum_s \nu_{\ell,s} \otimes \bar{\nu}_{\ell,s} = p^\mu \tilde{g}_{\mu\nu}(\nu_\ell) \gamma^\nu \,. \tag{4.24}$$

The general result for the energy loss rate, applicable to both processes in Fig. 4.1, reads as

$$\begin{aligned}\frac{dE_{\nu_i\to\nu_i\psi_f\bar{\psi}_f}}{dx} &= -\frac{G_F^2 k_1^6}{192\pi^3} f_e^2 \frac{c_V^2 + c_A^2}{672 n_s} (\delta_i - \delta_f) \\ &\times \Big[(75 - 53\sigma_i)(\delta_i - \delta_f)^2 + (122 - 77\sigma_i - 61\sigma_f + 16\sigma_i\sigma_f)(\delta_i - \delta_f)\delta_f \\ &\quad + 8(4 - 3\sigma_i - 3\sigma_f + 2\sigma_i\sigma_f)\delta_f^2 + 8\delta_{\text{int}}^2 \Big] \,. \end{aligned} \tag{4.25}$$

In Ref. [22], we have verified and checked compatibility with all formulas contained in Refs. [20] and [21]. This is important because it confirms that the model dependence of the results is only contained in the numerical prefactors, but not in the overall scaling of the results.

As outlined in Ref. [22], one can parameterize the results for NPCR as follows,

$$\Gamma_{\nu_i\to\nu_i\nu_f\bar{\nu}_f} = b\frac{G_F^2}{192\pi^3} k_1^5\,, \qquad \frac{dE_{\nu_i\to\nu_i\nu_f\bar{\nu}_f}}{dx} = -b'\frac{G_F^2}{192\pi^3} k_1^6\,. \tag{4.26}$$

For the CG spin sum, one obtains the following b coefficients,

$$b_{\text{CG}} = \frac{1}{7}(\delta_i - \delta_f)\left[(\delta_i - \delta_f)^2 + \frac{5}{3}\delta_f(\delta_i - \delta_f) + \frac{7}{15}\delta_f^2\right], \tag{4.27a}$$

$$b'_{\text{CG}} = \frac{25}{224}(\delta_i - \delta_f)\left[(\delta_i - \delta_f)^2 + \frac{112}{75}\delta_f(\delta_i - \delta_f) + \frac{32}{75}\delta_f^2\right]. \tag{4.27b}$$

For the BL spin sum, one obtains

$$b_{\text{BL}} = \frac{17}{210}(\delta_i - \delta_f)\left[(\delta_i - \delta_f)^2 + \frac{7}{17}\delta_{\text{int}}^2\right], \tag{4.28a}$$

$$b'_{\text{BL}} = \frac{11}{168}(\delta_i - \delta_f)\left[(\delta_i - \delta_f)^2 + \frac{4}{11}\delta_{\text{int}}^2\right]. \tag{4.28b}$$

Typically, one finds [22] numerical prefactors in these formulas are larger than those for LPCR by a factor of four or five. Also, NPCR has negligible threshold.

In papers of Stecker and Scully [10, 25, 26], the following bound is derived for the Lorentz-violating parameter of the electron-positron field alone (watch out for a difference in the conventions used for defining the δ_e parameter):

$$\delta_e \le 1.04 \times 10^{-20}\,. \tag{4.29}$$

The observation of very-high-energy neutrinos by IceCube, taking into consideration the LPCR process (but not NPCR!), implies that the Lorentz-violating parameter for neutrinos cannot be larger than (Ref. [10])

$$\delta_\nu \le 2.0 \times 10^{-20}\,. \tag{4.30}$$

This bound is based on the assumption that δ_e and δ_ν are different. Colloquially speaking, we can say that, if δ_ν were larger, then "Big Bird" (the 2 PeV specimen found in IceCube, see Refs. [27,28]) would have already decayed before it arrived at the IceCube detector. However, the full analysis requires Monte Carlo simulations involving astrophysical data and is much more involved [10,25,26].

Provided the Lorentz-violating parameters for the different neutrino mass eigenstates are different, low-energy neutrinos are affected by the decay and energy loss processes connected with NPCR, in view of a negligible threshold for NPCR. Typically, Numerical coefficients for NPCR are a factor of four or five larger than for LPCR, depending on the model used for the spin sums. This enhances the importance of the NPCR effect. Inspired by Eq. (4.30), we thus conjecture here that a full analysis of astrophysical data, using the NPCR process as a limiting factor for the observation of high-energy neutrinos, should yield a bound on the order of

$$|\delta_i - \delta_f| \leq \frac{1}{5^{1/3}} \times 2.0 \times 10^{-20} \sim 5.8 \times 10^{-21} \,, \tag{4.31}$$

where the prefactor takes into account the scaling of the effect with the δ parameter. Specifically, the decay and energy loss rates typically scale with the factor $(\delta_i - \delta_f)^3$. It would be very fruitful if this conjecture were to be checked against astrophysical data in an independent investigation.

4.4 An Attractive Scenario

At first, one might see a dilemma: Within a fully $SU(2)_L$ gauge-invariant theory, one necessarily has $\delta_\nu = \delta_e$ (see Ref. [23] for a detailed discussion), and so, the bound $\delta_\nu \leq 2.0 \times 10^{-20}$ given in Eq. (4.30) is not applicable, because the LPCR process does not exist. But then, they have to acknowledge that the bound $\delta_e \leq 1.04 \times 10^{-20}$ given in Eq. (4.29), which is derived for electrons, based on other physical processes, applies to the neutrino sector.

So, the dilemma is that either, one has to give up gauge invariance and uses different Lorentz-violating parameters for each of the three known particle generations, or, if one insists on gauge invariance, then this defeats part of the purpose of looking at the neutrino sector for Lorentz violation. This is because in the latter case, Lorentz-violating parameters for neutrinos and charged left-handed leptons within the same $SU(2)_L$ doublet are necessarily the same, and the tight bounds on Lorentz-violating parameters in the charged-fermion sector automatically apply to the neutrino sector as well. This observation has important consequences when examining the first-generation $SU(2)_L$ doublet, consisting of (ν_e, e_L). Electrons and positrons are stable particles, and small violations of Lorentz invariance would immediately lead to violations of causality on a macroscopic level (see Appendix A.2 of Ref. [18]). If we had to carry over all restrictions on Lorentz-violating electron parameters to the electron neutrino sector, then this would nullify all the motivations listed in Sec. 4.1 for investigating the first-generation neutrino sector.

On the contrary, If one accepts the necessity that different Lorentz-violating parameters should be used for each of the three known particle generations, then one needs to acknowledge that the parameter space for differential Lorentz-violation among neutrino mass eigenstates is restricted by additional constraints due to the NPCR process [22]. An attractive gauge-invariant scenario could still be found, as follows. Namely, one might observe that, as per the discussion in Appendix A.2 of Ref. [18], causality violations due to Lorentz violation are less severe for unstable particles, which decay and therefore are not amenable to the reliable transport of information. Part of the above sketched dilemma could thus be avoided as follows. One first observes that, as per the above argument, problems with respect to causality are less severe in the second-generation $SU(2)_L$ doublet (ν_μ, μ_L) and also in the third-generation $SU(2)_L$ doublet (ν_τ, τ_L), which are composed entirely of unstable particles. Full gauge invariance can be retained if we assume generation-dependent Lorentz-violating parameters δ_e, δ_μ, and δ_τ, for the three $SU(2)_L$ doublets, which could be encoded in modified Dirac matrices $\widetilde{\gamma}^i = \nu_f \gamma^i$ with $f = e, \mu, \tau$ [see Eq. (5) of Ref. [23]]. In the charged-fermion sector, we have nearly no mixing of mass and charge eigenstates. Let us then go into the high-energy regime where, where mass and flavor eigenstates, under the assumptions

$$\delta_\mu, \delta_\tau > 0\,, \qquad \delta_\mu \neq \delta_\tau\,, \qquad \delta_e = 0\,, \tag{4.32}$$

become equal. In this case, at high energy, one would have two neutrino mass eigenstates, which asymptotically approach the muon neutrino and tau neutrino flavor eigenstates at very high energy, decay into electron-positron pairs and (asymptotically) electron neutrinos, via LPCR and NPCR.

Of course, other scenarios and flavor and mass mixing phenomenologies are also possible, as discussed in Sec. IV B of Ref. [22]. In general, one could interpret the emergence of a specific predominant flavor composition of incoming super-high-energy cosmic neutrinos, consistent with one, and only one, specific mass eigenstate, as a signature of Lorentz violation. This is because a single, defined, oncoming mass eigenstate would be consistent with the two other mass eigenstates being "faster" and thus decaying into the single "slow" eigenstate.

In all discussed scenarios, one might find a conceivable explanation for the apparent cutoff in the cosmic neutrino spectrum at about 2 PeV, at the expense of reducing the allowed regime of Lorentz-violating δ parameters to the range of about 10^{-20}. In our "attractive scenario", one retains gauge invariance as outlined in Sec. 4 of Ref. [23] and still is able to account for a super-high-energy cutoff of the cosmic neutrino spectrum. Experimental confirmation or dismissal of this hypothesis will require better cosmic neutrino statistics at very high energies.

4.5 Conclusions

The existence of the NPCR process [see Fig. 4.1(b)] reveals a certain dilemma for Lorentz-violating neutrinos. Namely, under the hypothesis of a nonvanishing Lorentz-violating parameter δ, given as in Eq. (4.4), the virtuality

$$E^2 - p^2 = p^2(\nu^2 - 1) \approx E^2(\nu^2 - 1) = E^2\,\delta \tag{4.33}$$

of a neutrino becomes large for large energy, rendering a number of decay processes kinematically possible. Conversely, based on high-energy astrophysical observations, very strict bounds can be imposed on the Lorentz-violating parameters [see Eqs. (4.29), (4.30), and (4.31)].

Deep connections exist between Lorentz violation and gauge invariance. In Ref. [29], it is shown that spontaneous Lorentz violation can lead to an effective low-energy field theory with both Lorentz-breaking as well as gauge-invariance breaking terms. According to Refs. [29–40], even the photon could potentially be formulated as the Nambu-Goldstone boson linked to spontaneous Lorentz invariance violation. (This *ansatz* was originally formulated before electroweak unification.) For a broader view of this point, we refer to Appendix A of Ref. [23]. If one insists on the persistence of gauge invariance within the electroweak sector, then one has to acknowledge that bounds on Lorentz-violating parameters for charged leptons [e.g., Eq. (4.29)] also apply to the neutrino sector [thus lowering the bound otherwise given in Eq. (4.30) by a factor two, and further restricting the available parameter space for Lorentz-violating parameters in the neutrino sector]. Also, the assumption that $\delta_\nu = \delta_e$ would defeat the purpose of looking at neutrinos for Lorentz violation. If one insists on gauge invariance and still pursues the exploration of Lorentz violation in the neutrino sector, then more sophisticated considerations are required. Namely, one could potentially invoke flavor-dependent differential Lorentz violation across generations. In this case, flavor and mass eigenstates would become identical in the high-energy limit, and decay and energy loss processes could potentially contribute to an explanation for the apparent cutoff in the cosmic neutrino spectrum in the range of a few PeV (see Refs. [27, 28] and the discussion in Sec. 4.4).

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5 Antimatter Gravity: Second Quantization and Lagrangian Formalism

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Abstract. The application of the CPT theorem to an apple falling on Earth leads to the description of an anti-apple falling on anti–Earth (not on Earth). On the microscopic level, the Dirac equation in curved space-time simultaneously describes spin-1/2 particles and their antiparticles coupled to the same curved space-time metric (e.g., the metric describing the gravitational field of the Earth). On the macroscopic level, the electromagnetically and gravitationally coupled Dirac equation therefore describes apples and anti-apples, falling on Earth, simultaneously. A particle-to-antiparticle transformation of the gravitationally coupled Dirac equation therefore yields information on the behavior of "anti-apples on Earth". However, the problem is exacerbated by the fact that the operation of charge conjugation is much more complicated in curved as opposed to flat space-time. Our treatment is based on second-quantized field operators and uses the Lagrangian formalism. As an additional helpful result, prerequisite to our calculations, we establish the general form of the Dirac adjoint in curved space-time. On the basis of a theorem, we refute the existence of tiny, but potentially important, particle-antiparticle symmetry breaking terms whose possible existence has been investigated in the literature. Consequences for antimatter gravity experiments are discussed.

Povzetek. Iz izreka CPT za jabolko, ki pada na Zemljo, sledi izrek za anti-jabolko, ki pada na anti-Zemljo (ne na Zemljo). Na mikroskopskem nivoju Diracova enačba v ukrivljenem prostor-času hkrati opiše delce s spinom 1/2 in njihove antidelce sklopljene z isto metriko ukrivljenenega prostor-časa (metriko, ki opiše gravitacijsko polje na Zemlji). Na makroskopskem nivoju opiše Diracova enačba za fermione, ki interagirajo z elektromagnetnim in gravitacijskim poljem, hkrati opiše jabolko in anti-jabolko, ki padata na Zemljo. Transformacija delcev v antidelce v Diracovi enačbi ponudi informacijo o interakciji med "anti-jabolki in Zemljo". Operacija konjugacije naboja v ukrivljenem prostor-času je bolj zapletena kot v ravnem prostoru. Avtor uporabi Lagrangeov formalizem in obravnava fermione v drugi kvantizaciji. Avtor zatrdi, da simetrija delec-antidelec ni zlomljena, četudi so jo v literaturi že obravanavali. Obravnava poskuse z gravitacijo antisnovi.

Keywords: General Relativity, Antimatter Gravity, Antiparticles, CPT Symmetry, Spin Connection; Physics beyond the Standard Models

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5.1 Introduction

It is common wisdom in atomic physics that the Dirac equation describes particles and antiparticles simultaneously, and that the negative-energy solutions of the Dirac equation have to be reinterpreted in terms of particles that carry the opposite charge as compared to particles, and whose numerical value of the energy E is equal to the negative value of the physically observed energy [1]. Based on the Dirac equation, the existence of the positron was predicted, followed by its experimental detection in 1933, by Anderson [2]. If we did not reinterpret the negative-energy solutions of the Dirac equation, then the helium atom would be unstable against decay into a state where the two electrons perform quantum jumps into continuum states [3]. One of the electrons would jump into the positive-energy continuum, the other, into the negative-energy continuum, with the sum of the energies of the two continuum states being equal to the sum of the two bound-state energies of the helium atom from which the transition started [3–5].

The absolute necessity to reinterpret the negative-energy solutions of the Dirac equation as antiparticle wave functions, i.e., the necessity to interpret positive-energy and negative-energy solutions of one single equation as describing two distinct particles, hints at the possibility to use the Dirac equation as a bridge to the description of the gravitational interaction of antimatter. Namely, if the Dirac equation is being coupled to a gravitational field, then, since it describes particles and antiparticles simultaneously, the Dirac equation offers us an additional dividend: In addition to describing the gravitational interaction of particles, the Dirac equation automatically couples the antiparticle (the "anti-apple"), which is described by the same equation, to the gravitational field, too.

Corresponding investigations have been initiated in a series of recent publications [6–9]. One may ask whether the dynamics of particles and antiparticles differ in a central, static, gravitational field, in first approximation, but also, if there are any small higher-order effects breaking the particle-antiparticle symmetry under the gravitational interaction. The first of these questions has been answered in Refs. [6–8], with the result being that the Dirac particle and antiparticle behave exactly the same in a central gravitational field, due to a perfect particle-antiparticle symmetry which extends to the relativistic and curved-space-time corrections to the equations of motion.

In order to address the second question, it is necessary to perform the full particle-to-antiparticle symmetry transformation of the Dirac formalism, in an arbitrary (possibly dynamic) curved-space-time-background. This transformation is most stringently carried out on the level of the Lagrangian formalism. A preliminary result has recently been published in Ref. [9], where a relationship was established between the positive-energy and negative-energy solutions of the Dirac equation in an arbitrary dynamics curved-space-time-background. However, the derivation in Ref. [9] is based on a first-quantized formalism, which lacks the unified description in terms of the field operator. The field operator comprises *all* (as opposed to *any*) solution of the gravitationally (and electromagnetically) coupled Dirac equation. In general, a satisfactory description of antiparticles, in the field-theoretical context, necessitates a description in terms of particle- and antiparticle creation and annihilation processes, and therefore, the introduction of a field operator. In consequence, the investigation [9] is augmented here on the basis of a transformation of the entire Lagrangian density, which can be expressed in terms of the charge-conjugated (antiparticle) bispinor wave function, and generalized to the level of second quantization. The origin [9] of a rather disturbing minus sign which otherwise appears in the Lagrangian formalism upon charge conjugation in first quantization will be addressed. The use of the Lagrangian formalism necessitates a definition of the Dirac adjoint in curved space-times. As a spin-off result of the augmented investigations reported here, we find the general form of the Dirac adjoint in curved space-times, in the Dirac representation of γ matrices.

According to Ref. [10], the role of the CPT transformation in gravity needs to be considered with care: *A priori*, a CPT transformation of a physical system consisting of an apple falling on Earth would describe the fall of an anti-apple on anti-Earth. Key to our considerations is the fact that, on the microscopic level, the Dirac equation applies (for one and the same space-time metric) to both particles and antiparticles simultaneously (this translates, on the macroscopic level, to "apples" as well as "anti-apples"). This paper is organized as follows: We investigate the general form of the Dirac adjoint in Sec. 5.2, present our theorem in Sec. 5.3, and in Sec. 5.4, we provide an overview of connections to new forces and CPT violating parameters, Conclusions are reserved for Sec. 5.5.

5.2 Dirac Adjoint for Curved Space–Times

In order to properly write down the Lagrangian of a Dirac particle in a gravitational field, we first need to generalize the concept of the Dirac adjoint to curved space-times. We recall that the Dirac adjoint transforms with the inverse of the Lorentz transform as compared to the original Dirac spinor. A general spinor Lorentz transformation $S(\Lambda)$ is given as follows,

$$S(\Lambda) = \exp\left(-\frac{i}{4}\,\epsilon^{AB}\,\sigma_{AB}\right)\,, \qquad \sigma_{AB} = \frac{i}{2}\,\left[\gamma^A, \gamma^B\right]\,, \qquad A, B = 0, 1, 2, 3\,. \tag{5.1}$$

Note that the generator parameters $\epsilon^{AB} = -\epsilon^{BA}$, for local Lorentz transformations, can be coordinate-dependent. In the following, capital Roman letters $A, B, C, \dots = 0, 1, 2, 3$ refer to Lorentz indices in a local freely falling coordinate system. The (flat-space) Dirac matrices γ^A are assumed to be taken in the Dirac representation [1],

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & \mathbb{I}_{2\times 2} \end{pmatrix}\,, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}\,. \tag{5.2}$$

Here, the vector of Pauli spin matrices is denoted as $\vec{\sigma}$. In consequence, the spin matrices σ_{AB} are the flat-space spin matrices. The spin matrices fulfill the commutation relations

$$[\tfrac{1}{2}\sigma^{CD}, \tfrac{1}{2}\sigma^{EF}] = i\left(g^{CF}\tfrac{1}{2}\sigma^{DE} + g^{DE}\tfrac{1}{2}\sigma^{CF} - g^{CE}\tfrac{1}{2}\sigma^{DF} - g^{DF}\tfrac{1}{2}\sigma^{CE}\right)\,. \tag{5.3}$$

These commutation relations, we should note in passing, are completely analogous to those fulfilled by the matrices $\mathbb{M}_{AB}$ that generate (four-)vector local Lorentz transformations. As is well known, the latter have the components (denoted by indices C and D)

$$(\mathbb{M}_{AB})^{C}{}_{D} = g^{C}{}_{A}\, g_{DB} - g^{C}{}_{B}\, g_{DA} \,. \tag{5.4}$$

The vector local Lorentz transformation Λ with components $\Lambda^{C}{}_{D}$ is obtained as the matrix exponential

$$\Lambda^{C}{}_{D} = \left(\exp\left[\frac{1}{2}\,\epsilon^{AB}\,\mathbb{M}_{AB}\right]\right)^{C}{}_{D} \,. \tag{5.5}$$

The algebra fulfilled by the $\mathbb{M}$ matrices is well known to be

$$[\mathbb{M}^{CD}, \mathbb{M}^{EF}] = g^{CF}\,\mathbb{M}^{DE} + g^{DE}\,\mathbb{M}^{CF} - g^{CE}\,\mathbb{M}^{DF} - g^{DF}\,\mathbb{M}^{CE} \,. \tag{5.6}$$

The two algebraic relations (5.3) and (5.6) are equivalent if one replaces

$$\mathbb{M}^{CD} \to -\frac{i}{2}\,\sigma^{CD} \,, \tag{5.7}$$

which leads from Eq. (5.1) to Eq. (5.5). Under a local Lorentz transformation, a Dirac spinor transforms as

$$\psi'(x') = S(\Lambda)\,\psi(x) \,. \tag{5.8}$$

In order to write the Lagrangian, one needs to define the Dirac adjoint in curved space-time. In order to address this question, one has to remember that in flat-space-time, the Dirac adjoint $\overline{\psi}(x)$ is defined in such a way that is transforms with the inverse of the spinor Lorentz transform as compared to $\psi(x)$,

$$\overline{\psi}'(x') = \overline{\psi}(x)\,S(\Lambda^{-1}) = \overline{\psi}(x)\,[S(\Lambda)]^{-1} \,. \tag{5.9}$$

The problem of the definition of $\overline{\psi}(x)$ in curved space-time is sometimes treated in the literature in a rather cursory fashion [12]. Let us see if in curved space-time, we can use the *ansatz*

$$\overline{\psi}(x) = \psi^{+}(x)\,\gamma^{0} \,, \tag{5.10}$$

with the same flat-space γ^0 as is used in the flat-space Dirac adjoint. In this case,

$$\overline{\psi}'(x') = \psi^{+}(x')\,S^{+}(\Lambda)\,\gamma^{0} = \left(\psi^{+}(x')\,\gamma^{0}\right)\,\left[\gamma^{0}\,S^{+}(\Lambda)\,\gamma^{0}\right] \,, \tag{5.11}$$

To first order in the Lorentz generators ϵ_{AB}, we have indeed,

$$\gamma^{0}\,S^{+}(\Lambda)\,\gamma^{0} = 1 + \frac{i}{4}\,\epsilon^{AB}\,\gamma^{0}\,\sigma^{+}_{AB}\,\gamma^{0} = 1 + \frac{i}{4}\,\epsilon^{AB}\,\sigma_{AB} = [S(\Lambda)]^{-1} \,, \tag{5.12}$$

where we have used the identity

$$\begin{aligned} \sigma^{+}_{AB} &= -\frac{i}{2}\,[\gamma^{+}_{B}, \gamma^{+}_{A}] = -\frac{i}{2}\,\gamma^{0}\,[\gamma^{0}\gamma^{+}_{B}\gamma^{0},\ \gamma^{0}\gamma^{+}_{A}\gamma^{0}]\,\gamma^{0} \\ &= -\frac{i}{2}\,\gamma^{0}\,[\gamma_{B},\ \gamma_{A}]\,\gamma^{0} = -\gamma^{0}\,\sigma_{BA}\,\gamma^{0} = \gamma^{0}\,\sigma_{AB}\,\gamma^{0} \,. \end{aligned} \tag{5.13}$$

It is easy to show that Eq. (5.12) generalizes to all orders in the ϵ^{AB} parameters, which justifies our *ansatz* given in Eq. (5.10). The result is that the flat-space γ^0 matrix can be used in curved space, just like in flat space, in order to construct the Dirac adjoint. The Dirac adjoint spinor transforms with the inverse spinor representation of the Lorentz group [see Eq. (5.9)].

5.3 Lagrangian and Charge Conjugation

Equipped with an appropriate form of the Dirac adjoint in curved space-time, we start from the Lagrangian density [11–20]

$$\mathcal{L} = \overline{\psi}(x) \left[\overline{\gamma}^{\mu} \{ i (\partial_{\mu} - \Gamma_{\mu}) - e A_{\mu} \} - m_I \right] \psi(x) , \tag{5.14}$$

Here, the A_{μ} field describes the four-vector potential of the electromagnetic field, while the Γ_{μ} matrices describe the spin connection.

$$\Gamma_{\mu} = \frac{i}{4} \omega_{\mu}^{AB} \sigma_{AB} , \qquad \omega_{\mu}^{AB} = e_{\nu}^{A} \nabla_{\mu} e^{\nu B} , \qquad \nabla_{\mu} e^{\nu B} = \partial_{\mu} e^{\nu B} + \Gamma^{\nu}_{\mu\rho} e^{\rho B} . \tag{5.15}$$

For the form of the covariant coupling, we refer to Eqs. (3.129) and (3.190) of Ref. [11]. In the above equations, capital Roman indices $A, B, C, \dots = 0, 1, 2, 3$ refer to a freely falling coordinate system (a Lorentz index), while Greek indices $\mu, \nu, \rho, \dots = 0, 1, 2, 3$ refer to an external coordinate system (an Einstein index).

We shall attempt to derive the particle-antiparticle symmetry on the level of a transformation of the Lagrangian. In comparison to textbook treatments (see, e.g., pp. 89 ff. and 263 ff. of Ref. [21], p. 70 of Ref. [22], p. 66 of Ref. [23], pp. 89 ff. and 263 ff. of Ref. [23], p. 142 of Ref. [24], p. 218 of Ref. [25], p. 67 of Ref. [26], p. 116 of Ref. [27], p. 320 of Ref. [28], p. 153 of Ref. [1], and Chap. 7 of Ref. [29]), our derivation is much more involved in view of the appearance of the Γ_{μ} matrices which describe the gravitational coupling. In other words, we note that *none* of the mentioned standard textbooks of quantum field theory discuss the *gravitationally* coupled Dirac equation, and all cited descriptions are limited to the flat-space Dirac equation, where the role of the charge conjugation operation is much easier to analyze than in curved space.

The double-covariant coupling to both the gravitational as well as the electromagnetic field is given as follows,

$$\mathcal{D}_{\mu} = \partial_{\mu} - \Gamma_{\mu} + ie A_{\mu} = \nabla_{\mu} + ie A_{\mu} , \tag{5.16}$$

where $\nabla_{\mu} = \partial_{\mu} - \Gamma_{\mu}$ is the gravitational covariant derivative.

As a side remark, we note that gravitational spin connections $\Gamma_{\mu} = \frac{i}{4} \omega_{\mu}^{AB} \sigma_{AB}$ and other gauge-covariant couplings are unified in the so-called spin-charge family theory [30–34] which calls for a unification of all known interactions of nature in terms of an $SO(1, 13)$ overarching symmetry group. (In the current article, we use the spin connection matrices purely in the gravitational context.) The $SO(1, 13)$ has a 25-dimensional Lie group, with 13 boosts and 12 rotations in the internal space. This provides for enough Lie algebra elements to describe the Standard

Model interactions, and predict a fourth generation of particles. The spin-charge family theory is a significant generalization of Kaluza-Klein-type ideas [35,36].

In the context of the current investigations, though, we restrict ourselves to the gravitational spin connection matrices. In view of the (in general) nonvanishing space-time dependence of the Ricci rotation coefficients, we can describe the quantum dynamics of relativistic spin-1/2 particles on the basis of Eqs. (5.14) and (5.15). The σ_{AB} matrices defined in Eq. (5.15) represent the six generators of the spin-1/2 representation of the Lorentz group.

The Lagrangian (5.14) is Hermitian, and so

$$\mathcal{L} = \mathcal{L}^{+} = \psi^{+}(x) \left[(\overline{\gamma}^{\mu})^{+} \left\{ -i \overleftarrow{\partial}_{\mu} - e\, A_{\mu} \right\} - (-i)\, (\Gamma_{\mu})^{+}\, (\overline{\gamma}^{\mu})^{+} - m_{I} \right] \left[\overline{\psi}(x) \right]^{+} . \tag{5.17}$$

An insertion of γ^0 matrices under use of the identity $(\gamma^0)^2 = 1$ leads to the relation

$$\begin{aligned} \mathcal{L}^{+} = \psi^{+}(x)\, \gamma^0 & \left[\gamma^0\, (\overline{\gamma}^{\mu})^{+}\, \gamma^0 \left\{ -i \overleftarrow{\partial}_{\mu} - e\, A_{\mu} \right\} \right. \\ & \left. + i \left\{ \gamma^0\, (\Gamma_{\mu})^{+}\, \gamma^0 \right\} \gamma^0\, (\overline{\gamma}^{\mu})^{+}\, \gamma^0 - m_{I} \right] \gamma^0 \left[\overline{\psi}(x) \right]^{+} . \end{aligned} \tag{5.18}$$

Also, we recall that $\gamma^0\, (\Gamma_{\mu})^{+}\, \gamma^0 = -\Gamma_{\mu}$, because

$$\Gamma_{\mu}^{+} = -\frac{i}{4}\, \omega_{\mu}^{AB}\, \sigma_{AB}^{+} = -\frac{i}{4}\, \omega_{\mu}^{AB}\, \gamma^0\, \sigma_{AB}\, \gamma^0 = -\gamma^0\, \Gamma_{\mu} \gamma^0 . \tag{5.19}$$

So, the adjoint of the Lagrangian is

$$\mathcal{L}^{+} = \psi^{+}(x)\, \gamma^0 \left[\overline{\gamma}^{\mu} \left\{ -i \overleftarrow{\partial}_{\mu} - e\, A_{\mu} \right\} - i\, \Gamma_{\mu}\, \overline{\gamma}^{\mu} - m_{I} \right] \gamma^0 \left[\overline{\psi}(x) \right]^{+} . \tag{5.20}$$

Now, we use the relations $\psi^{+}(x)\, \gamma^0 = \overline{\psi}(x)$ and $\gamma^0 \left[\overline{\psi}(x) \right]^{+} = \psi(x)$, and arrive at the form

$$\mathcal{L}^{+} = \overline{\psi}(x) \left[\overline{\gamma}^{\mu} \left\{ -i \overleftarrow{\partial}_{\mu} - e\, A_{\mu} \right\} - i\, \Gamma_{\mu}\, \overline{\gamma}^{\mu} - m_{I} \right] \psi(x) . \tag{5.21}$$

Because $\mathcal{L}$ is a scalar, a transposition again does not change the Lagrangian, and we have

$$\left(\mathcal{L}^{+} \right)^{T} = \psi^{T}(x) \left[(\overline{\gamma}^{\mu})^{T} \left\{ -i \overrightarrow{\partial}_{\mu} - e\, A_{\mu} \right\} - i\, (\overline{\gamma}^{\mu})^{T}\, (\Gamma_{\mu})^{T} - m_{I} \right] \left[\overline{\psi}(x) \right]^{T} . \tag{5.22}$$

An insertion of the charge conjugation matrix $C = i\gamma^2\, \gamma^0$ (with the flat-space γ^2 and γ^0) leads to

$$\begin{aligned} \left(\mathcal{L}^{+} \right)^{T} = \psi^{T}(x)\, C^{-1} & \left[C\, (\overline{\gamma}^{\mu})^{T}\, C^{-1} \left\{ -i \overrightarrow{\partial}_{\mu} - e\, A_{\mu} \right\} \right. \\ & \left. - i\, C\, (\overline{\gamma}^{\mu})^{T}\, C^{-1}\, C\, \Gamma_{\mu}^{T}\, C^{-1} - m_{I} \right] C \left[\overline{\psi}(x) \right]^{T} . \end{aligned} \tag{5.23}$$

we use the identities $C\, (\overline{\gamma}^{\mu})^{T}\, C^{-1} = -\overline{\gamma}^{\mu}$, and $C\, (\Gamma_{\mu})^{T}\, C^{-1} = -\Gamma_{\mu}$. The latter of these can be shown as follows,

$$C\, \Gamma_{\mu}^{T}\, C^{-1} = \frac{i}{4} \left\{ \frac{i}{2} \omega_{\mu}^{AB}\, C \left[\gamma_{B}^{T}, \gamma_{A}^{T} \right] C^{-1} \right\} = \frac{i}{4} \left\{ \frac{i}{2} \omega_{\mu}^{AB} \left[-\gamma_{B}, -\gamma_{A} \right] \right\} = -\Gamma_{\mu} . \tag{5.24}$$

The result is the expression

$$\left(\mathcal{L}^{+}\right)^{\mathrm{T}} = \psi^{\mathrm{T}}(x)\, C^{-1} \left[(-\overline{\gamma}^{\mu}) \left\{-\mathrm{i}\, \overrightarrow{\partial}_{\mu} - e\, A_{\mu}\right\} - \mathrm{i}\, (-\overline{\gamma}^{\mu})\, (-\Gamma_{\mu}) - m_{\mathrm{I}}\right] C\, \left[\overline{\psi}(x)\right]^{\mathrm{T}} . \tag{5.25}$$

Now we express the result in terms of the charge-conjugate spinor $\psi^{\mathcal{C}}(x)$ and its adjoint $\overline{\psi^{\mathcal{C}}(x)}$ (further remarks on this point are presented in Appendix 5.7),

$$\psi^{\mathcal{C}}(x) = C\, \left[\overline{\psi}(x)\right]^{\mathrm{T}} , \qquad \overline{\psi^{\mathcal{C}}(x)} = -\psi^{\mathrm{T}}(x)\, C^{-1} , \tag{5.26}$$

where we use the identity $C^{-1} = -C$ (see also Appendix 5.6). The Lagrangian becomes

$$\begin{aligned} \mathcal{L} = \left(\mathcal{L}^{+}\right)^{\mathrm{T}} &= -\overline{\psi^{\mathcal{C}}(x)} \left[\overline{\gamma}^{\mu} \left\{\mathrm{i}\, \overrightarrow{\partial}_{\mu} + e\, A_{\mu}\right\} - \mathrm{i}\, \overline{\gamma}^{\mu}\, \Gamma_{\mu} - m_{\mathrm{I}}\right] \psi^{\mathcal{C}}(x) \\ &= -\, \overline{\psi^{\mathcal{C}}(x)}\, \left[\overline{\gamma}^{\mu} \{\mathrm{i}(\partial_{\mu} - \Gamma_{\mu}) + e\, A_{\mu}\} - m_{\mathrm{I}}\right]\, \psi^{\mathcal{C}}(x) . \end{aligned} \tag{5.27}$$

The Lagrangian given in Eq. (5.27) differs from (5.17) only with respect to the sign of electric charge, as is to be expected, and with respect to the replacement of the Dirac spinor $\psi(x)$ by its charge conjugation $\psi^{\mathcal{C}}(x)$. The overall minus sign is physically irrelevant as it does not influence the variational equations derived from the Lagrangian; besides, it finds a natural explanation in terms of the reinterpretation principle, if we interpret $\psi(x)$ as a Dirac wave function in first quantization.

Namely, there is a connection of the spatial integrals of the mass term, proportional to

$$J = \int d^3 r\, \overline{\psi}(x) \psi(x) = \int d^3 r\, \overline{\psi}(t, \vec{r})\, \psi(t, \vec{r}) = \int d^3 r\, \psi^{+}(t, \vec{r}) \gamma^0 \psi(t, \vec{r}) , \tag{5.28}$$

and the charge conjugate,

$$J^{\mathcal{C}} = \int d^3 r\, \overline{\psi}^{\mathcal{C}}(x) \psi^{\mathcal{C}}(x) = \int d^3 r\, \left(\psi^{\mathcal{C}}(t, \vec{r})\right)^{+} \gamma^0 \psi(t, \vec{r}) . \tag{5.29}$$

Both of the above integrals connect to the energy eigenvalue of the Dirac equation in the limit of time-independent fields (see Appendices 5.6 and 5.7). One can show that the energy eigenvalues of Dirac eigenstates ψ, in the limit of weak potentials and states composed of small momentum components, exactly correspond to the integrals J and $J^{\mathcal{C}}$ (up to a factor m_{I}). In turn, the dominant term in the Lagrangian in this limit is

$$\mathcal{L} \to -\overline{\psi}(x)\, m_{\mathrm{I}}\, \psi(x) = +\overline{\psi^{\mathcal{C}}(x)}\, m_{\mathrm{I}}\, \psi^{\mathcal{C}}(x) . \tag{5.30}$$

Because the integral $\int d^3 r\, \mathcal{L}$ equals $-J$ (or $+J^{\mathcal{C}}$), the sign change becomes evident: it is due to the fact that the states $\psi^{\mathcal{C}}$ describe antiparticle wave functions where the sign of the energy flips in comparison to particles. The matching of m_{I} to the gravitational mass can be performed in a central, static field [6,9], and results in the identification $m_{\mathrm{I}} = m_{\mathrm{G}}$, where m_{G} is the gravitational mass. The gravitational covariant derivative $\partial_{\mu} - \Gamma_{\mu}$ has retained its form in going from (5.17) to (5.27), in agreement with the perfect particle-antiparticle symmetry of the gravitational interaction. Because the above demonstration is general and holds for arbitrary (possibly dynamic) space-time background Γ, there is no room for a deviation of the gravitational interactions of antiparticles (antimatter) to deviate from those of matter. This has been demonstrated here on the basis of Lagrangian methods, supplementing a recent preliminary result [9].

In order to fully clarify the origin of the minus sign introduced upon charge conjugation, one consults Chaps. 2 and 3 of Ref. [1] and Chap. 7 of Ref. [29]. Namely, in second quantization, there is an additional minus sign incurred upon the charge conjugation, which restores the original sign pattern of the Lagrangian. According to Eq. (2.107) and (3.157) of Ref. [1], we can write the expansion of the free Dirac field operator as

$$\hat{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{m}{E} \left[a_s(\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x} + e^{ip\cdot x}\, v_s(\vec{p})\, b_s^+(\vec{p}) \right] . \tag{5.31}$$

The field operator is denoted by a hat in order to differentiate it from the Dirac wave function. The four-momentum is $p^\mu = (E, \vec{p})$, where $E = \sqrt{\vec{p}^2 + m^2}$ is the free Dirac energy, and $u_s(\vec{p})$ and $v_s(\vec{p})$ are the positive-energy and negative-energy spinors with spin projection s (onto the z axis). Furthermore, the particle annihilation operator $a_s(\vec{p})$ and the antiparticle creation operator $b_s^+(\vec{p})$, and their Hermitian adjoints, fulfill the commutation relations given in Eqs. (3.161) of Ref. [1],

$$\left\{ a_s(\vec{p}), a_{s'}^+(\vec{p}) \right\} = \frac{E}{m} (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}\,')\, \delta_{ss'} , \tag{5.32a}$$

$$\left\{ b_s(\vec{p}), b_{s'}^+(\vec{p}\,') \right\} = \frac{E}{m} (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}\,')\, \delta_{ss'} . \tag{5.32b}$$

The spinors are normalized according to Eq. (2.43a) of Ref. [1], i.e., they fulfill the relation $u_s^+(\vec{p})\, u_s(\vec{p}) = v_s^+(\vec{p})\, v_s(\vec{p}) = E/m$. For the charge conjugation in the second-quantized theory, it is essential that an additional minus sign is incurred in view of the anticommutativity of the field operators. Namely, without considering the interchange of the field operators, one would have, under charge conjugation, $J^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x) = \overline{\psi}^{\mathcal{C}}(x)\gamma^\mu\psi^{\mathcal{C}}(x) = J^{\mathcal{C}\mu}(x)$, i.e., the current would not change under charge conjugation which is intuitively inconsistent [see the remark following Eq. (4.618) of Ref. [29]]. However, for the field operator current (from here on, we denote field operators with a hat), we have $\hat{J}^\mu(x) = \hat{\overline{\psi}}(x)\gamma^\mu\hat{\psi}(x) = -\hat{\overline{\psi}}^{\mathcal{C}}(x)\gamma^\mu\hat{\psi}^{\mathcal{C}}(x) = -\hat{J}^{\mathcal{C}\mu}(x)$, because one has incurred an additional minus sign due to the restoration of the field operators into their canonical order after charge conjugation [see the remark following Eq. (7.309) of Ref. [29]].

In our derivation above, when one transforms to a second-quantized Dirac field (but keeps classical background electromagnetic field and a classical non-quantized curved-space-time metric), one starts from Eq. (5.21) as an equivalent, alternative formulation of Eq. (5.14). One observes that in going from Eq. (5.21) to (5.22), one has actually changed the order of the field operators in relation to the Dirac spinors. Restoring the original order, much in the spirit of Eq. (7.309) of

Ref. [29], one incurs an additional minus sign which ensures that

$$\begin{aligned}\hat{\mathcal{L}} &= \widehat{\overline{\psi}}(x)\, [\overline{\gamma}^{\mu}\{\mathrm{i}\,(\partial_{\mu} - \Gamma_{\mu}) - e\,A_{\mu}\} - m_{\mathrm{I}}]\; \hat{\psi}(x) \\ &= \widehat{\overline{\psi}}^{\mathcal{C}}(x)\; [\overline{\gamma}^{\mu}\{\mathrm{i}(\partial_{\mu} - \Gamma_{\mu}) + e\,A_{\mu}\} - m_{\mathrm{I}}]\; \hat{\psi}^{\mathcal{C}}(x)\,, \end{aligned} \tag{5.33}$$

exhibiting the effect of charge conjugation in the second-quantized theory—and restoring the overall sign of the Lagrangian. The theorem (5.33) shows that particles and antiparticles behave exactly the same in gravitational fields, but it does not imply, *a priori*, that $m_{\mathrm{I}} = m_{\mathrm{G}}$. The matching of the inertial mass m_{I} and the gravitational mass m_{G} most easily proceeds in a central, static field (Schwarzschild metric), as demonstrated in Sec. 3 of Ref. [9].

One should, at this stage, remember that experimental evidence, to the extent possible, supports the above derived symmetry relation. The only direct experimental result on antimatter and gravity comes, somewhat surprisingly, from the Supernova 1987A. Originating from the Large Magellanic Cloud, the originating neutrinos and antineutrinos eventually were detected on Earth. In view of their travel time of about 160,000 years, they were bent from a "straight line" by the gravity from our own galaxy. The gravitational bending changed the time needed to reach Earth by about 5 months. Yet, both neutrinos and antineutrinos reached Earth within the same 12 second interval, shows that neutrinos and antineutrinos fall similarly, to a precision of about 1 part in a million [37, 38]. In view of the exceedingly small rest mass of neutrinos, the influence of the mass term (even a conceivable tachyonic mass term) on the trajectory is negligible [39]. Yet, it is reassuring that experimental evidence, at this time, is consistent with Eq. (5.33).

5.4 Other Interpretations of Antimatter Gravity

In view of the symmetry relations derived in this article for the gravitationally and electromagnetically coupled Dirac equation, it is certain justified to ask about an adequate interpretation of antimatter gravity experiments. We have shown that canonical gravity cannot account for any deviations of gravitational interactions of matter versus antimatter. How could tests of antimatter "gravity" be interpreted otherwise? The answer to that question involves clarification of the question which "new" interactions could possibly mimic gravity. The criteria are as follows: *(i)* The "new" interaction would need to violate CPT symmetry. *(ii)* The "new" interaction would have to be a long-range interaction, mediated by a massless virtual particle.

One example of such an interaction would be induced if hydrogen atoms were to acquire, in addition to the electric charges of the constituents (electrons and protons), an additional "charge" $\eta\, e$, where e is the elementary charge, while antihydrogen atoms would acquire a charge $-\eta\, e$, where η is a small parameter. One could conjecture the existence of a small, CPT-violating "charge" $\eta e/2$ for electrons, protons, and neutrons, while positrons and antiprotons, and antineutrons, would carry a "charge" $-\eta e/2$. We will refer to this concept as the "η force" in the following. The difference in the gravitational force (acceleration due to the

Earth's field) felt by a hydrogen versus an antihydrogen atom is

$$F^{\eta}_{\overline{H}} - F^{\eta}_{H} = 2\eta \left[\frac{\eta}{2}(N_p + N_n + N_e)\right] \frac{e^2}{4\pi\epsilon R^2_{\oplus}} . \tag{5.34}$$

Here, $R_{\oplus}$ is the radius of the Earth, while N_p, N_n and N_e are the numbers of protons, neutrons and electrons in the Earth. The gravitational force on a falling antihydrogen atom is

$$F^{G}_{\overline{H}} = G \frac{m_p M_{\oplus}}{R^2_{\oplus}} . \tag{5.35}$$

Let us assume that an experiment establishes that $|F^{\eta}_{\overline{H}} - F^{\eta}_{H}| < \chi F^{G}_{\overline{H}}$, where χ is a measure of the deviation of the acceleration due to gravity+"η"-force for antihydrogen versus hydrogen. A quick calculation shows that this translates into a bound

$$\eta < 7.3 \times 10^{-19} \sqrt{\chi} . \tag{5.36}$$

Antimatter gravity tests thus limit the available parameter space for η, and could be interpreted in terms of corresponding limits on the maximum allowed value of η.

5.5 Conclusions

In the current paper, we have analyzed the particle-antiparticle symmetry of the gravitationally (and electromagnetically) coupled Dirac equation and come to the conclusion that a symmetry exists, for the second-quantized formulation, which precludes the existence particle-antiparticle symmetry breaking terms on the level of Dirac theory. In a nutshell, one might say the following: Just as much as the electromagnetically coupled Dirac equation predicts that antiparticles have the opposite charge as compared to particles (but otherwise behave exactly the same under electromagnetic interactions), the gravitationally coupled Dirac equation predicts that particles and antiparticles follow exactly the same dynamics in curved space-time, i.e., with respect to gravitational fields (in particular, they have the same gravitational mass, and there is no sign change in the gravitational coupling). In the derivation of our theorem (5.33), we use the second-quantized Dirac formalism, in the Lagrangian formulation. Our general result for the Dirac adjoint, communicated in Sec. 5.2, paves the way for the Lagrangian of the gravitationally coupled field, and its explicit form is an essential ingredient of our considerations.

Why is this interesting? Well, first, because the transformation of the gravitational force under the particle-to-antiparticle transformation has been discussed controversially in the literature [40–43]. In Ref. [10], it was pointed out that the role of the CPT transformation in gravity needs to be considered with care: It relates the fall of an apple on Earth to the fall of an anti-apple on anti-Earth, but not on Earth. The Dirac equation, colloquially speaking, applies to both apples as well as anti-apples on Earth, i.e., to particles and antiparticles in the same space-time metric. Second, our results have important consequences because one might have otherwise speculated about the existence of tiny violations of the particle-antiparticle symmetry, even on the level of the gravitationally coupled Dirac theory.

For example, in Ref. [44], it was claimed that the Dirac Hamiltonian for a particle in a central gravitational field, after a Foldy–Wouthuysen transformation which disentangles the particle from the antiparticle degrees of freedom, contains the term [see the last term on the first line of the right-hand side of Eq. (31)]

$$H \sim -\frac{\hbar}{2c}\vec{\Sigma}\cdot\vec{g}\,, \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} . \tag{5.37}$$

We here restore the factors $\hbar$ and c in order to facilitate the comparison to Ref. [44]. The term proportional to $\vec{\Sigma}\cdot\vec{g}$, where $\vec{g}$ is the acceleration due to gravity, would break parity, because $\vec{\Sigma}$ transforms as a pseudovector, while $\vec{g}$ transforms as a vector under parity. This aspect has given rise to discussion, based on the observation that an initially parity-even Hamiltonian (in a central field) should not give rise to parity-breaking terms after a disentangling of the effective Hamiltonians for particles and antiparticles [45, 46].

We should note that Ref. [44] was not the only place in the literature where the authors speculated about the existence of P, and CP–violating terms obtained after the identification of low-energy operators obtained from Dirac Hamiltonians in gravitational fields. E.g., in Eq. (46) of Ref. [47], spurious parity-violating, and CP-violating terms were obtained after a Foldy–Wouthuysen transformation; these terms would of course also violate particle-antiparticle symmetry.

In the context of the current discussion, the existence of terms proportional to $\vec{\Sigma}\cdot\vec{g}$, as given in Eq. (5.37), would also violate particle-antiparticle symmetry: This is because it lacks the universal prefactor $\beta = \gamma^0$, where

$$\beta = \begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & -\mathbb{I}_{2\times 2} \end{pmatrix} . \tag{5.38}$$

In fact, in the complete result (up to fourth order in the momenta) for the effective particle-antiparticle Hamiltonian in a central field, given in Eq. (21) of Ref. [6], all terms have a common prefactor β. The common prefactor β implies that, after the application of the reinterpretation principle for antiparticles, the effective Hamiltonians for particles and antiparticles in a central gravitational field (but without electromagnetic coupling) are exactly the same, and ensures the particle-antiparticle symmetry.

The absence of such parity-violating (and particle-antiparticle symmetry breaking) terms has meanwhile been confirmed in remarks following Eq. (15) of Ref. [48], in the text following Eq. (35) of Ref. [49], and also, in clarifying remarks given in the text following Eq. (7.33) of Ref. [50]. Further clarifying analyses can be found in Ref. [51] and in Ref. [52]. Related calculations have recently been considered in other contexts [50, 53, 54]. The question of whether such parity- and particle-antiparticle symmetry violating terms could exist in higher orders in the momentum expansion has been answered negatively in Ref. [7], but only for a static central gravitational field, and in Ref. [55], still negatively, for combined *static*, central gravitational and electrostatic fields.

However, the question regarding the absence of particle-antiparticle symmetry breaking terms for general, dynamic space-time backgrounds has not been answered conclusively in the literature up to this point, to the best of our knowledge. This has been the task of the current paper. In particular, our results imply a *no-go theorem regarding the possible emergence of particle-antiparticle-symmetry breaking gravitational, and combined electromagnetic-gravitational terms in general static and dynamic curved-space-time backgrounds.* Any speculation [44,47] about the re-emergence of such terms in a dynamic space-time background can thus be laid to rest. Concomitantly, we demonstrate that there are no "overlap" or "interference" terms generated in the particle-antiparticle transformation, between the gauge groups, namely, the $SO(1,3)$ gauge group of the local Lorentz transformations, and the $U(1)$ gauge group of the electromagnetic theory. This result implies both progress and, unfortunately, some disappointment, because the emergence of such terms would have been fascinating and would have opened up, quite possibly, interesting experimental opportunities. In our opinion, antimatter gravity experiments should be interpreted in terms of limits on CPT-violating parameters, such as the η parameter introduced in Sec. 5.4. This may be somewhat less exciting than a "probe of the equivalence principle for antiparticles" but still, of utmost value for the scientific community.

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5.6 APPENDIX: Sign Change of $\overline{\psi}\,\psi$ under Charge Conjugation

With the charge conjugation matrix $C = i\gamma^2\gamma^0$ (superscripts denote Cartesian indices), and the Dirac adjoint $\overline{\psi} = \psi^+\,\gamma^0$, we have

$$\psi^{\mathcal{C}} = C\,\overline{\psi}^{\mathrm{T}} = i\gamma^2\,\gamma^0\,\gamma^0\,\psi^* = i\gamma^2\,\psi^*\,. \tag{5.39}$$

We recall that the γ^2 (contravariant index, no square) matrix in the Dirac representation matrix is

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \left(\sigma^2\right)^+ = \sigma^2\,, \tag{5.40}$$

which implies that $\left(\gamma^2\right)^+ = -\gamma^2$. The Dirac adjoint of the charge conjugate is

$$\overline{\psi}^{\mathcal{C}} = \left(\psi^{\mathcal{C}}\right)^+\,\gamma^0 = \psi^{\mathrm{T}}(-i)\,\left(\gamma^2\right)^+\,\gamma^0 = \psi^{\mathrm{T}}(-i)\,(-\gamma^2)\,\gamma^0 = \psi^{\mathrm{T}}\,i\,\gamma^2\,\gamma^0\,. \tag{5.41}$$

This leads to a verification of the sign flip of the mass terms in the gravitationally coupled Lagrangian for antimatter, given in Eq. (5.27) [see also Eqs. (5.28) and (5.29)],

$$\overline{\psi}^{\mathcal{C}}\,\psi^{\mathcal{C}} = (\psi^{\mathrm{T}} i\gamma^2)\,\gamma^0\,(i\,\gamma^2\,\psi^*) = -(i)^2\psi^{\mathrm{T}}\,(\gamma^2)^2\,\gamma^0\psi^* = -\psi^{\mathrm{T}}\,\gamma^0\,\psi^* = -\overline{\psi}\,\psi\,. \tag{5.42}$$

Two useful identities *(i)* $\gamma^0\, C^+\, \gamma^0 = C$ and *(ii)* $C^{-1} = -C$ have been used in Sec. 5.3. These will be derived in the following. The explicit form of the γ^2 matrix in the Dirac representation implies that $\left(\gamma^2\right)^+ = -\gamma^2$. Based on this relation, we can easily show that

$$C^+ = \left(\mathrm{i}\,\gamma^2\,\gamma^0\right)^+ = -\mathrm{i}\,\gamma^0\,\left(\gamma^2\right)^+ = \mathrm{i}\,\gamma^0\,\gamma^2 = -\mathrm{i}\,\gamma^2\,\gamma^0 = -C\,. \tag{5.43}$$

The first identity $\gamma^0\, C^+\, \gamma^0 = C$ can now be shown as follows,

$$\gamma^0\, C^+\, \gamma^0 = \gamma^0\,\left[-\mathrm{i}\,\gamma^2\,\gamma^0\right]\,\gamma^0 = -\mathrm{i}\,\gamma^0\,\gamma^2 = \mathrm{i}\,\gamma^2\,\gamma^0 = C\,. \tag{5.44}$$

Furthermore, one has

$$C\,C^+ = C\,(-C) = \mathrm{i}\,\gamma^2\,\gamma^0\,\mathrm{i}\,\gamma^0\,\gamma^2 = -\left(\gamma^2\right)^2 = -\left(-\mathbb{I}_{4\times 4}\right) = \mathbb{I}_{4\times 4}\,, \tag{5.45}$$

so that

$$C^{-1} = C^+ = -C\,, \tag{5.46}$$

which proves, in particular, that $C^{-1} = -C$.

5.7 APPENDIX: General Considerations

A few illustrative remarks are in order. These concern the following questions: *(i)* To which extent do gravitational and electrostatic interactions differ for relativistic particles? This question is relevant because, in the nonrelativistic limit, in a central field, both interactions are described by potentials of the same functional form ("$1/R$ potentials"). *(ii)* Also, we should clarify why the integrals (5.28) and (5.29) represent the dominant terms in the evaluation of the Dirac particle energies, in the nonrelativistic limit.

After some rather deliberate and extensive considerations, one can show [8] that, up to corrections which combine momentum operators and potentials, the general Hamiltonian for a Dirac particle in a combined electric and gravitational field is

$$H_D = \vec{\alpha}\cdot\vec{p} + \beta\{m(1+\phi_G)\} + e\phi_C\,, \tag{5.47}$$

where ϕ_G is the gravitational, and ϕ_C is the electrostatic potential. Also, $\vec{\alpha}$ is the vector of Dirac α matrices, $\vec{p}$ is the momentum operator, and $\beta = \gamma^0$ is the Dirac β matrix. After a Foldy–Wouthuysen transformation [51], one sees that the gravitational interaction respects the particle-antiparticle symmetry, while the Coulomb potential does not, commensurate with the opposite sign of the charge for antiparticles. Question *(i)* as posed above can thus be answered with reference to the fact that, in leading approximation, the gravitational potential enters the Dirac equation as a scalar potential, modifying the mass term, while the electrostatic potential can be added to the free Dirac Hamiltonian $vec\alpha\cdot\vec{p} + \beta m$ by covariant coupling [1].

The second question posed above is now easy to answer: Namely, in the nonrelativistic limit, one has

$$\vec{\alpha}\cdot\vec{p} \to 0\,, \tag{5.48}$$

and furthermore, the gravitational and electrostatic potentials can be assumed to be weak against the mass term, at least for non-extreme Coulomb fields [56]. Under these assumptions, one has $H_D \to \beta m$, and the matrix element $\langle\psi|H_D|\psi\rangle$ assumes the form $\int d^3r\, \psi^+(\vec{r})\, \gamma^0\, m\psi(\vec{r})$ [see Eq. (5.28)].

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6 Main Properties of New Heavy Hadrons and the Luminosity of Hadronic Dark Matter

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Abstract. The origin and main properties of new heavy hadrons as dark matter candidates, are represented. Low-energy interactions of new hadrons with leptons and nucleons are described in the terms of effective vertexes. We consider the lowest excited levels of new mesons in the frame-work of the Heavy Quark Effective Theory. The effect of fine and hyper-fine splitting of excited states follows directly from this theory. We analyze phenomenological consequences of this effect as manifestation of dark matter particles.

Povzetek. Avtorja predstavita izvor in glavne lastnosti novih hadronov kot kandidatov za delce temne snovi. Nizkoenergijske interakcije novih hadronov z leptoni in nucleoni običajne snovi predstavita z efektivnimi vozlišči. Obravnavata nanjižja vzbujena stanja v okviru efektivne teorije težkih kvarkov, fini in hiperfini razcep vzbujenih stanj ter fenomenološke posledice za obravnavo teh delcev kot temne snovi.

Keywords: hadronic dark matter; hyperfine splitting; luminosity
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6.1 Introduction

The cald dark matter candidates usually are interpreted as stable weakly interacting massive particles (WIMP). Rigid experimental constraints on the cross-section of WIMP-nucleon interaction [1] exclude some variants of WIMPs. So, alternative scenarios are considered in literature, for example, the scenario with strongly interacting massive particle (SIMP) [2]- [7]. In these works, the scenario of hadronic dark matter realization was represented, where dark matter (DM) particles consists of new heavy and ordinary quarks. Such scenarios can be realized in the SM extensions with fourth generation [2, 3], in the chiral-symmetric models [7, 8], and in the extension with singlet quark [9].

Principal properties of hadronic DM particles of meson type were considered in Refs. [8–10], where it was shown that hadronic DM scenario is not excluded by EW and cosmochemical constraints. Low-energy interaction of hadronic dark matter (HDM) with ordinary matter was decribed in Refs. [11, 12]. There, Lagrangians

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of strong and weak interactions of new heavy mesons with ordinary light ones and gauge bosons were derived. It was shown in [12], that the effect of fine and hyperfine splitting manifests itself in the spectrum of new heavy mesons. Note, the existence of new heavy hadrons and their principal properties are the direct coseuqunsenses of high energy SM extensions. These extensions have independent meaning as variants of realization of grand unification theory. Application of this scenario to the description of DM is not obligatory, however, it gives the simplest and natural realization of hadronic DM scenario.

In this report, we consider in details the main properties of new heavy mesons and their phenomenological consequences. In Section 2, we describe the interaction of new quarks with the gauge bosons, electro-weak restrictions on the mass of these quarks and define the value of their mass. Low-energy interaction of new mesons with ordinary particles (leptons and nucleons) are considered in Section 3. The effects of fine and hyperfine splitting in the set of new heavy mesons are described in Section 4. Some conclusions are presented in Section 5.

6.2 New Heavy Stable Quarks

In the scenario with chiral extension of SM, new sets of the up and down quarks has the form:

$$Q = \{Q_R = \begin{pmatrix} U \\ D, \end{pmatrix}_R ; \;\; U_L, \;\; D_L\} \tag{6.1}$$

The structure of covariant derivatives is defined in standard way:

$$\begin{aligned} D_\mu Q_R =&(\partial_\mu - ig_1 Y_Q V_\mu - \frac{ig_2}{2}\tau_a V^a_\mu - ig_3 t_i G^i_\mu)Q_R; \\ D_\mu U_L =&(\partial_\mu - ig_1 Y_U V_\mu - ig_3 t_i G^i_\mu)U_L, \\ D_\mu D_L =&(\partial_\mu - ig_1 Y_D V_\mu - ig_3 t_i G^i_\mu)D_L. \end{aligned} \tag{6.2}$$

In the Eqs. (6.2), the values Y_A, $A = Q, U, D$, are the hypercharges and t_i are generators of $SU_C(3)$ -group. Here, gauge boson fields V^a_μ are superheavy chiral partner of standard fields. If we interprete the gauge field V_μ as standard $U(1)_Y$ one, then standard mixing of V_μ and V^3_μ is forbidden. Moreover, standard interpretation of the field V_μ and weak hypercharge $Y_Q = \bar{q}$ leads to wrong $V-A$ structure of photon interaction. These obstakles were considered in detail in Ref. [8], where hypercharge operator was redefined and vector-like interaction of new quarks was established:

$$L^{int}_Q = g_1 V_\mu \bar{Q}\gamma^\mu \hat{q} Q = g_1(c_w A_\mu - s_w Z_\mu)(q_U \bar{U}\gamma^\mu U - q_D \bar{D}\gamma^\mu D), \tag{6.3}$$

where the field V_μ is standard mixture of photon, A_μ, and boson, Z_μ. In expression (6.3), the values $c_w = \cos\theta_w$, $s_w = \sin\theta_w$, $g_1 c_w = e$ and θ_w is Weinberg angle.

In the scenario with singlet quark (SQ), new heavy quark, S, is a singlet with respect to $SU_W(2)$ weak group. The high-energy origin and low-energy phenomenology of singlet quark (SQ) were considered in many works (see, for example, [13]- [17] and references therein). The low-energy phenomenology of

SQ mainly is stipulated by effect of it's mixing with ordinary quarks. This mixing causes the appearence of flavor changing neutral currents (FCNC) and instability of SQ. Here, we consider the variant with stable SQ which have no the mixing with ordinary ones. Because the SQ strongly interacts with ordinary quarks, they form the bound states of type (Sq), (Sqq), (SSq). Here, we consider the main properties of two-quark, (Sq), meson states and the scenario, where the lightest neutral meson $M^0 = (\bar{S}q)$ is the DM candidate.

Further, we present the scenario of the SM extension with singlet quark S, which can be up, U, or down, D, type. The field S is singlet representation of $SU_W(2)$ group and has standard properties with respect to $U_Y(1)$ and color $SU_C(3)$ groups. Minimal Lagrangian of SQ interaction with the gauge bosons is:

$$L_S = i\bar{S}\gamma^{\mu}(\partial_{\mu} - ig_1 qV_{\mu} - ig_s t_a G^a_{\mu})S - M_S\bar{S}S. \tag{6.4}$$

In (6.4), hypercharge $Y/2 = q$ of singlet quark S, $t_a = \lambda_a/2$ are generators of $SU_C(3)$ -group, and M_S is mass of quark. Abelian part of the Lagrangian (6.4) describes the interactions of SQ with photon A and Z-boson:

$$L_S^{int} = g_1 qV_{\mu}\bar{S}\gamma^{\mu}S = qg_1(c_w A_{\mu} - s_w Z_{\mu})\bar{S}\gamma^{\mu}S. \tag{6.5}$$

Note, the interaction of SQ with Z-bosons has vector-like form.

The constraints on new fermions follow from the EW measurements of the vector boson polarizations. The contributions of new quarks into polarizations of gauge bosons γ, Z, W are described by Peskin-Takeuchi parameters (PT parameters). In our case, polarizations $\Pi_{ab}(0) = 0$ and PT parameters can be represented as follows:

$$\begin{aligned}S =& \frac{4s_w^2 c_w^2}{\alpha}[\frac{\Pi_{ZZ}(M_Z^2, M_Q^2)}{M_Z^2} - \frac{c_w^2 - s_w^2}{s_w c_w}\Pi'_{\gamma Z}(0, M_Q^2) - \Pi'_{\gamma\gamma}(0, M_Q^2)]; \;\; T = 0;\\ U =& -\frac{4s_w^2}{\alpha}[c_w^2\frac{\Pi_{ZZ}(M_Z^2, M_Q^2)}{M_Z^2} + 2s_w c_w \Pi'_{\gamma Z}(0, M_Q^2) + s_w^2 \Pi'_{\gamma\gamma}(0, M_Q^2)].\end{aligned} \tag{6.6}$$

In Eqs. (6.6), $\alpha = e^2/4\pi$, M_Q is mass of new quark and $\Pi_{ab}(p^2)$ are defined at $p^2 = M_Z^2$ and $p^2 = 0$. The values $\Pi_{ab}(p^2, M_Q^2)$ can be described by the expressions ($q = 2/3$):

$$\begin{aligned}\Pi_{ab}(p^2, M_Q^2) =& \frac{g_1^2}{9\pi^2}k_{ab}F(p^2, M_Q^2); \;\; k_{ZZ} = s_w^2, \; k_{\gamma\gamma} = c_w^2, \; k_{\gamma Z} = -s_w c_w;\\ F(p^2, M_Q^2) =& -\frac{1}{3}p^2 + 2M_Q^2 + 2A_0(M_Q^2) + (p^2 + 2M_Q^2)B_0(p^2, M_Q^2).\end{aligned} \tag{6.7}$$

By straightforward calculations we get rather simple expressions for PT parameters:

$$S = -U = \frac{ks_w^4}{9\pi}[-\frac{1}{3} + 2(1 + 2\frac{M_Q^2}{M_Z^2})(1 - \sqrt{\beta}\arctan\frac{1}{\sqrt{\beta}})]. \tag{6.8}$$

In Eq. (6.8), $\beta = 4M_Q^2/M_Z^2 - 1$, $k = 16(4)$ (SQ model) with the value of charge $q = 2/3(-1/3)$, and $k = 20$ in the chiral-symmetric model. We check that the values of PT parameters significantly less the experimental limits [18]:

$$S = 0.00 + 0.11(-0.10), \;\; U = 0.08 \pm 0.11, \;\; T = 0.02 + 0.11(-0.12). \tag{6.9}$$

So, the scenarios with singlet and mirror quarks satisfy to the experimental EW restrictions on new physics.

Additional EW restrictions follow from the flavor-changing neutral currents (FCNC). In the scenario, new quark does not mix with ordinary ones, FCNC are absent and there are no additional restrictions from the rare processes. Thus, the scenario with new heavy quarks is not excluded by precision EW restrictions. In Ref. [7], it was shown that the potential of new meson and nucleon interaction has repulsive character. So, the DM particles do not form the coupled states with nucleons. This effect makes it possible to escape strong cosmo-chemical constraints on the anomalous elements [7].

Quantum numbers, quark and isotopic structure of new hadrons are represented in Refs. [7,8], where their properties and evolution are briefly described. Here, we describe the principal properties of new mesons with structure of type (qQ), in particular, the mesons $M = (M^0, M^-)$. The mass M_0 of neutral component M^0 is defined from the equality of annihilation cross-sections at freez-out phase:

$$(\sigma(M)v_r)^{Mod} = (\sigma v_r)^{Exp} \tag{6.10}$$

In Eq. (6.10), the left part is model value of annihilation cross-section and the right part follows from the data on the recil abondence of DM, $(\sigma v_r)^{Exp} = 2{\cdot}10^{-9}\ GeV^{-2}$. The cross-section of annihilation $Q\bar{Q} \to gg, q\bar{q}$ was presented in [10]:

$$(\sigma(M))^{Mod} = \sigma(Q\bar{Q} \to gg, q\bar{q}) \approx \frac{44\pi}{9}\frac{\alpha_s^2}{M^2}. \tag{6.11}$$

Using the expression (6.11) and equality (6.10) we get the estimation of new quarks mass, $M \approx 10$ TeV. From this estimation, it follows that freezing out temperature $T_f \approx M/30 \approx 300$ GeV, i.e., it is much greater than the temperature of QCD phase transition, $T_{QCD} \approx 150$ MeV. So, the stage of hadronization of ordinary and new heavy hadrons begins much later the freezing out one. After phase transition new heavy quark Q combine with ordinary light quark q into new heavy Q-hadrons. In baryon asymmetrical Universe it is possible the forming of meson states $(q\bar{Q})$ and baryon states (qqQ) with unit electrical charge. Further, we consider the meson states only, while the more complicated states were considered in Ref. [10].

6.3 Interaction of DM with ordinary matter

Low-energy interaction of new hadrons with leptons is described by effective Lagranian in standard differential form [12]:

$$L^{eff}(WMM) = iG_{WM}U_{ik}W^{+\mu}(\bar{M}_{ui}\partial_\mu M_{dk} - \partial_\mu \bar{M}_{ui}M_{dk}) + h.c., \tag{6.12}$$

where $ui = u, c, t$; $dk = d, s, b$; U_{ik} are the element of CM matrix, $M_{ui} = (ui\bar{U})$, $M_{dk} = (dk\bar{U})$, and effective coupling constant $G_{WM} = g/2\sqrt{2}$. The value of G_{WM} is equal to the coupling constant in W-boson fundamental interaction with quarks. This is due to W-boson interacts with light standard quarks u, d only, it does not interact with heavy quark Q, which at low energy plays spectator role.

Low-energy Lagrangian of Z-boson interaction with new mesons can be represented in the form (6.12) too. However, in contrast to G_{WM}, effective coupling G_{ZM} is caused by interactions of Z with both quarks, Q and q, and the problem of coupling definition arises.

Inelastic scattering of leptons on the M-particles is described by t-channel diagram with W-boson in the intermediate state. Using Eq. (6.12) and standard vertex $We^-\nu_e$, by straightforward calculation, in approximation $m_e \ll m(M)$ and $|m(M^-) - m(M^0)| \ll m(M)$, where $m(M)$ is the mass of new mesons, we get the cross-section in the form [12]:

$$\sigma(l^- M^0 \to \nu_l M^-) \approx \frac{3g^4|U_{ud}|^2}{2^{10}\pi M_W^4} s(1 - \frac{\bar{m}^2}{s})^2, \tag{6.13}$$

where $\sqrt{s}$ is full energy in the CMS and $\bar{m}$ is mean mass of the doublet (M^0, M^-). Full process of lepton scattering on M^0 with account of final states is:

$$l^- M^0 \to \nu_l M^- \to \nu_l M^0 e^- \bar{\nu}_e. \tag{6.14}$$

So, in this process, neutrino with energy $E_\nu \sim E_l$ appears together with $e^-\bar{\nu}_e$ -pair. The cross-section of the process $\nu_l M^0 \to l^- M^+$ is described by the same expression (6.13).

Heavy DM particles are non-relativistic at the modern stage of evolution, they have an average velocity $\sim 10^{-3}$ with respect to Galaxy. From the kinematics of the heavy-light particles collisions, when $m \ll M$, it follows that momentum transfer is small (see comments below). In this case, the low-energy DM-nucleon interaction can be described by effective meson-exchange approach (see Ref. [11]). The nucleon-meson interaction was considered in [19] on the base of the gauge scheme realization of symmetry $U(1) \times SU(3)$. This scheme was developed and applied to the interaction of new heavy mesons with ordinary vector mesons [11,12]. Lagrangian which describes the interaction of nucleons and new M-mesons with ordinary vector mesons consists of two terms:

$$L_{NMV} = L_{NV} + L_{MV}. \tag{6.15}$$

In Eq. (6.15) the first term describes interaction of nucleon with standard light mesons:

$$\begin{aligned} L_{NV} &= g_\omega \omega_\mu(\bar{p}\gamma^\mu p + \bar{n}\gamma^\mu n) + \frac{1}{2} g\rho^0_\mu(\bar{p}\gamma^\mu p - \bar{n}\gamma^\mu n) \\ &+ \frac{1}{\sqrt{2}} g\rho^+_\mu \bar{p}\gamma^\mu n + \frac{1}{\sqrt{2}} g\rho^-_\mu \bar{n}\gamma^\mu p, \end{aligned} \tag{6.16}$$

where $g_\omega = \sqrt{3}g/2\sin\theta$, $g^2/4\pi \approx 3.4$ and $\sin\theta \approx 0.78$ [19]. The second term in Eq. (6.16) describes the interaction of M -particles with ordinary vector mesons [12]:

$$\begin{aligned} L_{MV} &= iG_{\omega M}\omega^\mu(\bar{M}^0 M^0_{,\mu} - \bar{M}^0_{,\mu} M^0 + M^+_{,\mu} M^- - M^+ M^-_{,\mu}) \\ &+ \frac{ig}{2}\rho^0_\mu(\bar{M}^0 M^0_{,\mu} - \bar{M}^0_{,\mu} M^0 + M^+_{,\mu} M^- - M^+ M^-_{,\mu}) \\ &+ \frac{ig}{\sqrt{2}}\rho^{+\mu}(\bar{M}^0 M^-_{,\mu} - \bar{M}^0_{,\mu} M^-) + \frac{ig}{\sqrt{2}}\rho^{-\mu}(M^+ M^0_{,\mu} - M^+_{,\mu} M^0). \end{aligned} \tag{6.17}$$

In Eq. (6.17), the coupling constant $G_{\omega M} = g_\omega/3$. In Ref. [11], it was shown that scalar mesons give very small contribution into NM interaction. The interactions of new mesons with ordinary pseudoscalar mesons (for instance, π-mesons) are absent due to parity conservation. This is an important property which differs new heavy hadrons from the standard baryons.

Low-energy scattering of nucleons on new mesons is described by t-channel diagrams with light vector and scalar mesons in the intermediate states. The diagrams with pseudoscalar mesons are absent at the tree level, while the contribution of scalar mesons is negligible. So, the dominant contribution into the cross-section gives the change by the vector mesons, ω and ρ mesons.

Now, we consider the kinematics of elastic scattering $MN \to MN$, where $M = (M^0, M^-)$ and $N = (p, n)$. In the case of non-relativistc particles, the maximal value of momentum transfer $Q^2 = -q^2$ is $Q^2_{max} = (pk)^2 \approx 4m_N^2 v_r^2$. So, $Q_{max} \approx m_N v_r \sim 10^{-3} m_N$, the value Q_{max} much less the mass of vector mesons m_v ($m_v \sim m_N$) and the meson-exchange model is relevant.

Using the vertexes from the Eqs. (6.16) and (6.17), we calculated the cross-section of the process $N_a M_b \to N_a M_b$ [11, 12]:

$$\sigma(N_a M_b \to N_a M_b) = \frac{g^4 m_p^2}{16\pi m_v^4}(1 + \frac{k_{ab}}{\sin^2\theta})^2, \tag{6.18}$$

where $N_a = (p, n)$, $M_b = (M^0, M^-)$, $g^2/4\pi \approx 3.4$, $\sin\theta = 1/\sqrt{3}$ and $k_{ab} = \pm 1$ for the case of proton, p, and neutron, n. From the Eq. (6.18) it follows that the value of cross-section is rather large, for example, $\sigma(pM^0 \to pM^0) \approx 0.9$ barn. Large cross-section of NM-scattering can cause noteceabl interaction of DM halo and galaxy. The problem of interaction between galaxies and their DM halo was considered in Ref. [20]. Analysis of the low-energy scattering $N_a M_b \to N_a M_b$ discover an important peculiarity of the NM-interaction. We show in Born approximation that potential of M-nucleon interaction at large distances ($d \sim m_\rho^{-1}$) has repulsive character [7, 10] and new heavy hadrons as DM-particles do not form coupled states with nucleon. This effect allows us to escape the problem of anomalous hydrogen and helium [7].

In spite of large cross-section of NM-scattering, the direct detection of hadronic DM by underground devices is difficult due to small free pass in ground, $l_{fr} < 1$ cm. So, we consider indirect constraints on hadronic DM which can impact on the parameters of big bang nucleosynthesis (BBN) and γ-spectrum of cosmic rays (CR) in the Galaxy [21]. In this work, the constraints were derived on the relation $R = \sigma(cm^2)/M_{DM}(g)$, where σ is cross-section of DM-baryon scattering (in cm^2) and M_{DM} is the mass of DM particle (in g). The constraints are as follow [21]:

$$\mathrm{BBN}: \quad R < 10^8\,cm^2 g^{-1}; \quad \mathrm{CR}: \quad R < 5 \cdot 10^{-3}\,cm^2 g^{-1}. \tag{6.19}$$

So, the second restriction is much more stringent and we compare it with model result. In our consideration, the value of mass is $M_{DM} \approx 10^4\mathrm{GeV} \approx 10^{-20}/0.56$ g, and cross-section $\sigma \sim 10^{-24} cm^2$. Thus, the model relation $R \approx 5.6 \cdot 10^{-5}\,cm^2 g^{-1}$ does not contradict to the CR restriction. The more proper measurements and constraints are considered in Ref. [22] for the case of cosmic ray interaction with

DM. The constrains were developed using NFW and Moore DM density profiles and new data from Fermi gamma ray space telescope. Here, we use the upper constraint with Moore profile (which is more stringent): $\sigma_{Nx} = 9.3 \cdot 10^{-30} m_x cm^2 GeV^{-1}$. At $m_x = 10^4$ GeV we get $\sigma < 10^{-25} cm^2$, which excludes the model estimation. Here, we note that we describe the DM-nucleon interaction in meson-exchange approach using the coupling constant, which was determined in low-energy hadrons interaction ($g^2/4\pi = 3.4$). Thus, the model assumption concern the value of coupling is not justified, and from the experiment we get the constraint on this parameter: $g^2/4\pi < 1$.

The processes of non-elastic scattering $N_a M_b \to N_c M_d$, where $N_a = (p, n)$ and $M_b = (M^0, M^-)$, are described by the kinematics of elastic scattering one. The dominant contribution into cross-section is caused by t -channel diagram with charged $\rho^{\pm}$-meson in the intermediate state. The expression for the cross-section explicitly indicates the presence of the threshold:

$$\sigma(N_a M_b \to N_c M_d) = \frac{g^4 m}{8\pi v_r m_v^4}\sqrt{2m}[E_a - \Delta_{ab}]^{1/2}, \tag{6.20}$$

where $E_a \approx m_a v_r^2/2$, $m(N_a) = m_a \approx m_b \approx m$, Δ_{ab} is the combination of mass-splitting $\Delta M = m(M^+) - m(M^0)$ and $\Delta m = m_n - m_p \approx 1.4$ MeV. The expression (6.20) can be represented in another form:

$$\sigma(N_a M_b \to N_c M_d) = \frac{g^4 m^2}{8\pi m_v^4}[1 - \frac{\Delta_{ab}}{E_p}]^{1/2}. \tag{6.21}$$

From (6.21) one can see that the process of scattering has the threshold $E_p^{thr} = \Delta_{ab}$, when $\Delta_{ab} > 0$. In Ref. [12], we present the expressions for the threshold in the case of basic reactions, naimly $pM^0 \to nM^+$, $nM^+ \to pM^0$, $nM^0 \to pM^-$ and $pM^- \to nM^0$.

6.4 Fine and hyperfine splitting in doublet of new heavy mesons

New heavy quarks possess strong QCD interaction, so they can form the coupled states - new heavy mesons, $(q\bar{Q})$, and fermions, (qqQ), (qQQ), (QQQ). Classification and the main properties of these hadrons were presented in [10] for the case of up and down type of new quark Q. The evolution of new hadrons was briefly considered in Ref. [7], where the process of burning out of heavy baryons was analyzed. Here, we represent the main properties of new mesons, M^0 and M^-, which can lead to the characteristic sygnals of the hadronic dark matter.

An important role in hadronic DM scenario plays the value of mass-splitting in the doublet of neutral, M^0, and charged, M^-, new heavy mesons. We define the value of mass-splitting as follows:

$$\Delta m = m(M^-) - m(M^0). \tag{6.22}$$

In the case of standard heavy-light (HL) mesons the value Δm is an order of MeV, besides, this value is positive for the case of D-meson (up heavy quark) and negative for the case of K- and B-mesons (down heavy quark). New heavy mesons M^0 and M^- are just the case of the heavy-light (HL) mesons, $m_Q \gg m_q$. From the heavy quark symmetry [23], it follows the analogy with standard HL mesons. So, for the case of up-type new mesons, $M^0 = (u\bar{U})$ and $M^- = (d\bar{U})$, we can assume that Δm is positive and $\Delta m \sim$ MeV. The instability condition of charged meson M^- leads to inequality $\Delta m > m_e$, where m_e is the mass of electron. Thus, charged partner of neutral DM particle has unique decay channel $M^- \to M^0 W^{*-} \to M^0 e^- \bar{\nu}_e$ with very small phase space in a final state. The expression for the width of charged meson is as follows [10]:

$$\Gamma(M^-) = \frac{G_F^2}{60\pi^3}|M_{ud}|^2(\Delta m^5 - m_e^5), \tag{6.23}$$

where M_{ud} is the element of CM matrix, which corresponds to the transition $d \to uW$. From the expression (6.23) one can see that at $\Delta m \to m_e$, the value of width $\Gamma(M^-) \to 0$, that is lifetime can be arbitrary large. For instance, at $\Delta m \sim 1$ MeV the lifetime $\tau \sim 10^5$ s. Thus, in the scenario with hadronic DM, new neutral meson M^0, as DM candidate, has charged metastable partner with the same mass. New heavy charged meson appears in the process of collision of DM with ordinary matter, leptons and nucleons (see the previous section).

Principal feature of hadronic DM scenario is the effect of hyperfine splitting of excited states of new heavy hadrons. In contrast to fine splitting, which is caused by change of light quark content ($d \to u$) and has the value an order of MeV, hyperfine splitting takes place for the mesons with the same quark content and has much less value (an order of keV). Further, we describe the effect of hyperfine splitting $\delta M_q = m(M_q^*) - m(M_q)$, where M_q^* is excited state of the meson M_q. Here, we consider the lowest excited states of the meson $M_q = (q\bar{U})$. In a direct analogy with the standard heavy-light (HL) mesons, $D_q = (cq)$ and $B_q = (\bar{b}q)$, we define the ground and excited states in the terms S_0^1 and S_1^1 (classification with quantum nubbers L_J^{2s+1}), or $\frac{1}{2}(0^-)$ and $\frac{1}{2}(1^+)$ (classification $I(J^P)$). Here, L, s, J, I and $P = (-1)^{1+L}$ are orbital momentum, spin, full momentum of the system, isospin and parity. The ground states $\frac{1}{2}(0^-)$ of the HL mesons we designate as D_q, B_q and M_q, while the excited states as D_q^*, B_q^* and M_q^*. Evaluation of the mass-splitting of the states M_q^* and M_q we carry out in analogy with standard splitting mechanism. The analogy is provided by the heavy quark symmetry which is the base of heavy quark effective theory (HQET). Heavy quark symmetry [23] leads to relations between the masses of excited states of B and D mesons [24]:

$$m(B_2) - m(B_1) \approx \frac{m_c}{m_b}(m(D_2) - m(D_1)), \tag{6.24}$$

where $m(B_k)$ and $m(D_k)$ are masses of B_k and D_k, m_c and m_b are masses of constituent quarks. The expression (6.24) successfully describes the relation of splitting between the lowest excited $\frac{1}{2}(1^-)$ and ground states $\frac{1}{2}(0^-)$ of B and D mesons:

$$\frac{m(B^*) - m(B)}{m(D^*) - m(D)} \approx \frac{m_c}{m_b} \longrightarrow 0.32 \approx 0.32\,(0.28). \tag{6.25}$$

In (6.25), we used $m(B^*) - m(B) = 45$ MeV and $m(D^*) - m(D) = 142$ MeV (see [18]), $m_c = 1.55$ GeV and $m_b = 4.88$ GeV [24]. The value of relation in bracket

(0.28) follows from the data $m_c = 1.32$ GeV and $M_b = 4.74$ GeV [18]. In order to evaluate the mass-splitting in the doublet of new mesons $M_q = (q\bar{U})$, we use the relation (6.25) and equality $m(U) \approx m(M_q) = M$. Using the value of mass $M = 10$ TeV, we get:

$$\frac{\delta m(M)}{\delta m(B)} = \frac{m(M^*) - m(M)}{m(B^*) - m(B)} \approx \frac{m_b}{M} \longrightarrow \delta m(M) \approx \delta m(B) \frac{m_b}{M} \approx 2\,\mathrm{KeV}. \quad (6.26)$$

Thus, we get very small mass-spltiting (hyperfine splitting) δm, which is much less the fine splitting, $\delta m \ll \Delta m$.

The excitation of hadronic DM particle can manifest itself in the processes of interaction of neutral meson M^0 with cosmic rays. Transition to the first excited state of the meson $M^0 = (u\bar{U})$ can be caused by the absorption of photons in keV range, which have the wavelength $\lambda \sim 10^{-9}$ cm. If we assume that the meson $M^0 = (u\bar{U})$ has the size an order of nucleon radius, $R_M \sim 10^{-13}$ cm, then $R_M \ll \lambda_{trans}$ and interaction of M^0 with photons is caused by multi-pole expansion of charge distribution in the system $(u\bar{U})$. So, the cross-section of γM^0 scattering is small and these neutral mesons manifest themselves as dark matter particles. At $\lambda_{trans} \ll R_M$ the cross-section of interaction γM^0 become large and dark matter is not absolutely "dark". Now, we consider a possible manifestation of keV-signal, which is caused by hyperfine splitting, in the spectrum of X-rays from the galaxy clusters. In Refs. [25,26], it was reported about emission line at $E \approx 3.5$ keV in a spectrum of galaxy center and galaxy clusters. Here, we should note that the existence in nature of superheavy-light mesons inevitably (in the framework of HQET) leads to hyperfine mass-splitting of ground and excited levels. Transitions between these states generate emission of photons with energy 3.5 keV when the mass of new heavy mesons $m(M) \approx 6$ TeV. This estimation in the framework of HQET follows from the Eq. (6.26) without refer to DM hypothesis.

6.5 Conclusion

High-energy extensions of SM, as a rule, contain heavy particles which possess conservative quantum number. In the extension with singlet quark, the conservation of baryon charge leads to the stability of the lightest new hadron which can be assumed as DM carrier. In this report, we present the main properties of new heavy hadrons and describe their low-energy interactions with ordinary leptons and nucleons. We considered some electro-weak and cosmological constrains on the new heavy quarks and hadrons. Excited states of these hadrons were considered in analogy with standard HL mesons. We show that there exist fine and hyperfine structure of excited levels which lead to weak or electromagnetic transitions. Such transitions appear in the processes of interaction of new heavy hadrons with ordinary particles and cosmic rays. This effect rises the problem of separation the terms "dark matter" and "hidden matter". It was noted that this problem becomes actual in the range of hard gamma rays.

In order to describe DM signals in hadronic processes, we developed and analyzed the low-energy model of new hadrons interaction with ordinary ones. The model of DM-nucleon interaction is based on the meson-exchange approach and realized in the frame of the gauge scheme realization of the $SU(3)$-symmetry. In the framework of this model, we derived analytical expressions for the cross-sections of elastic and inelastic collisions of nucleons and new heavy hadrons. These expressions will be used in the analysis and description of DM interactions with cosmic rays, interstellar gas and with Earth atmosphere. The most important signal of such interactions is the appearance of the metastable heavy charge paticles M^-. The scenario with hadronic DM provides a new aspect to the problem of interconnection of galaxies and their DM halos which can stipulate some peculiarities of galaxy formation. Effect of hyperfine splitting can explain the emission line at 3.5 keV in the spectrum of X-rays.

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7 How Far has so Far the Spin-Charge-Family Theory Succeeded To explain the Standard Model Assumptions, the Matter-Antimatter Asymmetry, the Appearance of the Dark Matter, the Second Quantized Fermion Fields. . . , Making Several Predictions

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Abstract. The assumptions of the *standard model*, which 50 years ago offered an elegant new step towards understanding basic fermion and boson fields, are still waiting for an explanation. The *spin-charge-family* theory is promising not only in explaining the *standard model* postulates but also in explaining the cosmological observations, like there are the appearance of the *dark matter*, of the *matter-antimatter asymmetry*, making several predictions. This theory assumes that the internal degrees of freedom of fermions (spins, handedness and all the charges) are described by the Clifford algebra objects in $d \geq (13+1)$-dimensional space. Fermions interact with only the gravity (the vielbeins and the two kinds of the spin connection fields, which manifest in $d = (3 + 1)$ as all the vector gauge fields as well as the scalar fields - the higgs and the Yukawa couplings). The theory describes the internal space of fermions with the Clifford objects which are products of odd numbers of γ^a objects, what offers the explanation for quantum numbers of quarks and leptons and anti-quarks and ani-leptons, with family included. In this talk I overview shortly the achievements of the *spin-charge-family* theory so far and in particular the explanation of the second quantization procedure offered by the description of the internal space of fermions with the anticommuting Clifford algebra objects of the odd character. The theory needs still to answer many open questions that it could be accepted as the next step beyond the *standard model*.

Povzetek. Privzetki *Standardnega Modela*, ki je pred 50 leti ponudil eleganten opis osnovnih fermionskh in bozonskih polj, so še vedno nepojasnjeni. *Teorija spinov-nabojev-družin* ponuja, poleg razlage privzetkov *Standardnega Modela*, tudi razlago nekaterih kozmoloških opažanj, kot je pojav *temne snovi*, *asimetrije snovi in antisnovi*, ponudi pa tudi več napovedi. Teorija privzame, da so notranje prostostne stopnje fermionov (spin, ročnost in vsi naboji) opisane z objekti Cliffordove algebre v prostoru z razsežnostjo $d \geq (13 + 1)$. Fermioni interagirajo samo z gravitacijskim poljem (s tetradami in dvema vrstama spinskih povezav), ki se v prostoru $d = (3 + 1)$ predstavi kot običajna gravitacija, kot vsa poznana vektorska umeritvena polja ter kot skalarna umeritvena polja, ki pojacnijo pojav Higgsovega skalarja in Yukawinih sklopitev. Notranje prostostne stopnje fermionov opisuje avtorica teorije s Cliffordovo algebro, ki ponudi razumevanje privzetkov za lastnosti kvarkov in leptonov in njihovih družin, v *Standardnem Modelu*. V predavanju avtorica na kratko predstavi dosedanje dosežke *Teorije spinov-nabojev-družin*, napovedi teorije ter tudi odprta vprašanja.

Poudarek predavanja je na ponudbi drugačne poti do druge kvantizacije fermionov kot je splošno privzeta Diracova. Opis notranjega prostora fermionov z objekti, ki antikomutirajo, pojasni antikomutacjske lastnosti fermionov v drugi kvantizaciji. Predstavi tudi odprta vprašanja, ki jih je potrebno rešiti, da bo teorija lahko sprejeta kot nov korak k razumevanju vesolja in osnovnih gradnikov vesolja.

Keywords: Beyond the standard model, Gravity as the only gauge fields, Kaluza-Klein-like theories, Higher dimensional spaces, Dark matter, Matter/antimatter asymmetry, Four families of quarks and leptons, Second quantization of fermion fields in Clifford and in Grassmann space, Explanation of the Dirac postulates

7.1 Introduction

Let us start with the motivation for the *spin-charge-family* theory.

The *standard model* offered an elegant new step towards understanding elementary fermion and boson fields by postulating (the inspiration came from the experiments):

a. The existence of massless fermion family members with the spins and charges in the fundamental representation of the groups, **a.i.** the quarks as colour triplets and colouress leptons, **a.ii** the left handed members as the weak doublets, **a.ii.** the right handed weak chargeless members, **a.iii.** the left handed quarks differing from the right handed leptons in the hyper charge, **a.iv.** all the right handed members differing among themselves in hyper charges, **a.v.** antifermions carry the corresponding anticharges of fermions and opposite handedness, **a.vi.** the number of massless families, determined by experiments (there is no right handed neutrino postulated, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino allowed).

b. The existence of massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the adjoint representations of the corresponding charged groups.

c. The existence of the massive weak doublet scalar higgs, **c.i.** carrying the weak charge $\pm\frac{1}{2}$ and the hypercharge $\pm\frac{1}{2}$ (as it would be in the fundamental representation of the two groups), **c.ii.** gaining at some step of the expanding universe the nonzero vacuum expectation value, **c.iii.** breaking the weak and the hyper charge and correspondingly breaking the mass protection, **c.iv.** taking care of the properties of fermions and of the weak bosons masses, **c.v.** as well as of the Yukawa couplings.

d. The presentation of fermions and bosons as second quantized fields.

e. The gravitational field in $d = (3 + 1)$ as independent gauge field.

The *standard model* assumptions have been confirmed without raising any doubts so far, but also by offering no explanations for the assumptions. The last among the fields postulated by the *standard model*, the scalar higgs, was detected in June 2012, the gravitational waves were detected in February 2016.

The *standard model* has in the literature several explanations, mostly with many new not explained assumptions. The most popular seem to be the grand unifying theories [14–30]. At least $SO(10)$ offers the explanation for the potulates from **a.i.** to **a.iv**, partly to **b.** — but does not explain the assumptions **a.v.** up to **a.vi.**, **c.** and **d.**, and does not connect gravity with gauge vector and scalar fields.

What questions should one ask to be able to find a trustworthy next step beyond the *standard models* of elementary particle physics and cosmology, which would offer understanding of not yet understood phenomena?

i. Where do fermions, quarks and leptons, originate and why do they differ from the boson fields in spins, charges and statistics?

ii. How can one describe the internal degrees of fermions to explain the Dirac's postulates of the second quantization?

iii. Why are charges of quarks and leptons so different, why have the left handed family members so different charges from the right handed ones and why does the handedness relate charges to anticharges?

iv. Where do families of quarks and leptons originate and how many families do exist?

v. Why do family members – quarks and leptons — manifest so different masses if they all start as massless?

vi. How is the origin of the scalar field (the Higgs's scalar) and the Yukawa couplings connected with the origin of families and how many scalar fields determine properties of the so far (and others possibly be) observed fermions and masses of weak bosons? (The Yukawa couplings certainly speak for the existence of several scalar fields with the properties of Higgs's scalar.) Why is the Higgs's scalar, or are all scalar fields of similar properties as the higgs, if there are several, doublets with respect to the weak and the hyper charge? Do possibly exist also scalar fields with the colour charges in the fundamental representation and where, if they are, do they manifest?

vii. Where do the so far observed (and others possibly non observed) vector gauge fields originate? Do they have anything in common with the scalar fields and the gravitational fields?

viii. Where does the *dark matter* originate?

ix. Where does the "ordinary" matter-antimatter asymmetry originate?

x. Where does the dark energy originate and why is it so small?

xi. What is the dimension of space? $(3+1)$?, $((d-1)+1)$?, ∞?

And many others.

My working hypotheses is that a trustworthy next step must offer answers to several open questions, the more answers to the above open questions the step covers the greater the possibilities of the theory being the right next step.

I am proposing the *spin-charge-family* theory [1–10], offering so far the answers from **i.** to **ix.** of the above questions; The more work is invested in this theory the more answers to the above open questions the theory offers.

Let me make in what follows a short introduction into the *spin-charge-family* theory to show briefly up the way the theory is offering the answers to the above mentioned open questions. A more detailed presentation of the theory and its achievements are presented in Sect. 7.2.

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [8, 31–38] due to the assumption that in $d \geq 5$ — in the *spin-charge-family* theory

$d \geq (13+1)$ — fermions interact with the gravity only [1], treating consequently all the vector gauge fields, the scalar gauge fields, and the gravity in an equivalent way, offering answers to the above questions **vi.** and **vii.**.

In the *spin-charge-family* theory the fermion internal space is described by the "basis vectors", which are the superposition of the odd products of the Clifford algebra objects. There are two kinds of the Clifford algebra objects [1,2,12,45,46]. In $d = (13+1)$-dimensional space the odd Clifford algebra objects of one kind offer in $d = (3+1)$ the description of the spins and all the charges of fermions and antifermions, since both — fermions and antifermions — appear in the same irreducible representation of one of the two Lorentz groups in the internal space of fermions, what consequently explains the connection among the spins, handedness and charges of fermions, answering the questions **i.** and **iii.**.

The other kind takes care of the family quantum numbers of fermions, distinguishing among different irreducible representations [3,4,7,9], and offering a part answer to **iv.**.

The creation operators, creating the single particle states, are tensor products of the superposition of the finite number of the Clifford odd "basis vectors" of the internal space and of the infinite basis in the momentum space. The "basis vectors" of the internal space transfer their oddness to the creation operators and correspondingly guarantees the oddness of the single fermion states, since the vacuum state has an even Clifford character.

The Hilbert space of fermions is formed from all possible tensor products of any number of single fermion creation operators, operating on the vacuum state [12].

The *spin-charge-family* theory offers correspondingly answers to the questions from **i.** to **iv.**, explaining the common origin of spins and charges of fermions and antifermions, of all the quantum numbers of quarks and leptons and antiquarks and antileptons postulated by the *standard model*, as well as of the origin of families. The theory explains as well the Dirac postulates of the second quantization of the ferrmion fields.

Fermions interact with the vielbeins and the two kinds of the spin connection fields, the gauge fields of the momenta and of the two kinds of the generators of the Lorentz transformations, determined by the two kinds of the Clifford algebra objects [3–10,12].

The spin connection fields of one kind manifest in $d = (3+1)$ as the vector gauge fields of the charges of fermions, as the gravitational fields and also as the scalar gauge fields [5], to which also the scalar fields which are the gauge field of the second kind of the spin connection fields contribute. These offer answers to the questions **vi.** and **vii.**, while explaining the common origin of the gravity, the vector gauge fields of the charges and the scalar gauge fields. The scalar gauge fields of

[1] Correspondingly the *spin-charge-family* theory shares with the Kaluza-Klein like theories their weak points, at least: **a.** Not yet solved the quantization problem of the gravitational field. **b.** The spontaneous break of the starting symmetry, which would at low energies manifest the observed almost massless fermions [32]. Concerning this second point we proved on the toy model of $d = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [68–70].

both origins — of both generators of the Lorentz transformations in internal space of fermions — determine the scalar higgs and the Yukawa couplings, which all are in the *standard model* postulated.

The two kinds of the Clifford algebra objects require the existence of the two groups of four families of quarks and leptons and antiquarks and antileptons. The two groups distinguish from each other with respect to the family quantum numbers and correspondingly with respect to the interaction with the different two groups of the scalar gauge fields, which determine masses of these two groups of families after the break of the weak and hyper charge symmetries. Consequently: **a.** To the observed three families of quarks and leptons and antiquarks and antileptons there must exist the fourth family [3,9,49,51,53,54]. **b.** The second group of the four families offers the explanation for the existence of the *d*ark matter [52,61].

The quantum numbers of the weak charge and the hyper charge of the scalar fields, taking care of the masses of the two groups of four families, depend on the space index of the scalar fields. The scalar fields with the space indexes 7 and 8 do carry the weak and the hyper charge as assumed by the *standard model*, explaining the origin of scalar higgs and Yukawa couplings [3,9,49,51,53,54], what adds the explanation to the question **vi.**.

There appear in the *spin-charge-family* the scalar fields with the space indexes $9-14$, which are the colour triplets [4,61]. They cause the transitions of antiquarks and antileptons into quarks and back. In the expanding universe under the non equilibrium conditions they offer the explanation of today's dominance of ordinary matter in the observed part of the universe.

It remains to tell how does in the *spin-charge-family* appear the spontaneous breaking of the starting symmetry in $d = (13+1)$, first with the appearance of the condensate of two right handed neutrinos [3,4,9], and then when scalar fields with space index $(7,8)$ obtain nonzero vacuum expectation values.

The detailed, although still short, presentation of the *spin-charge-family* theory is presented in Sects. 7.2and 7.2.1.

7.2 Short presentation of the *spin-charge-family* theory

The *spin-charge-family* theory assumes a simple starting action for fermions, coupled to only gravitational field in $d \geq (13+1)$-dimensional space through the vielbeins $f^{\alpha}{}_{a}$, the gauge fields of momenta, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the generators of the Lorentz transformations of the Clifford algebras, and with the internal space of fermions described by the anticommuting "basis vectors" of one of the two

Clifford algebras

$$
\begin{aligned}
\mathcal{A} &= \int d^d x \; E \; \frac{1}{2} \, (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
&\quad \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, , \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \, \{p_\alpha, E f^{\alpha}{}_a\}_- \, , \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \, , \\
R &= \frac{1}{2} \, \{f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^c{}_{b\beta})\} + h.c. \, , \\
\tilde{R} &= \frac{1}{2} \, \{f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^c{}_{b\beta})\} + h.c. \, . \qquad (7.1)
\end{aligned}
$$

Here [2] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$.

As written in the introduction, the tensor products of the superposition of the finite number of anticommuting "basis vectors" and of the infinite basis in the momentum space offer the description of the fermion creation and annihilation anticommuting operators. The creation and annihilation operators explain the Dirac postulates of the second quantized fermions, Sect. (7.2.1,7.2.1, 7.2.1).

The single fermion states manifest in $d = (3+1)$ space the spins and all the charges of the observed quarks and leptons and antiquarks and antileptons, Table 7.3, as well as families, Table 7.4, predicting the fourth family [49–51, 53, 54, 57, 58] to the observed three families and offering the explanation for the observed *dark matter* [52, 61].

The spin connection gauge fields manifest in $d = (3+1)$ as the ordinary gravity, the known vector gauge fields, the scalar gauge fields [5] with the properties of higgs explaining the higgses and the Yukawa couplings, predicting new vector and scalar fields, which offer explanation for the *dark matter* [52] and for *matter/antimatter asymmetry* [4].

To be in agreement with the observations in $d = (3+1)$ the manifold $M^{(13+1)}$ must break first into $M^{(7+1)} \times M^{(6)}$ (which manifests as $SO(7,1) \times SU(3) \times U(1)$), affecting both internal degrees of freedom - the one represented by γ^a and the one represented by $\tilde{\gamma}^a$ [3].

There is a scalar condensate (Table 7.5) of two right handed neutrinos with the family quantum numbers of the group of four families (the one which does not include the observed three families), Table 7.4, which bring masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [4].

---

[2] $f^{\alpha}{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

Since the left handed spinors couple differently (with respect to $M^{(7+1)}$) to scalar fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)}$ families [68]. The rest of families get heavy masses [3].

There is additional breaking of symmetry: The manifold $M^{(7+1)}$ breaks further to $M^{(3+1)}\times SU(2) \times SU(2)$ included in $M^{(4)}$. These electroweak break is caused by the scalar fields with the space index $(7,8)$. They carry due to the space index the weak charge and hyper charge [3,4].

I shall shortly present the influence of the breaks with the condensate and with the scalar fields (the electroweak break) when presenting properties of fermions and vector and scalar gauge fields in $d = (3+1)$.

7.2.1 Properties of fermion fields in the *spin-charge-family* theory

Let us start with the properties of the fermion fields in the *spin-charge-family* theory.

Fermion fields, which are the superposition of tensor products of the anticommuting "basis vectors" describing fermions internal degrees of freedom and of commuting basis in the momentum (coordinate) space, manifest the anticommuting properties already on the single fermion level [13], demonstrating that the first quantized fermions are the approximation to the second quantized fields.

There are two kinds of the anticommuting objects [1–3,9,12] — the Grassmann coordinates and correspondingly the Grassmann operators, θ^a and $\frac{\partial}{\partial\theta_a}$, and the Clifford coordinates/operators, γ^a and $\tilde{\gamma}^a$, expressible with one another. Either the Grassmann or the two Clifford algebras offer in d-dimensional space $2\cdot 2^d$ operators (the Grassmann algebra has 2^d-1 products of θ^a's and 2^d-1 products of $\frac{\partial}{\partial\theta_a}$'s and the identity, the two Clifford algebras have each 2^d-1 products of γ^a' and 2^d-1 products of $\tilde{\gamma}^a$'s and the identity) with the properties [12,13]

$$
\begin{aligned}
&\{\theta^a,\theta^b\}_+ = 0\,, \quad \{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+ = 0\,, \quad \{\theta_a,\frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,,\\
&(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,,\\
&\{\gamma^a,\gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+\,, \quad \{\gamma^a,\tilde{\gamma}^b\}_+ = 0\,,\\
&(\gamma^a)^\dagger = \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,,\\
&(a,b) = (0,1,2,3,5,\cdots,d)\,.
\end{aligned} \tag{7.2}
$$

The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$ is assumed.

The two algebras are expressible with one another

[3] A toy model [68,69] was studied in $d = (5+1)$ with the same action as in Eq. (7.1). The break from $d = (5+1)$ to $d = (3+1)\times$ an almost S^2 was studied. For a particular choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost S^2, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the $d = (13+1)$ case. This study is in progress quite some time.

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a}) , \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a}) ,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a) , \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a) . \tag{7.3}$$

Let me add the generators of the Lorentz transformations in both algebras

$$\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a}) , \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab} ,$$
$$S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) , \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a) ,$$
$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab} , \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0 ,$$
$$\{S^{ab}, \gamma^c\}_- = i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b) , \quad \{S^{ab}, \tilde{\gamma}^c\}_- = 0 ,$$
$$\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- = i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b) , \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0 , \tag{7.4}$$

The Grassmann algebra offers the description of the integer spin fermions, with the charges in the adjoint representations. Both Clifford algebras offer the description of the half integer spin fermions with charges in the fundamental representations. Both algebras, the Grassmann algebra and the two Clifford algebras, can be separated into odd and even parts with odd and even products of algebra elements.

While in the Grassmann algebra the Hermitian conjugated partners of products of θ^a's are the corresponding products of $\frac{\partial}{\partial\theta^a}$'s, Eq. (7.2), and opposite, in the Clifford algebras the Hermitian conjugated partners are less transparent, due to the relations $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ and $\tilde{\gamma}^{a\dagger} = \eta^{aa}\tilde{\gamma}^a$, Eq. (7.2).

In order to resolve the problem of the Hermitian conjugated partners in the Clifford case and also to be able to make predictions of the theory to be compared with the experimental results, let us arrange products of θ^a's as well as products of either γ^a's or $\tilde{\gamma}^a$'s into irreducible representations with respect to the Lorentz group with the generators [2] presented in Eq. (7.4) and to arrange the members of each irreducible representation to be eigenstates of the Cartan subalgebra

$$\mathbf{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, \cdots, \mathcal{S}^{d-1\;d} ,$$
$$S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d} ,$$
$$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d} . \tag{7.5}$$

The easiest way to achieve this is to find the eigenstates of each member of the Cartan subalgebras separately.

The observed fermions have the half integer spin and charges in the fundamental representations, and there are no fermions observed yet with the integer spins and charges in the adjoint representations. The *spin-charge-family* theory must correspondingly use the Clifford algebras. However, there are also no experimental evidences that there is any need for two independent representations offered by the two kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s.

Let us therefore start the discussion about the description of the internal space of fermions by taking into account the two Clifford algebras and let us leave the discussion on the Grassmann algebra for later, Ref. [13].

We can make a choice for the members of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebras of Eq. (7.5), taking into account Eq. (7.2). If S^{ab} and $\tilde{S}^{ab}$ represents each one of the ($\frac{d}{2}$ for even d) members of the Cartan subalgebra elements, we easily check that

$$
\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)}\,, & \stackrel{ab}{(k)} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, & (\stackrel{ab}{(k)})^2 &= 0\,, & \stackrel{ab}{(k)}^{\dagger} &= \eta^{aa} \stackrel{ab}{(-k)}\,,\\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, & (\stackrel{ab}{[k]})^2 &= \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}^{\dagger} &= \stackrel{ab}{[k]}\,,\\
\tilde{S}^{ab} \stackrel{ab}{(\tilde{k})} &= \frac{k}{2} \stackrel{ab}{(\tilde{k})}\,, & \stackrel{ab}{(\tilde{k})} &:= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, & (\stackrel{ab}{(\tilde{k})})^2 &= 0\,, & \stackrel{ab}{(\tilde{k})}^{\dagger} &= \eta^{aa} \stackrel{ab}{(-\tilde{k})}\,,\\
\tilde{S}^{ab} \stackrel{ab}{[\tilde{k}]} &= \frac{k}{2} \stackrel{ab}{[\tilde{k}]}\,, & \stackrel{ab}{[\tilde{k}]} &:= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, & (\stackrel{ab}{[\tilde{k}]})^2 &= \stackrel{ab}{[\tilde{k}]}\,, & \stackrel{ab}{[\tilde{k}]}^{\dagger} &= \stackrel{ab}{[\tilde{k}]}\,.
\end{aligned}
\tag{7.6}
$$

The notation $\stackrel{ab}{(k)}, \stackrel{ab}{[k]}, \stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[\tilde{k}]}$ is introduced to simplify the discussions. The Clifford "vectors" — nilpotents ($\stackrel{ab}{(k)}\stackrel{ab}{(k)} = 0$, $\stackrel{ab}{(\tilde{k})}\stackrel{ab}{(\tilde{k})} = 0$) and projectors ($\stackrel{ab}{[k]}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}$, $\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[\tilde{k}]} = \stackrel{ab}{[\tilde{k}]}$ — of both algebras are normalized up to a phase [2,12,13].

Both have half integer spins. The "eigenvalues" of the operator S^{03} for the "eigenvectors" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$, for example, are equal to $\pm\,\frac{i}{2}$, respectively, for the "vectors" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm\,\frac{i}{2}$, respectively, while all the rest "eigenvectors" have "eigenvalues" $\pm\,\frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}\,(\tilde{\gamma}^0 \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm\,\frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm\,\frac{i}{2}$, respectively, while all the rest "eigenvectors" have $k = \pm\,\frac{1}{2}$.

It is useful to know some additional relations among nilpotents and projectors, taken from Ref. [3]

$$
\begin{aligned}
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &= 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} &= \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} &= \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} &= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{(-k)} &= \eta^{aa} \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} &= 0\,.
\end{aligned}
\tag{7.7}
$$

The same relations are valid also if one replaces $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[k]}$ with $\stackrel{ab}{[\tilde{k}]}$.

The "basis vectors" are products of $\frac{d}{2}$ eigenvectors of all the Cartan subalgebra members. For the description of the internal space of fermions only those "basis vectors" which are products of an odd number of nilpotents, the rest can be projectors, are acceptable, since they *anticommute algebraically*, what we expect for the single fermion states of the second quantized fields.

To make clear what the anticommutation of the basis vectors mean, let us start with the first "basic vector", denoting it as $\hat{b}^{m=1\dagger}_{f=1}$, with f defining different irreducible representations and m a member in the representation f. Then its

Hermitian conjugated partner is $\hat{b}^{m}_{f} = (\hat{b}^{m\dagger}_{f})^{\dagger}$. Let us make a choice of the starting "basic vector" for the Clifford algebra of the kind γ^{a}'s with an odd products of the nilpotents

$$
\begin{aligned}
\hat{b}^{m=1\dagger}_{f=1} &:= \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{(+)}\overset{910}{(+)}\,\overset{1112}{[-]}\,\overset{1314}{[-]}\,\cdots\,\overset{d-3\,d-2}{[-]}\,\overset{d-1\,d}{[-]}\,,\\
(\hat{b}^{m=1\dagger}_{f=1})^{\dagger} = \hat{b}^{m=1}_{f=1} &= \overset{d-1\,d}{[-]}\,\overset{d-3\,d-2}{[-]}\,\cdots\,\overset{1314}{[-]}\,\overset{1112}{[-]}\,\overset{910}{(-)}\overset{78}{(-)}\overset{56}{[+]}\overset{12}{[+]}\overset{03}{(-i)}\,,
\end{aligned}
\tag{7.8}
$$

the rest products in $\cdots \overset{d-3\,d-2}{[-]}\,\overset{d-1\,d}{[-]}$ are assumed to be all projectors with $k=-1$, $[-]$. All the rest members of this irreducible representation are reachable by S^{ab}.

Let us see how do S^{ab}'s transform the "basis vectors".

$$
\begin{aligned}
S^{ac}\,\overset{ab}{(k)}\overset{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc}\,\overset{ab}{[-k]}\overset{cd}{[-k]}\,,\\
S^{ac}\,\overset{ab}{[k]}\overset{cd}{[k]} &= \frac{i}{2}\,\overset{ab}{(-k)}\overset{cd}{(-k)}\,,\\
S^{ac}\,\overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa}\,\overset{ab}{[-k]}\overset{cd}{(-k)}\,,\\
S^{ac}\,\overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc}\,\overset{ab}{(-k)}\overset{cd}{[-k]}\,,
\end{aligned}
\tag{7.9}
$$

We learn from Eq. (7.50) that S^{01} transforms $\hat{b}^{m=1\dagger}_{f=1}$ into, let us call it $\hat{b}^{m=2\dagger}_{f=1}$, $\hat{b}^{m=2\dagger}_{f=1} = \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[+]}\overset{78}{(+)}\overset{910}{(+)}\,\overset{1112}{[-]}\,\overset{1314}{[-]}\,\cdots\,\overset{d-3\,d-2}{[-]}\,\overset{d-1\,d}{[-]}$.

Application of all possible S^{dg} generates $2^{\frac{d}{2}-1}$ members of each Clifford odd irreducible representation. To each irreducible representation the Hermitian conjugated irreducible representation belongs.

The Hermitian conjugated partner of the starting "basic vector" of an odd product of nilpotents obviously belong to another irreducible representation, since it is not reachable by S^{ab}. Each S^{cd} namely transforms a pair of projectors into a pair of nilpotents, a pair of nilpotents into a pair of projectors, and a pair of a nilpotent and a projector into a pair of a projector and a nilpotens, changing in each member of a pair its k into $-k$. The Hermitian conjugation transforms in $\hat{b}^{m\dagger}_{f}$ an odd number of nilpotents, each carrying its own k, into the same number of nilpotents, each carrying then $-k$ [4].

From Eq. (7.50) we learn that the starting member $\hat{b}^{m=1\dagger}_{f=2}$ of the next irreducible representation can be obtained from $\hat{b}^{m=1\dagger}_{f=1}$ by replacing, for example, $\overset{03}{(+i)}\overset{12}{[+]}$ in $\hat{b}^{m=1\dagger}_{f=1}$ with $\overset{03}{[+i]}\overset{12}{(+)}$. This new "basis vector" does not belong to either the starting irreducible representation, or to the Hermitian conjugated partners of the starting irreducible representation, due to the way how it is creating: S^{01} transforms $\overset{03}{(+i)}\overset{12}{[+]}$ into $\overset{03}{[-i]}\overset{12}{(-)}$, the Hermitian conjugation transforms $\overset{03}{(-i)}\overset{12}{[+]}$.

[4] The "basis vectors" with an even number of nilpotents have in even dimensional spaces the property that there is one member of each representation which is self adjoint, the one which is the product of only projectors.

Exchanging all possible pairs in the starting "basis vector" by keeping the same k's while transforming a pair of nilpotents into a pair of projectors, a pair of projectors into a pair of nilpotents and a pair of a nilpotent and a projector into a pair of the projector and the nilpotent, we generate $2^{\frac{d}{2}-1}$ irreducible representations with $2^{\frac{d}{2}-1}$ members each.

The Hemitian conjugation then generates $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ partners to the $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations.

One can find that the algebraic product of $\hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$ is the same for all m of a particular irreducible representation f (since $\hat{b}^{m}_{f}(2\,S^{ab})^{\dagger} *_A (2\,S^{ab})\hat{b}^{m\dagger}_{f} = \hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$, due to the relation $(-2iS^{ab})^{\dagger}(-2iS^{ab}) = 1$).

Each irreducible representation contributes different algebraic product $\hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$.

For the representation of Eq. (7.8) the product $\hat{b}^{m=1}_{f=1} *_a \hat{b}^{m=1\dagger}_{f=1}$ is equal to
$|\psi_{oc}>|_{f=1} = \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[-]}\stackrel{910}{[-]}\stackrel{1112}{[-]}\stackrel{1314}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,.$
This can be checked by using Eq. (7.7).

Defining the vacuum state $|\psi_{oc}>$ for the vector space determined by γ^a's as a sum of all different products of $\sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f}$, $\forall$ m, and for $d = 2n+1$, one obtains

$$
\begin{aligned}
|\psi_{oc}> &= \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} \\
&+ \quad \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]} + \cdots |1>\,, \\
&\quad \text{for } d = 2(2n+1)\,. \qquad (7.10)
\end{aligned}
$$

Let me add that the application of any member of the Cartan subalgebras on the vacuum state, $S^{ab}|\psi_{oc}>= 0$, $\tilde{S}^{ab}|\psi_{oc}>= 0$, $\forall$ S^{ab} and $\tilde{S}^{ab}$ belonging to Cartan subalgeras of Eq. (7.5).

One finds that all the members of all the irreducible representations fulfill together with their Hermitian conjugated partners the relations

$$
\begin{aligned}
\hat{b}^{m}_{f}{}_{*_A}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,, \\
\hat{b}^{m\dagger}_{f}{}_{*_A}|\psi_{oc}> &= |\psi^{m}_{f}>\,, \\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,, \\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f}\}_{*_A+}|\psi_{oc}> &= \delta^{mm'}|\psi_{oc}>\,, \\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,, \qquad (7.11)
\end{aligned}
$$

for each f. $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_{f}$'s and $\hat{b}^{m'}_{f'}$'s among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(7.10).

The relations of Eq. (7.11) *almost* manifest the anticommutation relations for the second quantized fermion fields postulated by Dirac [67]. It is pointed out *almost*, since the relation

$$
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> = \delta^{mm'}\delta^{ff'}|\psi_{oc}> \qquad (7.12)
$$

is not fulfilled. There are, namely, besides $\hat{b}^{m}_{f}$, $2^{\frac{d}{2}-1}-1$ members of the Hermitian conjugated partners belonging each to a different irreducible representation, which

give a nonzero contribution — not an identity as $\hat{b}^{m}_{f}$ does — when multiplying $\hat{b}^{m\dagger}_{f}$ from the left hand side. $\hat{b}^{m}_{f'} *_A \hat{b}^{m\dagger}_{f} \neq 0$
for $2^{\frac{d}{2}-1}-1$ different $f' \neq f$, while $\hat{b}^{m}_{f} *_A \hat{b}^{m\dagger}_{f} = 1$.

Let me illustrate this on the example of $\hat{b}^{m=1\dagger}_{f=1}$ of Eq. (7.8). Besides $(\hat{b}^{m=1\dagger}_{f=1})^{\dagger} = \hat{b}^{m=1}_{f=1} = \stackrel{d-1\,d}{[-]}\;\stackrel{d-3\,d-2}{[-]}\cdots\stackrel{13\,14}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{9\,10}{(-)}\stackrel{78}{(-)}\stackrel{56}{[+]}\stackrel{12}{[+]}\stackrel{03}{(-i)}$ also

$$
\begin{aligned}
&\stackrel{d-1\,d}{[-]}\;\stackrel{d-3\,d-2}{[-]}\cdots\stackrel{13\,14}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{9\,10}{(-)}\stackrel{78}{(-)}\stackrel{56}{[+]}\stackrel{12}{(-)}\stackrel{03}{[+i]}\,,\\
&\stackrel{d-1\,d}{[-]}\;\stackrel{d-3\,d-2}{[-]}\cdots\stackrel{13\,14}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{9\,10}{(-)}\stackrel{78}{(-)}\stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{03}{[+i]}\,,\\
&\text{etc}
\end{aligned}
\tag{7.13}
$$

applied on $\hat{b}^{m=1\dagger}_{f=1}$, give a nonzero contributions.

But index f determine different irreducible representations and we can not expect that the algebraic anticommutation relations will be fulfilled also among different irreducible representations. Different irreducible representations should be treated in tensor products.

All the discussions about the Clifford algebra with γ^{a}'s, appearing after Eq. (7.7), can be as well repeated also for the Clifford algebra with $\tilde{\gamma}^{a}$'s.

The Dirac's postulates for the second quantized fermion fields include the infinite basis in momentum space, while we treated so far the finite dimensional internal space of fermions. Before extending the vector basis space by making the tensor product of internal space and the momentum space let us recognize that the observed quarks and leptons and antiquarks and antileptons do not at all suggest that there might be two different internal spaces, which could be described by two kinds of the Clifford algebra objects. Let us therefore first reduce the Clifford space by the postulate, which leave only γ^{a}'s as the algebra describing the internal degrees of freedom of fermions, while $\tilde{\gamma}^{a}$'s are used to give quantum numbers to different irreducible representations.

Reduction of the Clifford space It is needed to give to each irreducible representation of the Lorentz transformations in the internal space of fermions the quantum number, which will distinguish among the $2^{\frac{d}{2}-1}$ different irreducible representations. If we keep the Clifford algebra with γ^{a}'s to describe the internal space of fermions, then $\tilde{\gamma}^{a}$'s, or rather $\tilde{S}^{ab}$'s, can be used to determine "family" quantum number of each irreducible representation of the Lorentz algebra in the Clifford space of γ^{a}'s.

We want that all the relations among γ^{a}'s and $\tilde{\gamma}^{a}$'s, presented in Eq. (7.2), remain unchanged, while the eigenvalues of the Cartan subalgebra of $\tilde{S}^{ab}$ are expected to be changed.

The postulate [2,7,9,10,12,46]

$$
\tilde{\gamma}^{a} B = (-)^{B}\, i\, B \gamma^{a}\,, \tag{7.14}
$$

with $(-)^{B} = -1$, if B is a function of an odd product of γ^{a}'s, otherwise $(-)^{B} = 1$ [46], does just that [5]

[5] Eq. (7.14) requires that $\tilde{\gamma}^{a}(a_0 + a_b\gamma^{b} + a_{bc}\gamma^{b}\gamma^{c} + \cdots) = (ia_0\gamma^{a} + (-i)a_b\gamma^{b}\gamma^{a} + ia_{bc}\gamma^{b}\gamma^{c}\gamma^{a} + \cdots)$, what means that the relation $\tilde{\gamma}^{a}\, a_0 = ia_0\gamma^{a}$ is only one of the

It is not difficult to check that the relations in Eq. (7.2), concerning $\tilde{\gamma}^a$'s are still valid: $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \{\gamma^a, \tilde{\gamma}^b\}_+ = 0 \,, (\gamma^a)^\dagger = \eta^{aa}\,\gamma^a \,, (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a$.

After this postulate the vector space of $\tilde{\gamma}^a$'s is "frozen out". And also the Grassmann algebra space is now reduced to $\theta^a = \gamma^a$ and $\frac{\partial}{\partial\theta_a} = 0$ [6]. No vector space of $\tilde{\gamma}^a$'s exists any longer, what is in agreement with the observed properties of fermions. While the anticommutation relations among γ^a's and $\tilde{\gamma}^a$'s remain the same as in Eq. (7.2), it follows for the eigenvalues of $\tilde{S}^{ab}$

$$
\begin{aligned}
S^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)} \,, \qquad & \tilde{S}^{ab} \stackrel{ab}{(k)} &= \frac{k}{2} \stackrel{ab}{(k)} \,, \\
S^{ab} \stackrel{ab}{[k]} &= \frac{k}{2} \stackrel{ab}{[k]} \,, \qquad & \tilde{S}^{ab} \stackrel{ab}{[k]} &= -\frac{k}{2} \stackrel{ab}{[k]} \,,
\end{aligned}
\tag{7.15}
$$

showing that the eigenvalues of S^{ab} on the nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$ on the nilpotents and projectors of γ^a's. The members of the Cartan subalgebra of $\tilde{S}^{ab}$, Eq. (7.5), can now be used to give to the irreducible representations of S^{ab} the "family" quantum numbers.

Let me mention that if one arranges the space of odd products of γ^a's with respect to $\mathcal{S}^{ab}(= S^{ab} + \tilde{S}^{ab})$, these new "basis vector" will form multiplets with integer spins and charges in adjoint representations. Like the "basis vectors" expressed by Grassmann algebra do in Ref. [13], Table I, but this time with θ^a's replaced by γ^a's.

relations included into Eq. (7.14). Another relation, for example, is $\tilde{\gamma}^a\gamma^a = (-i)\gamma^a\gamma^a = -i\eta^{aa}$. One correspondingly finds $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b = 2\eta^{ab}$. $\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$. $\{\tilde{\gamma}^a, \gamma^a\}_+ = 0 = \tilde{\gamma}^a\gamma^a + \gamma^a\tilde{\gamma}^a = \gamma^a(-i)\gamma^a + \gamma^a i\gamma^a = 0$.

[6] Let me show how does the Grassmann space loose the Hermitian conjugated partners to θ^a's, while θ^a's become equal to γ^a's. My statement that Eq. (7.14) requires $\theta^a = \gamma^a$ and $\frac{\partial}{\partial\theta_a} = 0$ can be proved as follows. There are only two requirements which have to be analyzed in details, $\tilde{\gamma^a}(\alpha) = i\alpha\gamma^a$, α is any constant and $\tilde{\gamma}^a\gamma^a = -i\gamma^a\gamma^a$. Both relations apply on $|\psi_{oc}>$: In the Grassmann case the vacuum state is identity $|\,1>$, while in the Clifford algebra the vacuum state is the sum of even products of γ^a's as seen in Eq. (7.10), which applies on identity. Let us express γ^a's, $\tilde{\gamma}^a$'s and $|\psi_{oc}>$ in terms of θ^a's and $\frac{\partial}{\partial\theta_a}$ as written in Eq. (7.3). Eq. (7.3) requires that $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma^a} = i(\theta^a - \frac{\partial}{\partial\theta_a})$. Let us put these expressions into Eq. (7.14) and let $|\psi_{oc}>$ be expressed in terms of θ^a's. Taking into account that θ^a's applying on identity gives θ^a's back while $\frac{\partial}{\partial\theta_a}$ applying on identity gives zero, it follows that $|\psi_{oc}> = a_0 + a_{ab}\theta^a\theta^b + \cdots$, the rest of expansion is irrelevant for the proof. The constant α can be skipped, since constants appear in $|\psi_{oc}> = a_0 + a_{ab}\theta^a\theta^b + \cdots$ anyhow. The first relation $[\tilde{\gamma^a} = i\gamma^a]|\psi_o c>$, expressed with θ^a's and $\frac{\partial}{\partial\theta_a}$, reads: $i(\theta^a - \frac{\partial}{\partial\theta_a})(a_0 + a_{ab}\theta^a\theta^b + \cdots) = i(\theta^a + \frac{\partial}{\partial\theta_a})(a_0 + a_{ab}\theta^a\theta^b + \cdots)$. From this we find $i\theta^a a_0 = i a_0\theta^a$ and $i(-\frac{\partial}{\partial\theta_a})a_{ab}\theta^a\theta^b = i\frac{\partial}{\partial\theta_a}a_b\theta^a\theta^b$, requiring that $\frac{\partial}{\partial\theta_a} = 0$ (as an operator Hermitian conjugated to θ^a for $\forall$ a). These relation requires that the derivatives should not exist any longer, if the relation should hold. Then it follows from $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ that $\theta^a = \gamma^a$, which means that the Grassmann space has no meaning any longer, the only remaining space is the space of the Clifford products of odd number of γ^a's, on which γ^a's and $\tilde{\gamma^a}$'s operate: $[\tilde{\gamma^a} = i\gamma^a]|\psi_o c>$ and $[\tilde{\gamma^a}\gamma^b = -i\gamma^b\gamma^a]|\psi_{oc}>$. This complicates the proof .

It is useful to notice that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$\begin{aligned}
&\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]}, \\
&\gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}, \\
&\tilde{\gamma^a} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma^b} \stackrel{ab}{(k)}= -k \stackrel{ab}{[k]}, \\
&\tilde{\gamma^a} \stackrel{ab}{[k]}= \; i \stackrel{ab}{(k)}, \quad \tilde{\gamma^b} \stackrel{ab}{[k]}= -k\eta^{aa} \stackrel{ab}{(k)} \; .
\end{aligned} \tag{7.16}$$

Some additional applications of $\tilde{\gamma}^a$'s and $\tilde{S}^{ab}$'s on nilpotents and projectors expressed by the γ^a's can be found in App. 7.4.

Each irreducible representation has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (7.5). Now we can replace the fourth equation in Eq. (7.11) — $\{\hat{b}^m_f, \hat{b}^{m'\dagger}_f\}_{*A+}|\psi_{oc}>= \delta^{mm'}|\psi_{oc}>$ — with the relation in Eq. (7.12) — $\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*A+}|\psi_{oc}>= \delta^{mm'}\delta_{ff'}|\psi_{oc}>$.

Each family contributes in even dimensional spaces one summand of $\frac{d}{2}$ projectors to the vacuum state $|\psi_{oc}>$ of fermions.

Correspondingly the "basic vectors" and their Hermitian conjugated partners fulfill algebraically the anticommutation relations of Dirac's second quantized fermions: Different irreducible representations carry different "family" quantum numbers and to each "family" quantum number only one Hermitian conjugated partner with the same "family" quantum number belongs. Also each summand of the vacuum state, Eq. (7.10), belongs to a particular "family".

One can easily check that each "basic vector" $\hat{b}^{m\dagger}_f$, applied algebraically on $|\psi_{oc}>$, gives nonzero contribution on the summand in the odd number of γ^a's, determined by $\hat{b}^m_f\hat{b}^{m\dagger}_f$, which is the same for all m of particular f, representing therefore the corresponding state $|\psi^f_m>$, while on all other summands $\hat{b}^{m\dagger}_f$ gives zero, $\hat{b}^m_f$ applying on $|\psi_{oc}>$ gives zero for all f and all m.

To define creation and annihilation operators, which determine on the vacuum state the single fermion states, we ought to make the tensor products of the $2^{\frac{d}{2}-1}$ $\times\, 2^{\frac{d}{2}-1}$ "basis vectors", describing the internal space of fermions and of infinite basis of momenta.

The oddness of the products of the odd number of γ^a's guarantees the anticommuting properties of all the objects which include an odd number of γ^a's.

The creation and annihilation operators, derived as tensor products of the "basis vectors" and the basis in momenum space, will fulfill the Dirac's postulates of the second quantized fermions without postulating them, as Dirac did. They follow by themselves from the fact that the creation and annihilation operators are superposition of odd products of γ^a's.

Second quantized fermion fields Since the nonrelativistic quantum theory is an approximation of the relativistic second quantized field theory — as the relativistic classical physics is an approximation of the quantum physics, and as the nonrelativistic classical physics, which we use the most of time, is the approximation of the relativistic classical physics — let us try to recognize what properties should the single particle states have to form the Hilbert space of second quantized fields.

In the references [10, 12, 13] the properties of the single fermion states, the tensor products among which form the Hilbert space, are discussed in details. In this talk I am presenting this topic from the point of view of the *spin-charge-family*. This theory offers, as written in the introduction, the explanation for the appearance of the spin (and handedness in the case of massless fermions), of all the charges, as well as of the families fermions. The number of families depends on the way how does the symmetry of the space breaks from $d = (13 + 1)$ to $d = (3 + 1)$.

In Table 7.3 one irreducible representation of $SO(13 + 1)$ of one family (belonging to the one of the two groups of four families which includes the so far observed three families) is presented. The first "basis vector" describes the internal degrees of freedom of the right handed quark $\hat{u}^{c1\dagger}_{R}$, of the first family with $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \tilde{S}^{78})$ equal to $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, presented in Table 7.4 as $\hat{u}^{c1\dagger}_{R1}$. The "basis vector" $\hat{b}^{m=1\dagger}_{f=1}$, Eq. 7.8, represents for $d = (13 + 1)$ just this $\hat{u}^{c1\dagger}_{R1}$ quark, and $\hat{b}^{m=1}_{f=1}$ is its Hermitian conjugated partner.

The "basis vector" $\hat{b}^{m=1\dagger}_{f=2}$ represents for $d = (13+1)$ the right handed u-quark with all the properties of $\hat{u}^{c1\dagger}_{R1}$ except for the family quantum numbers, which are now equal to $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. One can read in Table 7.3 that the spin of this right handed quark $\hat{u}^{c1\dagger}_{R}$ is $+\frac{1}{2}$, the weak $SU(2)$ charge is zero, the colour charge is $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$. It carries the additional $SU(2)$ charge equal to $\frac{1}{2}$ and the "fermion" quantum number — τ^4 charge — equal to $\frac{1}{6}$.

When solving the equations of motion for free massless fermions, which follow from the action, presented in Eq. (7.1), under the assumption, that at low energies the momentum of this right handed quark is $p^a = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, the solution $s = 1$ is the superposition

$$\hat{\mathbf{u}}^{sf=1\dagger}_{R}(\vec{p}) = \beta(\hat{u}^{c1\dagger}_{R\uparrow} + \frac{p^1 + ip^2}{|p^0| + |p^3|}\hat{u}^{c1\dagger}_{R\downarrow})\,, \tag{7.17}$$

with $|p^0| = |\vec{p}|$, with $\uparrow, \downarrow$ denoting spin $\pm\frac{1}{2}$, respectively, and with $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ normalizing the state.

There are steps from the $d = (13 + 1)$ dimensional space to the step where momentum in higher dimensions do not contribute to dynamics in $d = (3 + 1)$, while the break of symmetry makes the internal degrees of freedom (spins and families) to manifest as the spin and charges as presented in Table 7.3 and families as presented at Table 7.4. One finds the detailed presentations in Ref.([3–5, 9, 49, 52, 70] and the references therein).

Let us here represent the general solutions of equations of motion for free massless fermions with the internal space of fermions described by the "basis vectors" $\hat{b}^{m\dagger}_{f}$, fulfilling the relations of Eq. (7.11), for each family f separately, and also with respect to different families, $\hat{b}^{m}_{f} {}_{*_A} \hat{b}^{m'\dagger}_{f} = \delta^{mm'} \delta_{ff'}$,

$$
\begin{aligned}
\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|} &\stackrel{\text{def}}{=} \sum_{m} c^{sf}{}_{m}\,(\vec{p}, |p^0| = |\vec{p}|)\;\hat{b}^{m\dagger}_{f}\,,\\
\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}, \vec{x}) &\stackrel{\text{def}}{=} (\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0 - \vec{p}\cdot\vec{x})})|_{|p^0|=|\vec{p}|}\,,\\
\sum_{m} (c^{sf*}{}_{m}(\vec{p}) \cdot c^{s'f'}{}_{m}(\vec{p}))|_{|p^0|=|\vec{p}|} &= \delta^{ss'} \delta_{ff'}\,,
\end{aligned}
\tag{7.18}
$$

s represents different orthonormalized solutions of the equations of motion, $c^{sf}{}_{m}(\vec{p}, |p^0| = |\vec{p}|)$ are coefficients, depending on momentum $|\vec{p}|$ with $|p^0| = |\vec{p}|$. For the case of the right handed u-quarks of Eq. (7.17) the two nonzero coefficients are β and $\beta \frac{p^1 + ip^2}{|p^0| + |p^3|}$.

Creation operators of an odd Clifford character $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ create the single particle states, $< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}}, \mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|}$, manifesting the oddness of the creation operators

$$
\begin{aligned}
< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}}, \mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} &= \int dp^0 \delta(p^0 - |\vec{p}|)\, \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-ip_a x^a} \;{*_A}\, |\psi_{oc}>\\
&= (\hat{\mathbf{b}}^{sf\dagger}(\vec{p}) \cdot\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} \;{*_A}\; |\psi_{oc}>\,,
\end{aligned}
\tag{7.19}
$$

with the property

$$
\begin{aligned}
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} \;< \psi^{s'\mathbf{f'}}(\vec{p'}, p'^0 = |\vec{p'}|)|x >< x||\,\psi^{\mathbf{sf}}(\vec{p}, p^0 = |\vec{p}|) >=\\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{ip'_a x^a}|_{p'^0=|\vec{p'}|}\, e^{-ip_a x^a}|_{p^0|=|\vec{p}|}\\
&\cdot < \psi_{oc}|\,(\hat{\mathbf{b}}^{s'f'}(\vec{p'})\;\hat{\mathbf{b}}^{sf\dagger}(\vec{p}))\,{}_{*_A}\, |\psi_{oc} >= \delta ss' \delta^{ff'} \delta(\vec{p} - \vec{p'})\,.
\end{aligned}
\tag{7.20}
$$

One further finds the single particle fermion states in the coordinate representation

$$
\begin{aligned}
&|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}}, \mathbf{x^0}) >= \int_{-\infty}^{+\infty} \frac{\mathbf{d^{d-1}p}}{(\sqrt{\mathbf{2\pi}})^{\mathbf{d-1}}}\, (\hat{\mathbf{b}}^{\mathbf{sf}\dagger}(\tilde{\mathbf{p}})\, \mathbf{e}^{-\mathbf{i}(\mathbf{p^0x^0} - \varepsilon\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}})}|_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} \;{*_\mathbf{A}}\; |\psi_{\mathbf{oc}} >=\\
&\sum_{m} \hat{b}^{m\dagger}_{f} |\psi_{oc}> \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, (c^{sf}{}_{m}(\vec{p})\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} =\\
&\sum_{m} \hat{b}^{m\dagger}_{f} |\psi_{oc} >\; c^{sf}{}_{m}(-i\frac{\partial}{\partial x_a}, |p^0| = |(-i\frac{\partial}{\partial x_a}|)\, \delta(\vec{x})\,,
\end{aligned}
\tag{7.21}
$$

where it is taken into account that $\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|}\, |\psi_{oc} >= \sum_m c^{sf}{}_{m}\;(\vec{p}, |p^0| = |\vec{p}|)\; \hat{b}^{m\dagger}_{f}$, Eq. (7.18), and that $\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\, e^{ip'_a x^a}\, e^{-ip_a x^a} = \delta(\vec{p} - \vec{p'})$. $\varepsilon = \pm 1$, depending on handedness and spin of solutions.

Taking into account the above derivations, leading to

$$
\int dp^0 \delta(p^0 - |\vec{p}|) e^{i(p^0x^0 - p^0x^0)} = 1
$$

and $<\psi_{oc}|\,\hat{\mathbf{b}}^{sf}(\vec{p},p^0)_{*_A}\,\hat{\mathbf{b}}^{s'f'\dagger}(\vec{p},p^0)\,|\psi_{oc}>=\delta^{ss'}\delta^{ff'}$, one finds

$$
\begin{aligned}
&<\psi^{\mathbf{sf}}(\vec{x},x^0)|\,\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>=\\
&=\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\int_{-\infty}^{+\infty}\delta(p^0-|\vec{p}|)<\psi^{\mathbf{sf}}(\vec{x},x^0)|\vec{p}><\vec{p}|\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>\\
&=\int_{-\infty}^{+\infty}\frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\,e^{-i\vec{p}\cdot\vec{x}}\,e^{i\vec{p}\cdot\vec{x}'}\int dp^0\,\delta(p^0-|\vec{p}|)\\
&<\psi_{oc}|\hat{\mathbf{b}}^{sf}\,(\vec{p},p^0)_{*_A}\,\hat{\mathbf{b}}^{s'f'\dagger}\,(\vec{p},p^0)_{\,*_A}\,|\psi_{oc}>=\\
&=\delta^{ss'}\,\delta_{ff'}\,\delta(\vec{x}-\vec{x}\,')\,.
\end{aligned}
\tag{7.22}
$$

The scalar product $<\psi^{\mathbf{sf}}(\vec{x},x^0)\,|\,\psi^{\mathbf{s'f'}}(\vec{x}\,',x^0)>$ has obviously the desired properties of the second quantized states.

The new creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$, which are generated on the tensor products of both spaces, internal and momentum, fulfill obviously the below anticommutation relations when applied on $|\psi_{oc}>$

$$
\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{+\,*_T}|\psi_{oc}> &=\delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p'})\,|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}\,(\vec{p'},\vec{x})\}_{+\,*_T}|\psi_{oc}> &=0\cdot|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}\,(\vec{p}\,',\vec{x})\}_{+\,*_T}|\psi_{oc}> &=0\cdot|\psi_{oc}>\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\,{}_{*_T}|\psi_{oc>} &=|\psi^{sf}(\vec{p},,\vec{x})>\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},,\vec{x})\,{}_{*_T}|\psi_{oc}> &=0\cdot|\psi_{oc}>\,,\\
|p^0| &=|\vec{p}|\,.
\end{aligned}
\tag{7.23}
$$

It is not difficult to show that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ and $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ manifest the same anticommutation relations also on tensor products of an arbitrary chosen products of sets of single fermion states [13].

Hilbert space of fermion fields The tensor products of any number of any sets of the single fermion creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ (fulfilling together with their Hermitian conjugated partners annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ the anticommutation relations of Eq. (7.23)) form the Hilbert space of the second quantized fermion fields. The number of the sets is infinite. The internal space, defined by $\hat{b}^m_f$, contributes in d-dimensional space for each chosen momentum $\vec{p}$ (and for a parameter $\vec{x}$) the finite number, $2^{2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}}$, of such sets, the total Hilbert space has, due to the infinite basis in the momentum (or coordinate) space, the infinite number of sets

$$
N_{\mathcal{H}}=\prod_{\vec{p}}^{\infty}2^{2^{d-2}}\,. \tag{7.24}
$$

The number operator is defined as

$$
\begin{aligned}
N^{sf}_{\vec{p}}&=\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\;*_T\;\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\\
N^{sf}_{\vec{p}}\,|\psi_{oc}>&=0\cdot|\psi_{oc}>\;.
\end{aligned}
\tag{7.25}
$$

The vacuum state contains no fermions.

The Clifford odd objects $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ demonstrate their oddness also with respect to the whole Hilbert space $\mathcal{H}$, that is with respect to any tensor product of members of any sets of creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,',\vec{x})$). Correspondingly the anticommutation relations follow also for the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})$ on $\mathcal{H}$

$$
\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p},\vec{x})\,,\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H} = \delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p}\,')\,\mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x})\,,\hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H} = 0\,\cdot\mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p},\vec{x}),\hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,',\vec{x})\}_{*_T+}\,\mathcal{H} = 0\,\cdot\mathcal{H}\,.
\end{aligned} \tag{7.26}
$$

I presented in this talk the derivation of the creation and annihilation operator of the second quantized fermion fields, which obey the Dirac's postulates for the second quantized fermion fields without postulating them, just by analyzing properties of creation and annihilation operators obtained as tensor products of the "basis vectors" of an odd Clifford algebra and of the basis in either momentum or coordinate space. In Ref. [10–13] the relation between the creation and annihilation operators, postulated by Dirac and the ones presented in this talk are discussed.

Properties of fermions in $\mathbf{d} = (3+1)$ This section follows quite a lot Refs. [3,4]. With respect to the last years I have not succeeded to improve much the part presented in this subsection. I have been working on the symmetries of the *spin-charge-family* theory and in particular on how can the theory, using the Clifford algebra to describe all the internal properties of fermions — spins, charges and families — help to explain the assumptions of the second quantized fermion fields. I shall therefore review the other achievements of the theory very briefly.

In Eq. (7.1) the starting action is presented for fermion and boson fields in $d = (13+1)$. In order that predictions of the *spin-charge-family* theory are in agreement with the observed properties of quarks and leptons and antiquarks and antileptons, of the vector gauge fields and of the scalar gauge fields (manfesting as the higgs and Yukawa couplings), the manifold $M^{(13+1)}$ ought to break first into $M^{(7+1)} \times M^{(6)}$ (which manifests as $SO(7,1) \times SU(3) \times U(1)$), affecting fermions, vector gauge fields and scalar gauge fields.

This first break is caused by the scalar condensate of two right handed neutrinos, presented in Table 7.5, Sect. 7.5 which interact with all the scalar gauge fields (with the gauge fields with the space index $(5,6,7,\cdots,14)$, as well as with those vector gauge fields (with the gauge fields with the space index $(0,1,2,3)$, which couple to the condensate. The only vector gauge fields which do not interact with the condensate and remain consequently massless are the weak charge, colour charge and hyper charge vector gauge fields.

Since the left handed fermions couple differently to scalar fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)}$ families [68]. The rest of families get heavy masses [7].

[7] A toy model [68,69] was studied in $d = (5+1)$ with the same action as in Eq. (7.1). The break from $d = (5+1)$ to $d = (3+1)\times$ an almost S^2 was studied. For a particular

The fermion families are arranged into twice two groups of massless four families, with respect to family quantum numbers as presented in Table 7.4 in Sect. 7.5, each group manifesting $SU(2)_{\subset SO(3,1)} \times SU(2)_{\subset SO(4)}$ symmetry, one group manifesting $SU(2)_L \times SU(2)_L$ symmetry, the other $SU(2)_R \times SU(2)_R$ symmetry.

The nonzero vacuum expectation values of the scalar fields with the space index $(7,8)$, which carry the weak and hyper charges, break the mass protection and make family massive [7,9].

The breaks of the staring symmetry make the spins in higher dimensions to manifest as charges in $d = (3+1)$.

The superposition of the Lorentz members of the Clifford algebra, manifesting in $d = (3+1)$ the spins, Eq. (7.52), charges, Eqs. (7.53, 7.54) and families, Eqs (7.55, 7.56). are presented in Sect. 7.5.

Let me rewrite the fermion part of the action, Eq. (7.1), by taking into account the degrees of freedom the action manifests in $d = (3+1)$ in the way that we can clearly see that the action does manifest in the low energy regime by the *standard model* required properties of fermions, of vector gauge fields and of scalar gauge fields [1–3,7,9,51–53,71,72].

$$
\begin{aligned}
\mathcal{L}_f = {} & \bar{\psi}\gamma^m (p_m - \sum_{A,i} g^{Ai}\tau^{Ai} A^{Ai}_m)\psi + \\
& \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\, \psi\} + \\
& \{\sum_{t=5,6,9,\ldots,14} \bar{\psi}\gamma^t p_{0t}\, \psi\},
\end{aligned}
\tag{7.27}
$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, with $m \in (0,1,2,3)$, $s \in (7,8)$, $(s',s") \in (5,6,7,8)$, (a,b) (appearing in $\tilde{S}^{ab}$) run within either $(0,1,2,3)$ or $(5,6,7,8)$, t runs $\in (5,\ldots,14)$, $(t',t")$ run either $\in (5,6,7,8)$ or $\in (9,10,\ldots,14)$. The spinor function ψ represents all family members of all the $2^{\frac{7+1}{2}-1} = 8$ families.

a. The first line of Eq. (7.27) determines in $d = (3+1)$ the kinematics and dynamics of fermion fields, coupled to the vector gauge fields [3,5,9]. The vector gauge fields are the superposition of the spin connection fields ω_{stm}, $m = (0,1,2,3)$, $(s,t) = (5,6,\cdots,13,14)$, the gauge fields of S^{st}. They are shortly presented in Sect. 7.34.

The operators τ^{Ai} ($\tau^{Ai} = \sum_{a,b} c^{Ai}{}_{ab}\, S^{ab}$, S^{ab} are the generators of the Lorentz transformations in the Clifford space of γ^a's) are presented in Eqs. (7.53, 7.54) of Sect. 7.5. They represent the colour charge, $\vec{\tau}^3$, the weak charge, $\vec{\tau}^1$, and the hyper charge, $Y = \tau^4 + \tau^{23}$, τ^4 is the fermion charge, originating in $SO(6) \subset SO(13,1)$, τ^{23} belongs together with $\vec{\tau^1}$ of $SU(2)_{weak}$ to $SO(4)$ group ($\subset SO(13+1)$).

choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost S^2, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made in the $d = (13+1)$ case. This study is in progress.

One fermion irreducible representation of the Lorentz group contains, as seen in Table 7.3, *quarks and leptons and antiquarks and antileptons*, belonging to the first family in Table 7.4. One can notice that the $SO(7,1)$ subgroup content of the $SO(13,1)$ group is the same for the quarks and leptons and the same for the antiquarks and antileptons. Quarks distinguish from leptons, and antiquarks from antileptons, only in the $SO(6) \subset SO(13,1)$ part, that is in the colour (τ^{33}, τ^{38}) part and in the fermion quantum number τ^4. The quarks distinguish from antiquarks, and leptons from antileptons, in the handedness, in the colour part and in the τ^4 part, explaining the relation between handedness and charges of fermions and antifermions [8].

The vector gauge fields, which interact with the condensate, presented in Table 7.5, become massive. The *vector gauge fields not interacting with the condensate — the weak, colour and hyper charged vector gauge fields — remain massless*, in agreement with by the *standard model* assumed gauge fields before the electroweak break of the mass protection,

After the electroweak break, caused by the scalar fields, the only conserved charges are the colour and the electromagnetic charge $Q = \tau^{13} + Y$, $Y = \tau^4 + \tau^{23}$.

b. The second line of Eq. (7.27) is the mass term, responsible in $d = (3+1)$ for the masses of fermions. The interaction of fermions with the superposition of the spin connection fields with the space index $s = (7,8)$, which gain nonzero vacuum expectation values, cause the electroweak break, bringing masses to fermions and antifermions and to the weak vector gauge fields. They are superposition of either $\omega_{s's's}$ or $\tilde{\omega}_{abs}$. *These scalar fields explain the appearance of the higgs and Yukawa couplings* of the *standard model*. Their properties are shortly presented in Subsect. 7.2.2.

These scalar gauge fields split into two groups of four families, one group manifesting the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),L)} \times \widetilde{SU}(2)_{(\widetilde{SO}(4),L)} \times U(1)$ — and the other the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),R)} \times \widetilde{SU}(2)_{(\widetilde{SO}(4),R)} \times U(1)$, Eq. (7.37). The scalar gauge fields, manifesting $SU(2)_{L,R} \times SU(2)_{L,R}$, are the superposition of the gauge fields $\tilde{\omega}_{abs}$, $s = (7,8)$, $(a,b) =$ either $(0,1,2,3)$ or $(5,6,7,8)$, manifesting as twice two triplets interacting each with one of the two groups of four families, presented in Table 7.4. The three $U(1)$ singlet scalar gauge fields are superposition of $\omega_{s't's}$, $s = (7,8)$, $(s',t') = (5,6,7,8,9,\cdots,14)$, with the sum of $S^{s't'}$ arranged into superposition of τ^{13}, τ^{23} and τ^4. The three triplets interact with both groups of quarks and leptons and antiquarks and antileptons.

Each of *the two groups have well defined symmetry of mass matrices, what limits the number of free parameters*.

To one of the groups of four families the observed quarks and leptons belong [51,54,57,58].

We predict the mixing matrices for quarks, taking as the input the masses of the fourth family, since the elements for the 3×3 submatrix of the 4×4

[8] Ref. [8] points out that the connection between handedness and charges for fermions and antifermions, both appearing in the same irreducible representation, explains the triangle anomalies in the *standard model* with no need to connect "by hand" the handedness and charges of fermions and antifermions.

mixing matrix are (far) not accurately enough measured, that we could predict masses of the fourth family quarks [8,51,54]. The newer are the experimental data the better is the agreement of the measured mixing matrix elements with our predictions [54,58] at least so far.

The stable of the upper four families offers the explanation for the *dark matter*appearance and it is so far in agreement with experimental evidences of the dark matter [52,61].

I discuss predictions of the *spin-charge-family* theory for the properties of the lower four families and of the *dark matter* in Sect. 7.3.

c. The third line of Eq. (7.27) represents the scalar fields, which cause transitions from antileptons and antiquarks into quarks and leptons and back, offering the explanation for the matter/antimatter asymmetry in the expanding universe at non equilibrium conditions [4]. They are colour triplets with respect to the space index equal to $(9, 10, 11, 12, 13, 14)$, while they carry the quantum numbers with respect to the superposition of $\mathcal{S}^{ab}$ in adjoint representations, as can be seen in Table 7.2 and in Fig. 7.1 of Subsect. 7.2.2. I discuss properties of these scalar fields, offered by the *spin-charge-family* theory, in Sect. 7.3.

7.2.2 Properties of vector and scalar gauge fields in *spin-charge-family* theory

In the starting action, Eq. (7.1), the second line represents the action for gauge fields in $d = (13+1)$-dimensional space, with the index $_{gf}$ denoting gauge fields, vector or scalar,

$$\begin{aligned}
\mathcal{A}_{gf} &= \int d^d x\, E\, (\alpha\, R + \tilde{\alpha}\, \tilde{R})\,, \\
R &= \frac{1}{2}\{f^{\alpha[a} f^{\beta b]}\, (\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\, \omega^{c}{}_{b\beta})\} + h.c.\,, \\
\tilde{R} &= \frac{1}{2}\{f^{\alpha[a} f^{\beta b]}\, (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\, \tilde{\omega}^{c}{}_{b\beta})\} + h.c.\,,
\end{aligned} \tag{7.28}$$

which in the *spin-charge-family* theory manifests after the break of the starting symmetry in $d = (3+1)$ as the action for all observed vector and scalar gauge fields. Here $f^{\beta}{}_{a}$ and $e^{a}{}_{\alpha}$ are vielbeins and inverted vielbeins respectively

$$e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}\,, \quad e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}_{b}\,, \tag{7.29}$$

$E = \det(e^{a}{}_{\alpha})$.

Varying the action of Eq. (7.28) with respect to the spin connection fields, the expression for the spin connection fields ω_{ab}^{e} follows

$$\begin{aligned}
\omega_{ab}{}^{e} = &\frac{1}{2E}\{e^{e}{}_{\alpha}\, \partial_{\beta}(E f^{\alpha}{}_{[a} f^{\beta}{}_{b]}) - e_{a\alpha}\, \partial_{\beta}(E f^{\alpha}{}_{[b} f^{\beta e]}) \\
&\qquad - e_{b\alpha} \partial_{\beta}(E f^{\alpha[e} f^{\beta}{}_{a]})\} \\
&+ \frac{1}{4}\{\bar{\Psi}(\gamma^{e}\, S_{ab} - \gamma_{[a} S_{b]}{}^{e})\Psi\} \\
&- \frac{1}{d-2}\{\delta^{e}_{a}[\frac{1}{E}\, e^{d}{}_{\alpha} \partial_{\beta}(E f^{\alpha}{}_{[d} f^{\beta}{}_{b]}) + \bar{\Psi}\gamma_{d} S^{d}{}_{b}\, \Psi] \\
&\qquad - \delta^{e}_{b}[\frac{1}{E} e^{d}{}_{\alpha} \partial_{\beta}(E f^{\alpha}{}_{[d} f^{\beta}{}_{a]}) + \bar{\Psi}\gamma_{d} S^{d}{}_{a}\, \Psi]\}\,.
\end{aligned} \tag{7.30}$$

If replacing S^{ab} in Eq. (7.30) with $\tilde{S}^{ab}$, the expression for the spin connection fields $\tilde{\omega}_{ab}{}^{e}$ follows.

In Ref. [5] it is proven that in spaces with the desired symmetry the vielbein can be expressed with the gauge fields, if only one of the two spin connection fields are present

$$f^{\sigma}{}_{m} = \sum_{A} \vec{\tau}^{A\sigma}\, \vec{\mathcal{A}}^{A}_{m}\,, \tag{7.31}$$

with

$$\begin{aligned}
\mathcal{A}^{Ai}_{m} &= \sum_{st} c^{Ai}{}_{st}\, \omega^{st}{}_{m}\,,\\
\tau^{Ai\sigma} &= \sum_{st} c^{Ai}{}_{st}\, (e_{s\tau}\, f^{\sigma}{}_{t} - e_{t\tau}\, f^{\sigma}{}_{s}) x^{\tau}\,,\\
\tau^{Ai} &= \sum_{st} c^{Ai}{}_{st}\, S^{st}\,.
\end{aligned} \tag{7.32}$$

If fermions are not present them spin connections of both kinds are uniquely determined by vielbeins, as can be noticed from Eq. (7.30). If fermions are present, carrying both — family members and family quantum numbers — then vielbeins and both kinds of spin connections are influenced by the presence of fermions, which carry different family and family members quantum numbers.

The scalar (gauge) fields, carrying the space index $s = (5, 6, \ldots, d)$, offer in the *spin-charge-family* for $s = (7, 8)$ the explanation for the origin of the Higgs's scalar and the Yukawa couplings of the *standard model*, while scalars with the space index $s = (9, 10, \ldots, 14)$ offer the explanation for the proton decay, as well as for the matter/antimatter asymmetry in the universe.

We use the notation

$$\begin{aligned}
\tau^{Ai} &= \sum_{a,b} c^{Ai}{}_{ab}\, S^{ab}\,,\\
\{\tau^{Ai}, \tau^{Bj}\}_{-} &= i\delta^{AB} f^{Aijk} \tau^{Ak}\,,\\
A^{Ai}_{a} &= \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_{a}\,,
\end{aligned} \tag{7.33}$$

$a = m = (0, 1, 2, 3)$ for vector gauge fields and $a = s = (5, 6, \ldots, 14)$ for scalar aguge fields.

The explicit expressions for $c^{Ai}{}_{ab}$, and correspondingly for τ^{Ai}, and A^{Ai}_{a}, are written in Sect. 7.5.

Vector gauge fields in $d = (3 + 1)$ In the *spin-charge-family* theory there are besides the gravity, the colour and the weak $SU(2)_{I}$ vector gauge fields, also the second $SU(2)_{II}$ and the $U(1)_{\tau^4}$ vector gauge fields. The $U(1)_{\tau^4}$ vector gauge field is the vector gauge field of $\tau^4 (= -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}))$ - the fermion charge. The hyper charge vector gauge field of the *standard model* is the superposition of the third component of the second $SU(2)_{II}$ vector gauge fields and the $U(1)_{\tau^4}$ vector gauge field ($A^{Y}_{m} = \cos\theta_2 A^{\tau^4}_{m} + \sin\theta_2 A^{23}_{m}$, θ_2 is the angle of the break of the $SU(2)_{II} \times U(1)_{\tau^4}$ symmetry to $U(1)_Y$ at the scale $\geq 10^{16}$ or higher, [9] and references therein). After the appearance of the condensate, presented in Table 7.5, there are namely only the gravity, the colour, the weak $SU(2)_I$ and the $U(1)_Y$ hyper charge vector gauge fields, which remain massless. The two components of the second $SU(2)_{II}$ vector gauge fields and the superposition $A^{Y'}_m = -\sin\theta_2 A^{\tau^4}_m + \cos\theta_2 A^{23}_m$, which is the gauge field of $Y'(= -\tan^2\theta_2 \tau^4 + \tau^{23})$ gain high masses due to the interaction with the condensate. All the vector gauge fields are expressible with the spin connection fields ω_{stm},

$$A^{Ai}_m = \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_m\,. \tag{7.34}$$

Let me present expressions for the two $SU(2)$ vector gauge fields, $SU(2)_I$ and $SU(2)_{II}$

$$\begin{aligned}
\vec{\mathcal{A}}^1_m &= \vec{A}^1_m = (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m})\,,\\
\vec{\mathcal{A}}^2_m &= \vec{A}^2_m = (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m})\,.
\end{aligned} \tag{7.35}$$

The reader can similarly construct all the other vector gauge fields from the coefficients for the corresponding charges, or find the expressions in Refs. [4,7,9] and references therein.

The electroweak break, caused by the non zero expectation values of the scalar gauge fields, carrying the space index $s = (7, 8)$, makes the weak and the hyper charge massive. The only vector gauge fields which remains massless are the electromagnetic and the colour vector gauge fields — the observed two.

Scalar gauge fields in d $= (3+1)$ There are in the *spin-charge-family* theory scalar fields taking care of the masses of quarks and leptons: They have the space index $s = (7, 8)$ and carry with respect to the space index the weak charge $\tau^{13} = \pm\frac{1}{2}$ and the hyper charge $Y = \mp\frac{1}{2}$. With respect to $\tau^{Ai} = \sum_{ab} c^{Ai}{}_{ab} S^{ab}$ and $\tilde{\tau}^{Ai} = \sum_{ab} c^{Ai}{}_{ab} \tilde{S}^{ab}$ they carry charges and family charges in adjoint representations, Table 7.1, Eq. (7.39).

There are scalar fields transforming antileptons and antiquarks into quarks and leptons and back. They carry space index $s = (9, 10, \ldots, 14)$, They are with respect to the space index colour triplets, while they carry charges τ^{Ai} and $\tilde{\tau}^{Ai}$ in adjoint representations.

The infinitesimal generators $\mathcal{S}^{ab}$, which apply on the spin connections ω_{bde} $(= f^{\alpha}{}_e\ \omega_{bd\alpha})$ and $\tilde{\omega}_{\tilde{b}\tilde{d}e}$ $(= f^{\alpha}{}_e\ \tilde{\omega}_{\tilde{b}\tilde{d}\alpha})$, on either the space index e or any of the indices $(b, d, \tilde{b}, \tilde{d})$, as follows

$$\mathcal{S}^{ab}\, A^{d\ldots e\ldots g} = i\,(\eta^{ae}\, A^{d\ldots b\ldots g} - \eta^{be}\, A^{d\ldots a\ldots g})\,, \tag{7.36}$$

(see Section IV. and Appendix B in Ref. [9]).

Scalar gauge fields determining scalar higgs and Yukawa couplings

Let me introduce a common notation A^{Ai}_s for all the scalar gauge fields with $s = (7, 8)$, independently of whether they originate in ω_{abs} — in this case Ai

$= (Q, Q', Y')$ - or in $\tilde{\omega}_{\tilde{a}\tilde{b}s}$ — in this case all the family quantum numbers of all eight families contribute. All these gauge fields contribute to the masses of the quarks and leptons and the antiquarks and antileptons after gaining nonzero vacuum expectation values.

$$\begin{aligned}
&A^{Ai}_{s} \text{ represents } (\, A^{Q}_{s}\,, A^{Q'}_{s}\,, A^{Y'}_{s}\,, \vec{\tilde{A}}^{\tilde{1}}_{s}\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{s}\,, \vec{\tilde{A}}^{\tilde{2}}_{s}\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{s}\,)\,,\\
&\tau^{Ai} \text{ represents } (Q,\, Q',\, Y',\, \vec{\tilde{\tau}}^{1},\, \vec{\tilde{N}}_{L},\, \vec{\tilde{\tau}}^{2},\, \vec{\tilde{N}}_{R})\,.
\end{aligned} \tag{7.37}$$

Here τ^{Ai} represent all the operators, which apply on the fermions. These scalars, the gauge scalar fields of the generators τ^{Ai} and $\tilde{\tau}^{Ai}$, are expressible in terms of the spin connection fields (Ref. [9], Eqs. (10, 22, A8, A9)).

Let me demonstrate [9] that all the scalar fields with the space index $(7, 8)$ carry with respect to this space index the weak and the hyper charge $(\mp\frac{1}{2}, \pm\frac{1}{2})$, respectively. This means that all these scalars have properties as required for the Higgs in the *standard model*.

We need to know the application of the operators τ^{13} $(= \frac{1}{2}(\mathcal{S}^{56} - \mathcal{S}^{78})$, Y $(= \tau^{4} + \tau^{23})$ and Q $(= \tau^{13} + Y)$, Eq (7.53, 7.54, 7.58), with $\mathcal{S}^{ab}$ defined in Eq. (7.36), on the scalar fields with the space index $s = (7, 8)$.

To compare the properties of the scalar fields with those of the Higgs's scalar of the *standard model* let the scalar fields be eigenstates of $\tau^{13} = \frac{1}{2}(\mathcal{S}^{56} - \mathcal{S}^{78})$.

I rewrite for this purpose the second line of Eq. (7.27) as follows, ignoring the momentum p_s, $s = (5, 6, \ldots, d)$, since it is expected that solutions with nonzero momenta in higher dimensions do not contribute to the masses of fermion fields at low energies in $d = (3+1)$. We pay correspondingly no attention to the momentum p_s , $s \in (5, \ldots, 8)$, when having in mind the lowest energy solutions, manifesting at low energies.)

$$\begin{aligned}
&\sum_{s=(7,8),A,i} \bar{\psi}\,\gamma^{s}\,(-\tau^{Ai}\,A^{Ai}_{s}\,)\,\psi =\\
&-\bar{\psi}\{\stackrel{78}{(+)}\;\tau^{Ai}\,(A^{Ai}_{7} - i\,A^{Ai}_{8}) + \stackrel{78}{(-)}\,(\tau^{Ai}\,(A^{Ai}_{7} + i\,A^{Ai}_{8})\}\psi\,,\\
&\stackrel{78}{(\pm)} = \frac{1}{2}\,(\gamma^{7} \pm\, i\gamma^{8}\,)\,,\quad A^{Ai}_{\stackrel{78}{(\pm)}} := (A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,,
\end{aligned} \tag{7.38}$$

with the summation over A and i performed, since A^{Ai}_{s} represent the scalar fields $(A^{Q}_{s}, A^{Q'}_{s}, A^{Y'}_{s})$ determined by $\omega_{s',s'',s}$ and those determined by $(\tilde{\omega}_{a,b,s}$ $\tilde{A}^{\tilde{4}}_{s}$, $\vec{\tilde{A}}^{\tilde{1}}_{s}$, $\vec{\tilde{A}}^{\tilde{2}}_{s}$, $\vec{\tilde{A}}^{\tilde{N}_{R}}_{s}$ and $\vec{\tilde{A}}^{\tilde{N}_{L}}_{s}$).

The application of the operators τ^{13}, Y $(Y = \frac{1}{2}(\mathcal{S}^{56} + \mathcal{S}^{78}) - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14}))$ and Q on the scalar fields $(A^{Ai}_{7} \mp i\,A^{Ai}_{8})$ with respect to the space index $s = (7, 8)$, by taking into account Eq. (7.36) to make the application of the generators $\mathcal{S}^{ab}$ on the space indexes, gives

$$\begin{aligned}
\tau^{13}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= \pm\,\frac{1}{2}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,,\\
Y\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= \mp\,\frac{1}{2}\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8})\,,\\
Q\,(A^{Ai}_{7} \mp i\,A^{Ai}_{8}) &= 0\,.
\end{aligned} \tag{7.39}$$

Since τ^4, Y, τ^{13} and τ^{1+}, τ^{1-} give zero if applied on (A^{Q}_{s}, $A^{Q'}_{s}$ and $A^{Y'}_{s}$) with respect to the quantum numbers (Q, Q', Y'), and since Y and τ^{13} commute with the family quantum numbers, one sees that the scalar fields A^{Ai}_{s} (=(A^{Q}_{s}, A^{Y}_{s}, $A^{Y'}_{s}$, $\tilde{A}^{\tilde{4}}_{s}$, $\tilde{A}^{\tilde{Q}}_{s}$, $\vec{\tilde{A}}^{\tilde{1}}_{s}$, $\vec{\tilde{A}}^{\tilde{2}}_{s}$, $\vec{\tilde{A}}^{\tilde{N}_R}_{s}$, $\vec{\tilde{A}}^{\tilde{N}_L}_{s}$)), rewritten as $A^{Ai}_{\substack{78\\(\pm)}} = (A^{Ai}_{7} \mp i\, A^{Ai}_{8})$, are eigenstates of τ^{13} and Y, having the quantum numbers of the *standard model* Higgs' scalar.

These superposition of $A^{Ai}_{\substack{78\\(\pm)}}$ are presented in Table 7.1 as two doublets with respect to the weak charge τ^{13}, with the eigenvalue of τ^{23} (the second $SU(2)_{II}$ charge) equal to either $-\frac{1}{2}$ or $+\frac{1}{2}$, respectively. The operators $\tau^{1\boxplus\atop\boxminus} = \tau^{11} \pm i\tau^{12}$

name	superposition	τ^{13}	τ^{23}	spin	τ^4	Q
$A^{Ai}_{\substack{78\\(-)}}$	$A^{Ai}_{7} + iA^{Ai}_{8}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
$A^{Ai}_{\substack{56\\(-)}}$	$A^{Ai}_{5} + iA^{Ai}_{6}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1
$A^{Ai}_{\substack{78\\(+)}}$	$A^{Ai}_{7} - iA^{Ai}_{8}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0
$A^{Ai}_{\substack{56\\(+)}}$	$A^{Ai}_{5} - iA^{Ai}_{6}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	+1

Table 7.1. The two scalar weak doublets, one with $\tau^{23} = -\frac{1}{2}$ and the other with $\tau^{23} = +\frac{1}{2}$, both with the "fermion" quantum number $\tau^4 = 0$, are presented. In this table all the scalar fields carry besides the quantum numbers determined by the space index also the quantum numbers A and i from Eq. (7.37). The table is taken from Ref. [9].

$$\tau^{1\boxplus\atop\boxminus} = \frac{1}{2}[(\mathcal{S}^{58} - \mathcal{S}^{67}) \ {\boxminus\atop\boxplus}\ i\,(\mathcal{S}^{57} + \mathcal{S}^{68})]\,, \tag{7.40}$$

transform one member of a doublet from Table 7.1 into another member of the same doublet, keeping τ^{23} ($= \frac{1}{2}\,(\mathcal{S}^{56} + \mathcal{S}^{78})$) unchanged, clarifying the above statement.

It is not difficult to show that the scalar fields $A^{Ai}_{\substack{78\\(\pm)}}$ are *triplets* as the gauge fields of the *family quantum numbers* ($\vec{\tilde{N}}_R$, $\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^2$, $\vec{\tilde{\tau}}^1$; Eqs. (7.55, 7.56, 7.36)) or singlets as the gauge fields of $Q = \tau^{13} + Y$, $Q' = -\tan^2\vartheta_1 Y + \tau^{13}$ and $Y' = -\tan^2\vartheta_2\tau^4 + \tau^{23}$.

Let us do this for $\tilde{A}^{\tilde{N}_L i}_{\substack{78\\(\pm)}}$ and for $A^{Q}_{\substack{78\\(\pm)}}$, taking into account Eq. (7.52) (where we replace S^{ab} by $\mathcal{S}^{ab}$) and Eq. (7.36), and recognizing that $\tilde{A}^{\tilde{N}_L \boxplus\atop\boxminus}_{\substack{78\\(\pm)}} = \tilde{A}^{\tilde{N}_L 1}_{\substack{78\\(\pm)}} \ {\boxminus\atop\boxplus}\ i\,\tilde{A}^{\tilde{N}_L 2}_{\substack{78\\(\pm)}}$.

$$\begin{aligned}
\tilde{A}^{\tilde{N}_L \boxplus\atop\boxminus}_{\substack{78\\(\pm)}} &= \{(\tilde{\omega}_{23\substack{78\\(\pm)}} + i\,\tilde{\omega}_{01\substack{78\\(\pm)}}) \ {\boxminus\atop\boxplus}\ i\,(\tilde{\omega}_{31\substack{78\\(\pm)}} + i\,\tilde{\omega}_{02\substack{78\\(\pm)}})\}\,,\\
\tilde{A}^{\tilde{N}_L 3}_{\substack{78\\(\pm)}} &= (\tilde{\omega}_{12\substack{78\\(\pm)}} + i\,\tilde{\omega}_{03\substack{78\\(\pm)}})\,,\\
A^{Q}_{\substack{78\\(\pm)}} &= \omega_{56\substack{78\\(\pm)}} - (\omega_{9\,10\substack{78\\(\pm)}} + \omega_{11\,12\substack{78\\(\pm)}} + \omega_{13\,14\substack{78\\(\pm)}})\,.
\end{aligned}$$

One finds

$$
\begin{aligned}
&\tilde{N}_L^3\, \tilde{A}^{\tilde{N}_L \substack{\boxplus \\ \square}}_{\substack{78 \\ (\pm)}} = \substack{\boxplus \\ \square}\, \tilde{A}^{\tilde{N}_L \substack{\boxplus \\ \square}}_{\substack{78 \\ (\pm)}}\,, \quad \tilde{N}_L^3\, \tilde{A}^{\tilde{N}_L 3}_{\substack{78 \\ (\pm)}} = 0\,, \\
&Q\, A^{Q}_{\substack{78 \\ (\pm)}} = 0\,.
\end{aligned}
\tag{7.41}
$$

with $Q = \mathcal{S}^{56} + \tau^4 = \mathcal{S}^{56} - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14})$, and with τ^4 defined in Eq. (7.54), if replacing S^{ab} by $\mathcal{S}^{ab}$ from Eq. (7.36). Similarly one finds properties with respect to the Ai quantum numbers for all the scalar fields $A^{Ai}_{\substack{78 \\ (\pm)}}$.

After the appearance of the condensate (Table 7.5), which breaks the $SU(2)_{II}$ symmetry and brings masses to all the scalar fields, the weak $\vec{\tau}^1$ and the hyper charge Y remain the conserved charges.

At the electroweak scale the scalar gauge fields with the space index $(7, 8)$, with the Lagrange density

$$
\begin{aligned}
\mathcal{L}_{sg} = E \sum_{A,i} \{ & (p_m\, A^{Ai}_s)^\dagger\, (p^m\, A^{Ai}_s) - (-\lambda^{Ai} + (m'_{Ai})^2))\, A^{Ai\dagger}_s A^{Ai}_s \\
& + \sum_{B,j} \Lambda^{AiBj}\, A^{Ai\dagger}_s A^{Ai}_s\, A^{Bj\dagger}_s A^{Bj}_s \}\,,
\end{aligned}
\tag{7.42}
$$

gain nonzero vacuum expectation values and cause the electroweak break [9]. The above Lagrange density needs to be studied. At this stage is just postulated.

The two groups of four families became massive. The mass matrices manifest either $\widetilde{SU}(2)_{\widetilde{SO}(3,1)L} \times \widetilde{SU}(2)_{\widetilde{SU}(4)L} \times U(1)$ symmetry, this is the case for the lower four families of the eight families, presented in Table 7.4, or $\widetilde{SU}(2)_{\widetilde{SO}(3,1)R} \times \widetilde{SU}(2)_{\widetilde{SU}(4)R} \times U(1)$ symmetry, this is the case for the higher four families, presented in Table 7.4. The same three $U(1)$ singlet fields contribute to the masses of both groups, the two $SU(2)$ triplet fields are for each of the two groups different, although manifesting the same symmetries.

The mass matrix of family member — quarks and leptons — are 4×4 matrices. The observed three families of quarks and leptons form the 3×3 submatrices of the 4×4 matrices. The symmetry of the mass matrices, manifesting in all orders [57], limits the number of free parameters.

All the scalars, the two triplets and the three singlets, are doublets with respect to the weak charge, contributing to the weak and the hyper charge of the fermions so that they transform the right handed members into the left handed onces.

$$
\mathcal{M}^{\alpha} = \begin{pmatrix} -a_1 - a & e & d & b \\ e & -a_2 - a & b & d \\ d & b & a_2 - a & e \\ b & d & e & a_1 - a \end{pmatrix}^{\alpha}, \tag{7.43}
$$

with α representing family members — quarks and leptons of left and right handedness [49–51, 53, 54, 58].

---

[9] The expression for the Lagrange density of Eq. (7.42) is only estimated, more or less guessed, I have no estimate yet for the constants.

The mass matrices of the upper four families have the same symmetry as the mass matrices of the lower four families, but the scalar fields determining the masses of the upper four families have different properties (nonzero vacuum expectation values, masses and coupling constants) than those of the lower four, giving to quarks and leptons of the upper four families much higher masses in comparison with the lower four families of quarks and leptons, what offers the explanation for the appearance of the *dark matter*, studied at Refs. [52, 61].

Scalar fields transforming antiquarks and antileptons into quarks and leptons

I follow in this part to a great deal similar part in Ref. [3].

To the matter-antimatter asymmetry the terms contribute, which cause transitions from antileptons into quarks and from antiquarks into quarks and back. These are terms included into the third line of Eq. (7.27). Let me rewrite this part of the fermion action

$$\mathcal{L}_{f'} = \psi^{\dagger}\gamma^{0}\gamma^{t}\Big\{ \sum_{t=(9,10,\dots 14)} \big[p_{t} - (\frac{1}{2}S^{s's"}\omega_{s's"t} + \frac{1}{2}S^{t't''}\omega_{t't"t} + \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt})\big]\Big\}\,\psi,$$

as follows

$$\begin{aligned}
\mathcal{L}_{f"} = \psi^{\dagger}\gamma^{0}(-)\{\sum_{+,-}\sum_{(t\,t')} \overset{tt'}{(\oplus)}\;\cdot \\
&[\tau^{2+}A^{2+}_{\substack{tt'\\(\oplus)}} + \tau^{2-}A^{2-}_{\substack{tt'\\(\oplus)}} + \tau^{23}A^{23}_{\substack{tt'\\(\oplus)}} + \tau^{1+}A^{1+}_{\substack{tt'\\(\oplus)}} + \tau^{1-}A^{1-}_{\substack{tt'\\(\oplus)}} + \tau^{13}A^{13}_{\substack{tt'\\(\oplus)}} \\
&+ \tilde{\tau}^{2+}\tilde{A}^{2+}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{2-}\tilde{A}^{2-}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{23}\tilde{A}^{23}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{1+}\tilde{A}^{1+}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{1-}\tilde{A}^{1-}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{13}\tilde{A}^{13}_{\substack{tt'\\(\oplus)}} \\
&+ \tilde{N}^{+}_{R}\tilde{A}^{N_R+}_{\substack{tt'\\(\oplus)}} + \tilde{N}^{-}_{R}\tilde{A}^{N_R-}_{\substack{tt'\\(\oplus)}} + \tilde{N}^{3}_{R}\tilde{A}^{N_R3}_{\substack{tt'\\(\oplus)}} + \tilde{N}^{+}_{L}\tilde{A}^{N_L+}_{\substack{tt'\\(\oplus)}} + \tilde{N}^{-}_{L}\tilde{A}^{N_L-}_{\substack{tt'\\(\oplus)}} + \tilde{N}^{3}_{L}\tilde{A}^{N_L3}_{\substack{tt'\\(\oplus)}} \\
&+ \sum_{i}\tau^{3i}A^{3i}_{\substack{tt'\\(\oplus)}} + \tau^{4}A^{4}_{\substack{tt'\\(\oplus)}} + \sum_{i}\tilde{\tau}^{3i}\tilde{A}^{3i}_{\substack{tt'\\(\oplus)}} + \tilde{\tau}^{4}\tilde{A}^{4}_{\substack{tt'\\(\oplus)}}\,]\}\psi\,,
\end{aligned} \tag{7.44}$$

where (t, t') run in pairs over $[(9, 10), (11, 12), (13, 14)]$ and the summation must go over $+$ and $-$ of $\substack{tt'\\(\oplus)}$.

In Eq. (7.44) the relations below are used

$$
\begin{aligned}
\sum_{t,s',s''} \gamma^{t}\, \frac{1}{2} S^{s's"}\, \omega_{s's"t} &= \sum_{+,-} \sum_{(t\,t')} \overset{tt'}{(\oplus)}\; \frac{1}{2} S^{s's"}\, \omega_{s"s"\overset{tt'}{(\oplus)}}\,,\\
\omega_{s"s"\overset{tt'}{(\oplus)}} := \omega_{s"s"\overset{tt'}{(\pm)}} &= (\omega_{s's"t} \mp i\, \omega_{s's"t'})\,,\\
\overset{tt'}{(\oplus)} &:= = \frac{1}{2}\,(\gamma^{t} \pm \gamma^{t'})\,,\\
\overset{tt'}{(\oplus)}\; \frac{1}{2} S^{s's"}\, \omega_{s"s"\overset{tt'}{(\oplus)}} &= \overset{tt'}{(\oplus)}\; \{\tau^{2+}\, A^{2+}_{\overset{tt'}{(\oplus)}} + \tau^{2-}\, A^{2-}_{\overset{tt'}{(\oplus)}} + \tau^{23}\, A^{23}_{\overset{tt'}{(\oplus)}}\\
&\quad + \tau^{1+}\, A^{1+}_{\overset{tt'}{(\oplus)}} + \tau^{1-}\, A^{1-}_{\overset{tt'}{(\oplus)}} + \tau^{13}\, A^{13}_{\overset{tt'}{(\oplus)}}\, \}\,,\\
A^{2\boxplus}_{\overset{tt'}{(\oplus)}} &= (\omega_{58\overset{tt'}{(\oplus)}} + \omega_{67\overset{tt'}{(\oplus)}}) \boxplus i(\omega_{57\overset{tt'}{(\oplus)}} - \omega_{68\overset{tt'}{(\oplus)}})\,,\\
A^{23}_{\overset{tt'}{(\oplus)}} &= (\omega_{56\overset{tt'}{(\oplus)}} + \omega_{78\overset{tt'}{(\oplus)}})\,,\\
A^{1\boxplus}_{\overset{tt'}{(\oplus)}} &= (\omega_{58\overset{tt'}{(\oplus)}} - \omega_{67\overset{tt'}{(\oplus)}}) \boxplus i(\omega_{57\overset{tt'}{(\oplus)}} + \omega_{68\overset{tt'}{(\oplus)}})\,,\\
A^{13}_{\overset{tt'}{(\oplus)}} &= (\omega_{56\overset{tt'}{(\oplus)}} - \omega_{78\overset{tt'}{(\oplus)}})\,,\\
(t\,t') &\in ((9\,10), (11\,12), (13\,14))\,.
\end{aligned}
\tag{7.45}
$$

The rest of expressions in Eq. (7.45) are obtained in a similar way. They are presented in Eq. (7.62).

The scalar fields with the scalar index $s = (9, 10, \cdots, 14)$, presented in Table 7.2, carry one of the triplet colour charges and the "fermion" charge equal to twice the quark "fermion" charge, or the antitriplet colour charges and the "antifermion" charge. They carry in addition the quantum numbers of the adjoint representations originating in S^{ab} or in $\tilde{S}^{ab}$ [10].

If the antiquark $\bar{u}^{\bar{c}2}_{L}$, from the line 43 presented in Table 7.3, with the "fermion" charge $\tau^4 = -\frac{1}{6}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the hyper charge $Y (= \tau^4 + \tau^{23} =) -\frac{2}{3}$ and the electromagnetic charge $Q (= Y + \tau^{13} =) -\frac{2}{3}$ submits the $A^{2\boxminus}_{\overset{9\,10}{(\oplus)}}$ scalar field, it transforms into u^{c3}_{R} from the line 17 of Table 7.3, carrying the quantum numbers $\tau^4 = \frac{1}{6}$, $\tau^{13} = 0$, $\tau^{23} = \frac{1}{2}$, $(\tau^{33}, \tau^{38}) = (0, -\frac{1}{\sqrt{3}})$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$. These two quarks, d^{c1}_{R} and u^{c3}_{R} can bind together with u^{c2}_{R} from the 9^{th} line of the same table (at low enough energy, after the electroweak transition, and if they belong to a superposition with the left handed partners to the first family) -into the colour chargeless baryon - a proton. This transition is presented in Figure 7.1.

The opposite transition at low energies would make the proton decay.

[10] Although carrying the colour charge in one of the triplet or antitriplet states, these fields can not be interpreted as superpartners of the quarks since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.

field	prop.	τ^4	τ^{13}	τ^{23}	(τ^{33}, τ^{38})	Y	Q	$\tilde{\tau}^4$	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$
$A^{1\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{11\,12\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{11\,12\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{13\,14\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(0, \oplus\frac{1}{\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{13\,14\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(0, \oplus\frac{1}{\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{2\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	$\boxplus 1$	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3} + \boxplus 1$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{23}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{1\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	$\boxplus 1$	0	0	0
$\tilde{A}^{13}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{2\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	$\boxplus 1$	0	0
$\tilde{A}^{23}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{N_L\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	$\boxplus 1$	0
$\tilde{A}^{N_L 3}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\tilde{A}^{N_R\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	$\boxplus 1$
$\tilde{A}^{N_R 3}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$A^{3i}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\boxplus 1 + \oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$A^{4}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\cdots$												
$\vec{A}^3_m$	vector	0	0	0	octet	0	0	0	0	0	0	0
A^4_m	vector	0	0	0	0	0	0	0	0	0	0	0

Table 7.2. Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in Eq. (7.27), are presented. The space degrees of freedom contribute one of the triplets values to the colour charge of all these scalar fields. These scalars are with respect to the two $SU(2)$ charges, (τ^{13} and $\vec{\tau}^2$), and the two $\widetilde{SU}(2)$ charges, ($\vec{\tilde{\tau}}^1$ and $\vec{\tilde{\tau}}^2$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "fermion" number (τ^4) of the quarks. The quantum numbers of the two vector gauge fields, the colour and the $U(1)_{II}$ ones, are added.

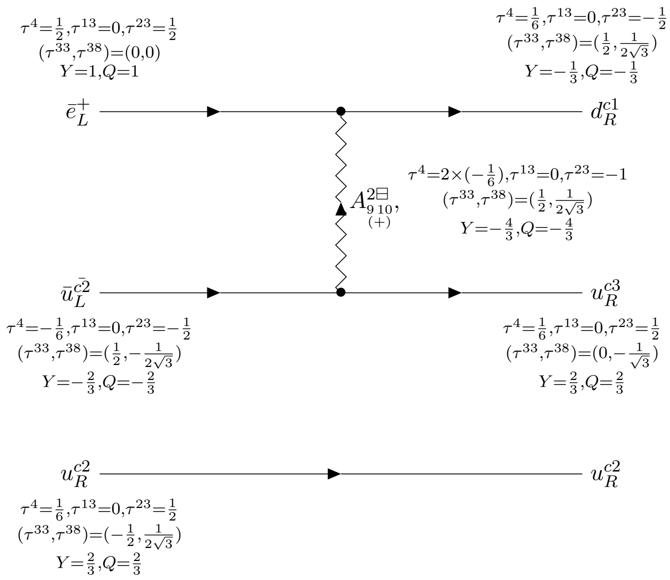

Fig. 7.1. The birth of a "right handed proton" out of an positron $\bar{e}_L{}^{+}$, antiquark $\bar{u}_L^{\bar{c2}}$ and quark (spectator) u_R^{c2}. The family quantum number can be any.

7.3 Achievements and conclusions

It remains to point out the achievements of the *spin-charge-family* theory so far and tell the open problems.

Achievements:

a. The simple starting action, Eq. (7.1), with the Clifford algebra used to describe the internal space of fermions, which in $d \geq (13+1)$ interact with the vielbeins and the two kinds of spin connection fields, offers **a.i.** that one irreducible representation of the Lorentz algebra in internal space manifests in $d = (3+1)$ all the fermions and antifermions with the spins and charges of the *standard model*, **a.ii.** that eight irreducible representations define in $d = (3+1)$ (after the reduction of the Clifford algebra from two kinds to only one kind) two times four families, **a.iii.** that the two kinds of the spin connection fields manifest in $d = (3+1)$ all the vector gauge fields of the *standard model*, **a.iv.** that the scalar fields with respect to $d = (3+1)$, carrying the weak and the hyper charge $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively, forming two groups of scalar fields manifesting each the $SU(2) \times SU(2) \times U(1)$ symmetry, offer the explanation for the Higgs's scalar and Yukawa couplings of the *standard model* giving masses to two groups of four families — the lower four families predicting the fourth family of quarks and leptons to the observed three, the stable of the upper four families offering explanation for the *dark matter*, **a.v.** that both groups of four families together spread masses from almost zero to $\geq 10^{16}$ GeV, **a.vi.** that the scalar gauge fields manifesting as colour triplets and antitriplets offer the explanation for the matter/antimatter asymmetry of the ordinary matter.

b. The decision to describe the internal space of fermions with the Clifford odd algebra enables to define the creation operators as tensor products of finite number of "basis vectors" of internal space and infinite basis in ordinary space applying on the vacuum state, which fulfill together the their Hermitian conjugated annihilation operators the anticommutation relations postulated by Dirac for the second quantized fields. The single fermion states have therefore by themselves the anticommuting character. Tensor products of any number and any kind of the single fermion creation operators define the second quantized fermion fields forming the whole Hilbert space.

Predictions:

The *spin-charge-family* theory offers several explanations as discuss in Sects. 7.1 and 7.2 and also several predictions.

A. Prediction of the fourth family to the observed three families, Subsect. 7.2.1. Taking into account the experimental data for masses of the observed families of quarks and the corresponding mixing matrix we fit 6 parameters of the two quark mass matrices, presented in Eq. (7.43), to twice 3 measured massess of quarks and to 6 measured parameters of the mixing matrix.

Althrough any accurate 3×3 submatrix of the 4×4 unitary matrix determines the 4×4 matrix completely, neither the quark nor the lepton mixing matrix is measured accurately enough that it would be possible to determine three complex phases of the 4×4 quark mixing matrix and the mixing matrix elements of the fourth family quarks to the other three family members. We therefore assume that mass matrices are symmetric and real, while making a choice for the masses of the fourth family.

Results are presented for two choices of $m_{u_4} = m_{d_4}$, Ref. [54], [arxiv:1412.5866]:
1. $m_{u_4} = 700$ GeV, $m_{d_4} = 700$ GeV.....new_1
2. $m_{u_4} = 1\,200$ GeV, $m_{d_4} = 1200$ GeV.....new_2

$$
|V_{(ud)}| = \begin{pmatrix}
exp_n & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & \\ \hline
new_1 & 0.97423(4) & 0.22539(7) & 0.00299 & \mathit{0.00776(1)} \\
new_2 & 0.97423[5] & 0.22538[42] & 0.00299 & \mathbf{0.00793[466]} \\ \hline
exp_n & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & \\ \hline
new_1 & 0.22534(3) & 0.97335 & 0.04245(6) & \mathit{0.00349(60)} \\
new_2 & 0.22531[5] & 0.97336[5] & 0.04248 & \mathbf{0.00002[216]} \\ \hline
exp_n & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & \\ \hline
new_1 & 0.00667(6) & 0.04203(4) & 0.99909 & \mathit{0.00038} \\
new_2 & 0.00667 & 0.04206[5] & 0.99909 & \mathbf{0.00024[21]} \\ \hline
new_1 & \mathit{0.00677(60)} & \mathit{0.00517(26)} & \mathit{0.00020} & \mathit{0.99996} \\
new_2 & \mathbf{0.00773} & \mathbf{0.00178} & \mathbf{0.00022} & \mathbf{0.99997[9]}
\end{pmatrix} . \tag{7.46}
$$

One can see that the above results for the mixing matrices of the lower three families are in agreement with what Ref. [55] requires, namely that
$V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, and $V_{u_3 d_4} < V_{u_1 d_4}$.

Since we have not yet fit the mass matrix of Eq. (7.43) to the newest experimental data [56], which appear after our Bled 2020, the evaluation for our 4×4 quark mixing matrix with the new data and correspondingly a new prediction is not yet offered.

Let me repeat the discussion of Ref. [58] that the existence of the fourth family to the observed three is still not in disagreement with the latest experimental data although some phenomenologists say different.

B. The *spin-charge-family* theory predicts in the low energy regime (up to 10^{16} GeV or higher) the existence of two decoupled groups of four families, which at the electroweak break become massive [52]. The stable family of the upper group of four families (with almost zero Yukawa couplings to the lower group of four families) is the candidate for the dark matter, Subsect. 7.2.1.

I review here briefly the estimations done in Ref. [52]. We used the simple hydrogen-like model to evaluate properties of the fifth family heavy baryons, taking into account that for masses of the order of a few TeV or larger the force among the constituents of the fifth family baryons is determined mostly by one gluon exchange. The fifth family neutron is estimated as the most stable nucleon. The "nuclear interaction" among these baryons is found to have very interesting properties. We studied scattering amplitudes among fifth family neutrons and with the ordinary matter.

We followed the behaviour of the fifth family quarks and antiquarks in the plasma of the expanding universe, through the freezing out procedure, solving the Boltzmann equations, through the colour phase transition, while forming neutrons, up to the present dark matter, taking into account the cosmological evidences, the direct experimental evidences and all others known properties of the dark matter.

The cosmological evolution suggested the limits for the masses of the fifth family quarks

$$10 \text{ TeV} < m_{q_5} c^2 < \text{a few} \cdot 10^2 \text{ TeV} \tag{7.47}$$

and for the scattering cross sections

$$10^{-8} \, \text{fm}^2 < \sigma_{c_5} < 10^{-6} \, \text{fm}^2 \,, \tag{7.48}$$

while the measured density of the dark matter does not put much limitation on the properties of heavy enough clusters.

The direct measurements limit the fifth family quark mass to

$$\text{several } 10 \, \text{TeV} < m_{q_5} c^2 < 10^5 \, \text{TeV} \,. \tag{7.49}$$

We also find that our fifth family baryons of the mass of several 10 TeV/c^2 have for a factor more than 100 times too small scattering amplitude with the ordinary matter to cause a measurable heat flux on the Earth's surface.

C. The *spin-charge-family* theory predicts several scalar fields with the weak and the hyper charge of the Higg's scalar $(\pm\frac{1}{2}, \mp\frac{1}{2})$ — two triplets and three singlets — offering the explanation for the existence of the Higgs's scalar and Yukawa couplings, Subsect. 7.2.2.

The additional two triplets and the same three singlets determine properties of the upper four families of quarks and leptons, Subsect. 7.2.2.

D. The *spin-charge-family* theory predicts several scalar fields which are colour triplets or antitriplets, offering the explanation for the matter/antimatter asymmetry in the (nonequilibrium) expanding universe as well as the proton decay [4], Subsect. 7.2.2.
E. The mass matrices of the two fourth family groups are close to democratic one, causing spreading of the fermion masses from 10^{-8} MeV to 10^{16} GeV or even higher.

I conclude by saying that there are still a lot of open problems to be solved. Some of them are common to the other theories, like the Kaluza-Klein-like theories, the others require to extract as much as possible from the offer of the theory. We need collaborators, since the more work is put into the *spin-charge-family* theory the more explanations for the observed phenomena follow.

7.4 APPENDIX: Useful relations

From Eq. (7.16) it follows

$$
\begin{aligned}
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, &\quad \tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[k]}\stackrel{cd}{[k]}\,,\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,, &\quad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]} &= -\frac{i}{2} \stackrel{ab}{(k)}\stackrel{cd}{(k)}\,,\\
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, &\quad \tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \stackrel{ab}{[k]}\stackrel{cd}{(k)}\,,\\
S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,, &\quad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)} &= \frac{i}{2}\eta^{cc} \stackrel{ab}{(k)}\stackrel{cd}{[k]}\,.
\end{aligned}
\tag{7.50}
$$

By using Eq. (7.14) one finds the relations

$$
\begin{aligned}
&\stackrel{ab}{\tilde{(k)}}\stackrel{ab}{(k)} = 0\,, \quad &&\stackrel{ab}{\tilde{(-k)}}\stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{\tilde{(k)}}\stackrel{ab}{[k]} = i \stackrel{ab}{(k)}\,, \quad &&\stackrel{ab}{\tilde{(k)}}\stackrel{ab}{[-k]} = 0\,,\\
&\stackrel{ab}{\tilde{[k]}}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad &&\stackrel{ab}{\tilde{[-k]}}\stackrel{ab}{(k)} = 0\,,\\
&\stackrel{ab}{\tilde{[k]}}\stackrel{ab}{[k]} = 0\,, \quad &&\stackrel{ab}{\tilde{[-k]}}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,.
\end{aligned}
\tag{7.51}
$$

7.5 APPENDIX: One irreducible representation of the internal space and families described by the Clifford algebra γ^a

Below the subgroups of the starting groups $SO(13,1)$ and $\widetilde{SO}(13,1)$ are presented, manifesting in $d = (3+1)$ the spins, charges and families of fermions in the *spin-charge-family* theory. Table 7.3, representing one $SO(13,1)$ irreducible representation of fermions — quarks and leptons and antiquarks and antileptons — uses these expressions.

a.i. The generators of the two $SU(2)$ ($\subset SO(3,1) \subset SO(7,1) \subset SO(13,1)$) groups, describing spins of fermions

$$\vec{N}_{\pm}(=\vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm i S^{01}, S^{31} \pm i S^{02}, S^{12} \pm i S^{03}), \tag{7.52}$$

are presented.

a.ii. The generators of the two $SU(2)$ ($SU(2) \subset SO(4) \subset SO(7,1) \subset SO(13,1)$) groups, describing the two kinds of weak charges of fermions

$$\begin{aligned}
\vec{\tau}^{1} &:= \frac{1}{2}(S^{58} - S^{67},\, S^{57} + S^{68},\, S^{56} - S^{78}), \\
\vec{\tau}^{2} &:= \frac{1}{2}(S^{58} + S^{67},\, S^{57} - S^{68},\, S^{56} + S^{78}),
\end{aligned} \tag{7.53}$$

are presented.

a.iii. The $SU(3)$ and $U(1)$ subgroups of $SO(6) \subset SO(13,1)$, describing the colour charge and the "fermion" charge of fermions

$$\begin{aligned}
\vec{\tau}^{3} := &\ \frac{1}{2}\{S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, \\
&S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, \\
&S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}, \\
\tau^{4} := &-\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}),
\end{aligned} \tag{7.54}$$

are presented.

b.i. The two $\widetilde{SU}(2)$ subgroups of $\tilde{SO}(3,1)$ ($\subset \widetilde{SO}(7,1) \subset \widetilde{SO}(13,1)$), describing families of fermions

$$\vec{\tilde{N}}_{L,R} := \frac{1}{2}(\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03}), \tag{7.55}$$

are presented.

b.ii. The two $\widetilde{SU}(2)$ subgroups of $\widetilde{SO}(4)$ ($\subset \widetilde{SO}(7,1) \subset \widetilde{SO}(13,1)$), describing families of fermions

$$\begin{aligned}
\vec{\tilde{\tau}}^{1} &:= \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67},\, \tilde{S}^{57} + \tilde{S}^{68},\, \tilde{S}^{56} - \tilde{S}^{78}), \\
\vec{\tilde{\tau}}^{2} &:= \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67},\, \tilde{S}^{57} - \tilde{S}^{68},\, \tilde{S}^{56} + \tilde{S}^{78}),
\end{aligned} \tag{7.56}$$

are presented.

b.iii. The group $\tilde{U}(1)$, the subgroup of $\widetilde{SO}(6)$ ($\subset \widetilde{SO}(13,1)$), describing family quantum numbers of fermions

$$\tilde{\tau}^{4} := -\frac{1}{3}(\tilde{S}^{9\,10} + \tilde{S}^{11\,12} + \tilde{S}^{13\,14}), \tag{7.57}$$

are presented.

c. Relations among the hyper, weak and the second $SU(2)$ charges

$$\begin{aligned}
&Y := \tau^4 + \tau^{23}\,, \quad Y' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}\,, \quad Q := \tau^{13} + Y\,, \quad Q' := -Y \tan^2 \vartheta_1 + \tau^{13} \quad ,\\
&\tilde{Y} := \tilde{\tau}^4 + \tilde{\tau}^{23}\,, \quad \tilde{Y}' := -\tilde{\tau}^4 \tan^2 \vartheta_2 + \tilde{\tau}^{23}\,, \quad \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}\,, \quad \tilde{Q}' = -\tilde{Y} \tan^2 \vartheta_1 + \tilde{\tau}^{13} \quad ,
\end{aligned} \tag{7.58}$$

are presented.

Below are some of the above expressions written in terms of nilpotents and projectors

$$\begin{aligned}
&N^{\pm}_{+} = N^1_{+} \pm i\, N^2_{+} = -\stackrel{03}{(\mp i)}\stackrel{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^1_{-} \pm i\, N^2_{-} = \stackrel{03}{(\pm i)}\stackrel{12}{(\pm)}\,,\\
&\tilde{N}^{\pm}_{+} = -\stackrel{03}{\tilde{(\mp i)}}\stackrel{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \stackrel{03}{\tilde{(\pm i)}}\stackrel{12}{\tilde{(\pm)}}\,,\\
&\tau^{1\pm} = (\mp)\ \stackrel{56}{(\pm)}\stackrel{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\ \stackrel{56}{(\mp)}\stackrel{78}{(\mp)}\,,\\
&\tilde{\tau}^{1\pm} = (\mp)\ \stackrel{56}{\tilde{(\pm)}}\stackrel{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\ \stackrel{56}{\tilde{(\mp)}}\stackrel{78}{\tilde{(\mp)}}\ .
\end{aligned} \tag{7.59}$$

i		$^a\hat{b}_i^{\dagger}$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
1	$\hat{u}^{c1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$\hat{u}^{c1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$\hat{d}^{c1\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\hat{d}^{c1\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$\hat{d}^{c1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]} \vert \stackrel{56}{(-)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$\hat{d}^{c1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)} \vert \stackrel{56}{(-)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$\hat{u}^{c1\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]} \vert \stackrel{56}{[+]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$\hat{u}^{c1\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)} \vert \stackrel{56}{[+]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$\hat{u}^{c2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$\hat{u}^{c2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$\hat{d}^{c2\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$\hat{d}^{c2\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$\hat{d}^{c2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]} \vert \stackrel{56}{(-)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$\hat{d}^{c2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)} \vert \stackrel{56}{(-)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$\hat{u}^{c2\dagger}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]} \vert \stackrel{56}{[+]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$\hat{u}^{c2\dagger}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)} \vert \stackrel{56}{[+]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$\hat{u}^{c3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$\hat{u}^{c3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{[+]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$\hat{d}^{c3\dagger}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$\hat{d}^{c3\dagger}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \vert \stackrel{56}{(-)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Continued on next page

i		$a\hat{b}_i^\dagger$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
21	$\hat{d}_L^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$\hat{d}_L^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$\hat{u}_L^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$\hat{u}_L^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	$\hat{\nu}_R^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	$\hat{\nu}_R^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	$\hat{e}_R^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	$\hat{e}_R^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	$\hat{e}_L^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	$\hat{e}_L^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	$\hat{\nu}_L^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	$\hat{\nu}_L^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\hat{\bar{d}}_L^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\hat{\bar{d}}_L^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\hat{\bar{u}}_L^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\hat{\bar{u}}_L^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\hat{\bar{d}}_R^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\hat{\bar{d}}_R^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\hat{\bar{u}}_R^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\hat{\bar{u}}_R^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\hat{\bar{d}}_L^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\hat{\bar{d}}_L^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\hat{\bar{u}}_L^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\hat{\bar{u}}_L^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\hat{\bar{d}}_R^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\hat{\bar{d}}_R^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\hat{\bar{u}}_R^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\hat{\bar{u}}_R^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\hat{\bar{d}}_L^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\hat{\bar{d}}_L^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\hat{\bar{u}}_L^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\hat{\bar{u}}_L^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\hat{\bar{d}}_R^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\hat{\bar{d}}_R^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\hat{\bar{u}}_R^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\hat{\bar{u}}_R^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\hat{\bar{e}}_L^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\hat{\bar{e}}_L^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\hat{\bar{\nu}}_L^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\hat{\bar{\nu}}_L^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\hat{\bar{\nu}}_R^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\hat{\bar{\nu}}_R^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{(-)}\stackrel{78}{(+)}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\hat{\bar{e}}_R^\dagger$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\hat{\bar{e}}_R^\dagger$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\mid\stackrel{56}{[+]}\stackrel{78}{[-]}\parallel\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 7.3. The left handed ($\Gamma^{(13,1)} = -1$), multiplet of creation operators of fermions — the members of one fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented in the massless basis as products of nilpotents and projectors. The multiplet contains the left handed ($\Gamma^{(3+1)} = -1$ weak ($SU(2)_{I}$) charged ($\tau^{13} = \pm\frac{1}{2}$, ($\vec{\tau}^{1} = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}$)) and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, $\vec{\tau}^{2} = \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}$)) quarks and leptons and the right handed ($\Gamma^{(3+1)} = 1$), weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($(\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, ($\vec{\tau}^{3} = \frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}$)), carrying the "fermion charge" ($\tau^{4} = \frac{1}{6}, = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$. The colourless leptons carry the "fermion charge" ($\tau^{4} = -\frac{1}{2}$). In the same multiplet of creation operators the left handed weak ($SU(2)_{I}$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_{I}$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons are included. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Anti-quarks are antitriplets, carrying the "fermion charge" ($\tau^{4} = -\frac{1}{6}$). The anti-colourless antileptons carry the "fermion" charge ($\tau^{4} = \frac{1}{2}$). $Y = (\tau^{23} + \tau^{4})$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The creation operators of opposite charges (antifermion creation operators) are reachable from the particle ones besides by S^{ab} also by the application of the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}_{\mathcal{N}}$, presented in Refs. [65,66]. The reader can find this Weyl representation also in Refs. [4,9,71,72] and in the references therein.

Table 7.3 represents in the *spin-charge-family* theory the creation operators for observed *quarks and leptons and antiquarks and antileptons* for a particular family, Table (7.4). Hermitian conjugation of the creation operators of Table 7.3 generates the corresponding annihilation operators, fulfilling together with the creation operators anticommutation relations for fermions of Eq. (7.23).

The condensate of two right handed neutrinos with the family quantum numbers of the upper four families, causing the break of the starting symmetry $SO(13,1)$ into $SO(7,1) \times SU(3) \times U(1)$, is presented in Table 7.5.

7.6 APPENDIX: Expressions for scalar fields in term of $\omega_{s's''s}$ and $\tilde{\omega}_{abs}$

The scalar fields, responsible for masses of the family members and of the heavy bosons [6,7] after gaining nonzero vacuum expectation values and triggering the electroweak break, are presented in the second line of Eq. (7.27). These scalar fields are included in the covariant derivatives as $-\frac{1}{2}\, S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $s \in (7,8)$, $(a,b), \in (0,\ldots,3), (5,\ldots,8)$.

One can express the scalar fields carrying the quantum numbers of the subgroups of the family groups, expressed in terms of $\tilde{\omega}_{abs}$ (they contribute to mass matrices of quarks and leptons and to masses of the heavy bosons), if taking into account Eqs. (7.55, 7.56, 7.58),

$$
\begin{aligned}
\sum_{a,b} -\frac{1}{2}\tilde{S}^{ab}\,\tilde{\omega}_{abs} &= -(\vec{\tilde{\tau}}^{\tilde{1}}\,\vec{\tilde{A}}^{\tilde{1}}_{s} + \vec{\tilde{N}}_{\tilde{L}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{s} + \vec{\tilde{\tau}}^{\tilde{2}}\,\vec{\tilde{A}}^{\tilde{2}}_{s} + \vec{\tilde{N}}_{\tilde{R}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{s})\,,\\
\vec{\tilde{A}}^{\tilde{1}}_{s} &= (\tilde{\omega}_{58s} - \tilde{\omega}_{67s},\, \tilde{\omega}_{57s} + \tilde{\omega}_{68s},\, \tilde{\omega}_{56s} - \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{s} &= (\tilde{\omega}_{23s} + i\,\tilde{\omega}_{01s},\, \tilde{\omega}_{31s} + i\,\tilde{\omega}_{02s},\, \tilde{\omega}_{12s} + i\,\tilde{\omega}_{03s})\,,\\
\vec{\tilde{A}}^{\tilde{2}}_{s} &= (\tilde{\omega}_{58s} + \tilde{\omega}_{67s},\, \tilde{\omega}_{57s} - \tilde{\omega}_{68s},\, \tilde{\omega}_{56s} + \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{s} &= (\tilde{\omega}_{23s} - i\,\tilde{\omega}_{01s},\, \tilde{\omega}_{31s} - i\,\tilde{\omega}_{02s},\, \tilde{\omega}_{12s} - i\,\tilde{\omega}_{03s})\,,\\
&\quad (s \in (7,8))\,. \qquad (7.60)
\end{aligned}
$$

Scalars, expressed in terms of ω_{abc} (contributing as well to the mass matrices of quarks and leptons and to masses of the heavy bosons) follow, if using Eqs. (7.53,

					$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$	$\tilde{\tau}^4$
I	$\hat{u}^{c1\dagger}_{R\,1}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,1}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,2}$	$\overset{03}{[+i]}\overset{12}{(+)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,2}$	$\overset{03}{[+i]}\overset{12}{(+)}\mid\overset{56}{[+]}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,3}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(+)}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,3}$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{(+)}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,4}$	$\overset{03}{[+i]}\overset{12}{(+)}\mid\overset{56}{(+)}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,4}$	$\overset{03}{[+i]}\overset{12}{(+)}\mid\overset{56}{(+)}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,5}$	$\overset{03}{[+i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,5}$	$\overset{03}{[+i]}\overset{12}{[+]}\mid\overset{56}{[+]}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,6}$	$\overset{03}{(+i)}\overset{12}{(+)}\mid\overset{56}{[+]}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,6}$	$\overset{03}{(+i)}\overset{12}{(+)}\mid\overset{56}{[+]}\overset{78}{[+]}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,7}$	$\overset{03}{[+i]}\overset{12}{[+]}\mid\overset{56}{(+)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,7}$	$\overset{03}{[+i]}\overset{12}{[+]}\mid\overset{56}{(+)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,8}$	$\overset{03}{(+i)}\overset{12}{(+)}\mid\overset{56}{(+)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{(-)}\;\overset{13\,14}{(-)}$	$\hat{\nu}^{\dagger}_{R\,8}$	$\overset{03}{(+i)}\overset{12}{(+)}\mid\overset{56}{(+)}\overset{78}{(+)}\Vert\overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

Table 7.4. Eight families of creation operators of $\hat{u}^{c1\dagger}_R$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 7.3 — and of the colourless right handed neutrino $\hat{\nu}^{\dagger}_R$ — of spin $\frac{1}{2}$, appearing in the 25^{th} line of Table 7.3 — are presented in the left and in the right column, respectively. Table is taken from [9]. Families belong to two groups of four families, one (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eq. (7.52). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (7.59). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$) (Eq. (7.59)) transform $\hat{u}^{\dagger}_{1R}$ to all the members of one family of the same colour. The same generators transform equivalently the right handed neutrino $\hat{\nu}^{\dagger}_{1R}$ to all the colourless members of the same family.

state	S^{03}	S^{12}	τ^{13}	τ^{23}	τ^4	Y	Q	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{\tau}^4$	$\tilde{Y}$	$\tilde{Q}$	$\tilde{N}_L^3$	$\tilde{N}_R^3$
$(\vert\nu_{1R}^{VIII}>_1\ \vert\nu_{2R}^{VIII}>_2)$	0	0	0	1	−1	0	0	0	1	−1	0	0	0	1
$(\vert\nu_{1R}^{VIII}>_1\ \vert e_{2R}^{VIII}>_2)$	0	0	0	0	−1	−1	−1	0	1	−1	0	0	0	1
$(\vert e_{1R}^{VIII}>_1\ \vert e_{2R}^{VIII}>_2)$	0	0	0	−1	−1	−2	−2	0	1	−1	0	0	0	1

Table 7.5. The condensate of the two right handed neutrinos ν_R, with the quantum numbers of the VIII^{th} family, coupled to spin zero and belonging to a triplet with respect to the generators τ^{2i}, is presented, together with its two partners. The condensate carries $\vec{\tau}^1 = 0$, $\tau^{23} = 1$, $\tau^4 = -1$ and $Q = 0 = Y$. The triplet carries $\tilde{\tau}^4 = -1$, $\tilde{\tau}^{23} = 1$ and $\tilde{N}_R^3 = 1$, $\tilde{N}_L^3 = 0$, $\tilde{Y} = 0$, $\tilde{Q} = 0$. The family quantum numbers of quarks and leptons are presented in Table 7.4.

7.54, 7.58)

$$\begin{aligned}
\sum_{s',s''} -\frac{1}{2} S^{s's''}\,\omega_{s's''s} &= -(g^{23}\,\tau^{23}\,A^{23}_s + g^{13}\,\tau^{13}\,A^{13}_s + g^4\,\tau^4\,A^4_s)\,,\\
g^{13}\,\tau^{13}\,A^{13}_s + g^{23}\,\tau^{23}\,A^{23}_s + g^4\,\tau^4\,A^4_s &= g^Q\,QA^Q_s + g^{Q'}\,Q'A^{Q'}_s + g^{Y'}\,Y'\,A^{Y'}_s\,,\\
A^4_s &= -(\omega_{9\,10\,s} + \omega_{11\,12\,s} + \omega_{13\,14\,s})\,,\\
A^{13}_s &= (\omega_{56s} - \omega_{78s})\,,\quad A^{23}_s = (\omega_{56s} + \omega_{78s})\,,\\
A^Q_s &= \sin\vartheta_1\,A^{13}_s + \cos\vartheta_1\,A^Y_s\,,\\
A^{Q'}_s &= \cos\vartheta_1\,A^{13}_s - \sin\vartheta_1\,A^Y_s\,,\\
A^{Y'}_s &= \cos\vartheta_2\,A^{23}_s - \sin\vartheta_2\,A^4_s\,,\\
&\quad (s \in (7,8))\,.
\end{aligned} \tag{7.61}$$

Scalar fields from Eq. (7.60) interact with quarks and leptons and antiquarks and antileptons through the family quantum numbers, while those from Eq. (7.61) interact through the family members quantum numbers. In Eq. (7.61) the coupling constants are explicitly written in order to see the analogy with the gauge fields of the *standard model*.

Expressions for the vector gauge fields in terms of the spin connection fields and the vielbeins, which correspond to the colour charge ($\vec{A}^3_m$), the $SU(2)_{II}$ charge ($\vec{A}^2_m$), the weak $SU(2)_I$ charge ($\vec{A}^1_m$) and the $U(1)$ charge originating in $SO(6)$ ($\vec{A}^4_m$), can be found by taking into account Eqs. (7.53, 7.54). Equivalently one finds the vector gauge fields in the "tilde" sector, or one just uses the expressions from Eqs. (7.61, 7.60), if replacing the scalar index s with the vector index m.

The expression for $\sum_{tab} \gamma^t\ \frac{1}{2}\,\tilde{S}^{ab}\,\tilde{\omega}_{abt}$, used in Eq. (7.45) ($\tilde{S}^{ab}$ are the infinitesimal generators of either $\widetilde{SO}(3,1)$ or $\widetilde{SO}(4)$, while $\tilde{\omega}_{abt}$ belong to the corresponding gauge fields with $t = (9,\ldots,14)$), and obtained by using Eqs. (7.55 -

7.59), are

$$\begin{aligned}
&\sum_{abt} \gamma^t \frac{1}{2} \tilde{S}^{ab}\, \tilde{\omega}_{abt} = \sum_{+-tt'ab} \overset{tt'}{(\oplus)}\, \frac{1}{2} \tilde{S}^{ab}\, \tilde{\omega}_{ab\, \underset{(\oplus)}{tt'}} = \\
&\sum_{+-tt'} \overset{tt'}{(\oplus)}\, \{ \tilde{\tau}^{2+}\, \tilde{A}^{2+}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{2-}\, \tilde{A}^{2-}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{23}\, \tilde{A}^{23}_{\underset{(\oplus)}{tt'}} + \\
&\tilde{\tau}^{1+}\, \tilde{A}^{1+}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{1-}\, \tilde{A}^{1-}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{13}\, \tilde{A}^{13}_{\underset{(\oplus)}{tt'}} + \\
&\tilde{N}^{+}_{R}\, \tilde{A}^{N_R+}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{-}_{R}\, \tilde{A}^{N_R-}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{3}_{R}\, \tilde{A}^{N_R3}_{\underset{(\oplus)}{tt'}} + \\
&\tilde{N}^{+}_{L}\, \tilde{A}^{N_L+}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{-}_{L}\, \tilde{A}^{N_L-}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{3}_{L}\, \tilde{A}^{N_L3}_{\underset{(\oplus)}{tt'}} \}, \\
&\tilde{A}^{N_R\boxplus}_{\underset{(\oplus)}{tt'}} = (\tilde{\omega}_{23\underset{(\oplus)}{tt'}} - i\, \tilde{\omega}_{01\underset{(\oplus)}{tt'}}) \boxplus i(\tilde{\omega}_{31\underset{(\oplus)}{tt'}} - i\, \tilde{\omega}_{02\underset{(\oplus)}{tt'}}), \\
&\tilde{A}^{N_R3}_{\underset{(\oplus)}{tt'}} = (\tilde{\omega}_{12\underset{(\oplus)}{tt'}} - i\, \tilde{\omega}_{03\underset{(\oplus)}{tt'}}), \\
&\tilde{A}^{N_L\boxplus}_{\underset{(\oplus)}{tt'}} = (\tilde{\omega}_{23\underset{(\oplus)}{tt'}} + i\, \tilde{\omega}_{01\underset{(\oplus)}{tt'}}) \boxplus i(\tilde{\omega}_{31\underset{(\oplus)}{tt'}} + i\, \tilde{\omega}_{02\underset{(\oplus)}{tt'}}), \\
&\tilde{A}^{N_R3}_{\underset{(\oplus)}{tt'}} = (\tilde{\omega}_{12\underset{(\oplus)}{tt'}} + i\, \tilde{\omega}_{03\underset{(\oplus)}{tt'}}).
\end{aligned} \tag{7.62}$$

The term $\sum_{tt't''} \gamma^t\, \frac{1}{2}\, S^{t't"}\, \omega_{t't"t}$ in Eq. (7.27) can be rewritten with respect to the generators $S^{t't"}$ and the corresponding gauge fields $\omega_{s's"t}$ as one colour octet scalar field and one $U(1)_{II}$ singlet scalar field (Eq. 7.54)

$$\begin{aligned}
\gamma^t\, \frac{1}{2}\, S^{t"t'"}\, \omega_{t"t'"t} &= \sum_{+,-} \sum_{(t\,t')} \overset{tt'}{(\oplus)}\, \{ \vec{\tau}^3 \cdot \vec{A}^3_{\underset{(\oplus)}{tt'}} + \tau^4 \cdot A^4_{\underset{(\oplus)}{tt'}} \}, \\
&(t\,t') \in ((9\,10), 11\,12), 13\,14))\,.
\end{aligned} \tag{7.63}$$

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8 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space, New Way of Second Quantization of Fermions — Part I

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Abstract. Both algebras, Clifford and Grassmann, offer ″basis vectors″ for describing the internal degrees of freedom of fermions [5, 6, 12]. The oddness of the ″basis vectors″, transfered to the creation operators, which are tensor products of the finite number of ″basis vectors″ and the infinite number of momentum basis, and to their Hermitian conjugated partners annihilation operators, offers the second quantization of fermions without postulating the conditions proposed by Dirac [1–3], enabling the explanation of the Dirac's postulates. But while the Clifford fermions manifest the half integer spins — in agreement with the observed properties of quarks and leptons and antiquarks and antileptons — the ″Grassmann fermions″ manifest the integer spins. In Part I properties of the creation and annihilation operators of integer spins ″Grassmann fermions″ are presented and the proposed equations of motion solved. The anticommutation relations of second quantized integer spin fermions are shown when applying on the vacuum state as well as when applying on the Hilbert space of the infinite number of ″Slater determinants″ with all the possibilities of empty and occupied ″fermion states″. In Part II the conditions are discussed under which the Clifford algebras offer the appearance of the second quantized fermions, enabling as well the appearance of families. In both parts, Part I and Part II, the relation between the Dirac way and our way of the second quantization of fermions is presented.

Povzetek. Avtorja obravnavata Cliffordovo in Grassmannovo algebro. Obe ponudita ″bazne vektorje″ za opis notranjega prostora fermionov [5, 6, 12]. ″Bazni vektorji″, ki antikomutirajo, poskrbijo za antikomutacijske lastnosti kreacijskih operatorjev, ki so tenzorski produkti končnega števila teh ″baznih vektorjev″ in neskončnega števila vektorjev običajnega prostora ter njihovih hermitsko konjugiranih anihilacijskih operatorjev. Antikomutatorji teh kreacijskih in anihilacijskih operatorjev izpolnjujejo vse pogoje, ki jih za drugo kvantizacijo fermionov postulira Dirac [1–3]. Predlagana pot avtorjev do druge kvantizacije stanj fermionskih polj pojasni Diracove postulate druge kvantizacije. Cliffordovi fermioni nosijo polceloštevilski spin — kar se ujema z opaženimi lastnostmi kvarkov in leptonov ter antikvarkov in antileptonov — "Grassmannovi fermioni" pa nosijo celoštevilski spin. Prvi del članka predstavi lastnosti kreacijskih in anihilacijskih operatorjev za ″Grassmannove fermione″, ko delujejo na vakuumsko stanje in tudi, ko delujejo na neskončno število "Slaterjevih determinant" ″Grassmannovih fermionskih″ stanj vsemi možnostmi zasedenosti teh stanj. V drugem delu obravnavata avtorja pogoje, pri katerih Cliffordove algebre ponudijo opis fermionov v drugi kvantizaciji hkrati s pojavom družin fermionov. V obeh delih primerjata Diracovo pot z njuno potjo do druge kvantizacije fermionov.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Explanation of the Dirac postulates, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

8.1 Introduction

In a long series of works we, mainly one of us N.S.M.B. ([5–12, 15] and the references therein), have found phenomenological success with the model named by N.S.M.B the *spin-charge-family* theory, with fermions, the internal degrees of freedom of which is describable with the Clifford algebra of all linear combinations of products of γ^a's in $d = (13 + 1)$ (may be with d infinity), interacting with only gravity. The spins of fermions from higher dimensions, $d > (3 + 1)$, manifest in $d = (3 + 1)$ as charges of the *standard model*, gravity in higher dimensions manifest as the *standard model* gauge vector fields as well as the scalar Higgs and Yukawa couplings.

There are two anticommuting kinds of algebras, the Grassmann algebra and the Clifford algebra (of two independent subalgebras), expressible with each other. The Grassmann algebra, with elements θ^a, and their Hermitian conjugated partners $\frac{\partial}{\partial\theta^a}$ [12], can be used to describe the internal space of fermions with the integer spins and charges in the adjoint representations, the two Clifford algebras, we call their elements γ^a and $\tilde{\gamma}^a$, can each of them be used to describe half integer spins and charges in fundamental representations. The Grassmann algebra is equivalent to the two Clifford algebras and opposite.

The two papers explain how do the oddness of the internal space of fermions manifests in the single particle wave functions, relating the oddness of the wave functions to the corresponding creation and annihilation operators of the second quantized fermions, in the Grassmann case and in the Clifford case, explaining therefore the postulates of Dirac for the second quantized fermions. We also show that the requirement that the Clifford odd algebra represents the observed quarks and leptons and antiquarks and antileptons reduces the Clifford algebra for the factor of two, reducing at the same time the Grassmann algebra, disabling the possibility for the integer spin fermions.

In this paper it is demonstrated how do the Grassmann algebra — in Part I — and the two kinds of the Clifford algebras — in Part II — if used to describe the internal degrees of freedom of fermions, take care of the second quantization of fermions without postulating anticommutation relations [1–3]. Either the odd Grassmann algebra or the odd Clifford algebra offer namely the appearance of the creation operators, defined on the tensor products of the "basis vectors" of the internal space and of the momentum space basis. These creation operators, together with their Hermitian conjugated partners anihilation operators, inherit oddness from the "basis vectors" determined by the odd Grassmann or the odd Clifford algebras, fulfilling correspondingly, the anticommutation relations postulated by Dirac for the second quantized fermions, if they apply on the corresponding vacuum state, Eq. (8.7) (defined by the sum of products of all the annihilation times the corresponding Hermitian conjugated creation operators). Oddness of the "basis vectors", describing the internal space of fermions, guarantees the oddness of all the objects entering the tensor product.

In d-dimensional Grassmann space of anticommuting coordinates θ^a's, $i = (0, 1, 2, 3, 5, \cdots, d)$, there are 2^d "basis vectors", which are superposition of products of θ^a. One can arrange them into the odd and the even irreducible representations with respect to the Lorentz group. There are as well derivatives with respect to θ^a's, $\frac{\partial}{\partial \theta_a}$'s, taken in Ref. [12] as, up to a sign, Hermitian conjugated to θ^a's, ($\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial \theta_a}$, $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$), which form again 2^d "basis vectors". Again half of them odd and half of them even (the odd Hermitian conjugated to odd products of θ^a's, the even Hermitian conjugated to the even products of θ^a's). Grassmann space offers correspondingly $2 \cdot 2^d$ degrees of freedom.

There are two kinds of the Clifford "basis vectors", which are expressible with θ^a and $\frac{\partial}{\partial \theta_a}$: $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$ [6,13,14]. They are, up to η^{aa}, Hermitian operators. Each of these two kinds of the Clifford algebra objects has 2^d operators. "Basis vectors" of Clifford algebra have together again $2 \cdot 2^d$ degrees of freedom.

There is the *odd algebra* in all three cases, θ^a's, γ^a's, $\tilde{\gamma}^a$'s, which if used to generate the creation and annihilation operators for fermions, and correspondingly the single fermion states, leads to the Hilbert space of second quantized fermions obeying the anticommutation relations of Dirac [1] without postulating these relations: the anticommutation properties follow from the oddness of the "basis vectors" in any of these algebras.

Let us present steps which lead to the second quantized fermions:

i. The internal space of a fermion is described by either Clifford or Grassmann algebra of an odd Clifford character (superposition of an odd number of Clifford "coordinates" (operators) γ^a's or of an odd number of Clifford "coordinates" (operators) $\tilde{\gamma}^a$'s) or of an odd Grassmann character (superposition of an odd number of Grassmann "coordinates" (operators) θ^a's).

ii. The eigenvectors of all the (chosen) Cartan subalgebra members of the corresponding Lorentz algebra are used to define the "basis vectors" in the odd part of internal space of fermions. (The Cartan subalgebra is in all three cases chosen in the way to be in agreement with the ordinary choice.) The algebraic application of this "basis vectors" on the corresponding vacuum state (either Clifford $|\psi_{oc}>$, defined in Eq. (18) of Part II, or Grassmann $|\phi_{og}>$, Eq. (8.7), which is in the Grassmann case just the identity) generates the "basis states", describing the internal degrees of freedom of fermions. The members of the "basis vectors" manifest together with their Hermitian conjugated partners properties of creation and annihilation operators which anticommute, Eq. (8.11) in Part I and Eq. (18) in Part II, when applying on the corresponding vacuum state, due to the algebraic properties of the odd products of the algebra elements.

iii. The plane wave solutions of the corresponding Weyl equations (either Clifford, Eq. (23) or Grassmann, Eq. (8.21)) for free massless fermions are the tensor products of the superposition of the members of the "basis vectors" and of the momentum basis. The coefficients of the superposition correspondingly depend on a chosen momentum $\vec{p}$, with $|p^0| = |\vec{p}|$, for any of continuous many moments $\vec{p}$.

iv. The creation operators defined on the tensor products, $*_T$, of superposition of finite number of "basis vectors" defining the final internal space and of the infinite (continuous) momentum space, Eq. (24) in the Clifford case and Eq. (8.22) in the Grassmann case, have infinite basis.

v. Applied on the vacuum state these creation operators form anticommuting single fermion states of an odd Clifford/Grassmann character.

vi. The second quantized Hilbert space $\mathcal{H}$ consists of "Slater determinants" with no single particle state occupied (with no creation operators applying on the vacuum state), with one single particle state occupied (with one creation operator applying on the vacuum state), with two single particle states occupied (with two creation operator applying on the vacuum state), and so on. "Slater determinants" can as well be represented as the tensor product multiplication of all possible single particle states of any number.

vii. The creation operators together with their Hermitian conjugated partners annihilation operators fulfill, due to the oddness of the "basis vectors", while the momentum part commutes, the anticommutation relations, postulated by Dirac for second quantized fermion fields, not only when they apply on the vacuum state, but also when they apply on the Hilbert space $\mathcal{H}$, Eq. (39) in the Clifford case and Eq. (8.34) in the Grassmann case. In the Clifford case this happens only after "freezing out" half of the Clifford space, as it is shown in Part II, Sect. 2.2, what brings besides the correct anticommutation relations also the "family" quantum number to each irreducible representation of the Lorentz group of the remaining internal space.

The oddness of the creation operators forming the single fermion states of an odd character, transfers to the application of these creation operators on the Hilbert space of the second quantized fermions in the Clifford and in the Grassmann case.

viii. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac's postulates for the second quantized fermions.

In the subsection 8.1.1 of this section we discuss in a generalized way our assumption, that the oddness of the "basis vectors" in the internal space transfer to the corresponding creation and annihilation operators determining the second quantized single fermion states and correspondingly the Hilbert space of the second quantized fermions.

We present in Sect. 8.2 properties of the Grassmann odd (as well as, for our study of anticommuting "Grassmann fermions" not important, the Grassmann even) algebra and of the chosen "basis vectors" for even ($d = (2(2n+1), 4n)$, n is an integer) dimensional space-time, $d = (d-1)+1$, and illustrate anticommuting "basis vectors" on the case of $d = (5+1)$, Subsect. 8.2.1, chapter *A.b.*.

We define the action for the integer spin "Grassmann fermions" in Subsect. 8.2.2. Solutions of the corresponding equations of motion, which are the tensor products of finite number of "basis vectors" and of infinite number of basis in momentum space, define the creation operators depending on internal quantum numbers and on $\vec{p}$ in d-dimensional space-time. We illustrate the corresponding superposition of ”basis vectors”, solving the equation of motion in $d = (5+1)$ in chapter *B.a.*.

We present in Sect. 8.3 the Hilbert space $\mathcal{H}$ of the tensor multiplication of one fermion creation operators of all possible single particle states of an odd character and of any number, representing ”Slater determinants” with no ”Grassmann fermion” state occupied with ”Grassmann fermions”, with one ”Grassmann fermion” state occupied, with two ”Grassmann fermion” states occupied, up to the ”Slater determinant” with all possible ”Grassmann fermion” states of each of infinite number of momentum $\vec{p}$ occupied. The Hilbert space $\mathcal{H}$ is the tensor product $\prod_{\infty} \otimes_N$ of finite number of $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$, for (continues) infinite possibilities for $\vec{p}$.

On $\mathcal{H}$ the creation and annihilation operators manifest the anticommutation relations of second quantized ”fermions” without any postulates. These second quantized ”fermion” fields, manifesting in the Grassmann case an integer spin, offer in d-dimensional space, $d > (3+1)$, the description of the corresponding charges in adjoint representations. We follow in this paper to some extent Ref. [12].

In Subsect. 8.3.3 relation between the by Dirac postulated creation and annihilation operators and the creation and annihilation operators presented in this Part I — for integer spins ”Grassmann fermions” — are discussed.

In Sect. 8.4 we comment on what we have learned from the second quantized ”Grassmann fermion” fields with integer spin when internal degrees of freedom are described with Grassmann algebra and compare these recognitions with the recognitions, which the Clifford algebra is offering, discussions on which appear in Part II.

In Part II we present in equivalent sections properties of the two kinds of the Clifford algebras and discuss conditions under which odd products of odd elements (operators), γ^a and $\tilde{\gamma}^a$'s of the two Clifford algebras, demonstrate the anticommutation relations required for the second quantized fermion fields on the Hilbert space $\mathcal{H} = \prod_{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, this time with the half integer spin, offering in d-dimensional space, $d > (3+1)$, the description of charges, as well as the appearance of families of fermions [12], both needed to describe the properties of the observed quarks and leptons and antiquarks and antileptons, appearing in families.

In Part II we discuss relations between the Dirac way of second quantization with postulates and our way using Clifford algebra.

This paper is a part of the project named the *spin-charge-family* theory of one of the authors (N.S.M.B.), so far offering the explanation for all the assumptions of the *standard model*, with the appearance of the scalar fields included.

The Clifford algebra offers in even d-dimensional spaces, $d \geq (13+1)$ indeed, the description of the internal degrees of freedom for the second quantized fermions with the half integer spins, explaining all the assumptions of the *standard model*: The appearance of charges of the observed quarks and leptons and their families, as well as the appearance of the corresponding gauge fields, the scalar fields, explaining the Higgs scalar and the Yukawa couplings, and in addition the appearance of the dark matter, of the matter/antimatter asymmetry, offering several predictions [5–11, 15, 16].

8.1.1 Our main assumption and definitions

In this subsection we clarify how does the main assumption of Part I and Part II, *the decision to describe the internal space of fermions with the "basis vectors" expressed with the superposition of odd products of the anticommuting members of the algebra*, either the Clifford one or the Grassmann one, acting algebraically, $*_A$, on the internal vacuum state $|\psi_o>$, relate to the creation and annihilation anticommuting operators of the second quantized fermion fields.

To appreciate the need for this kind of assumption, let us first have in mind that algebra with the product $*_A$ is only present in our work, usually not in other works, and thus has no well known physical meaning. It is at first a product by which you can multiply two internal wave functions B_i and B_j with each other,

$$C_k = B_i *_A B_j \,,$$
$$B_i *_A B_j = \mp B_j *_A B_i \,,$$

the sign $\mp$ depends on whether B_i and B_j are products of odd or even number of algebra elements: The sign is $-$ if both are (superposition of) odd products of algebra elements, in all other cases the sign is $+$.

Let $\mathbf{R}^{d-1}$ define the external spatial or momentum space. Then the tensor product $*_T$ extends the internal wave functions into the wave functions $\mathbf{C}_{\vec{p},\,i}$ defined in both spaces

$$\mathbf{C}_{\vec{p},\,i} = |\vec{p}> *_T |B_i>\,,$$

where again B_i represent the superposition of products of elements of the anticommuting algebras, in our case either θ^a or γ^a or $\tilde{\gamma}^a$, used in this paper.

We can make a choice of the orthogonal and normalized basis so that $<\mathbf{C}_{\vec{p},i}|\mathbf{C}_{\vec{p'},j}> = \delta(\vec{p}-\vec{p'})\,\delta_{ij}$. Let us point out that either B_i or $\mathbf{C}_{\vec{p},\,i}$ apply algebraically on the vacuum state, $B_i *_A |\psi_o>$ and $\mathbf{C}_{\vec{p},\,i} *_A |\psi_o>$.

Usually a product of single particle wave functions is not taken to have any physical meaning in as far as most physicists simply do not work with such products at all.

To give to the algebraic product, $*_A$, and to the tensor product, $*_T$, defined on the space of single particle wave functions, the physical meaning, we postulate the connection between the anticommuting/commuting properties of the "basis vectors", expressed with the odd/even products of the anticommuting algebra elements and the corresponding creation operators, creating second quantized single fermion/boson states

$$
\begin{aligned}
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_A |\psi_o> = |\psi_{\vec{p},i}>\,,\\
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_T |\psi_{\vec{p'},j}> = 0\,,\\
&\qquad\qquad \text{if } \vec{p} = \vec{p'} \text{ and } i = j\,,\\
&\text{in all other cases} \quad \text{it follows}\\
&\hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},\,j}} *_A |\psi_o> = \mp\, \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p'},\,j}} *_T \hat{b}^{\dagger}_{\mathbf{C}_{\vec{p},\,i}} *_A |\psi_o>\,,
\end{aligned}
$$

with the sign $\pm$ depending on whether $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ have both an odd character, the sign is $-$, or not, then the sign is $+$.

To each creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ its Hermitian conjugated partner represents the annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$

$$\begin{aligned}\hat{b}_{\mathbf{c}_{\vec{p}},i} &= (\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i})^{\dagger}\,,\\ \text{with the} \quad &\text{property}\\ \hat{b}_{\mathbf{c}_{\vec{p}},i} *_A |\psi_o> &= 0\,,\\ \text{defining the} \quad &\text{vacuum state as}\\ |\psi_o>: &= \sum_i (B_i)^{\dagger} *_A B_i \,|\, I>\end{aligned}$$

where summation i runs over all different products of annihilation operator $\times$ its Hermitian conjugated creation operator, no matter for what $\vec{p}$, and $|\, I>$ represents the identity, $(B_i)^{\dagger}$ represents the Hermitian conjugated wave function to B_i.

Let the tensor multiplication $*_T$ denotes also the multiplication of any number of single particle states, and correspondingly of any number of creation operators.

What further means that to each single particle wave function we define the creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$, applying in a tensor product from the left hand side on the second quantized Hilbert space — consisting of all possible products of any number of the single particle wave functions — adding to the Hilbert space the single particle wave function created by this particular creation operator. In the case of the second quantized fermions, if this particular wave function with the quantum numbers and $\vec{p}$ of $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ is already among the single fermion wave functions of a particular product of fermion wave functions, the action of the creation operator gives zero, otherwise the number of the fermion wave functions increases for one. In the boson case the number of boson second quantized wave functions increases always for one.

If we apply with the annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$ on the second quantized Hilbert space, then the application gives a nonzero contribution only if the particular products of the single particle wave functions do include the wave function with the quantum number i and $\vec{p}$.

In a Slater determinant formalism the single particle wave functions define the empty or occupied places of any of infinite numbers of Slater determinants. The creation operator $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ applies on a particular Slater determinant from the left hand side. Jumping over occupied states to the place with its i and $\vec{p}$. If this state is occupied, the application gives in the fermion case zero, in the boson case the number of particles increase for one. The particular Slater determinant changes sign in the fermion case if $\hat{b}^{\dagger}_{\mathbf{c}_{\vec{p}},i}$ jumps over odd numbers of occupied states. In the boson case the sign of the Slater determinant does not change.

When annihilation operator $\hat{b}_{\mathbf{c}_{\vec{p}},i}$ applies on particular Slater determinant, it is jumping over occupied states to its own place. giving zero, if this space is empty and decreasing the number of occupied states of this space is occupied. The Slater determinant changes sign in the fermion case, if the number of occupied states before its own space is odd. In the boson case the sign does not change.

Let us stress that choosing antisymmetry or symmetry is a choice which we make when treating fermions or bosons, respectively, namely the choice of using oddness or evenness of basis vectors, that is the choice of using odd products or even products of algebra anticummuting elements.

To describe the second quantized fermion states we make a choice of the basis vectors, which are the superposition of the odd numbers of algebra elements, of both Clifford and Grassmann algebras.

The creation operators and their Hermitian conjugation partners annihilation operators therefore in the fermion case anticommute. The single fermion states, which are the application of the creation operators on the vacuum state $|\psi_o>$, manifest correspondingly as well the oddness. The vacuum state, defined as the sum over all different products of annihilation $\times$ the corresponding creation operators, have an even character.

Let us end up with the recognition:

One usually means antisymmetry when talking about Slater-determinants because otherwise one would not get determinants.

In the present paper [5–7, 13] the choice of the symmetrizing versus antisymmetrizing relates indeed the commutation versus anticommutation with respect to the a priori completely different product $*_A$, of anticommuting members of the Clifford or Grassmann algebra. The oddness or evenness of these products transfer to quantities to which these algebras extend.

8.2 Properties of Grassmann algebra in even dimensional spaces

In Grassmann d-dimensional space there are d anticommuting operators θ^a, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,,$$
$$\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}\,, (a, b) = (0, 1, 2, 3, 5, \cdots, d)\,. \tag{8.1}$$

Defining [12]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,,$$
it follows
$$(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,. \tag{8.2}$$

The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$ is assumed.

It appears useful to arrange 2^d products of θ^a into irreducible representations with respect to the Lorentz group with the generators [6]

$$\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,. \tag{8.3}$$

2^{d-1} members of the representations have an odd Grassmann character (those which are superposition of odd products of θ^a's). All the members of any particular odd irreducible representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

If we exclude the self adjoint identity there is $(2^{d-1}-1)$ members of an even Grassmann character, they are even products of θ^a's. All the members of any particular even representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

The Hermitian conjugated 2^{d-1} odd partners of odd representations of θ^a's and $(2^{d-1}-1)$ even partners of even representations of θ^a's are reachable from odd and even representations, respectively, by the application of Eq. (8.2).

It appears useful as well to make the choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra as follows

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\;d}\,, \tag{8.4}$$

and choose the members of the irreducible representations of the Lorentz group to be the eigenvectors of all the members of the Cartan subalgebra of Eq. (8.4)

$$\begin{aligned}
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) &= k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,\\
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) &= 0\,,\\
&\text{or}\\
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,\frac{i}{k}\theta^a\theta^b &= 0\,,
\end{aligned} \tag{8.5}$$

with $k^2 = \eta^{aa}\eta^{bb}$. The eigenvector $\frac{1}{\sqrt{2}}\,(\theta^0 \mp \theta^3)$ of $\mathbf{S}^{03}$ has the eigenvalue $k = \pm i$, the eigenvalues of all the other eigenvectors of the rest of the Cartan subalgebra members, Eq. (8.4), are $k = \pm 1$.

We choose the "basis vectors" to be products of odd nilpotents $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and of even objects $\frac{i}{k}\theta^a\theta^b$, with eigenvalues $k = \pm i$ and 0, respectively.

Let us check how does $\mathbf{S}^{ac} = i(\theta^a \frac{\partial}{\partial\theta_c} - \theta^c \frac{\partial}{\partial\theta_a})$ transform the product of two "nilpotents" $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and $\frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d)$. Taking into account Eq. (8.3) one finds that $\mathcal{S}^{ac}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,\frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d) = -\frac{\eta^{aa}\eta^{cc}}{2k}\,(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)$. $\mathcal{S}^{ac}$ transforms the product of two Grassmann odd eigenvectors of the Cartan subalgebra into the superposition of two Grassmann even eigenvectors.

"Basis vectors" have an odd or an even Grassmann character, if their products contain an odd or an even number of "nilpotents", $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, respectively. "Basis vectors" are normalized, up to a phase, in accordance with Eq. (8.38) of 8.5.

The Hermitian conjugated representations of (either an odd or an even) products of θ^a's can be obtained by taking into account Eq. (8.2) for each "nilpotent"

$$\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)^\dagger = \eta^{aa}\,\frac{1}{\sqrt{2}}(\frac{\partial}{\partial\theta_a} + \frac{\eta^{aa}}{-ik}\frac{\partial}{\partial\theta_b})\,,$$
$$(\frac{i}{k}\,\theta^a\theta^b)^\dagger = \frac{i}{k}\,\frac{\partial}{\partial\theta_a}\,\frac{\partial}{\partial\theta_b}\,. \tag{8.6}$$

Making a choice of the identity for the vacuum state,

$$|\phi_{og}> = |\,1>\,, \tag{8.7}$$

we see that algebraic products — we shall use a dot , $\cdot$, or without a dot for an algebraic product of eigenstates of the Cartan subalgebra forming "basis vectors" and $*_A$ for the algebraic product of "basis vectors" — of different θ^a's, if applied on such a vacuum state, give always nonzero contributions,

$$(\theta^0 \mp \theta^3)\cdot(\theta^1 \pm i\theta^2)\cdots(\theta^{d-1} \mp \theta^d)|\,1>\neq \text{zero},$$

(this is true also, if we substitute any of nilpotents $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ or all of them with the corresponding even operators $(\frac{i}{k}\,\theta^a\theta^b)$; in the case of odd Grassmann irreducible representations at least one nilpotent must remain). The Hermitian conjugated partners, Eq. (8.6), applied on $|\,1>$, give always zero

$$(\frac{\partial}{\partial\theta_0} \mp \frac{\partial}{\partial\theta_3})\cdot(\frac{\partial}{\partial\theta_1} \pm i\frac{\partial}{\partial\theta_2})\cdots(\frac{\partial}{\partial\theta_{d-1}} \pm i\frac{\partial}{\partial\theta_d})|\,1>= 0.$$

Let us notice the properties of the odd products θ^a's and of their Hermitian conjugated partners:

i. Superposition of products of different θ^a's, applied on the vacuum state $|\,1>$, give nonzero contribution. To create on the vacuum state the "fermion" states we make a choice of the "basis vectors" of the odd number of θ^a's, arranging them to be the eigenvectors of all the Cartan subalgebra elements, Eq. (8.4).

ii. The Hermitian conjugated partners of the "basis vectors", they are products of derivatives $\frac{\partial}{\partial\theta_a}$'s, give, when applied on the vacuum state $|\,1>$, Eq. (8.7), zero. Each annihilation operator annihilates the corresponding creation operator.

iii. The algebraic product, $*_A$, of a "basis vector" by itself gives zero, the algebraic anticommutator of any two "basis vectors" of an odd Grassmann character (superposition of an odd products of θ^a's) gives zero ("basis vectors" of the two decuplets in Table 8.1 and the "basis vector" of Eq. (8.13) $\frac{1}{2}(\theta^0 \mp \theta^3)$, for example, demonstrate this property).

iv. The algebraic application of any annihilation operator on the corresponding Hermitian conjugated "basis vector" gives identity, on all the rest of "basis vectors" gives zero. Correspondingly the algebraic anticommutators of the creation operators and their Hermitian conjugated partners, applied on the vacuum state, give identity, all the rest anticommutators of creation and annihilation operators applied on the vacuum state, give zero.

v. Correspondingly the "basis vectors" and their Hermitian conjugated partners, applied on the vacuum state $|\,1>$, Eq. (8.7), fulfill the properties of creation and annihilation operator, respectively, for the second quantized "fermions" on the level of one "fermion" state.

8.2.1 Grassmann "basis vectors"

We construct 2^{d-1} Grassmann odd "basis vectors" and $2^{d-1}-1$ (we skip self adjoint identity, which we use to describe the vacuum state $|1>$) Grassmann even "basis vectors" as superposition of odd and even products of θ^a's, respectively. Their Hermitian conjugated 2^{d-1} odd and $2^{d-1}-1$ even partners are, according to Eqs. (8.2, 8.6), determined by the corresponding superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, respectively [1].

A.a. Grassmann anticommuting "basis vectors" with integer spins

Let us choose in $d = 2(2n+1)$-dimensional space-time, n is a positive integer, the starting Grassmann odd "basis vector" $\hat{b}^{\theta 1\dagger}_1$, which is the eigenvector of the Cartan subalgebra of Eqs. (8.4, 8.5) with the egenvalues $(+i, +1, +1, \cdots, +1)$, respectively, and has the Hermitian conjugated partner equal to $(\hat{b}^{\theta 1\dagger}_1)^\dagger = \hat{b}^{\theta 1}_1$,

$$\begin{aligned}\hat{b}^{\theta 1\dagger}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \\ &\cdots (\theta^{d-1} + i\theta^d)\,, \\ \hat{b}^{\theta 1}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}} \left(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^d}\right)\cdots\left(\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3}\right).\end{aligned} \tag{8.8}$$

In the case of $d = 4n$, n is a positive integer, the corresponding starting Grassmann odd "basis vector" can be chosen as

$$\begin{aligned}\hat{b}^{\theta 1\dagger}_1 &:= \left(\frac{1}{\sqrt{2}}\right)^{\frac{d}{2}-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots \\ &\cdots(\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d\,.\end{aligned} \tag{8.9}$$

All the rest of "basis vectors", belonging to the same irreducible representation of the Lorentz group, follow by the application of $\mathbf{S}^{ab}$'s.

We denote the members i of this starting irreducible representation k by $\hat{b}^{\theta k\dagger}_i$ and their Hermitian conjugated partners by $\hat{b}^{\theta k}_i$, with $k = 1$.

"Basis vectors", belonging to different irreducible representations $k = 2$, will be denoted by $\hat{b}^{\theta 2\dagger}_j$ and their Hermitian conjugated partners by $\hat{b}^{\theta 2}_j = (\hat{b}^{\theta k\dagger}_j)^\dagger$.

$\mathbf{S}^{ac}$'s, which do not belong to the Cartan subalgebra, transform step by step the two by two "nilpotents", no matter how many "nilpotents" are between the chosen two, up to a constant, as follows:

$\mathbf{S}^{ac}\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\cdots\frac{1}{\sqrt{2}}(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d) \propto -\frac{\eta^{aa}\eta^{cc}}{2k}(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)\cdots$,

leaving at each step at least one "nilpotent" unchanged, so that the whole irreducible representation remains odd.

The superposition of $\mathbf{S}^{bd}$ and $i\mathbf{S}^{bc}$ transforms $-\frac{\eta^{aa}\eta^{cc}}{2k}(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)$ into $\frac{1}{\sqrt{2}}(\theta^a - \frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c - \frac{\eta^{cc}}{ik'}\theta^d)$, and not into $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c - \frac{\eta^{cc}}{ik'}\theta^d)$ or into $\frac{1}{\sqrt{2}}(\theta^a - \frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d)$.

[1] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case, where the Hermitian conjugated operators follow by taking into account Eq. (8.2). In the Clifford case $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is proportional to $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{i(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

Therefore we can start another odd representation with the "basis vector" $\hat{b}^{\theta 2\dagger}_{1}$ as follows

$$\hat{b}^{\theta 2\dagger}_{1} := (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \cdots (\theta^{d-1} + i\theta^d) ,$$
$$(\hat{b}^{\theta 2\dagger}_{1})^{\dagger} = \hat{b}^{\theta 1}_{2} := (\frac{1}{\sqrt{2}})^{\frac{d}{2}} (\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}}) \cdots (\frac{\partial}{\partial\theta^{0}} - \frac{\partial}{\partial\theta^{3}}) . \qquad (8.10)$$

The application of $\mathbf{S}^{ac}$'s determines the whole second irreducible representation $\hat{b}^{\theta 2\dagger}_{j}$.

One finds that each of these two irreducible representations has $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ members, Ref. [12].

Taking into account Eq. (8.1), it follows that odd products of θ^a's anticommute and so do the odd products of $\frac{\partial}{\partial\theta_a}$'s.

Statement 1: The oddness of the products of θ^a's guarantees the anticommuting properties of all objects which include odd number of θ^a's.

One further sees that $\frac{\partial}{\partial\theta^a}\theta^b = \eta^{ab}$, while $\frac{\partial}{\partial\theta_a}|1>= 0$, and $\theta^a|1>= \theta^a|1>$. and $\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A+}$ = We can therefore conclude

$$\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A+}|\, 1 > = \delta_{ij}\, \delta^{kl}\, |\, 1 > ,$$
$$\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l}_{j}\}_{*_A+}|\, 1 > = 0 \,\cdot |\, 1 > ,$$
$$\{\hat{b}^{\theta k\dagger}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A+}\, |\, 1 > = 0 \,\cdot |\, 1 > ,$$
$$\hat{b}^{\theta k}_{j} \, *_A \, |\, 1 > = 0 \,\cdot |\, 1 > , \qquad (8.11)$$

where $\{\hat{b}^{\theta k}_{i}, \hat{b}^{\theta l\dagger}_{j}\}_{*_A+} = \hat{b}^{\theta k}_{i} *_A \hat{b}^{\theta l\dagger}_{j} + \hat{b}^{\theta l}_{j} *_A \hat{b}^{\theta k\dagger}_{i}$ is meant.

These anticommutation relations of the "basis vectors" of the odd Grassmann character, manifest on the level of the Grassmann algebra the anticommutation relations required by Dirac [1] for second quantized fermions.

The "Grassmann fermion basis states" can be obtained by the application of creation operators $\hat{b}^{\theta k\dagger}_{i}$ on the vacuum state $|1>$

$$|\phi^{k}_{o\, i} > = \hat{b}^{\theta k\dagger}_{i}\, |1> . \qquad (8.12)$$

We use them to determine the internal space of "Grassmann fermions" in the tensor product $*_T$ of these "basis states" and of the momentum space, when looking for the anticommuting single particle "Grassmann states", which have, according to Eq. (8.5), an integer spin, and not half integer spin as it is the case for the so far observed fermions.

A.b. Illustration of anticommuting "basis vectors" in $d = (5+1)$*-dimensional space*

Let us illustrate properties of Grassmann odd representations for $d = (5+1)$-dimensional space.

Table 8.1 represents two decuplets, which are "egenvectors" of the Cartan subalgbra ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{56}$), Eq. (8.4), of the Lorentz algebra $\mathbf{S}^{ab}$. The two decuplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line of both decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the eigenvectors of $\mathbf{S}^{03}$ and $\mathbf{S}^{12}$ alone, while the eigenvector of $\mathbf{S}^{56}$ has, as a "spectator", the eigenvalue either $+1$ (the first triplet in both decuplets) or -1 (the second triplet in both decuplets). Each of the two decuplets contains also one "fourplet" with the "charge" $\mathbf{S}^{56}$ equal to zero $((7^{th}, 8^{th}, 9^{th}, 10^{th})$ lines in each of the two decuplets (Table II in Ref. [6])).

Paying attention on the eigenvectors of $\mathbf{S}^{03}$ alone one recognizes as well even and odd representations of $SO(1,1)$: $\theta^0\theta^3$ and $\theta^0 \pm \theta^3$, respectively.

The Hermitian conjugated "basis vectors" follow by using Eq. (8.6) and is for the first "basis vector" of Table 8.1 equal to $(-)^2(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$. One correspondingly finds that when $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ applies on $(\frac{1}{\sqrt{2}})^3(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)$ the result is identity. Application of $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ on all the rest of "basis vectors" of the decuplet I as well as on all the "basis vectors" of the decuplet II gives zero. "Basis vectors" are orthonormalized with respect to Eq. (8.38). Let us notice that $\frac{\partial}{\partial\theta_a}$ on a "state" which is just an identity, $|1>$, gives zero, $\frac{\partial}{\partial\theta_a}|1>= 0$, while $\theta^a|1>$, or any superposition of products of θ^a's, applied on $|1>$, gives the "vector" back.

One easily sees that application of products of superposition of θ^a's on $|1>$ gives nonzero contribution, while application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on $|1>$ gives zero.

The two by $\mathbf{S}^{ab}$ decoupled Grassmann decuplets of Table 8.1 are the largest two irreducible representations of odd products of θ^a's. There are 12 additional Grassmann odd "vectors", arranged into irreducible representations of six singlets and six sixplets

$$\begin{aligned}
&(\frac{1}{2}(\theta^0 \mp \theta^3), \frac{1}{2}(\theta^1 \pm i\theta^2), \frac{1}{2}(\theta^5 \pm i\theta^6),\\
&\frac{1}{2}(\theta^0 \mp \theta^3)\theta^1\theta^2\theta^5\theta^6, \frac{1}{2}(\theta^1 \pm i\theta^2)\theta^0\theta^3\theta^5\theta^6, \frac{1}{2}(\theta^5 \pm i\theta^6)\theta^0\theta^3\theta^1\theta^2). \qquad (8.13)
\end{aligned}$$

The algebraic application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on the corresponding Hermitian conjugated partners, which are products of superposition of θ^a's, leads to the identity for either even or odd Grassmann character [2].

Besides 32 Grassmann odd eigenvectors of the Grassmann Cartan subalgebra, Eq. (8.4), there are $(32-1)$ Grassmann "basis vectors", which we arrange into irreducible representations, which are superposition of even products of θ^a's. The even self adjoint operator identity (which is indeed the normalized product of all the annihilation times $*_A$ creation operators) is used to represent the vacuum state.

It is not difficult to see that Grassmann "basis vectors" of an odd Grassmann character anticommute among themselves and so do odd products of superposition of $\frac{\partial}{\partial\theta^a}$'s, while equivalent even products commute.

The Grassmann odd algebra (as well as the two odd Clifford algebras) offers, due to the oddness of the internal space giving oddness as well to the elements of the tensor products of the internal space and of the momentum space, the description of the anticommuting second quantized fermion fields, as postulated by Dirac. But the Grassmann "fermions"

---

[2] We shall see in Part II that the vacuum states are in the Clifford case, similarly as in the Grassmann case, for both kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s, sums of products of the annihilation $\times$ its Hermitian conjugated creation operators, and correspondingly self adjoint operators, but they are not the identity.

I	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1	1	1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	1	1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1	1	1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1	1	-1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	1	-1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1	1	-1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0	1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0	1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	1	0

II	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\gamma^{(5+1)}$	$\gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1	-1	-1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	-1	-1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1	-1	-1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1	-1	1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	-1	1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1	-1	1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0	-1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0	-1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	-1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	-1	0

Table 8.1. The two decuplets, the odd eigenvectors of the Cartan subalgebra, Eq. (8.4), ($\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}$, for $SO(5,1)$) of the Lorentz algebra in Grassmann $(5+1)$-dimensional space, forming two irreducible representations, are presented. Table is partly taken from Ref. [12]. The "basis vectors" within each decuplet are reachable from any member by $\mathbf{S}^{ab}$'s and are decoupled from another decuplet. The two operators of handedness, $\Gamma^{((d-1)+1)}$ for $d=(6,4)$, are invariants of the Lorentz algebra, Eq. (8.40), $\Gamma^{(5+1)}$ for the whole decuplet, $\Gamma^{(3+1)}$ for the "triplets" and "fourplets".

carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

A.c. Grassmann commuting "basis vectors" with integer spins

Grassmann even "basis vectors" manifest the commutation relations, and not the anticommutation ones as it is the case for the Grassmann odd "basis vectors". Let us use in the Grassmann even case, that is the case of superposition of an even number of θ^a's in $d=2(2n+1)$, the notation $\hat{a}^{\theta k\dagger}_j$, again chosen to be eigenvectors of the Cartan subalgebra, Eq. (8.4), and let us start with one representative

$$\hat{a}^{\theta 1\dagger}_j := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6) \cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,. \tag{8.14}$$

The rest of "basis vectors", belonging to the same Lorentz irreducible representation, follow by the application of $\mathbf{S}^{ab}$. The Hermitian conjugated partner of $\hat{a}^{\theta 1 \dagger}_1$ is $\hat{a}^{\theta 1}_1 = (\hat{a}^{\theta 1 \dagger}_1)^\dagger$

$$\hat{a}^{\theta 1}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} \frac{\partial}{\partial \theta^d} \frac{\partial}{\partial \theta^{d-1}} (\frac{\partial}{\partial \theta^{d-3}} - i \frac{\partial}{\partial \theta^{d-2}}) \cdots (\frac{\partial}{\partial \theta^0} - \frac{\partial}{\partial \theta^3}) \,. \tag{8.15}$$

If $\hat{a}^{\theta k \dagger}_j$ represents a Grassmann even creation operator, with index k denoting different irreducible representations and index j denoting a particular member of the k^{th} irreducible representation, while $\hat{a}^{\theta k}_j$ represents its Hermitian conjugated partner, one obtains by taking into account Sect. 8.2, the relations

$$\begin{aligned}
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k' \dagger}_j\}_{*_A -} |1> &= \delta_{ij}\, \delta^{kk'}\, |1>, \\
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k'}_j\}_{*_A -} |1> &= 0 \cdot |1>, \\
\{\hat{a}^{\theta k \dagger}_i, \hat{a}^{\theta k' \dagger}_j\}_{*_A -} |1> &= 0 \cdot |1>, \\
\hat{a}^{\theta k}_i *_A |1> &= 0 \cdot |1>, \\
\hat{a}^{\theta k \dagger}_i *_A |1> &= |\phi^k_{e\,i}> \,.
\end{aligned} \tag{8.16}$$

Equivalently to the case of Grassmann odd "basis vectors" also here $\{\hat{a}^{\theta k}_i, \hat{a}^{\theta l \dagger}_j\}_{*_A -} = \hat{a}^{\theta k}_i *_A \hat{a}^{\theta l \dagger}_j - \hat{a}^{\theta l}_j *_A \hat{a}^{\theta k \dagger}_i$ is meant.

8.2.2 Action for free massless "Grassmann fermions" with integer spin [12]

In the Grassmann case the "basis vectors" of an odd Grassmann character, chosen to be the eigenvectors of the Cartan subalgebra of the Lorentz algebra in Grassmann space, Eq. (8.4), manifest the anticommutation relations of Eq. (8.11) on the algebraic level.

To compare the properties of creation and annihilation operators for "integer spin fermions", for which the internal degrees of freedom are described by the odd Grassmann algebra, with the creation and annihilation operators postulated by Dirac for the second quantized fermions depending on the quantum numbers of the internal space of fermions and on the momentum space, we need to define the tensor product $*_T$ of the odd "Grassmann basis states", described by the superposition of odd products of θ^a's (with the finite degrees of freedom) and of the momentum (or coordinate) space (with the infinite degrees of freedom), taking as the basis the tensor product of both spaces.

Statement 2: For deriving the anticommutation relations for the "Grassmann fermions", to be compared to anticommutation relations of the second quantized fermions, we need to define the tensor product of the Grassmann odd "basis vectors" and the momentum space

$$\mathbf{basis}_{(p^a, \theta^a)} = |p^a> *_T |\theta^a> \,. \tag{8.17}$$

We need even more, we need to find the Lorentz invariant action for, let say, free massless "Grassmann fermions" to define such a "basis", that would manifest the relation $|p^0| = |\vec{p}|$. We follow here the suggestion of one of us (N.S.M.B.) from Ref. [12].

$$
\begin{aligned}
\mathcal{A}_G &= \int d^d x \; d^d\theta \; \omega \{\phi^\dagger \gamma^0_G \frac{1}{2}\,\theta^a p_a \phi\} + \text{h.c.}\,,\\
\omega &= \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k)\,,
\end{aligned}
\tag{8.18}
$$

with $\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta_a})$, $(\gamma^a_G)^\dagger = \gamma^a_G$, for each $a = (0, 1, 2, 3, 5, \cdots, d)$. We use the integral over θ^a coordinates with the weight function ω from Eq. (8.38, 8.39). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G, which takes care of the Lorentz invariance. Namely

$$
\begin{aligned}
\mathbf{S}^{ab\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\,\mathbf{S}^{ab}\,,\\
\mathbf{S}^{\dagger}\,(1 - 2\theta^0 \frac{\partial}{\partial \theta^0}) &= (1 - 2\theta^0 \frac{\partial}{\partial \theta^0})\,\mathbf{S}^{-1}\,,\\
\mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab} + \mathbf{S}^{ab})}\,,
\end{aligned}
\tag{8.19}
$$

while θ^a, $\frac{\partial}{\partial \theta_a}$ and p^a transform as Lorentz vectors.

The Lagrange density is up to the surface term equal to [3]

$$
\begin{aligned}
\mathcal{L}_G &= \frac{1}{2}\phi^\dagger \gamma^0_G (\theta^a - \frac{\partial}{\partial \theta_a})\,(\hat{p}_a \phi)\\
&= \frac{1}{4}\{\phi^\dagger \gamma^0_G\,(\theta^a - \frac{\partial}{\partial \theta_a})\,\hat{p}_a \phi - \\
&\quad (\hat{p}_a \phi^\dagger)\gamma^0_G (\theta^a - \frac{\partial}{\partial \theta_a})\phi\}\,,
\end{aligned}
\tag{8.20}
$$

leading to the equations of motion [4]

$$
\frac{1}{2}\gamma^0_G\,(\theta^a - \frac{\partial}{\partial \theta_a})\,\hat{p}_a\,|\phi> \,= 0\,,
\tag{8.21}
$$

as well as the the "Klein-Gordon" equation,

$$
(\theta^a - \frac{\partial}{\partial \theta_a})\,\hat{p}_a\,(\theta^b - \frac{\partial}{\partial \theta_b})\,\hat{p}_b\,|\phi> = 0 = \hat{p}_a \hat{p}^a\,|\phi>\,.
$$

The eigenstates ϕ of equations of motion for free massless "Grassmann fermions", Eq. (8.21), can be found as the tensor product, Eq.(8.17) of the superposition of 2^{d-1} Grassmann odd "basis vectors" $\hat{b}^{\theta k\dagger}_i$ and the momentum space, represented by plane waves, applied on the vacuum state $|\,1>$. Let us remind

---

[3] Taking into account the relations $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$, $\gamma^0_G = -i\eta^{aa}\gamma^a\tilde{\gamma}^a$ the Lagrange density can be rewritten as $\mathcal{L}_G = -i\frac{1}{2}\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,(\hat{p}_a\phi) = -i\frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\phi - \hat{p}_a\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\phi\}$.

[4] Varying the action with respect to $\phi^\dagger$ and ϕ it follows: $\frac{\partial \mathcal{L}_G}{\partial \phi^\dagger} - \hat{p}_a\,\frac{\partial \mathcal{L}_G}{\partial \hat{p}_a \phi^\dagger} = 0 = \frac{-i}{2}\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\,\phi$, and $\frac{\partial \mathcal{L}_G}{\partial \phi} - \hat{p}_a\,\frac{\partial \mathcal{L}_G}{\partial(\hat{p}_a\phi)} = 0 = \frac{i}{2}\hat{p}_a\,\phi^\dagger\gamma^0_G\,\tilde{\gamma}^a$.

that the "basis vectors" are the "eigenstates" of the Cartan subalgebra, Eq. (8.4), fulfilling (on the algebraic level) the anticommutation relations of Eq. (8.11). And since the oddness of the Grassmann odd "basis vectors" guarantees the oddness of the tensor products of the internal part of "Grassmann fermions" and of the plane waves, we expect the equivalent anticommutation relations also for the eigenstates of the Eq. (8.21), which define the single particle anticommuting states of "Grassmann fermions".

The coefficients, determining the superposition, depend on momentum p^a, $a = (0, 1, 2, 3, 5, \ldots, d)$, $|p^0| = |\vec{p}|$, of the plane wave solution $e^{-ip_a x^a}$.

Let us therefore define the new creation operators and the corresponding single particle "Grassmann fermion" states as the tensor product of two spaces, the Grassmann odd "basis vectors" and the momentum space basis

$$\begin{aligned}
\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) &\stackrel{\text{def}}{=} \sum_i c^{ks}{}_i(\vec{p})\, \hat{b}^{\theta k\dagger}_i \,, \qquad |p^0| = |\vec{p}|\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}) &\stackrel{\text{def}}{=} \hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) \cdot e^{-ip_a x^a}\,, \quad |p^0| = |\vec{p}|\,,\\
< x|\phi^{ks}_{tot}(\vec{p}) > &= \hat{\underline{\mathbf{b}}}^{\theta ks\dagger}_{tot}(\vec{p})\,|\,1>\,, \qquad |p^0| = |\vec{p}|\,,
\end{aligned} \tag{8.22}$$

with s representing different solutions of the equations of motion and k different irreducible representations of the Lorentz group, $\vec{p}$ denotes the chosen vector $(p^0, \vec{p})$ in momentum space.

One has further

$$|\phi^{ks}(x^0, \vec{x}) > = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \hat{\underline{\mathbf{b}}}^{\theta ks\dagger}(\vec{p})|_{|p^0|=|\vec{p}|}|\,1> \tag{8.23}$$

The orthogonalized states $|\phi^{ks}(\vec{p}) >$ fulfill the relation

$$\begin{aligned}
< \phi^{ks}(\vec{p})|\phi^{k's'}(\vec{p}') > &= \delta^{kk'}\, \delta_{ss'}\, \delta_{pp'}\,, \qquad |p^0| = |\vec{p}|\,,\\
< \phi^{k's'}(x^0, \vec{x}')|\phi^{ks}(x^0, \vec{x}) > &= \delta^{kk'}\, \delta_{ss'}\, \delta_{\vec{x}',\vec{x}}\,,
\end{aligned} \tag{8.24}$$

where we assumed the discretization of momenta $\vec{p}$ and coordinates $\vec{x}$.

In even dimensional spaces ($d = 2(2n+1)$ and $4n$) there are 2^{d-1} Grassmann odd superposition of "basis vectors", which belong to different irreducible representations, among them twice $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ of the kind presented in Eqs. (8.8, 8.9) and discussed in the chapter *A.b.* of the subsect. 8.2.1 and in Table 8.1 for a particular case $d = (5+1)$. The illustration for the superposition $\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ is presented, again for $d = (5+1)$, in chapter *B.a.*.

We introduced in Eq. (8.22) the creation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ as the tensor product of the "basis vectors" of Grassmann algebra elements and the momentum basis. The Grassmann algebra elements transfer their oddness to the tensor products of these two basis. Correspondingly must $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ together with their Hermitian conjugated annihilation operators $(\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}))^\dagger = \hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ fulfill the the anticommutation relations equivalent to the anticommutation relations of

Eq. (8.11)

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= \delta^{kk'}\,\delta_{ss'}\delta(\vec{p}-\vec{p}\,')\,|1>\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= 0\ \cdot |1>\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T +}|1> &= 0\ \cdot |1>\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})\ *_T\ |1> &= 0\ \cdot |1>\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{8.25}$$

k labels different irreducible representations of Grassmann odd "basis vectors", s labels different — orthogonal and normalized — solutions of equations of motion and $\vec{p}$ represent different momenta fulfilling the relation $(p^0)^2 = (\vec{p})^2$. Here we allow continuous momenta and take into account that

$$<\ 1|\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')|1> = \delta^{kk'}\delta^{ss'}\delta(\vec{p}-\vec{p}\,')\,, \tag{8.26}$$

in the case of continuous values of $\vec{p}$ in even d-dimensional space.

For each momentum $\vec{p}$ there are 2^{d-1} members of the odd Grassmann character, belonging to different irreducible representations. The plane wave solutions, belonging to different $\vec{p}$, are orthogonal, defining correspondingly ∞ many degrees of freedom for each of 2^{d-1} "fermion" states, defined by $\hat{\underline{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$, when applying on the vacuum state $|1>$, Eq. (8.7).

With the choice of the Grassmann odd "basis vectors" in the internal space of "Grassmann fermions" and by extending these "basis states" to momentum space to be able to solve the equations of motion, Eq. (8.21), we are able to define the creation operators $\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ of the odd Grassmann character, which together with their Hermitian conjugated partners annihilation operators, fulfill the anticommutation relations of Eq. (8.25), manifesting the properties of the second quantized fermion fields. Anticommutation properties of creation and annihilation operators are due to the odd Grassmann character of the "basis vectors".

To define the Hilbert space of all possible "Slater determinants" of all possible occupied and empty fermion states and to discuss the application of $\hat{\underline{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{k\, s\dagger}_{tot}(\vec{p})$ on "Slater determinants", let us see what the anticommutation relations, presented in Eq. (8.25), tell. We realize from Eq. (8.25) the properties

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k'\,s'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k'\,s'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k'\,s'}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k'\,s'}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k'\,s'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{\theta k'\,s'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p})\,,\\
\text{if at least one of}\,(k,s,\vec{p}) &\quad \text{distinguishes from}(k',s',\vec{p}\,')\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})|\,1> &= |\,1>\,,\\
\hat{\underline{\mathbf{b}}}^{\theta k\,s}_{tot}(\vec{p})|\,1> &= 0\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{8.27}$$

From the above relations we recognize how do the creation and annihilation operators apply on "Slater determinants" of empty and occupied states, the later determined by $\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})$:

i. The creation operator $\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})$ jumps over the creation operator defining the occupied state, which distinguish from the jumping creation one in at least one of $(k, s, \vec{p})$, changing sign of the "Slater determinant" every time, up to the last step when it comes to its own empty state, the one with its quantum numbers $(k, s, \vec{p})$, occupying this empty state, or if this state is already occupied, gives zero.

ii. The annihilation operator changes sign of the "Slater determinant" when ever jumping over the occupied state carrying different internal quantum numbers (k, s) or $\vec{p}$, unless it comes to the occupied state with its own $(k, s, \vec{p})$, emptying this state or, if this state is empty, gives zero.

We show in Part II that the Clifford odd "basis vectors" describe fermions with the half integer spin, offering as well the corresponding anticommutation relations, explaining Dirac's postulates for second quantized fermions.

We discuss in Sect. 8.3 the properties of the "Slater determinants" of the occupied and empty "Grassmann fermion states", created by $\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})$.

In Subsect. B.a. we present one solution of the equations of motion for free massless "Grassmann fermions".

B.a. Plane wave solutions of equations of motion, Eq. (8.21), *in* $d = (5+1)$*-dimensional space*

One of such plane wave massless solutions of the equations of motion in $d = (5+1)$-dimensional space for momentum $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, $p^0 = |p^0|$, is the superposition of "basis vectors", presented in Table 8.1 among the first three members of the first decuplet, $k = I$. This particular solution $\hat{\underline{\mathbf{b}}}^{\theta k\,s\dagger}_{tot}(\vec{p})$ carries the spin $\mathcal{S}^{12} = 1$ ("up") and the "charge"

$\mathcal{S}^{56} = 1$ (both from the point of view of $d = (3+1)$)

$$\begin{aligned}
&\hat{\underline{\mathbf{b}}}^{\theta 1 1\dagger}_{tot}(\vec{p}) := \beta\, (\frac{1}{\sqrt{2}})^2 \{ \frac{1}{\sqrt{2}}\, (\theta^0 - \theta^3)(\theta^1 + i\theta^2) \\
&- \frac{2(|p^0| - |p^3|)}{p^1 - ip^2}\, (\theta^0\theta^3 + i\theta^1\theta^2) \\
&- (\frac{(p^1 + ip^2)^2}{(|p^0| + |p^3|)^2})\, \frac{1}{\sqrt{2}} (\theta^0 + \theta^3)(\theta^1 - i\theta^2) \} \\
&\times (\theta^5 + i\theta^6) \cdot e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}\,, \qquad |p^0| = |\vec{p}|\,,
\end{aligned}$$

β is the normalization factor. The notation $\hat{\underline{\mathbf{b}}}^{\theta 1 1\dagger}_{tot}(\vec{p})$ means that the creation operator represents the plane wave solution of the equations of motion, Eq. (8.21), for a particular $|p^0| = |\vec{p}|$.

Applied on the vacuum state the creation operator defines the second quantized single particle state of particular momentum $\vec{p}$. States, carrying different $\vec{p}$, are orthogonal (due to the orthogonality of the plane waves of different momenta and due to the orthogonality of $\hat{\underline{\mathbf{b}}}^{\theta k' s'\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ with respect to k and s, Eqs. (8.24, 8.26, 8.25)).

More solutions can be found in [12] and the references therein.

8.3 Hilbert space of anticommuting integer spin "Grassmann fermions"

The Grassmann odd creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$, with $|p^0| = |\vec{p}|$, are defined on the tensor products of 2^{d-1} "basis vectors", defining the internal space of integer spin "Grassmann fermions", and on infinite basis states of momentum space for each component of $\vec{p}$, chosen so that they solve for particular $(\vec{p})$ the equations of motion, Eq. (8.21). They fulfill together with their Hermitian conjugated annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ the anticommutation relations of Eq. (8.25).

These creation operators form the Hilbert space of "Slater determinants", defining for each "Slater determinant" places with either empty or occupied "Grassmann fermion" states.

Statement 3: Introducing the tensor product multiplication $*_T$ of any number of single "Grassmann fermion" states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})$ and with the identity included, applying on the vacuum state of any $(k, s, \vec{p})$), we generate the Hilbert space of the second quantized "Grassmann fermion" fields.

It is straightforward to recognize that the above definition of the Hilbert space is equivalent to the space of "Slater determinants" of all possible empty or occupied states of any momentum and any quantum numbers describing the internal space. The identity in this tensor product multiplication, for example, represents the "Slater determinant" of no single fermion state present.

The 2^{d-1} Grassmann odd creation operators of particular momentum $\vec{p}$, if applied on the vacuum state $|\,1>$, Eq. (8.7), define 2^{d-1} states. Since any state can be occupied or empty, the Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$ consists correspondingly of

$$N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-1}}\,. \tag{8.28}$$

"Slater determinants", namely the one with no occupied state, those with one occupied state, those with two occupied states, up to the one with all 2^{d-1} states occupied.

The total Hilbert space $\mathcal{H}$ of anticommuting integer spin "Grassmann fermions" consists of infinite many "Slater determinants" of particular $\vec{p}$, $\mathcal{H}_{\vec{p}}$, due to infinite many degrees of freedom in the momentum space

$$\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}\,, \tag{8.29}$$

with the infinite number of degrees of freedom

$$N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-1}}\,. \tag{8.30}$$

8.3.1 "Slater determinants" of anticommuting integer spin "Grassmann fermions" of particular momentum $\vec{p}$

Let us write down explicitly these $2^{2^{d-1}}$ contributions to the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$, using the notation that $\mathbf{0^k_{s\vec{p}}}$ represents the unoccupied state $\underline{\hat{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p})|1>$ (of the s^{th} solution of the equations of motion belonging to the k^{th} irreducible representation), while $\mathbf{1^k_{s\vec{p}}}$ represents the corresponding occupied state.

The number operator is according to Eq. (8.11) and Eq. (8.27) equal to

$$\begin{aligned} N^{\theta k s}_{\vec{p}} &= \underline{\hat{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}) *_T \underline{\hat{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\,,\\ N^{\theta ks}_{\vec{p}} *_T 0^k_{s\vec{p}} &= 0\,, \quad N^{\theta ks}_{\vec{p}} *_T 1^k_{s\vec{p}} = 1\,. \end{aligned} \tag{8.31}$$

Let us simplify the notation so that we count for $k = 1$ empty states as $\mathbf{0_{r\vec{p}}}$, and occupied states as $\mathbf{1_{r\vec{p}}}$, with $r = (1, \dots, s^1_{max})$, for $k = 2$ we count $r = s^1_{max} + 1, \dots, s^1_{max} + s^2_{max}$, up to $r = 2^{d-1}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$\begin{aligned} &|\mathbf{0_{1\vec{p}}}, \mathbf{0_{2\vec{p}}}, \mathbf{0_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >\,, \quad |\mathbf{1_{1\vec{p}}}, \mathbf{0_{2\vec{p}}}, \mathbf{0_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >,\\ &|\mathbf{0_{1\vec{p}}}, \mathbf{1_{2\vec{p}}}, \mathbf{0_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >\,, \quad |\mathbf{0_{1\vec{p}}}, \mathbf{0_{2\vec{p}}}, \mathbf{1_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >,\\ &\qquad\qquad\vdots\\ &|\mathbf{1_{1\vec{p}}}, \mathbf{1_{2\vec{p}}}, \mathbf{0_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >\,, \quad |\mathbf{1_{1\vec{p}}}, \mathbf{0_{2\vec{p}}}, \mathbf{1_{3\vec{p}}}, \dots, \mathbf{0_{2^{d-1}\vec{p}}} >,\\ &\qquad\qquad\vdots\\ &|\mathbf{1_{1\vec{p}}}, \mathbf{1_{2\vec{p}}}, \mathbf{1_{3\vec{p}}}, \dots, \mathbf{1_{2^{d-1}\vec{p}}} >\,, \end{aligned} \tag{8.32}$$

with a part with none of states occupied ($N_{r\vec{p}} = 0$ for all $r = 1, \dots, 2^{d-1}$), with a part with only one of states occupied ($N_{r\vec{p}} = 1$ for a particular $r = 1, \dots, 2^{d-1}$ while $N_{r'\vec{p}} = 0$ for all the others $r' \neq r$), with a part with only two of states occupied ($N_{r\vec{p}} = 1$ and $N_{r'\vec{p}} = 1$, where r and r' run from $1, \dots, 2^{d-1}$), and so on. The last part has all the states occupied.

Taking into account Eq. (8.27) is not difficult to see that the creation operator $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized "fermions".

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p})\}_{*_T +} \mathcal{H}_{\vec{p}} &= \delta^{kk'}\,\delta_{ss'} \mathcal{H}_{\vec{p}}\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s'}_{tot}(\vec{p})\}_{*_T +}\, \mathcal{H}_{\vec{p}} &= 0\,\cdot\,\mathcal{H}_{\vec{p}}\,,\\
\{\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p})\}_{*_T +}\, \mathcal{H}_{\vec{p}} &= 0\,\cdot\,\mathcal{H}_{\vec{p}}\,.
\end{aligned} \tag{8.33}$$

The proof for the above relations easily follows if taking into account that, when ever the creation or annihilation operator jumps over an odd products of occupied states, the sign changes. Then one sees that the contribution of the application of $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p})\, \mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\, \mathcal{H}_{\vec{p}}$.

If the creation and annihilation operators are Hermitian conjugated to each other, the result of

$$\{\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) + \hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})\}\mathcal{H}_{\vec{p}} = \mathcal{H}_{\vec{p}}$$

is the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operates on its own half of $\mathcal{H}_{\vec{p}}$. Jumping together over even number of occupied states $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})$ do not change the sign of particular "Slater determinant". (Let us add that $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ reduces for particular k and s the Hilbert space $\mathcal{H}_{\vec{p}}$ for a factor $\frac{1}{2}$, and so does $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

8.3.2 "Slater determinants" of Hilbert space of anticommuting integer spin "fermions"

The total Hilbert space of anticommuting "fermions" is the infinite product of the Hilbert spaces of particular $\vec{p}$, $\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, Eq. (8.29), represented by infinite numbers of "Slater determinants" $N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-1}}$, Eq. (8.30). The notation $\otimes_N$ is to point out that creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})$, which origin in superposition of odd number of θ^a's, keep their odd character also in the tensor products of the internal and momentum space, as well as in the "Slater determinants", in which creation operators determine the occupied states.

The application of creation operators $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p})$ and their Hermitian conjugated annihilation operators $\hat{\underline{\mathbf{b}}}^{\theta k s}_{tot}(\vec{p})$ on the Hilbert space $\mathcal{H}$ has the property, manifested in Eq. (8.27), leading to the conclusion that the application of $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p'}) *_T \mathcal{H}$ is not zero if at least one of $(k, s, \vec{p})$ is not equal to $(k', s', \vec{p'})$, while $\hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p'}) *_T \mathcal{H} + \hat{\underline{\mathbf{b}}}^{\theta k' s' \dagger}_{tot}(\vec{p'}) *_T \hat{\underline{\mathbf{b}}}^{\theta k s \dagger}_{tot}(\vec{p}) *_T \mathcal{H} = 0$ for any

$(k, s, \vec{p})$ and any $(k', s', \vec{p'})$, what is not difficult to prove when taking into account Eq. (8.27).

One can easily show that the creation operators $\underline{\hat{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})$ and the annihilation operators $\underline{\hat{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}\,')$ fulfill equivalent anticommutation on the whole Hilbert space of infinity many "Slater determinants" as they do on the Hilbert space $\mathcal{H}_{\vec{p}}$.

$$\begin{aligned}
\{\underline{\hat{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p})\,, \underline{\hat{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\mathcal{H} &= \delta^{kk'}\;\delta_{ss'}\delta(\vec{p}-\vec{p}\,')\,\mathcal{H}\,,\\
\{\underline{\hat{\mathbf{b}}}^{\theta k\, s}_{tot}(\vec{p}), \underline{\hat{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\,\mathcal{H} &= 0\;\cdot\mathcal{H}\,,\\
\{\underline{\hat{\mathbf{b}}}^{\theta k\, s\dagger}_{tot}(\vec{p})\,, \underline{\hat{\mathbf{b}}}^{\theta k'\, s'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\,\mathcal{H} &= 0\;\cdot\mathcal{H}\,.
\end{aligned} \tag{8.34}$$

Creation operators, $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define the second quantized fermion states.

8.3.3 Relations between creation operators $\underline{\hat{\mathbf{b}}}^{\theta\, k\, s\dagger}_{tot}(\vec{p})$ in the Grassmann odd algebra and the creation operators postulated by Dirac

Creation operators $\underline{\hat{\mathbf{b}}}^{\theta\, k\, s\dagger}_{tot}(\vec{p})$ define the second quantized "fermion" fields of integer spins.

Since the second quantized Dirac fermions have the half integer spin, the "Grassmann fermions", the internal degrees of which is described by the Grassmann odd algebra, have the integer spin. The comparison between the second quantized fields of Dirac and those presented in this Part I of the paper can only be done on a rather general level. We leave therefore the detailed comparison of the creation and annihilation operators for fermions with half integer spins between those postulated by Dirac and the ones following from the Clifford odd algebra presented in Part II to Subsect. 3.4 of Part II.

Here we discuss only the relations among appearance of the creation and annihilation operators offered by the Grassmann odd algebra and those postulated by Dirac. In both cases we treat only $d = (3+1)$-dimensional space, that is we solve the equations of motion for $p^a = (p^0, p^1, p^2, p^3)$ (in the case that $d > 4$ the rest of space demonstrates the charges in $d = (3+1)$, when $p^a = (p^0, p^1, p^2, p^3, 0, 0, \ldots, 0)$).

It is pointed out in what follows that both internal spaces — either the internal space postulated by Dirac or the internal space offered by the Grassmann algebra — are finite dimensional, as also the internal space offered by the Clifford algebra is finite dimensional.

In the Dirac case the second quantized states are in $d = (3+1)$ dimensions postulated as follows

$$\underline{\Psi}^{s\dagger}(x^0, \vec{x}) = \sum_{i, \vec{p}_k} \hat{\mathbf{a}}^{\dagger}_i(\vec{p}_k)\, u^s_i(\vec{p}_k)\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})}\,. \tag{8.35}$$

$v^s_i(\vec{p}_k)$ $(= u^s_i\, e^{-i(p^0x^0)-\varepsilon\vec{p}\cdot\vec{x}})$ are the two left handed $(\Gamma^{(3+1)} = -1)$ and the two right handed $(\Gamma^{(3+1)} = 1$, Eq. (B.3)) two-component column matrices, representing the four solutions s of the Weyl equation for free massless fermions of particular momentum $|\vec{p}_k| = |p^0_k|$ ([2], Eqs. (20-49) - (20-51)), the factor $\varepsilon = \pm 1$ depends on the product of handedness and spin.

$\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ are by Dirac postulated creation operators, which together with annihilation operators $\hat{\mathbf{a}}_i(\vec{p}_k)$, fulfill the anticommutation relations ([2], Eqs. (20-49) - (20-51)),

$$\{\hat{\mathbf{a}}^\dagger_i(\vec{p}_k), \hat{\mathbf{a}}^\dagger_j(\vec{p}_l)\}_{*_T+} = 0 = \{\hat{\mathbf{a}}_i(\vec{p}_k), \hat{\mathbf{a}}_j(\vec{p}_l)\}_{*_T+}\,,$$
$$\{\hat{\mathbf{a}}_i(\vec{p}_k), \hat{\mathbf{a}}^\dagger_j(\vec{p}_l)\}_{*_T+} = \delta_{ij}\delta_{\vec{p}_k \vec{p}_l}\,, \tag{8.36}$$

in the case of discretized momenta for a fermion in a box. Creation operators and annihilation operators, $\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ and $\hat{\mathbf{a}}_i(\vec{p}_k)$, are postulated to have on the Hilbert space of all "Slater determinants" these anticommutation properties.

To be able to relate the creation operators of Dirac $\hat{\mathbf{a}}^\dagger_i(\vec{p}_k)$ with $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ from Eq. (8.34), let us remind the reader that $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ is a superposition of basic vectors $\hat{b}^{\theta k\dagger}_i$ with the coefficients $c^{ks}{}_i(\vec{p})$, which depend on the momentum $\vec{p}$, Eq. (8.22) ($\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}) = \sum_i c^{ks}{}_i(\vec{p})\, \hat{b}^{\theta k\dagger}_i$), so that $\hat{\underline{\mathbf{b}}}^{\theta k s\dagger}_{tot}(\vec{p}_k)$ $(= \sum_i c^{ks}{}_i(\vec{p})\, \hat{b}^{\theta k\dagger}_i\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})})$ solves the equations of motion for free massless "Grassmann fermions" for plane waves, while $|p^0| = |\vec{p}|$.

We treat in this subsection the Grassmann case in $(3+1)$-dimensional space only, without taking care on different irreducible representations k as well as on charges, in order to be able to relate the creation and annihilation operators in Grassmann space with the Dirac's ones. In this case the odd Grassmann creation operators are expressible with the "basic vectors", which are fourplets, presented in Table 8.1 on the 7^{th} up to the 10^{th} lines, the same on both decuplets, neglecting $\theta^5\theta^6$ contribution. (They have handedness in $d = (3+1)$ equal zero.)

Let us rewrite creation operators in the Dirac case so that their expressions resemble the expression for the creation operators

$$\hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k) = \sum_i c^s{}_i(\vec{p})\, \hat{b}^{\theta\dagger}_i\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})},$$

leaving out the index of the irreducible representation.

$$\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k) \stackrel{\text{def}}{=} \sum_i \hat{\mathbf{a}}^\dagger_i(\vec{p}_k)\, u^s_i(\vec{p}_k)\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})} \stackrel{\text{def}}{=} \sum_i \alpha^s_i(\vec{p}_k)\, \hat{a}^\dagger_i\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}$$

to be compared with

$$\hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k) = \sum_i c^s{}_i(\vec{p})\, \hat{b}^{\theta\dagger}_i\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}\,. \tag{8.37}$$

We define in the Dirac case two creation operators: $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ and $\hat{a}^\dagger_i$. Since $\underline{\Psi}^{s\dagger}(x^0, \vec{x}) = \sum_{\vec{p}_k} \hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$, Eq. (8.35), we realize that the two expressions

$$u^s_i(\vec{p}_k)\, \hat{\mathbf{a}}^\dagger_i(\vec{p}_k) \quad \text{and} \quad \alpha^s_i(\vec{p}_k)\, \hat{a}^\dagger_i$$

describe the same degrees of freedom.

These new creation operators $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ can not be related directly to $\hat{\underline{\mathbf{b}}}^{\theta s\dagger}_{tot}(\vec{p}_k)$, since the first ones describe the second quantized fields of the half integer spin fermions, while the later describe the second quantized integer spin "fermion" fields. However, both fulfill the anticomutation relations of Eq. (8.34).

The reader can notice that the creation operators $\hat{a}^\dagger_i$ do not depend on $\vec{p}$ as also $\hat{b}^{\theta\dagger}_i$ do not, both describing the internal degrees of freedom, while $\alpha^s_i(\vec{p}_k)\,\hat{a}^\dagger_i$ and $\alpha^s_i(\vec{p}_k)\,\hat{b}^{\theta\dagger}_i$ do.

The creation and annihilation operators of Dirac fulfill obviously the anticommutation relations of Eq. (8.34). To see this we only have to replace $\hat{\underline{\mathbf{b}}}^{\theta h\, s\dagger}_{tot}(\vec{p})$ by $\hat{\underline{\mathbf{a}}}^{h\, s\dagger}_{tot}(\vec{p})$ by taking into account relation of Eq. (8.37).

Creation and annihillation operators of the Dirac second quantized fermions with half integer spins are in Part II, in Subsect. III.D, related to the corresponding ones, offered by the Clifford algebra. Relating the creation and annihilation operators offered by the Clifford algebra objects with the Dirac's ones ensures us that the Clifford odd algebra explains the Dirac's postulates.

8.4 Conclusions

We learn in this Part I paper, that in d-dimensional space the superposition of odd products of θ^a's exist, Eqs. (8.8, 8.10, 8.9), chosen to be the eigenvectors of the Cartan subalgebra, Eq. (8.5), which together with their Hermitian conjugated partners, odd products of $\frac{\partial}{\partial\theta_a}$'s, Eqs. (8.2, 8.8, 8.6), fulfill on the algebraic level on the vacuum state $|\phi_o>=|\,1>$, Eq. (8.25), the requirements for the anticommutation relations for the Dirac's fermions.

The creation operators defined on the tensor products of internal space of "Grassmann basis vectors" (of finite number of basis states) and of momentum space (with infinite number of basis states), arranged to be solutions of the equation of motion for free massless "Grassmann fermions", Eq. (8.21), form the infinite dimensional Hilbert space of "Slater determinants" of (continuous) infinite number of momenta, with $2^{2^{d-1}}$ possibilities for each momentum $\vec{p}$, Eq. (8.34)). These creation operators and their Hermitian conjugated partners fulfill on the Hilbert space the anticommutation relations postulated by Dirac for second quantized fermion fields.

We demonstrate the way of deriving second quantized integer fermion fields.

In the subsection 8.1.1 we clarify the relation between our description of the internal space of fermions with "basis vectors", manifesting oddness and transferring the oddness to the corresponding creation and annihilation operators of second quantized fermions, to the ordinary second quantized creation and annihilation operators from a slightly different point of view.

Since the creation and annihilation operators, which are superposition of odd products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s, respectively, anticommute algebraically when applying on the vacuum state, Eq. (8.11, 8.12) (while the corresponding even products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s commute, Eq. (8.16)), it follows that also creation operators, defined on tensor products of the finite number of "basis vectors" (describing the internal degrees of freedom of "Grassmann fermions") and on infinite basis of momentum space, together with their Hermitian conjugated partners annihilation operators, fulfill the anticommutation relations of Eq. (8.34). The use of the Grassmann odd algebra to describe the internal space of "Grassmann fermions" offers the anticommutation relations without postulating them: on the (simple) vacuum state as well as on the Hilbert space of infinite number of "Slater determinants" of all possible single particle states, empty or occupied, of the second quantized integer spin "fermion" fields. Correspondingly we second quantized "fermion fields" without postulating commutation relations of Dirac.

The internal "basis vectors" are chosen to be eigenvectors of the Cartan subalgebra operators in the way that the symmetry agrees with the properties of usual Dirac's creation and annihilation operators of second quantized fermions — in the Clifford case for half integer spin, while in the "Grassmann fermions" for the integer spins.

The "Grassmann fermions" carry the spin and charges, originated in $d \geq 5$, in the adjoint representations. "Grassmann fermions" offer no families, what means that there is no available operators, which would connect different irreducible representations of the Lorentz group (without breaking symmetries).

No elementary "Grassmann fermions" with the spins and charges in the adjoint representations have been observed, and since the observed quarks and leptons and anti-quarks and anti-leptons have half integer spins, charges in the fundamental representations and appear in families, it does not seem possible for the future observation of the integer spin elementary "Grassmann fermions", especially not since Eq. (19) in Part II demonstrates that the reduction of space in Clifford case, needed for the appearance of second quantized half integer fermions, reduces also the Grassmann space, leaving no place for second quantized "Grassmann fermions" with the integer spin.

In Part II two kinds of operators are studied; There are namely two kinds of the Clifford algebra objects, $\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})$ and $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})$, which anticommute, $\{\gamma^a, \tilde{\gamma}^a\}_+ = 0$ ($\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$), and offer correspondingly two kinds of independent representations.

Each of these two kinds of independent representations can be arranged into irreducible representations with respect to the two Lorentz generators — $S^{ab} = \frac{i}{4}\,(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and $\tilde{S}^{ab} = \frac{i}{4}\,(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$. All the Clifford irreducible representations of any of the two kinds of algebras are independent and disconnected.

The two Dirac's actions in d-dimensional space for free massless fermions ($\mathcal{A} = \int\, d^dx\, \frac{1}{2}\,(\psi^\dagger\gamma^0\,\gamma^a\, p_a\psi) + h.c.$ and $\tilde{\mathcal{A}} = \int\, d^dx\, \frac{1}{2}\,(\psi^\dagger\tilde{\gamma}^0\,\tilde{\gamma}^a\, p_a\psi) + h.c.$) lead to the equations of motion, which have the solutions in both kinds of algebras for an odd Clifford character (they are superposition of an odd products of γ^a's and $\tilde{\gamma}^a$'s, respectively), forming on the tensor product of finite number of "basis vectors" describing the internal space and of the infinite number of basis of momentum space, the creation and annihilation operators, which only "almost" anticommute, while the Grassmann odd creation and annihilation operators do anticommute. Although "vectors" of one irreducible representation of an odd Clifford algebra character, anticommute among themselves and so do their Hermitian conjugated partners in each of the two kinds of the Clifford algebras, the anticommutation relations among creation and annihilation operators in each of the two Clifford algebras separately, do not fulfill the requirement, that only the anticommutator of a creation operator and its Hermitian conjugated partner gives a nonzero contribution.

The decision, the postulate, Eq. (12), that only one kind of the Clifford algebra objects — we make a choice of γ^a — describes the internal space of fermions, while the second kind — $\tilde{\gamma}^a$ in this case — does not, and consequently determine "family" quantum numbers which distinguish among irreducible representations of S^{ab}, solves the problems:

a. Creation operators and their Hermitian conjugated partners, which are odd products of superpositions of γ^a, applied on the vacuum state, fulfill on the algebraic level the anticommutation relations, and the creation and annihilation operators creating the second quantized Clifford fermion fields fulfill all the requirements, which Dirac postulated for fermions.

b. Different irreducible representations with respect to S^{ab} carry now different "family" quantum numbers determined by $\frac{d}{2}$ commuting operators among $\tilde{S}^{ab}$.

c. The operators of the Lorentz algebra, which do not belong to the Cartan subalgebra, connect different irreducible representations of S^{ab}.

The above mentioned decision, Eq. (19) in Part II, obviously reduces the degrees of freedom of the odd (and even) Clifford algebra, while opening the possibility for the appearance of "families", as well as for the explanation for the Dirac's second quantization postulates. This decision, reducing as well the degrees of freedom of Grassmann algebra, disables the existence of the integer spin "fermions" as elementary particles.

Let us point out again at the end that when the internal part of the single particle wave function anticommute under the algebra product $*_A$, then this implies that the wave functions with such internal part anticommute under the extension of $*_A$ to the (full) single particle wave functions and so do anticommute the corresponding creation and annihilation operators what manifests also on the properties of the Hilbert space.

The anticommuting single fermion states manifest correspondingly the oddness already on the level of the first quantization.

8.5 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [6] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta>=<\theta|\mathbf{B}>^{\dagger}$,

$$<\mathbf{b}|\theta>=\sum_{k=0}^{d} b_{a_1\ldots a_k}\theta^{a_1}\cdots\theta^{a_k},$$

by requiring

$$\begin{aligned}
&\{d\theta^a,\theta^b\}_+ = 0\,,\quad \int d\theta^a = 0\,,\quad \int d\theta^a\theta^a = 1\,,\\
&\int d^d\theta\;\theta^0\theta^1\cdots\theta^d = 1\,,\\
&d^d\theta = d\theta^d\ldots d\theta^0\,,\quad \omega = \prod_{k=0}^{d}(\frac{\partial}{\partial\theta_k}+\theta^k)\,,
\end{aligned}\tag{8.38}$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function [6] $\omega = \prod_{k=0}^{d}(\frac{\partial}{\partial\theta_k} + \theta^k)$ to define the scalar product in Grassmann space $< \mathbf{B}|\mathbf{C} >$

$$
\begin{aligned}
< \mathbf{B}|\mathbf{C} > &= \int d^d\theta^a \;\; \omega < \mathbf{B}|\theta >< \theta|\mathbf{C} > \\
&= \sum_{k=0}^{d} \int b^*_{b_1\dots b_k} c_{b_1\dots b_k} \,. \qquad (8.39)
\end{aligned}
$$

To define norms in Clifford space Eq. (8.38) can be used as well.

8.6 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\varepsilon_{a_1 a_2 \dots a_{d-1}\, a_d}\, S^{a_1 a_2} \cdot S^{a_3 a_4} \cdots S^{a_{d-1} a_d} \,, \qquad (8.40)$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case S^{ab} is defined in Eq. (8.3), while in the Clifford case Eq. (8.40) simplifies, if we take into account that $S^{ab}|_{a\neq b} = \frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b} = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2} \quad \prod_a \quad (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n \,. \qquad (8.41)$$

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9 Understanding the Second Quantization of Fermions in Clifford and in Grassmann Space, New Way of Second Quantization of Fermions — Part II

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Abstract. We present in Part II the description of the internal degrees of freedom of fermions by the superposition of odd products of the Clifford algebra elements, either γ^a's or $\tilde{\gamma}^a$'s [1–3], which determine with their oddness the anticommuting properties of the creation and annihilation operators of the second quantized fermion fields in even d-dimensional space-time, as we do in Part I of this paper by the Grassmann algebra of θ^a's and $\frac{\partial}{\partial\theta_a}$'s. We discuss: **i.** The properties of the two kinds of the odd Clifford algebras, forming two independent spaces, both expressible with the Grassmann coordinates θ^a's and their derivatives $\frac{\partial}{\partial\theta_a}$'s [2,7,8]. **ii.** The freezing out procedure of one of the two kinds of the odd Clifford objects, enabling that the remaining Clifford objects determine with their oddness in the tensor products of the finite number of the Clifford basis vectors and the infinite number of momentum basis, the creation and annihilation operators carrying the family quantum numbers and fulfilling the anticommutation relations of the second quantized fermions: on the vacuum state, and on the whole Hilbert space defined by the sum of infinite number of "Slater determinants" of empty and occupied single fermion states. **iii.** The relation between the second quantized fermions as postulated by Dirac [19–21] and the ones following from our Clifford algebra creation and annihilation operators, what offers the explanation for the Dirac postulates.

Povzetek. V drugem delu prispevka predstavita avtorja opis notranjega prostora fermionov v sodo razsežnih prostorih s superpozicijo lihih produktov elementov Cliffordove algebre, bodisi γ^a ali $\tilde{\gamma}^a$ [1–3]. Lihi značaj teh produktov določa antikomutacijske lastnosti kreacijskih in anihilacijskih operatorjev fermionskih stanj v drugi kvantizaciji brez Diracovih postulatov. (V prvem delu prispevka sta predstavila fermionske prostotne stopnje z Grassmannovimi koordinatami θ^a in $\frac{\partial}{\partial\theta_a}$). Obravnavata: **i.** Lastnosti dveh vrst lihih Cliffordovih objektov, ki tvorita neodvisna prostora. Obe Cliffordovi algebri sta izrazljivi z Grassmannovimi koordinatami θ^a in njihovimi odvodi $\frac{\partial}{\partial\theta_a}$ [2, 7, 8]. **ii.** Pokaěta, da četudi imajo vektorski produkti končnega števila lihih Cliffordovih produktov in (zvezno) neskončnega števila baz̆ičnih vektorjev v običajnem prostoru antikomutacijski značaj kot ga Dirac predpiše za fermione v drugi kvantizaciji v vsaki od Cliffordih algeber posebej, pa ustreže predlagani opis fermionov opazljivim lastnostim fermionov šele, ko s postulatom zagotovita, da samo ena od algeber opiše notranji prostor fermionov, operatorji preostale algebre pa določajo kvantna števila družin — na vakuumskem stanju in na celem Hilbertovem prostoru, ki ga določa vsota neskočnega števila "Slaterjevih determinant" praznih in zasedenih enofermionskih stanj, ki imajo vsa lihi značaj. **iii.** Relacijo med Diracovimi postulati za fermione v drugi kvantizaciji [19–21] in obravnavano potjo do drude kvantizacije, ki pojasni Diracove privzetke.

Keywords: Second quantization of fermion fields in Clifford and in Grassmann space, Spinor representations in Clifford and in Grassmann space, Explanation of the Dirac postulates, Kaluza-Klein-like theories, Higher dimensional spaces, Beyond the standard model

9.1 Introduction

In a long series of works we, mainly one of us N.S.M.B. ([1–3, 10–15] and the references therein), have found phenomenological success with the model named by N.S.M.B the *spin-charge-family* theory, with fermions, the internal space of which is describable as superposition of odd products of the Clifford algebra elements γ^a's in $d = (13+1)$ (may be with d infinity), interacting with only gravity. The spins of fermions from higher dimensions, $d > (3+1)$, manifest in $d = (3+1)$ as charges of the *standard model*, the gravity originating in higher dimensions manifest as the *standard model* vector gauge fields and the scalar Higgs explaining the Yukawa couplings.

There are two kinds of anticommuting algebras, the Grassmann algebra and the Clifford algebra, the later with two independent subalgebras. The Grassmann algebra, with elements θ^a, and their Hermitian conjugated partners $\frac{\partial}{\partial\theta^a}$ [3], describes fermions with the integer spins and charges in the adjoint representations, the two Clifford algebras, we call their elements γ^a and $\tilde{\gamma}^a$, can each of them be used to describe half integer spins and charges in the fundamental representations. The Grassmann algebra is expressible with the two Clifford algebras and opposite.

The two papers explain how do the oddness of the internal space of fermions manifests in the single particle wave functions, relating the oddness of the wave functions to the corresponding creation and annihilation operators of the to the second quantized fermions, in the Grassmann case and in the Clifford case, explaining therefore the postulates of Dirac for the second quantized fermions.

We learn in Part I of this paper, that in d-dimensional space 2^{d-1} superposition of odd products of d θ^a's exist, chosen to be the eigenvectors of the Cartan subalgebra, Eq. (4) of Part I, and arranged in tensor products with the momentum space to be solutions of the equation of motion for free massless "fermions", Eq. (21) of Part I.

The creation operators, defined as the tensor products of the superposition of the finite number of "basis vectors" in Grassmann space, guaranteeing the oddness of operators, and of the infinite basis in momentum space, form — applied on the vacuum state — the second quantized states of integer spin "Grassmann fermions". The creation operators fulfill together with their Hermitian conjugated partners annihilation operators (based on the internal space of odd products of $\frac{\partial}{\partial\theta_a}$'s) all the requirements of the anticommutation relations postulated by Dirac for fermions: **i.** on the simple vacuum state $|1>$ (Eqs. (7,11) of Part I), **ii.** on the Hilbert space $\mathcal{H}$ $(= \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$, with the number of empty and occupied single fermion states for particular $\vec{p}$ equal to $2^{2^{d-1}}$) of infinite many "Slater determinants" of all possible empty and occupied single fermion states (with the infinite number of possibilities of moments for each of 2^{d-1} internal degrees of freedom), Eqs. (25, 34) of Part I.

While the creation and annihilation operators, which are superposition of odd products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s, respectively, anticommute on the vacuum state $|\phi_o>=|1>$, Eq. (7,11), the superposition of even products of θ^a's and $\frac{\partial}{\partial\theta_a}$'s, respectively, commute, Eq. (16) of Part I.

The superposition of odd products of γ^a's and their Hermitian conjugated partners, as well as of odd products of $\tilde{\gamma}^a$'s and their Hermitian conjugated partners, on the corresponding vacuum states, Eq. (9.18), anticommute. Since the tensor products of the "basis vectors" determining the internal space of Clifford fermions and of the basis in momentum space manifest oddness of the internal space, no postulates of anticommutation relations as in the Dirac second quantization proposal is needed also for Clifford fermions with the internal space described by one of the two Clifford objects (in Subsect. 9.2.2 we make a choice of γ^a's). The oddness of the " basis vectors", defining the internal space of fermions, transfers to the creation and annihilation operators forming the second quantized single fermion states in the Clifford and the Grassmann space.

The "Grassmann fermions" have integer spins, and spins in the part with $d \geq 5$ manifesting as charges in $d = (3+1)$, in adjoint representations, Table I in Part I. There is no operator which would connect different irreducible representations of the corresponding Lorentz group. There are no elementary fermions with integer spin observed so far either.

The Clifford fermions, describing the internal space with γ^a's, have half integer spins and spins in the part with $d \geq 5$ manifesting as charges in $d = (3+1)$ in fundamental representations [10–12,15,17,18]. The operators $\tilde{S}^{ab}$ $(= \frac{i}{4}\{\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\}_-)$ connect, after the reduction of the Clifford algebra degrees of freedom by a factor of 2, Subsect. 9.2.2, different irreducible representations of the Lorentz group S^{ab} $(= \frac{i}{4}\{\gamma^a\gamma^b - \gamma^b\gamma^a\}_-)$ and determine "family" quantum numbers. All in agreement with the observed families of quarks and leptons.

In Part II the properties of the two kinds of the Clifford algebras objects, γ^a's and $\tilde{\gamma}^a$'s, are discussed. Both are expressible with θ^a's and $\frac{\partial}{\partial\theta_a}$'s ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$ [2,7,8]), and both are, up to a constant $\eta^{aa} = (1,-1,-1,\ldots,-1)$, Hermitian operators. Each of these two kinds of the Clifford algebra objects of an odd Clifford character (superposition of odd number of products of either γ^a's or $\tilde{\gamma}^a$'s, respectively) has 2^{d-1} members, together again $2\cdot 2^{d-1}$ members, the same as in the case of "Grassmann fermions".

These two internal spaces, described by the two Clifford algebras, are independent, each of them with their own generators of the Lorentz transformations, Eq. (9.3), and their corresponding Cartan subalgebras, Eq. (9.4).

In each of these two internal spaces there exist $2^{\frac{d}{2}-1}$ "basis vectors" in $2^{\frac{d}{2}-1}$ irreducible representations, chosen to be ''eigenvectors" of the corresponding Cartan subalgebra elements, Eq. (9.5), and having the properties of creation and annihilation operators (the Hermitian conjugated partners of the creation operators) on the vacuum state: **i.** The application of any creation operator on the vacuum state, Eq. (9.18), gives nonzero contribution, while the application of any annihilation operator on the vacuum state gives zero contribution. **ii.** Within each of these two spaces all the annihilation operators anticommute among themselves and all the creation operators anticommute among themselves. **iii.** The vacuum state is a superposition of products of the annihilation operators with their Hermitian conjugated partners creation operators, like in the Grassmann case. The Clifford vacuum states, Eq. (9.18), are not the identity like in the Grassmann case, Eq. (19) in Part I.

However, there is not only the anticommutator of the creation operator and its Hermitian conjugated partner, which gives the nonzero contribution on the vacuum state in each of the two spaces — what in the Grassmann algebra is the case, and what the postulates of Dirac require. There are, namely, the additional $(2^{\frac{d}{2}-1}-1)$ members of the same irreducible representation, to which the Hermitian conjugated partner of the creation operator belongs, giving the nonzero anticommutator with this creation operator on the vacuum state (Eq. (9.11) in Subsect. 9.2.1 illustrates such a case).

And, there is no operators, which would connect different irreducible representations in each of the two Clifford algebras and correspondingly there is no "family" quantum number for each irreducible representation, needed to describe the observed quarks and leptons. (Let the reader be reminded that also the Grassmann algebra has no operators, which would connect different irreducible representations. The Dirac's second quantization postulates do not take care of charges and families of fermions, both can be treated and incorporated into the second quantization postulates as quantum numbers of additional groups as proposed by the *standard model*.) We solve these problems with the requirement, presented in Eq. (9.12): $\tilde{\gamma}^a B = (-)^B \, i \, B\gamma^a$, with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [8].

We present in the subsection 9.1.1 of this section a short overview of steps, which lead to the second quantized fermions in the Clifford space, offering the explanation for the Dirac's postulates. In the subsection 9.1.2 we discuss our assumption, that the oddness of the "basis vectors" in the internal space transfer to the corresponding creation and annihilation operators determining the second quantized single fermion states and correspondingly the Hilbert space of the second quantized fermions, in a generalized way.

We present in Sect. 9.2 the properties of the Clifford algebra "basis vectors" in the space of d γ^a's and in the space of d $\tilde{\gamma}^a$'s. In Subsect. 9.2.1 we discuss properties of the "basis vectors" of half integer spin. In Subsect. 9.2.2 we discuss conditions, under which operators of one of these two kinds of the Clifford algebra objects demonstrate by themselves the anticommutation relations required for the second quantized "fermions", manifesting the half integer spins, offering the explanation for the spin and charges of the observed quarks and leptons and anti-quarks and anti-leptons and also for their families [1–3, 10–15, 17].

In Subsect. 9.2.3 we generate the basis states manifesting the family quantum numbers.

In Subsect. 9.2.4 the superposition of "basis vectors", solving the Weyl equation, are constructed, forming creation operators depending on the momenta and fulfilling with their Hermitian conjugated partners the anticommutation relations for the second quantized fermions.

We illustrate in Sect. 9.2.5 properties of the Clifford odd "basis vectors" in $d = (5+1)$-dimensional space, and extending the internal space in a tensor product to momentum space, we present also the superposition solving the Weyl equation, and correspondingly present creation and annihilation operators depending on the momentum $\vec{p}$.

We present in Sect. 9.3 the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ as "Slater determinants": i. with no "fermions" occupying any of the 2^{d-2} fermion states, ii. with one "fermion" occupying one of the 2^{d-2} fermion states, iii. with two "fermions" occupying the 2^{d-2} fermion states,..., up to the "Slater determinant" with all possible fermion states of a particular $\vec{p}$ occupied by "fermions". The total Hilbert space $\mathcal{H}$ is then the tensor product $\prod_{\infty} \otimes_N$ of infinite number of $\mathcal{H}_{\vec{p}}$. On $\mathcal{H}$ the tensor products of creation and annihilation operators (solving the equations of motion for free massless fermions) manifest the anticommutation relations of second quantized "fermions" without any postulates. We also illustrate the application of the tensor products of creation and annihillation operators on $\mathcal{H}$ in a simple toy model.

In Subsect. 9.3.4 the correspondence between our way and the Dirac way of second quantized fermions is presented, demonstrating that our way does explain the Dirac's postulates.

In Sect. 9.4 we note that the present work is the part of the project named the *spin-charge-family* theory of one of the two authors of this paper (N.S.M.B.).

In Sect. 9.5 we comment on what we have learned from the second quantized integer spins "fermions", with the internal degrees of freedom described with Grassmann algebra, manifesting (from the point of view of $d = (3+1)$) charges in the adjoint representations and compare these recognitions with the recognitions, which the Clifford algebra is offering for the description of fermions, appearing in families of the irreducible representations of the Lorentz group in the internal — Clifford — space, with half integer spins and charges and family quantum numbers in the fundamental representations [1–3, 10–15].

9.1.1 Steps leading to second quantized Clifford fermions

We claim that when the internal part of the single particle wave functions anticommute under the Clifford algebra product $*_A$, then the wave functions with such internal part, extended with a tensor product to momentum space, anticommute as well, and so do anticommute the creation and annihillation operators, creating and annihilating the extended fermion states, assuming that the oddness of the algebra of the wave function extends to the creation and annihilation operators as presented in Subsect. 9.1.2.

If the internal part commute with respect to $*_A$ then the corresponding wave functions and the creation operators commute as well.

Let us present steps which lead to the second quantized Clifford fermions, when using the odd Clifford algebra objects to define their internal space:

i. The superposition of an odd number of the Clifford algebra elements, either of γ^a's or of $\tilde{\gamma}^a$, each with $2 \cdot (2^{\frac{d}{2}-1})^2$ degrees of freedom, is used to describe the internal space of fermions in even dimensional spaces.

ii. The "basis vectors" — the superposition of an odd number of Clifford algebra elements — are chosen to be the "eigenvectors" of the Cartan subalgebras, Eq. (9.4), of the corresponding Lorentz algebras, Eq. (9.3), in each of the two algebras.

iii. There are two groups of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding Lorentz group, for either γ^a's or for $\tilde{\gamma}^a$ algebras, each member of one group has its Hermitian conjugated partner in another group.

Making a choice of one group of "basis vectors" (for either γ^a's or for $\tilde{\gamma}^a$) to be creation operators, the other group of "basis vectors" represents the annihilation operators. The creation operators anticommute among themselves and so do anticommute annihilation operators.

iv. The vacuum state is then (for either γ^a's or for $\tilde{\gamma}^a$'s algebras) the superposition of products of annihilation $\times$ their Hermitian conjugated partners the creation operators.

The application of the creation operators on the vacuum state forms the "basis states" in each of the two spaces. The application of the annihilation operators on the vacuum state gives zero, Subsect. 9.1.2.

v. The requirement that application of $\tilde{\gamma}^a$ on γ^a gives $-i\eta^{aa}$, and the application of $\tilde{\gamma}^a$ on identity gives $i\eta^{aa}$ and that only γ^a's are used to determine the internal space of half integer fermions, Eq. (9.2.2), reduces the dimension of the Clifford algebra for a factor of two, enabling that the Cartan subalgebra of $\tilde{S}^{ab}$'s determines the "family" quantum numbers of each irreducible representation of S^{ab}'s, Eq. (9.3), and correspondingly also of their Hermitian conjugated partners.

vi. The tensor products of superposition of the finite number of members of the "basis vectors" and the infinite dimensional momentum basis, chosen to solve the Weyl equations for free massless half integer spin fermions, determine the creation and (their Hermitian conjugated partners) annihilation operators, which depend on the momenta $\vec{p}$, while $|p^0| = |\vec{p}|$ ($p^a = (p^0, p^1, p^2, p^3, p^5, \ldots, p^d)$), manifesting the properties of the observed fermions. These creation and annihilation operators fulfill on the Hilbert space all the requirements for the second quantized fermions, postulated by Dirac, Eq. (9.28) [19–21].

vii. The second quantized Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular $\vec{p}$ is a tensor product of creation operators of a particular $\vec{p}$, defining "Slater determinants" with no single particle state occupied (with no creation operators applying on the vacuum state), with one single particle state occupied (with one creation operator applying on the vacuum state), with two single particle states occupied, and so on, defining in d-dimensional space $2^{(2^{\frac{d}{2}-1})^2}$ dimensional space for each $\vec{p}$.

viii. Total Hilbert space is the infinite product ($\otimes_N$) of $\mathcal{H}_{\vec{p}}$: $\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}$. The notation $\otimes_N$ is to point out that odd algebraic products of the Clifford γ^a's operators anticommute no matter for which $\vec{p}$ they define the orthonormalized superposition of "basis vectors", solving the equations of motion as the orthonormalized plane wave solutions with $p^0 = |\vec{p}|$ and that the anticommutation character keeps also in the tensor product of internal basis and momentum basis.

Since the momentum space belonging to different $\vec{p}$ satisfy the "orthogonality" relations, the creation and annihilation operators determined by $\vec{p}$ anticommute with the creation and annihilation operators determined by any other $\vec{p}\,'$. This means that in what ever way the Hilbert space $\mathcal{H}$ is arranged, the sign is changed whenever a creation or an annihilation operator, applying on the Hilbert space $\mathcal{H}$, jumps over odd number of occupied states. No postulates for the second quantized fermions are needed in our odd Clifford space with creation and annihilation operators carrying the family quantum numbers.

x. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac's postulates for the second quantized fermions.

9.1.2 Our main assumption and definitions

(This subsection is the same as the one of Part I.)

In this subsection we clarify how does the main assumption of Part I and Part II: *the decision to describe the internal space of fermions with the "basis vectors" expressed with the superposition of odd products of the anticommuting members of the algebra*, either the Clifford one or the Grassmann one, acting algebraically, $*_A$, on the internal vacuum state $|\psi_o>$, relate to the creation and annihilation anticommuting operators of the second quantized fermion fields.

To appreciate the need for this kind of assumption, let us first have in mind that algebra with the product $*_A$ is only present in our work, usually not in other works, and thus has no well known physical meaning. It is at first a product by which you can multiply two internal wave functions B_i and B_j with each other,

$$C_k = B_i *_A B_j \,,$$
$$B_i *_A B_j = \mp B_j *_A B_i \,,$$

the sign $\mp$ depends on whether B_i and B_j are products of odd or even number of algebra elements: The sign is $-$ if both are (superposition of) odd products of algebra elements, in all other cases the sign is $+$.

Let $\mathbf{R}^{d-1}$ define the external spatial or momentum space. Then the tensor product $*_T$ extends the internal wave functions into the wave functions $\mathbf{C}_{\vec{p},\,i}$ defined in both spaces

$$\mathbf{C}_{\vec{p},\,i} = |\vec{p}> *_T |B_i>\,,$$

where again B_i represent the superposition of products of elements of the anticommuting algebras, in our case either θ^a or γ^a or $\tilde{\gamma}^a$, used in this paper.

We can make a choice of the orthogonal and normalized basis so that $<\mathbf{C}_{\vec{p},i}|\mathbf{C}_{\vec{p'},j}>= \delta(\vec{p}\vec{p'})\,\delta_{ij}$. Let us point out that either B_i or $\mathbf{C}_{\vec{p},\,i}$ apply algebraically on the vacuum state, $B_i *_A |\psi_o>$ and $\mathbf{C}_{\vec{p},\,i} *_A |\psi_o>$.

Usually a product of single particle wave functions is not taken to have any physical meaning in as far as most physicists simply do not work with such products at all.

To give to the algebraic product, $*_A$, and to the tensor product, $*_T$, defined on the space of single particle wave functions, the physical meaning, we postulate the connection between the anticommuting/commuting properties of the "basis vectors", expressed with the odd/even products of the anticommuting algebra elements and the corresponding creation operators, creating second quantized single fermion/boson states

$$
\begin{aligned}
&\hat{b}^{\dagger}_{C_{\vec{p},i}} *_A |\psi_o> = |\psi_{\vec{p},i}>\,,\\
&\hat{b}^{\dagger}_{C_{\vec{p},i}} *_T |\psi_{\vec{p'},j}> = 0\,,\\
&\qquad \text{if}\, \vec{p} = \vec{p'} \,\text{and}\, i = j\,,\\
&\text{in all other cases} \quad \text{it follows}\\
&\hat{b}^{\dagger}_{C_{\vec{p},i}} *_T \hat{b}^{\dagger}_{C_{\vec{p'},j}} *_A |\psi_o> = \mp \hat{b}^{\dagger}_{C_{\vec{p'},j}} *_T \hat{b}^{\dagger}_{C_{\vec{p},i}} *_A |\psi_o>\,,
\end{aligned}
$$

with the sign $\pm$ depending on whether $\hat{b}^{\dagger}_{C_{\vec{p},i}}$ have both an odd character, the sign is $-$, or not, then the sign is $+$.

To each creation operator $\hat{b}^{\dagger}_{C_{\vec{p},i}}$ its Hermitian conjugated partner represents the annihilation operator $\hat{b}_{C_{\vec{p},i}}$

$$
\begin{aligned}
&\hat{b}_{C_{\vec{p},i}} = (\hat{b}^{\dagger}_{C_{\vec{p},i}})^{\dagger}\,,\\
&\text{with the} \quad \text{property}\\
&\hat{b}_{C_{\vec{p},i}} *_A |\psi_o> = 0\,,\\
&\text{defining the} \quad \text{vacuum state as}\\
&|\psi_o>: = \sum_i (B_i)^{\dagger} *_A B_i\,|\,I>
\end{aligned}
$$

where summation i runs over all different products of annihilation operator $\times$ its Hermitian conjugated creation operator, no matter for what $\vec{p}$, and $|\,I>$ represents the identity, $(B_i)^{\dagger}$ represents the Hermitian conjugated wave function to B_i.

Let the tensor multiplication $*_T$ denotes also the multiplication of any number of single particle states, and correspondingly of any number of creation operators.

What further means that to each single particle wave function we define the creation operator $\hat{b}^{\dagger}_{C_{\vec{p},i}}$, applying in a tensor product from the left hand side on the second quantized Hilbert space — consisting of all possible products of any number of the single particle wave functions — adding to the Hilbert space the single particle wave function created by this particular creation operator. In the case of the second quantized fermions, if this particular wave function with the quantum numbers and $\vec{p}$ of $\hat{b}^{\dagger}_{C_{\vec{p},i}}$ is already among the single fermion wave functions of a particular product of fermion wave functions, the action of the creation operator gives zero, otherwise the number of the fermion wave functions increases for one. In the boson case the number of boson second quantized wave functions increases always for one.

If we apply with the annihilation operator $\hat{b}_{C_{\vec{p},i}}$ on the second quantized Hilbert space, then the application gives a nonzero contribution only if the particular products of the single particle wave functions do include the wave function with the quantum number i and $\vec{p}$.

In a Slater determinant formalism the single particle wave functions define the empty or occupied places of any of infinite numbers of Slater determinants. The creation operator $\hat{b}^{\dagger}_{c_{\vec{p},i}}$ applies on a particular Slater determinant from the left hand side. Jumping over occupied states to the place with its i and $\vec{p}$. If this state is occupied, the application gives in the fermion case zero, in the boson case the number of particles increase for one. The particular Slater determinant changes sign in the fermion case if $\hat{b}^{\dagger}_{c_{\vec{p},i}}$ jumps over odd numbers of occupied states. In the boson case the sign of the Slater determinant does not change.

When annihilation operator $\hat{b}_{c_{\vec{p},i}}$ applies on particular Slater determinant, it is jumping over occupied states to its own place, giving zero, if this space is empty and decreasing the number of occupied states, if this space is occupied. The Slater determinant changes sign in the fermion case, if the number of occupied states before its own space is odd. In the boson case the sign does not change.

Let us stress that choosing antisymmetry or symmetry is a choice which we make when treating fermions or bosons, respectively, namely the choice of using oddness or evenness of basis vectors, that is the choice of using odd products or even products of algebra anticummuting elements.

To describe the second quantized fermion states we make a choice of the basis vectors, which are the superposition of the odd numbers of algebra elements, of both Clifford and Grassmann algebras.

The creation operators and their Hermitian conjugation partners annihilation operators therefore in the fermion case anticommute. The single fermion states, which are the application of the creation operators on the vacuum state $|\psi_o>$, manifest correspondingly as well the oddness. The vacuum state, defined as the sum over all different products of annihilation $\times$ the corresponding creation operators, have an even character.

Let us end up with the recognition:

One usually means antisymmetry when talking about Slater-<u>determinants</u> because otherwise one would not get determinants.

In the present paper [1,2,7,10] the choice of the symmetrizing versus antisymmetrizing relates indeed the commutation versus anticommutation with respect to the a priori completely different product $*_A$, of anticommuting members of the Clifford or Grassmann algebra. The oddness or evenness of these products transfer to quantities to which these algebras extend.

9.2 Properties of Clifford algebra in even dimensional spaces

We can learn in Part I that in d-dimensional space of anticommuting Grassmann coordinates (and of their Hermitian conjugated partners — derivatives), Eqs. (2,6) of Part I, there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — both are expressible in terms of θ^a and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [2].

$$\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,$$
$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \tag{9.1}$$

offering together $2 \cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$.

Taking into account Eqs. (1,2) of Part I ($\{\theta^a, \theta^b\}_+ = 0$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, $\{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ = \delta_{ab}$, $\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial\theta_a}$ and $(\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a$) one finds

$$
\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ \,, \\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,, \\
&\quad (\gamma^a)^\dagger = \eta^{aa}\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a\,,
\end{aligned}
\tag{9.2}
$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.

It follows for the generators of the Lorentz algebra of each of the two kinds of the Clifford algebra operators, S^{ab} and $\tilde{S}^{ab}$, that:

$$
\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,, \\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,, \\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,, \\
\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- &= i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,, \\
\{S^{ab}, \tilde{\gamma}^c\}_- &= 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,,
\end{aligned}
\tag{9.3}
$$

where $\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})$, Eq. (3) of Part I.

Let us make a choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra for each of the two kinds of the operators of the Clifford algebra, S^{ab} and $\tilde{S}^{ab}$, equivalent to the choice of Cartan subalgebra of $\mathbf{S}^{ab}$ in the Grassmann case, Eq. (4) in Part I,

$$
\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d}\,, \\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d}\,.
\end{aligned}
\tag{9.4}
$$

Representations of γ^a and representations of $\tilde{\gamma}^a$ are independent, each with twice $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ irreducible representations of an odd Clifford character and with twice $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$irreducible representations of an even Clifford character in even dimensional spaces.

We make a choice for the members of the irreducible representations of the two Lorentz groups to be the "eigenvectors" of the corresponding Cartan subalgebra of Eq. (9.4), taking into account Eq. (9.2),

$$
\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned}
\tag{9.5}
$$

The Clifford "vectors" — nilpotents and projectors — of both algebras are normalized, up to a phase, with respect to Eq. (9.45) of 9.6. Both have half integer spins.

The "eigenvalues" of the operator S^{03}, for example, for the "vector" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$ are equal to $\pm \frac{i}{2}$, respectively, for the "vector" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm \frac{i}{2}$, respectively, while all the rest "vectors" have "eigenvalues" $\pm \frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}\,(\tilde{\gamma^0} \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm \frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm \frac{i}{2}$, respectively, while all the rest "vectors" have $k = \pm \frac{1}{2}$.

To make discussions easier let us introduce the notation for the "eigenvectors" of the two Cartan subalgebras, Eq. (9.4), Ref. [2,7].

$$
\begin{aligned}
&\stackrel{ab}{(k)}:=\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,, &&\stackrel{ab}{(k)}^{\dagger}=\eta^{aa}\stackrel{ab}{(-k)}\,, &&(\stackrel{ab}{(k)})^2=0\,,\\
&\stackrel{ab}{[k]}:=\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\,, &&\stackrel{ab}{[k]}^{\dagger}=\stackrel{ab}{[k]}\,, &&(\stackrel{ab}{[k]})^2=\stackrel{ab}{[k]}\,,\\
&\stackrel{ab}{(\tilde{k})}:=\frac{1}{2}(\tilde{\gamma}^a+\frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, &&\stackrel{ab}{(\tilde{k})}^{\dagger}=\eta^{aa}\stackrel{ab}{(-\tilde{k})}\,, &&(\stackrel{ab}{(\tilde{k})})^2=0\,,\\
&\stackrel{ab}{[\tilde{k}]}:=\frac{1}{2}(1+\frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, &&\stackrel{ab}{[\tilde{k}]}^{\dagger}=\stackrel{ab}{[\tilde{k}]}\,, &&(\stackrel{ab}{[\tilde{k}]})^2=\stackrel{ab}{[\tilde{k}]}\,,
\end{aligned}
\tag{9.6}
$$

with $k^2 = \eta^{aa}\eta^{bb}$. Let us notice that the "eigenvectors" of the Cartan subalgebras are either projectors

$$(\stackrel{ab}{[k]})^2=\stackrel{ab}{[k]}\,,\qquad (\stackrel{ab}{[\tilde{k}]})^2=\stackrel{ab}{[\tilde{k}]}\,,$$

or nilpotents

$$(\stackrel{ab}{(k)})^2=0\,,\qquad (\stackrel{ab}{(\tilde{k})})^2=0\,.$$

We pay attention on even dimensional spaces, $d = 2(2n + 1)$ or $d = 4n$, $n \geq 0$.

The "basis vectors", which are products of $\frac{d}{2}$ either of nilpotents or of projectors or of both, are "eigenstates" of all the members of the Cartan subalgebra, Eq. (9.4), of the corresponding Lorentz algebra, forming $2^{\frac{d}{2}-1}$ irreducible representations with $2^{\frac{d}{2}-1}$ members in each of the two Clifford algebras cases.

The "basis vectors" of Eq. (9.7) are "eigenvectors" of all the Cartan subalgebra members, Eq. (9.4), in $d = 2(2n + 1)$-dimensional space of γ^a's. The first one is the product of nilpotents only and correspondingly a superposition of an odd products of γ^a's. The second one belongs to the same irreducible representation as the first one, if it follows from the first one by the application of S^{01}, for example.

$$
\begin{aligned}
&\stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\;, \qquad \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\;,\\
&\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{d-1\,d}{[-]}\;.
\end{aligned}
\tag{9.7}
$$

One finds for their Hermitian conjugated partners, up to a sign,

$$
\begin{aligned}
&\stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)}\;, \qquad \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{(-)}\;,\\
&\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\cdots\stackrel{d-1\,d}{[-]}\;.
\end{aligned}
$$

The "basis vectors" form an orthonormal basis within each of the irreducible representations or among irreducible representations, like the product of the following annihilation and the corresponding creation operator:
$\stackrel{d-1\,d}{(-)} \cdots \stackrel{12}{(-)}\stackrel{03}{(-i)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)} \cdots \stackrel{d-1\,d}{(+)} = 1$, while all the algebraic products, which do not relate the annihilation operators with their Hermitian conjugated creation operators, give zero.

Usually the operators γ^a's are represented as matrices. We use γ^a's here to form the basis. One can find in Ref. [9] how does the application of γ^a's on the basis defined in $d = (3+1)$ look like.

9.2.1 Clifford "basis vectors" with half integer spin

In the Grassmann case the 2^{d-1} odd and 2^{d-1} even Grassmann operators, which are superposition of either odd or even products of θ^a's, are well distinguishable from their 2^{d-1} odd and 2^{d-1} even Hermitian conjugated operators, which are superposition of odd and even products of $\frac{\partial}{\partial \theta_a}$'s, Eq. (6) in Part I.

In the Clifford case the relation between "basis vectors" and their Hermitian conjugated partners (made of products of nilpotents ($\stackrel{ab}{(k)}$ or $\stackrel{ab}{(\tilde{k})}$) and projectors ($\stackrel{ab}{[k]}$ or $\stackrel{ab}{[\tilde{k}]}$), Eq. (9.6), are less transparent (although still easy to be evaluated). This can be noticed in Eq. (9.6), since $\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger$ is $\eta^{aa}\,\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{i\,(-k)}\gamma^b)$, while $(\frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b))^\dagger = \frac{1}{\sqrt{2}}(1 + \frac{i}{k}\gamma^a\gamma^b)$ is self adjoint. (This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.)

One easily sees that in even dimensional spaces, either in $d = 2(2n+1)$ or in $d = 4n$, the Clifford odd "basis vectors" (they are products of an odd number of nilpotents and an even number of projectors) have their Hermitian conjugated partners in another irreducible representation, since Hermitian conjugation changes an odd number of nilpotents (changing at the same time the handedness of the "basis vectors"), while the generators of the Lorentz transformations change two nilpotents at the same time (keeping the handedness unchanged).

The Clifford even "basis vectors" have an even number of nilpotents and can have an odd or an even number of projectors. Correspondingly an irreducible representation of an even "basis vector" can be a product of projectors only and therefore is self adjoint.

Let us recognize the properties of the nilpotents and projectors. The relations are taken from Ref. [10].

$$
\begin{aligned}
\stackrel{ab}{(k)}\stackrel{ab}{(k)} &= 0\,, \qquad & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} &= \eta^{aa}\,\stackrel{ab}{[k]}\,,\\
\stackrel{ab}{[k]}\stackrel{ab}{[k]} &= \stackrel{ab}{[k]}\,, \qquad & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} &= 0\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &= 0\,, \qquad & \stackrel{ab}{[k]}\stackrel{ab}{(k)} &= \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[-k]} &= \stackrel{ab}{(k)}\,, \qquad & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} &= 0\,.
\end{aligned}
\tag{9.8}
$$

The same relations are valid also if one replaces $\stackrel{ab}{(k)}$ with $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[k]}$ with $\stackrel{ab}{[\tilde{k}]}$, Eq. (9.6).

Taking into account Eq. (9.8) one recognizes that the product of annihilation and the creation operator from Eq. (9.7), $\stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}$, applied on a vacuum state — defined as a sum of products of all annihilation $\times$ their Hermitian conjugated partner creation operators from all irreducible representations, $\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\stackrel{78}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \cdots$, Eq. (9.18), gives a nonzero contribution, but is not the only one for a chosen creation operator. There are several other choices, like

$$\stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,$$
$$\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,$$

which also give nonzero contributions.

Let us recognize:

i. The two Clifford spaces, the one spanned by γ^a's and the second one spanned by $\tilde{\gamma}^a$'s, are independent vector spaces, each with 2^d "vectors".

ii. The Clifford odd "vectors" (the superposition of products of odd numbers of γ^a's or $\tilde{\gamma}^a$'s, respectively) can be arranged for each kind of the Clifford algebras into two groups of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding Lorentz group. The two groups are Hermitian conjugated to each other.

iii. Different irreducible representations are indistinguishable with respect to the "eigenvalues" of the corresponding Cartan subalgebra members.

iv. The Clifford even part (made of superposition of products of even numbers of γ^a's and $\tilde{\gamma}^a$'s, respectively) splits as well into twice $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group. One member of each Clifford even representation, the one which is the product of projectors only, is self adjoint. Members of one irreducible representation are with respect to the Cartan subalgebra indistinguishable from all the other irreducible representations.

v. The $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations are orthogonal to one another and so are orthogonal their corresponding Hermitian conjugated partners. For illustration of the orthogonality one can look at Table 9.1, and recognize that any "basis vector" of the first four multiplets of *odd I*, if multiplied from the left hand side or from the right hand side with any other "basis vector" from the rest three "families" of *odd I* get zero when taking into account Eq. (9.8). One can repeat this also for any "basis vectors" of all the "families" of *odd I*, as well as among all the "basis vectors" within *odd II*. Generalization to any even dimension d is straightforward.

vi. Denoting "basis vectors" by $\hat{b}^{m\dagger}_f$, (where f defines different irreducible representations and m a member in the representation f), and their Hermitian conjugate partners by $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$, let us start for $d = 2(2n+1)$ with

$$\begin{aligned}\hat{b}^{m=1\dagger}_{f=1} &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,\\ (\hat{b}^{m=1\dagger}_{f=1})^\dagger = \hat{b}^{m=1}_{f=1} &:= \stackrel{d-1\,d}{(-)}\cdots\stackrel{12}{(-)}\stackrel{03}{(-i)}\,,\end{aligned} \tag{9.9}$$

and making a choice of the vacuum state $|\psi_{oc}>$ as a sum of all the products of $\hat{b}^m_f \cdot \hat{b}^{m\dagger}_f$ for all $f = (1, 2, \cdots, 2^{\frac{d}{2}-1})$, one recognizes for the "basis vectors" of an odd Clifford character for each of the two Clifford algebras the properties

$$\begin{aligned}
\hat{b}^m_f{}_{*_A} |\psi_{oc}> &= 0\, |\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_f{}_{*_A} |\psi_{oc}> &= |\psi^m_f>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A +} |\psi_{oc}> &= 0\, |\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m\dagger}_f\}_{*_A +} |\psi_{oc}> &= |\psi_{oc}>\,.
\end{aligned} \tag{9.10}$$

$*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m'}_{f'}$ among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(9.18), which takes into account Eq. (9.2). All the products of Clifford algebra elements are up to now the algebraic ones and so are also the products in Eq. (9.10). Since we use here anticommutation relations, we pointed out with $*_A$ this algebraic character of the products, to be later distinguished from the tensor product $*_T$, when the creation and annihilattion operators are defined on an extended basis, which is the tensor product of the superposition of the "basis vectors" of the Clifford space and of the momentum basis, applying on the Hilbert space of "Slater determinants". The tensor product $*_T$ is used as well as the product mapping a pair of the fermion wave functions in to two fermion wave functions and further to many fermion wave functions — that is to the extended algebra of many fermion system.

Obviously, $\hat{b}^{m\dagger}_f$ and $\hat{b}^m_f$ have on the level of the algebraic products, when applying on the vacuum state $|\psi_{oc}>$, *almost* the properties of creation and annihilation operators of the second quantized fermions in the postulates of Dirac, as it is discussed in the next items. We illustrate properties of "basis vectors" and their Hermitian conjugated partners on the example of $d = (5+1)$-dimensional space in Subsect. 9.2.5.

vii. a. There is, namely, the property, which the second quantized fermions should fulfill in addition to the relations of Eq. (9.10). The anticommutation relations of creation and annihilation operators should be:

$$\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A +} |\psi_{oc}> = \delta^{mm'} \delta_{ff'} |\psi_{oc}>\,. \tag{9.11}$$

For any $\hat{b}^m_f$ and any $\hat{b}^{m'\dagger}_{f'}$ this is not the case; besides $\hat{b}^{m=1}_{f=1} = \overset{d-1\, d}{(-)} \cdots \overset{56}{(-)}\overset{12}{(-)}\overset{03}{(-i)}$, for example, also

$$\hat{b}^{m'}_{f'} = \overset{d-1\, d}{(-)} \cdots \overset{56}{(-)}\overset{12}{[+]}\overset{03}{[+i]}\,,$$

and several others give, when applied on $\hat{b}^{m=1\dagger}_{f=1}$, nonzero contributions. There are namely $2^{\frac{d}{2}-1} - 1$ too many annihilation operators for each creation operator, which give, applied on the creation operator, nonzero contribution.

vii. b. To use the Clifford algebra objects to describe second quantized fermions, representing the observed quarks and leptons as well as the antiquarks and antileptons [3,10–15,17], *the families should exist*.

vii. c. The operators should exist, which connect one irreducible representation of fermions with all the other irreducible representations.

vii. d. Two independent choices for describing the internal degrees of freedom of the observed quarks and leptons are not in agreement with the observed properties of fermions.

We solve these problems, cited in vii. a., vii. b., vii. c. and vii. d., by reducing the degrees of freedom offered by the two kinds of the Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s, making a choice of one — γ^a's — to describe the internal space of fermions, and using the other one — $\tilde{\gamma}^a$'s — to describe the "family" quantum number of each irreducible representation of S^{ab}'s in space defined by γ^a's.

9.2.2 Reduction of the Clifford space by the postulate

The creation and annihilation operators of an odd Clifford algebra of both kinds, of either γ^a's or $\tilde{\gamma}^a$'s, would obviously obey the anticommutation relations for the second quantized fermions, postulated by Dirac, at least on the vacuum state, which is a sum of all the products of annihilation times, $*_A$, the corresponding creation operators, provided that each of the irreducible representations would carry a different quantum number.

But we know that a particular member m has for all the irreducible representations the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either γ^a's or $\tilde{\gamma}^a$'s), Eq. (9.6).

The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces, let us make a choice of $\tilde{\gamma}^a$'s, *and use* $\tilde{\gamma}^a$'s *to define the "family" quantum number for each irreducible representation of the vector space of* γ^a's, while *keeping the relations of* Eq. (9.2) *unchanged:* $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $(\gamma^a)^\dagger = \eta^{aa}\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\tilde{\gamma}^a$, $(a,b) = (0,1,2,3,5,\cdots,d)$.

We therefore *postulate*:

Let $\tilde{\gamma}^a$'s operate on γ^a's as follows [2,3,8,14,15]

$$\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\,, \tag{9.12}$$

with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [8].

After this postulate the vector space of $\tilde{\gamma}^a$'s is correspondingly "frozen out". No vector space of $\tilde{\gamma}^a$'s needs to be taken into account any longer, in agreement with the observed properties of fermions. This solves the problems vii.a - vii. d. of Subsect. 9.2.1.

Taking into account Eq. (9.12) we can check that:

a. Relations of Eq. (9.2) remain unchanged [1].

b. Relations of Eq. (9.3) remain unchanged [2].

[1] Let us show that the relation $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}$ remains valid when applied on B, if B is either an odd or an even product of γ^a's: $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\, \gamma^c = -i\,(\tilde{\gamma}^a\gamma^c\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^a) = -i\, i\,\gamma^c(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c$, while $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\, \gamma^c\gamma^d = i\,(\tilde{\gamma}^a\gamma^c\gamma^d\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^d\gamma^a) = i(-i)\,\gamma^c\gamma^d(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c\gamma^d$. The relation is valid for any γ^c and γ^d, even if $c = d$.

[2] One easily checks that $\tilde{\gamma}^{a\dagger}\gamma^c = -i\gamma^c\gamma^{a\dagger} = -i\eta^{aa}\gamma^c\gamma^a = \eta^{aa}\tilde{\gamma}^a\gamma^c = -i\eta^{aa}\gamma^c\gamma^a$.

c. The eigenvalues of the operators S^{ab} and $\tilde{S}^{ab}$ on nilpotents and projectors of γ^a's are after the reduction of Clifford space

$$S^{ab} \stackrel{ab}{(k)}= \frac{k}{2} \stackrel{ab}{(k)}, \qquad \tilde{S}^{ab} \stackrel{ab}{(k)}= \frac{k}{2} \stackrel{ab}{(k)},$$
$$S^{ab} \stackrel{ab}{[k]}= \frac{k}{2} \stackrel{ab}{[k]}, \qquad \tilde{S}^{ab} \stackrel{ab}{[k]}= -\frac{k}{2} \stackrel{ab}{[k]}, \tag{9.13}$$

demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$, so that $\tilde{S}^{ab}$ can be used to denote irreducible representations of S^{ab} with the "family" quantum number, what solves the problems vii. b. and vii. c. of Subsect. 9.2.1.

d. We further recognize that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]},$$
$$\gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)},$$
$$\tilde{\gamma^a} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma^b} \stackrel{ab}{(k)}= -k \stackrel{ab}{[k]},$$
$$\tilde{\gamma^a} \stackrel{ab}{[k]}= \ i \stackrel{ab}{(k)}, \quad \tilde{\gamma^b} \stackrel{ab}{[k]}= -k\eta^{aa} \stackrel{ab}{(k)} \ . \tag{9.14}$$

e. One finds, using Eq. (9.12),

$$\stackrel{ab}{(\tilde{k})} \stackrel{ab}{(k)} = 0, \quad \stackrel{ab}{(\tilde{-k})} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]},$$
$$\stackrel{ab}{(\tilde{k})} \stackrel{ab}{[k]} = i \stackrel{ab}{(k)}, \quad \stackrel{ab}{(\tilde{k})} \stackrel{ab}{[-k]}= 0,$$
$$\stackrel{ab}{[\tilde{k}]} \stackrel{ab}{(k)} = \stackrel{ab}{(k)}, \quad \stackrel{ab}{[\tilde{-k}]} \stackrel{ab}{(k)}= 0,$$
$$\stackrel{ab}{[\tilde{k}]} \stackrel{ab}{[k]} = 0, \quad \stackrel{ab}{[\tilde{-k}]} \stackrel{ab}{[k]}= \stackrel{ab}{[k]} \ . \tag{9.15}$$

f. From Eq. (9.14) it follows

$$
\begin{aligned}
S^{ac} \overset{ab}{(k)}\overset{cd}{(k)} &= -\frac{i}{2}\eta^{aa}\eta^{cc} \overset{ab}{[-k]}\overset{cd}{[-k]}\,, \\
\tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{(k)} &= \frac{i}{2}\eta^{aa}\eta^{cc} \overset{ab}{[k]}\overset{cd}{[k]}\,, \\
S^{ac} \overset{ab}{[k]}\overset{cd}{[k]} &= \frac{i}{2} \overset{ab}{(-k)}\overset{cd}{(-k)}\,, \\
\tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{[k]} &= -\frac{i}{2} \overset{ab}{(k)}\overset{cd}{(k)}\,, \\
S^{ac} \overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \overset{ab}{[-k]}\overset{cd}{(-k)}\,, \\
\tilde{S}^{ac} \overset{ab}{(k)}\overset{cd}{[k]} &= -\frac{i}{2}\eta^{aa} \overset{ab}{[k]}\overset{cd}{(k)}\,, \\
S^{ac} \overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc} \overset{ab}{(-k)}\overset{cd}{[-k]}\,, \\
\tilde{S}^{ac} \overset{ab}{[k]}\overset{cd}{(k)} &= \frac{i}{2}\eta^{cc} \overset{ab}{(k)}\overset{cd}{[k]}\;.
\end{aligned}
\tag{9.16}
$$

g. Each irreducible representation has now the ″family″ quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (9.4). Correspondingly the creation and annihilation operators fulfill algebraically the anticommutation relations of Dirac second quantized fermions: Different irreducible representations carry different ″family″ quantum numbers and to each ″family″ quantum member only one Hermitian conjugated partner with the same ″family″ quantum number belong. Also each summand of the vacuum state, Eq. (9.18), belongs to a particular ″family″. This solves the problem vii. a. of Subsect. 9.2.1.

The anticommutation relations of Dirac fermions are therefore fulfilled on the vacuum state, Eq. (9.18), on the algebraic level, without postulating them. They follow by themselves from the fact that the creation and annihilation operators are superposition of odd products of γ^a's.

Statement 1: The oddness of the products of γ^a's guarantees the anticommuting properties of all objects which include odd number of γ^a's.

We shall show in Subsect. 9.2.4 of this section, and in Sect. 9.3, that the same relations are valid also on the Hilbert space of all the second quantized fermions states, with the creation operators defined on the tensor product of ″basis vectors″ of the Clifford algebra and on the basis of the momentum space, where the Hilbert space is defined with the creation operators of all possible momenta of all possible ″Slater determinants″ applying on $|\psi_{oc}>$.

Let us write down the anticommutation relations of Clifford odd ″basic vectors″, representing the creation operators and of the corresponding annihilation operators again.

$$
\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A +}\, |\psi_{oc}> &= \delta^{m m'}\, \delta_{f f'}\, |\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A +}\, |\psi_{oc}> &= 0\,\cdot\, |\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A +}\, |\psi_{oc}> &= 0\,\cdot\, |\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f}\,{}_{*_A} |\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\hat{b}^{m}_{f}\,{}_{*_A} |\psi_{oc}> &= 0\,\cdot\, |\psi_{oc}>\,,
\end{aligned}
\tag{9.17}
$$

with (m, m') denoting the "family" members and (f, f') denoting "families", $*_A$ represents the algebraic multiplication of $\hat{b}^{m}_{f}$ with the vacuum state $|\psi_{oc}>$ of Eq.(9.18) and among themselves, taking into account Eq. (9.2).

h. The vacuum state for the vector space determined by γ^a's remains unchanged $|\psi_{oc}>$, Eq. (80) of Ref. [3], it is a sum of the products of any annihilation operator with its Hermitian conjugated partner of any family.

$$
\begin{aligned}
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-1\;d}{[-]}\\
&+ \quad \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]}\cdots\overset{d-1\;d}{[-]} + \cdots |1>\,,\\
&\quad \text{for } d = 2(2n+1)\,,\\
|\psi_{oc}> &= \overset{03}{[-i]}\overset{12}{[-]}\overset{35}{[-]}\cdots\overset{d-3\;d-2}{[-]}\overset{d-1\;d}{[+]}\\
&+ \quad \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]}\cdots\overset{d-3\;d-2}{[-]}\overset{d-1\;d}{[+]} + \cdots |1>\,,\\
&\quad \text{for } d = 4n\,,
\end{aligned}
\tag{9.18}
$$

n is a positive integer.

i. Taking into account the relation among θ^a in Eq. (9.1) and Eq. (9.12), requiring that $\tilde{\gamma}^a a_0 = i a_0 \gamma^a$, leads to $\frac{\partial}{\partial \theta_a} = 0$, and further to

$$
\theta^a = \gamma^a\,. \tag{9.19}
$$

Eq. (9.12)) namely requires: $\tilde{\gamma}^a(a_0 + a_b\gamma^b + a_{bc}\gamma^b\gamma^c + \cdots) = (i a_0\gamma^a + (-i) a_b \gamma^b\gamma^a + i a_{bc}\gamma^b\gamma^c\gamma^a + \cdots)$, what means that Eq. (9.19) is only one of the relations [3] The application of $\tilde{\gamma}^a$ depends on the space on which it applies.

The Hermitian conjugated part of the space in the Grassmann case is "freezed out" together with the "vector" space of $\tilde{\gamma}^a$'s.

9.2.3 Clifford fermions with families

Let us make a choice of the starting creation operator $\hat{b}^{1\dagger}_{1}$ of an odd Clifford character and of its Hermitian conjugated partner in $d = 2(2n+1)$ and $d = 4n$,

[3] Another relation, for example, is $\tilde{\gamma}^a\gamma^a = (-i)\gamma^a\gamma^a = -i\eta^{aa}$. One also has $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b = 2\eta^{ab}$. $\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$. $\{\tilde{\gamma}^a, \gamma^a\}_+ = 0 = \tilde{\gamma}^a\gamma^a + \gamma^a\tilde{\gamma}^a = \gamma^a(-i\gamma^a + \gamma^a i\gamma^a = 0$.

respectively, as follows

$$
\begin{aligned}
\hat{b}^{1\dagger}_{1} &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\;d-2}{(+)}\;\;\stackrel{d-1\;d}{(+)}\;,\\
(\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1} &:= \stackrel{d-1\;d}{(-)}\;\;\stackrel{d-3\;d-2}{(-)}\cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,,\\
d &= 2(2n+1)\,,\\
\hat{b}^{1\dagger}_{1} &:= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots \stackrel{d-3\;d-2}{(+)}\;\;\stackrel{d-1\;d}{[+]}\;,\\
(\hat{b}^{1\dagger}_{1})^{\dagger} = \hat{b}^{1}_{1} &:= \stackrel{d-1\;d}{[+]}\;\;\stackrel{d-3\;d-2}{(-)}\cdots \stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,,\\
d &= 4n\,.
\end{aligned}
\tag{9.20}
$$

All the rest "vectors", belonging to the same Lorentz representation, follow by the application of the Lorentz generators S^{ab}'s.

The representations with different "family" quantum numbers are reachable by $\tilde{S}^{ab}$, since, according to Eq. (9.16), we recognize that $\tilde{S}^{ac}$ transforms two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into two projectors $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$, without changing k ($\tilde{S}^{ac}$ transforms $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$ into $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$, as well as $\stackrel{ab}{[k]}\stackrel{cd}{(k)}$ into $\stackrel{ab}{(k)}\stackrel{cd}{[k]}$). All the "family" members are reachable from one member of a new family by the application of S^{ab}'s.

In this way, by starting with the creation operator $\hat{b}^{1\dagger}_{1}$, Eq. (9.20), $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members follow.

Let us find the starting member of the next "family" to the "family" of Eq. (9.20) by the application of $\tilde{S}^{01}$

$$
\begin{aligned}
\hat{b}^{1\dagger}_{2} &:= \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)}\cdots \stackrel{d-3\;d-2}{(+)}\;\;\stackrel{d-1\;d}{(+)}\;,\\
\hat{b}^{1}_{2} &:= \stackrel{d-1\;d}{(-)}\;\;\stackrel{d-3\;d-2}{(-)}\cdots \stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{01}{[+i]}\;.
\end{aligned}
\tag{9.21}
$$

The corresponding annihilation operators, that is the Hermitian conjugated partners of $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members, following from the starting creation operator $\hat{b}^{1\dagger}_{1}$ by the application of S^{ab}'s — the family members — and the application of $\tilde{S}^{ab}$ — the same family member of another family — can be obtained by Hermitian conjugation.

The creation and annihilation operators of an odd Clifford character, expressed by nilpotents and projectors of γ^a*'s, obey anticommutation relations of* Eq. (9.17), *without postulating the second quantized anticommutation relations* as we explain in Subsect. 9.2.2.

The even partners of the Clifford odd creation and annihilation operators follow by either the application of γ^a on the creation operators, leading to $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members, or with the application of $\tilde{\gamma}^a$ on the creation operators, leading to another group of the Clifford even operators, again with the $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members.

It is not difficult to recognize, that each of the Clifford even "families", obtained by the application of γ^a or by $\tilde{\gamma}^a$ on the creation operators, contains one selfadjoint operator, which is the product of projectors only, contributing as a summand to the vacuum state, Eq. (9.18).

9.2.4 Action for free massless Clifford fermions with half integer spin and solutions of Weyl equations

To relate the creation operators, expressed with the Clifford odd "basis vectors", and the creation operators, creating the second quantized fermions, we define the tensor products of the finite number of odd Clifford "basis vectors" and infinite basis of momentum space. To compare properties of our creation operators of the second quantized fermions with those of Dirac, the solution of the equations of motion of the Weyl (for massless free fermions) or of the Dirac equations are appropriate.

The Lorentz invariant action for a free massless fermion in Clifford space is well known

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^{\dagger}\gamma^0 \, \gamma^a p_a \psi) + \text{h.c.} \, , \tag{9.22}$$

$p_a = i \, \frac{\partial}{\partial x^a}$, leading to the equation of motion

$$\gamma^a p_a |\psi> = 0 \, , \tag{9.23}$$

and to the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> = p^a p_a |\psi>= 0 \, ,$$

γ^0 appears in the action to take care of the Lorentz invariance of the action.

Our Clifford algebra "basis vectors" offer the description of only the internal degrees of freedom of fermions (in $d = (3+1)$ the "basis vectors" offers the description of only the spin and family degrees of freedom, in $d \geq 5$ also of the charges [4, 10, 11, 15] and the references therein).

We need to extend the internal degrees of freedom (offering final number — $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ — of basis vectors of the odd products of γ^a) to the momentum or coordinate space with (infinite number of) basis.

Statement 2: For deriving the anticommutation relations for the Clifford fermions, to be compared with the anticommutation relations of the second quantized fermions, we need to define the tensor product of the Clifford odd "basis vectors" and the momentum space

$$\mathbf{basis}_{(p^a, \gamma^a)} = |p^a> *_T |\gamma^a> \, .$$

The new state vector space is the tensor product of the internal space of fermions and the space of momenta or coordinates. All states have an odd Clifford character due to oddness of the internal space.

Solutions of Eq. (9.23) for free massless fermions of momentum $p^a, a = (0, 1, 2, 3, 5, \ldots, d)$ are superposition of "basis vectors" $\hat{b}^{m\dagger}_f$, expressed by operators γ^a, where f denotes a "family" and m a "family" member quantum number, Eqs. (9.20, 9.21), and of plane waves in the case of free, in our case, massless fermions. The equations of motion require that $|p^0| = |\vec{p}|$. Correspondingly it follows

$$
\begin{aligned}
< x|\psi^{\mathbf{sf}}(\tilde{\mathbf{p}},\mathbf{p^0}) > |_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} &= \int dp^0 \delta(p^0 - |\vec{p}|)\, \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-ip_a x^a} \; *_A |\psi_{oc}> \\
&= (\hat{\mathbf{b}}^{sf\dagger}(\vec{p}) \cdot \; e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})})|_{p^0=|\vec{p}|} \; *_A \; |\psi_{oc}>, \\
&\quad \text{where we define}, \\
\hat{\mathbf{b}}^{sf\dagger}(\vec{p})|_{p^0=|\vec{p}|} &\stackrel{\text{def}}{=} \sum_m c^{sf}{}_m \; (\vec{p}, |p^0| = |\vec{p}|) \; \hat{b}^{m\dagger}_f, \\
|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}},\mathbf{x^0}) > &= \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} \; (\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|} \; *_A \; |\psi_{oc}>,
\end{aligned}
\tag{9.24}
$$

s represents different orthonormalized solutions of the equations of motion, $\varepsilon = \pm 1$, depending on handedness and spin of solutions, $c^{sf}{}_m(\vec{p}, |p^0| = |\vec{p}|)$ are coefficients, depending on momentum $|\vec{p}|$ with $|p^0| = |\vec{p}|$, while $*_A$ denotes the algebraic multiplication of the "basis vectors" $\hat{b}^{m\dagger}_f$ on the vacuum state $|\psi_{oc}>$, Eq. (9.17).

An illustration of $\hat{\mathbf{b}}^{sf\dagger}(\vec{p})$ is presented in Subsect. 9.2.5.

Since the "basis vectors" in internal space of fermions are orthogonal according to Eq. (9.10) ($\{\hat{b}^m_f \, {}_{*_A}\, , \hat{b}^{m'\dagger}_{f'} \, {}_{*_A}\}_+|\psi_{oc}>= \hat{b}^m_f \, {}_{*_A} \, \hat{b}^{m'\dagger}_{f'} \, {}_{*_A}|\psi_{oc}>$),

$$
\begin{aligned}
&\hat{b}^m_f \, {}_{*_A} \, \hat{b}^{m'\dagger}_{f'} \, {}_{*_A}|\psi_{oc}>= \delta^{mm'}\delta_{ff'}\,|\psi_{oc}>, \\
&\text{it follows for particular } \vec{p}\,, p^0 = |\vec{p}|, \text{ that} \\
&\sum_m c^{sf*}{}_m(\vec{p}, |p^0| = |\vec{p}|) \;\; c^{s'f'}{}_m(\vec{p}, |p^0| = |\vec{p}|) = \delta^{ss'}\delta_{ff'}\,, \\
&\text{leading to} \\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} \; < \psi^{s'\mathbf{f}'}(\vec{p'}, p'^0 = |\vec{p'}|)|x >< x||\, \psi^{\mathbf{sf}}(\vec{p}, p^0 = |\vec{p}|) >= \\
&\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{ip'_a x^a}|_{p'^0=|\vec{p'}|}\, e^{-ip_a x^a}|_{p^0|=|\vec{p}|} \\
&\cdot < \psi_{oc}|\,(\hat{\mathbf{b}}^{s'f'}(\vec{p'})\; \hat{\mathbf{b}}^{sf\dagger}(\vec{p}))\,{}_{*_A}\,|\psi_{oc}>= \; \delta ss'\delta^{ff'}\delta(\vec{p} - \vec{p'})\,,
\end{aligned}
\tag{9.25}
$$

while we take into account that $\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}}\; e^{ip'_a x^a}\; e^{-ip_a x^a} \; = \delta(\vec{p} - \vec{p'})$.

Let us now evaluate the scalar product $< \psi^{\mathbf{sf}}(\vec{x}, x^0)\,|\, \psi^{\mathbf{s'f'}}(\vec{x}\,', x^0) >$, taking into account that the scalar product is evaluated at a time x^0 and correspondingly using the relation

$$
\begin{aligned}
&< \psi^{\mathbf{sf}}(\vec{x}, x^0)|\, \psi^{\mathbf{s'f'}}(\vec{x}\,', x^0) >= \delta^{ss'}\; \delta_{ff'}\, \delta(\vec{x} - \vec{x}\,') = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int \frac{dp'^0}{\sqrt{2\pi}} \delta(p^0 - p'^0) \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} \delta(p^0 - |\vec{p}|)\, \delta(p'^0 - |\vec{p'}|) \\
&< \psi_{oc}|(\hat{\mathbf{b}}^{s'f'}(\vec{p'}, p'^0)\, \hat{\mathbf{b}}^{sf\dagger}(\vec{p}, p^0))\,{}_{*_A}\,|\psi_{oc} > e^{ip'_a x'^a}\; e^{-ip_a x^a} \; = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}}\; \delta(p^0 - |\vec{p'}|) \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\; \delta(p^0 - |\vec{p}|) \\
&< \psi_{oc}|(\hat{\mathbf{b}}^{sf}(\vec{p}, p^0)\, \hat{\mathbf{b}}^{s'f'\dagger}(\vec{p'}, p^0))\,{}_{*_A}\,|\psi_{oc} > e^{i(p^0x^0 - \vec{p}\cdot\vec{x})}\; e^{-i(p^0x'^0 - \vec{p'}\cdot\vec{x})}\,.
\end{aligned}
\tag{9.26}
$$

The scalar product $< \psi^{\mathbf{sf}}(\vec{x}, x^0) \,|\, \psi^{\mathbf{s'f'}}(\vec{x}\,', x^0) >$ has obviously the desired properties of the second quantized states.

Let us define the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, which determine, when applying on the vacuum state, the fermion states, Eq. (9.24),

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &\stackrel{\text{def}}{=} \hat{\mathbf{b}}^{sf\dagger}(\vec{p})\, e^{-i(p^0x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) &= (\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}))^{\dagger} = \hat{\mathbf{b}}^{sf}(\vec{p})\, e^{i(p^0x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,|\psi_{oc}> &= |\,\psi^{\mathbf{sf}}(\vec{p}, p^0=|\vec{p}|) > \,.
\end{aligned} \tag{9.27}$$

In Eq. (9.27) $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ creates on the vacuum state $|\psi_{oc}>$ the single fermion states. We can multiply, using the tensor product $*_T$ multiplication this time, an arbitrary number of such single particle states, what means that we multiply an arbitrary number of creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p'}) *_T \cdots *_T \hat{\underline{\mathbf{b}}}^{s''f''\dagger}_{tot}(\vec{p''})$, applying on $|\psi_{oc}>$, which gives nonzero contributions, provided that they distinguish among themselves in at least one of the properties $(s, f, \vec{p})$, in the internal space quantum numbers (s, f) or in momentum part $\vec{p}$, due to the orthonormal property of plane waves.

The space of all such functions, which one can form - including the identity - represents the second quantized Hilbert space. We present these tensor products as "Slater determinants" of occupied and empty states in Section 9.3.

Due to anticommutation relations of any two of creation operators

$$\{\hat{\mathbf{b}}^{sf\dagger}(\vec{p})\,,\,\hat{\mathbf{b}}^{s'f'}(\vec{p})\}_{+}\,|\psi_{oc}>= \delta^{ff'}\delta^{ss'}\,|\psi_{oc}>,$$

Eqs. (9.17, 9.24), while plane waves form the orthonormal basis in the momentum representation, Eq. (9.25), the new creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, which are are generated on the tensor products of both spaces, internal and momentum, fulfill the anticommutation relations when applied on $|\psi_{oc}>$.

$$\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,')\}_{+}\,{*_T}|\psi_{oc}> &= \delta^{ss'}\,\delta_{ff'}\,\delta(\vec{p}-\vec{p'})\,|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})\}_{+}\,{*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,,\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{+}\,{*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,{*_T}|\psi_{oc>} &= |\psi^{sf}(\vec{p})>\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,{*_T}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{9.28}$$

It is not difficult to show that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ manifest the same anticommutation relations also on tensor products of an arbitrary chosen set of single fermion states, what we discuss in Sect. 9.3.

Therefore, with the choice of the Clifford odd "basis states" to describe the internal space of fermions (we can proceed equivalently in the Grassmann case) and using the tensor product of the internal space and the momentum or coordinate space to solve the equations of motion, we derive the anticommutation relations among creation operators $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $(\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}))^{\dagger} = \hat{\mathbf{b}}^{sf}(\vec{p})\, e^{i(p^0x^0 - \vec{p},\cdot\vec{x})} = \underline{\hat{\mathbf{b}}}^{sf}_{tot}(\vec{p})$, with $|p^0| = |\vec{p}|$. While application of $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ on $|\psi_{oc}>$ generates the single fermion state, the application of $\underline{\hat{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ gives zero.

We shall demonstrate in Sect. 9.3 that there is $\{\underline{\hat{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})\,,\, \underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\}_{+}$, which when applied on the Hilbert space of the second quantized fermions (that is on tensor products of all single fermion states, or equivalently on all possible "Slater determinants"), gives zero when at least one of $(s', f', \vec{p'})$ differ from $(s, f, \vec{p})$, while $\{\underline{\hat{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\, \underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\}_{+}$ applied on the Hilbert space, gives the whole Hilbert space back.

Taking into account the last line of Eq. (9.24) and Eqs. (9.26,9.27), the creation operators $\underline{\Psi}^{\dagger}$ follow, which determine, when applying on the vacuum state $|\psi_{oc}>$, the fermion fields $|\psi^{\mathbf{sf}}(\tilde{\mathbf{x}}, \mathbf{x^0})>$, depending on coordinates at particular time x^0

$$
\begin{aligned}
\underline{\Psi}^{sf\dagger}(\vec{x}, x^0) &\overset{\text{def}}{=} \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})_{|p^0|=|\vec{p}|}\,, \\
\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,|\psi_{oc}> &= |\,\psi^{\mathbf{sf}}(\vec{x}, x^0)>\,, \\
\{\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,, \underline{\Psi}^{s'f'}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= \delta^{ss'}\delta^{ff'}\delta(\vec{x} - \vec{x'})|\psi_{oc}>\,, \\
\{\underline{\Psi}^{sf}(\vec{x}, x^0)\,, \underline{\Psi}^{s'f'}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= 0\,. \\
\{\underline{\Psi}^{sf\dagger}(\vec{x}, x^0)\,, \underline{\Psi}^{s'f'\dagger}(\vec{x'}, x^0)\}_{+}\,|\psi_{oc}> &= 0\,,
\end{aligned} \tag{9.29}
$$

where $\underline{\Psi}^{\dagger}(\vec{x}, x^0)$ and $\underline{\Psi}^{sf}(\vec{x}, x^0)$ are creation and annihilation partners, respectively, Hermitian conjugated to each other, in the coordinate representation, presenting the creation and annihilation operators of the second quantized fields.

The application of the creation operators $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})_{|p^0|=|\vec{p}|}$ and $\underline{\Psi}^{\dagger}(\vec{x}, x^0)$ and their Hermitian conjugated partners on the Hilbert space of fermion fields will be discussed in Sect. 9.3.

Dirac uses the Lagrange and Hamilton formalism for fermion fields and assuming that the second quantized states should anticommute to describe fermions, he derives the anticommuting creation and annihillation operators. In Subsect. 9.3.4 we compare the Dirac anticommutation relations with our way of deriving anticommutation relations for second quantized fields in details.

In Subsect. 9.2.5 the properties of creation and annihilation operators, $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\underline{\hat{\mathbf{b}}}^{sf}_{tot}(\vec{p})$, respectively, described by the odd Clifford algebra objects in $d = (5+1)$-dimensional space are discussed.

9.2.5 Illustration of Clifford fermions with families in $d = (5+1)$-dimensional space

We illustrate properties of the Clifford odd, and correspondingly anticommuting, creation and their Hermitian conjugated partners annihilation operators, belonging to $2^{\frac{6}{2}-1} = 4$ "families", each with $2^{\frac{6}{2}-1} = 4$ members in $d = (5+1)$-dimensional space. The spin in the fifth and the sixth dimension manifests as the charge in $d = (3+1)$.

In Table 9.1 the "basis vectors" of odd and even Clifford character are presented. They are "eigenvectors" of the Cartan subalgebras, Eq. (9.4).

Half of the Clifford odd "basis vectors" are (chosen to be) creation operators $\hat{b}^{m\dagger}_{f}$, denoted in table by *odd I*, appearing in four "families", $f = (1(a), 2(b), 3(c), 4(d))$. The rest half of the Clifford odd "basis vectors" are their Hermitian conjugated partners $\hat{b}^{m}_{f}$, presented in *odd II* part and denoted with the corresponding "family" and family members (a_m, b_m, c_m, d_m) quantum numbers.

The normalized vacuum state is the product of $\hat{b}^{m}_{f} \cdot \hat{b}^{m\dagger}_{f}$ — this product is the same for each member of a particular family and different for different families — summed over four families

$$
\begin{aligned}
|\psi_{oc}> = \frac{1}{\sqrt{2^{\frac{6}{2}-1}}} \, & (\overset{03}{[-i]}\overset{12}{[-1]}\overset{56}{[-1]} + \overset{03}{[+i]}\overset{12}{[+1]}\overset{56}{[-1]} \\
& + \overset{03}{[+i]}\overset{12}{[-1]}\overset{56}{[+1]} + \overset{03}{[-i]}\overset{12}{[+1]}\overset{56}{[+1]}) . \qquad (9.30)
\end{aligned}
$$

One easily checks, by taking into account Eq. (9.15), that the creation operators $\hat{b}^{m\dagger}_{f}$ and the annihilation operators $\hat{b}^{m}_{f}$ fulfill the anticommutation relations of Eq (9.17).

The summands of the vacuum state $|\psi_{oc}>$ appear among selfadjoint members of *even I* part of Table 9.1, each of summands belong to different "family" [4].

All the Clifford even "families" with "family" members of Table 9.1 can be obtained as algebraic products, $*_A$, of the Clifford odd "vectors" of the same table.

Let us find the solutions of the Weyl equation, Eq. (9.23), taking into account four basis creation operators of the first family, $f = 1(a)$, in Table 9.1. Assuming that moments in the fifth and the sixth dimensions are zero, $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, the following four plane wave solutions for positive energy, $p^0 = |\vec{p}|$, can be found, two with the positive charge $\frac{1}{2}$ and with spin S^{12} either equal to $\frac{1}{2}$ or to $-\frac{1}{2}$, and two with the negative charge $-\frac{1}{2}$ and again with S^{12} either $\frac{1}{2}$ or $-\frac{1}{2}$.

$$
\begin{aligned}
& \text{Clifford odd creation operators in } d = (5+1) \\
& p^0 = |p^0| \,, \quad S^{56} = \frac{1}{2} \,, \quad \Gamma^{(3+1)} = 1 \,, \\
& \left(\underline{\hat{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}) = \beta \left(\overset{03}{(+i)}\,\overset{12}{(+)} \,|\, \overset{56}{(+)} + \frac{p^1 + i p^2}{|p^0| + |p^3|} \, \overset{03}{[-i]}\,\overset{12}{[-]} \,|\, \overset{56}{(+)} \right) \right) \cdot \\
& \quad e^{-i(|p^0| x^0 - \vec{p} \cdot \vec{x})} \,, \\
& \left(\underline{\hat{\mathbf{b}}}^{21\dagger}_{tot}(\vec{p}) = \beta^* \left(\overset{03}{[-i]}\,\overset{12}{[-]} \,|\, \overset{56}{(+)} - \frac{p^1 - i p^2}{|p^0| + |p^3|} \, \overset{03}{(+i)}\,\overset{12}{(+)} \,|\, \overset{56}{(+)} \right) \right) \cdot \\
& \quad e^{-i(|p^0| x^0 + \vec{p} \cdot \vec{x})} \,,
\end{aligned}
$$

[4] If we would make a choice for creation operators the "families" with the "family" members of *odd II* of Table 9.1, instead of "families" with the "family" members of *odd I*, then their Hermitian conjugated partners would be the "families" with the "family" members in *odd I*. The vacuum state would be the sum of products of annihilation operators of *odd I* times the creation operators of *odd II* and would be the sum of selfadjoint members appearing in *even II*.

odd I	m	f = 1(a) $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	f = 2(b) $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	f = 3(c) $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	f = 4(d) $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		03 12 56	03 12 56	03 12 56	03 12 56					
	1	(+i)(+)(+)	[+i][+](+)	[+i](+)[+]	(+i)[+][+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	[−i][−](+)	(−i)(−)(+)	(−i)[−][+]	[−i](−)[+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	[−i](+)[−]	(−i)[+][−]	(−i)(+)(−)	[−i][+](−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	(+i)[−][−]	[+i](−)[−]	[+i][−](−)	(+i)(−)(−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1
odd II		03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	03 12 56 $_{f\,m}$	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		(−i)(+)(+) d_4	[−i][+](+) d_3	[−i](+)[+] d_2	(−i)[+][+] d_1	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
		[+i][−](+) c_4	(+i)(−)(+) c_3	(+i)[−][+] c_2	[+i](−)[+] c_1	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
		[+i](+)[−] b_4	(+i)[+][−] b_3	(+i)(+)(−) b_2	[+i][+](−) b_1	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
		(−i)[−][−] a_4	[−i](−)[−] a_3	[−i][−](−) a_2	(−i)(−)(−) a_1	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1

even I	m	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	−1	−1
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	−1	1
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	1
even II	m	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	−1

Table 9.1. $2^d = 64$ "eigenvectors" of the Cartan subalgebra, Eq. (9.4), of the Clifford odd and even algebras in $d = (5+1)$ are presented, divided into four groups, each group with four "families", each "family" with four "family" members. Two of four groups are sums of an odd number of γ^a's. The "basis vectors", $\hat{b}^{m\dagger}_f$, Eqs. (9.20, 9.21), in *odd I* group, belong to four "families" $(f = 1(a), 2(b), 3(c), 4(d))$ with four members $(m = 1, 2, 3, 4)$, having their Hermitian conjugated partners, $\hat{b}^m_f$, among "basis vectors" of the *odd II* part, denoted by the corresponding "family" and "family" members (a_m, b_m, c_m, d_m) quantum numbers. The "family" quantum numbers, the eigenvalue of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, of $\hat{b}^{m\dagger}_f$ are written above each "family". The two groups with the even number of γ^a's, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. There are members in each group, which are products of projectors only. Numbers — 03 12 56 — denote the indexes of the corresponding Cartan subalgebra members $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, Eq. (9.4). In the columns $(7, 8, 9)$ the eigenvalues of the Cartan subalgebra members (S^{03}, S^{12}, S^{56}), Eq. (9.4), are presented. The last two columns tell the handedness of $d = (5+1)$, $\Gamma^{(5+1)}$, and of $d = (3+1)$, $\Gamma^{(3+1)}$, respectively, defined in Eq.(9.48).

$$\text{Clifford odd creation operators in } d = (5+1)$$

$$p^0 = |p^0|, \quad S^{56} = -\frac{1}{2}, \quad \Gamma^{(3+1)} = -1,$$

$$\left(\underline{\hat{\mathbf{b}}}^{31\dagger}_{tot}(\vec{p}) \;=\; -\beta \left(\stackrel{03}{[-i]} \stackrel{12}{(+)} \mid \stackrel{56}{[-]} + \frac{p^1 + ip^2}{|p^0| + |p^3|} \stackrel{03}{(+i)} \stackrel{12}{[-]} \mid \stackrel{56}{[-]} \right) \right) \cdot$$

$$e^{-i(|p^0|x^0 + \vec{p}\cdot\vec{x})},$$

$$\left(\underline{\hat{\mathbf{b}}}^{41\dagger}_{tot}(\vec{p}) \;=\; -\beta^* \left(\stackrel{03}{(+i)} \stackrel{12}{[-]} \mid \stackrel{56}{[-]} - \frac{p^1 - ip^2}{|p^0| + |p^3|} \stackrel{03}{[-i]} \stackrel{12}{(+)} \mid \stackrel{56}{[-]} \right) \right) \cdot$$

$$e^{-i(|p^0|x^0 - \vec{p}\cdot\vec{x})}, \tag{9.31}$$

Index $^{s=(1,2,3,4)}$ counts different solutions of the Weyl equations, index $^{f=1}$ denotes the family quantum number, all solutions belong to the same family, while $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ takes care that the corresponding states are normalized.

All four superposition of $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})|_{p^0=|\vec{p}|} = \sum_m c^{sf=1}{}_m(\vec{p},|p^0|=|\vec{p}|)\ \hat{b}^{m\dagger}_{f=1}\ e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}$, with $m=(1,2)$ for the first two states, and with $m=(3,4)$ for the second two states, Table 9.1, $s=(1,2,3,4)$, are orthogonal and correspondingly normalized, fulfilling Eq. (9.25).

9.3 Hilbert space of Clifford fermions

The Clifford odd creation operators $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, with $|p^0|=|\vec{p}|$, are defined in Eq. (9.27) on the tensor products of the $(2^{\frac{d}{2}-1})^2$ "basis vectors" (describing the internal space of fermion fields) and of the (continuously) infinite number of basis in the momentum space. The solutions of the Weyl equation, Eq. (9.23), are plane waves of particular momentum $\vec{p}$ and with energy related to momentum, $|p^0|=|\vec{p}|$.

The creation operator $\underline{\hat{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ defines, when applied on the vacuum state $|\psi_{oc}>$, the s^{th} of the $2^{\frac{d}{2}-1}$ plane wave solutions of a particular momentum $\vec{p}$ belonging to the f^{th} of the $2^{\frac{d}{2}-1}$ "families". They fulfill together with the Hermitian conjugated partners annihilation operators $\underline{\hat{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ the anticommutation relations of Eq. (9.28).

These creation operators form the Hilbert space of "Slater determinants", defining for each "Slater determinant" the "space" for any of the single particle fermion states of an odd Clifford character, due to the oddness of the "basis vector" of an odd Clifford character. Each of these "spaces" can be empty or occupied. Correspondingly there is the "Slater determinant" with all the "spaces" empty, the "Slater determinants" with only one of the "spaces" occupied, any one, and all the rest empty, the "Slater determinants" with two "spaces" occupied, any two, and all the rest empty, and so on.

These "Slater determinant" of all possible occupied and empty states can be explained as well if introducing the tensor multiplication of single fermion states of any quantum number and any momentum, with the constant included.

Statement 3: Introducing the tensor product multiplication $*_T$ of any number of Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\underline{\hat{\mathbf{b}}}^{s\,f\,\dagger}_{tot}(\vec{p})$) of any $(s,f,\vec{p})$ we generate the Hilbert space of Clifford fermions.

The Hilbert space of a particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, contains the finite number of "Slater determinants". The number of "Slater determinants" is in d-dimensional space equal to

$$N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}}\,. \tag{9.32}$$

The total Hilbert space of anticommuting fermions is the product $\otimes_N$ of the Hilbert spaces of particular $\vec{p}$

$$\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}\,. \tag{9.33}$$

The total Hilbert space $\mathcal{H}$ is correspondingly infinite and contains $N_{\mathcal{H}}$ "Slater determinants"

$$N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-2}} \,. \tag{9.34}$$

Before starting to comment the application of the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and annihilation $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ operators on the Hilbert space $\mathcal{H}$ (described with all possible "Slater determinants" of all possible occupied and empty fermion states of all possible $(s, f, \vec{p})$, or by the tensor products of all possible single fermion states of all possible $(s, f, \vec{p})$, with the identity included) let us discuss properties of creation and annihilation operators, the anticommutation relations of which are presented in Eq. (9.28).

The creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})$, having an odd Clifford character, anticommute, manifesting the properties as follows

$$\begin{aligned}
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') &= -\hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,,\\
\text{if at least one of } (s, f, \vec{p}) \quad &\text{is different from } (s', f', \vec{p}\,')\,,\\
\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) &= 0\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) &= 1\ (\text{identity})\,,\\
\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})|\psi_{oc}> &= 0\,.
\end{aligned} \tag{9.35}$$

The above relations, leading from the commutation relations of Eq. (9.28), determine the rules of the application of creation and annihilation operators on "Slater determinants":

i. The creation operator $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ jumps over the creation operators determining the occupied state of another kind (that is over the occupied state distinguishing from the jumping creation one in any of the internal quantum numbers (s, f) or in $\vec{p}$) up to the last step when it comes to its own empty state with the quantum numbers (f, s) and $\vec{p}$, occupying this empty state, or, if this state is already occupied, gives zero. Whenever $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ jumps over an occupied state changes the sign of the "Slater determinant".

ii. The annihilation operator changes the sign whenever jumping over the occupied state carrying different internal quantum numbers (s, f) or $\vec{p}$, unless it comes to the occupied state with its own internal quantum numbers (s, f) and its own $\vec{p}$, emptying this state, or, if this state is empty, gives zero.

Let us point out that the Clifford odd creation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, and annihilation operators, $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p'})$, fulfill the anticommutation relations of Eq. (9.28) for any $\vec{p}$ and any (s, f) due to the anticommuting character (the Clifford oddness) of the "basis vectors", $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m}_{f}$, Eqs. (9.20, 9.21), what means that the anticommuting character of creation and annihilation operators is not postulated.

The total Hilbert space $\mathcal{H}$ has infinite number of degrees of freedom (of "Slater determinants") due to the infinite number of Hilbert spaces $\mathcal{H}_{\vec{p}}$ of particular $\vec{p}$, $\mathcal{H} = \prod^{\infty}_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}$, while the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ has the finite dimension $2^{2^{d-2}}$.

In Subsects. 9.3.1, 9.3.2, 9.3.3 the properties of Hilbert spaces are discussed in more details.

9.3.1 Application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on Hilbert space of Clifford fermions of particular $\vec{p}$

The 2^{d-2} Clifford odd creation operators of particular momentum $\vec{p}$, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}, p^0)$, with the property $|p^0| = |\vec{p}|$, each representing the s^{th} solution of Eq. (9.23) for a particular family f, fulfill together with the (Hermitian conjugated partners) annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ the anticommutation relations of Eq. (9.28), the application of which on the Hilbert space of "Slater determinants" are discussed in Eq. (9.35) and in the text below this equation.

The Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$ consists correspondingly of $2^{2^{d-2}}$ "Slater determinants". Let us write down explicitly these $2^{2^{d-2}}$ contributions to the Hilbert space $\mathcal{H}_{\vec{p}}$ of a particular momentum $\vec{p}$, using the notation that $\mathbf{0^{sf}}_{\vec{p}}$ represents the unoccupied state $|\psi^{sf}(\vec{p}, p^0) > |_{|p^0|=|\vec{p}|} = \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})|_{|p^0|=|\vec{p}|} \, |\psi_{oc} >$ of the s^{th} solution of the equations of motion for the f^{th} family and the momentum $|p^0| = |\vec{p}|$), Eq. (9.24), while $\mathbf{1^{sf}}_{\vec{p}}$ represents the corresponding occupied state.

The number operator is defined as

$$
\begin{aligned}
N^{sf}_{\vec{p}} &= \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) \, *_T \, \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) \,, \\
N^{sf}_{\vec{p}} \, |\psi_{oc}> &= 0 \cdot |\psi_{oc}> \,, \qquad N^{sf}_{\vec{p}} \, *_T \, \mathbf{0^{sf}}_{\vec{p}} = 0 \,, \\
N^{sf}_{\vec{p}} \, *_T \, \mathbf{1^{sf}}_{\vec{p}} &= 1 \cdot \mathbf{1^{sf}}_{\vec{p}} \,, \quad N^{sf}_{\vec{p}} \, *_T \, N^{sf}_{\vec{p}} \, *_T \, \mathbf{1^{sf}}_{\vec{p}} = 1 \cdot \mathbf{1^{sf}}_{\vec{p}} \,.
\end{aligned}
\tag{9.36}
$$

One can check the above relations on the example of $d = (5 + 1)$, with the "basis vectors" for $f = 1$ presented in Table 9.2 and with the solution for Weyl equation, Eq. (9.23), presented in Eq. (9.31).

i	$f = 1(a)$	Her. con. $f = 1(a)$
1	$\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}$	$\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(-)}$
2	$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}$	$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(-)}$
3	$\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]}$	$\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{[-]}$
4	$\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]}$	$\overset{03}{(-i)}\overset{12}{[-]}\overset{56}{[-]}$

Table 9.2. The four creation operators of the irreducible representation *odd I* from Table 9.1, $d = (5 + 1)$, $f = 1(a)$. together with their Hermitian conjugated partners are presented (up to a phase).

Let us write down the Hilbert space of second quantized fermions $\mathcal{H}_{\vec{p}}$, using the simplified notation as in Part I, Sect. III.A., counting for $f = 1$ empty states as $\mathbf{0_{rp}}$, and occupied states as $\mathbf{1_{rp}}$, with $r = (1, \dots, 2^{\frac{d}{2}-1})$, for $f = 2$ we count $r = 2^{\frac{d}{2}-1} + 1, \cdots, 2^{d-2}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$
\begin{aligned}
&|\mathbf{0_{1p}}, \mathbf{0_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} > |_1 \quad, \\
&|\mathbf{1_{1p}}, \mathbf{0_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} > |_2 \quad, \\
&|\mathbf{0_{1p}}, \mathbf{1_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} > |_3 \quad, \\
&|\mathbf{0_{1p}}, \mathbf{0_{2p}}, \mathbf{1_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} > |_4 \quad, \\
&\vdots \\
&|\mathbf{1_{1p}}, \mathbf{1_{2p}}, \mathbf{0_{3p}}, \dots, \mathbf{0_{2^{d-2}p}} > |_{2^{d-2}+2} \quad, \\
&\vdots \\
&|\mathbf{1_{1p}}, \mathbf{1_{2p}}, \mathbf{1_{3p}}, \dots, \mathbf{1_{2^{d-2}p}} > |_{2^{2^{d-2}}} \quad,
\end{aligned}
\tag{9.37}
$$

with a part with none of states occupied ($N_{rp} = 0$ for all $r = 1, \dots, 2^{d-2}$), with a part with only one of states occupied ($N_{rp} = 1$ for a particular $r = (1, \dots, 2^{d-2})$, while $N_{r'p} = 0$ for all the others $r' \neq r$), with a part with only two of states occupied ($N_{rp} = 1$ and $N_{r'p} = 1$, where r and r' run from $(1, \dots, 2^{d-2})$, and so on. The last part has all the states occupied.

It is not difficult to see that the creation and annihilation operators, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized Clifford fermions.

$$
\begin{aligned}
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p})\}_{*_T+} \mathcal{H}_{\vec{p}} &= \delta^{ss'}\,\delta^{ff'} \mathcal{H}_{\vec{p}}\,, \\
\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p})\}_{*_T+}\, \mathcal{H}_{\vec{p}} &= 0\,\cdot \mathcal{H}_{\vec{p}}\,, \\
\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}), \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p})\}_{*_T+}\, \mathcal{H}_{\vec{p}} &= 0\,\cdot \mathcal{H}_{\vec{p}}\,.
\end{aligned}
\tag{9.38}
$$

The proof for the above relations easily follows if one takes into account that whenever the creation or annihilation operator jumps over an odd products of occupied states the sign of the "Slater determinant" changes due to the oddness of the occupied states, while states, belonging to different $\vec{p}$ are orthogonal [5], see Eq. (9.35) and the text below this equation. Then one sees that the contribution of the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}) *_T$ on $\mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T$ on $\mathcal{H}_{\vec{p}}$.

If the creation and annihilation operators are Hermitian conjugated to each other, the result follows

$$
(\,\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) + \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,) *_T \mathcal{H}_{\vec{p}} = \mathcal{H}_{\vec{p}}\,,
$$

manifesting that this application of $\mathcal{H}_{\vec{p}}$ gives the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operates on their own half of $\mathcal{H}_{\vec{p}}$. Jumping together over an even

[5] The orthogonality of the states are even easier to be visualized since the two delta functions at $\vec{x}$ and at $\vec{x}\,'$, $\vec{x} \neq \vec{x}\,'$ are obviously orthogonal.

number of occupied states, $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and $\hat{\mathbf{b}}^{sf\dagger}_{tot}(\vec{p})$ do not change the sign of the particular "Slater determinant". (Let us add that $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ reduces for the particular s and f the Hilbert space $\mathcal{H}_{\vec{p}}$ for the factor $\frac{1}{2}$, and so does $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

Let us repeat that the number of "Slater determinants" in the Hilbert space of particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, in d-dimensional space is finite and equal to $N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}}$.

9.3.2 Application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on total Hilbert space $\mathcal{H}$ of Clifford fermions

The total Hilbert space of anticommuting fermions is the infinite product of the Hilbert spaces of particular $\vec{p}$, Eq. (9.33), $\mathcal{H} = \prod^{\infty}_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}$.

Due to the Clifford odd character of creation and annihilation operators, Eq. (9.28), and the orthogonality of the plane waves belonging to different momenta $\vec{p}$, it follows that $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') *_T \mathcal{H} \neq 0, \vec{p} \neq \vec{p}\,'$, while $\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}) *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') + \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,') *_T \hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})\} *_T \mathcal{H} = 0, \vec{p} \neq \vec{p}\,'$. This can be proven if taking into account Eq. (9.35). For "plane wave solutions" of equations of motion in a box the momentum $\vec{p}$ is discretized, otherwise is continuous. The number of "Slater determinants" in the Hilbert space $\mathcal{H}$ in d-dimensional space is infinite (in both cases) $N_{\mathcal{H}} = \prod^{\infty}_{\vec{p}} 2^{2^{d-2}}$.

Since the creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and the annihilation operators $\hat{\underline{\mathbf{b}}}^{s'f'}_{tot}(\vec{p}\,')$ fulfill for particular $\vec{p}$ the anticommutation relations on $\mathcal{H}_{\vec{p}}$, Eq. (9.38), and since the momentum states, the plane wave solutions, are orthogonal, and correspondingly the creation and annihilation operators defined on the tensor products of the internal basis and the momentum basis, representing fermions, anticommute, Eq. (9.28) (the Clifford odd objects $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ demonstrate their oddness also with respect to $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}\,')$), the anticommutation relations follow also for the application of $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ on $\mathcal{H}$

$$
\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})\,, \hat{\underline{\mathbf{b}}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = \delta^{ss'}\, \delta_{ff'}\, \delta(\vec{p} - \vec{p}\,')\, \mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}), \hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = 0\, \cdot \mathcal{H}\,,\\
&\{\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}), \hat{\mathbf{b}}^{s'f'\dagger}_{tot}(\vec{p}\,')\}_{*_T+}\, \mathcal{H} = 0\, \cdot \mathcal{H}\,.
\end{aligned}
\tag{9.39}
$$

9.3.3 Illustration of $\mathcal{H}$ in d = (1 + 1)

Let us illustrate the properties of $\mathcal{H}$ and the application of the creation operators on $\mathcal{H}$ in $d = (1 + 1)$ dimensional space in a toy model with two discrete momenta (p^1_1, p^1_2). Generalization to many momenta is straightforward.

The internal space of fermions contains only one creation operator, one "basis vector" $\hat{b}^1_1 = \overset{01}{(+i)}$, one family member $m = 1$ of the only family $f = 1$. Correspondingly the creation operators $\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^1_i)|_{|p^0|=|p^1_i|} :\, = \overset{01}{(+i)}\;\; e^{-i(p^0 x^0 - p^1_i x^1)}|_{|p^1_i|=|p^0_i|}$ distinguish only in momentum space of the fermion degrees of freedom. Their Hermitian conjugated annihilation operators are $\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^1_i)_{|p^0|=|\vec{p}^1_i|}$, while the vacuum state is $|\psi_{oc}>=\stackrel{01}{(-i)}\cdot\stackrel{01}{(+i)}=\stackrel{01}{[-i]}$.

The whole Hilbert space for this toy model has correspondingly four "Slater determinants", numerated by $|\quad>_i, i=(1,2,3,4)$

$$(|\mathbf{0}_{p_1}\mathbf{0}_{p_2}>|_1\,,\;|\mathbf{1}_{p_1}\mathbf{0}_{p_2}>|_2\,,\;|\mathbf{0}_{p_1}\mathbf{1}_{p_2}>|_3\,,\;|\mathbf{1}_{p_1}\mathbf{1}_{p_2}>|_4)\,,$$

$\mathbf{0}_{p^1_i}$ represents an empty state and $\mathbf{1}_{p^1_i}$ the occupied state. Let us evaluate the application of $\{\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^1_1)\,,\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^1_2)\}_{*_T+}$ on the Hilbert space $\mathcal{H}$. It follows

$$\begin{aligned}
&\{\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^1_1)\,,\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^1_2)\}_{*_T+}\mathcal{H}=\\
&\hat{\underline{\mathbf{b}}}^{11}_{tot}(\vec{p}^1_1)\;*_T\;(|\mathbf{0}_{p_1}\mathbf{1}_{p_2}>|_{1\to3}\,,\,-|\mathbf{1}_{p_1}\mathbf{1}_{p_2}>|_{2\to4})+\\
&\hat{\underline{\mathbf{b}}}^{11\dagger}_{tot}(\vec{p}^1_2)\;*_T\;(|\mathbf{0}_{p_1}\mathbf{0}_{p_2}>_{2\to1}\,,\,+|\mathbf{0}_{p_1}\mathbf{1}_{p_2}>_{4\to3})=\\
&(-|\mathbf{0}_{p_1}\mathbf{1}_{p_2}>_{2\to4\to3}+|\mathbf{0}_{p_1}\mathbf{1}_{p_2}>_{2\to1\to3})\;=0\,.
\end{aligned}$$

9.3.4 Relation between second quantized fermions of Dirac and second quantized fermions originated in odd Clifford algebra

The Clifford odd creation operators $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ obey the anticommutation relations of Eq. (9.39) — on the vacuum state $|\psi_{oc}>$, Eq. (9.18), and on the whole Hilbert space $\mathcal{H}$, Eq. (9.39). Creation operators, $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define second quantized fermion states.

Let us relate here the Dirac's second quantization relations and the relations between creation operators $\hat{\underline{\mathbf{b}}}^{sf}_{tot}(\vec{p})$ and their Hermitian conjugated partners annihilation operators, without paying attention on the charges and family quantum numbers, since Dirac's creation operators do not pay attention on these two kinds of quantum numbers. We shall relate vectors in $d=(3+1)$ of both origins.

In the Dirac case the second quantized field operators are in $d=(3+1)$ dimensions postulated as follows

$$\underline{\Psi}^{hs\dagger}(\vec{x},x^0)=\sum_{m,\vec{p}_k}\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k)\,v^{hs}_m(\vec{p}_k)\,. \tag{9.40}$$

$v^{hs}_m(\vec{p}_k)=u^{hs}_m(\vec{p_k})\,e^{-i(p^0x^0-\varepsilon\vec{p}_k\cdot\vec{x})}$ are the two left handed ($\Gamma^{(3+1)}=-1=h$) and the two right handed ($\Gamma^{(3+1)}=1=h$, Eq. (B.3)) two-component column matrices, $m=(1,2)$, representing the twice two solutions s of the Weyl equation for free massless fermions of particular momentum $|\vec{p}_k|=|p^0_k|$ ([20], Eqs. (20-49) - (20-51)), the factor $\varepsilon=\pm1$ depends on the product of handedness and spin.

$\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k)$ are by Dirac postulated creation operators, which together with the annihilation operators $\hat{\mathbf{a}}^h_m(\vec{p}_k)$, fulfill the anticommutation relations ([20], Eqs. (20-49) - (20-51))

$$\begin{aligned}
&\{\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k),\,\hat{\mathbf{a}}^{h'\dagger}_n(\vec{p}_l)\}_{*_T+}=0=\{\hat{\mathbf{a}}^h_m(\vec{p}_k),\,\hat{\mathbf{a}}^{h'}_n(\vec{p}_l)\}_{*_T+}\,,\\
&\{\hat{\mathbf{a}}^h_m(\vec{p}_k),\,\hat{\mathbf{a}}^{h'\dagger}_n(\vec{p}_l)\}_{*_T+}=\delta_{mn}\delta^{hh'}\delta_{\vec{p}_k\vec{p}_l}
\end{aligned} \tag{9.41}$$

in the case of discretized momenta for a fermion in a box. (Massive fermions are represented by four vectors which are the superposition of both handedness.)

Let us present the two "basis vectors" $\hat{b}^{h\dagger}_m$, $m = (1, 2)$, h representing left and right handedness, in the internal space of fermions in $d = (3 + 1)$, described by the Clifford odd algebra, representing the creation operators of one particular family (f not shown in this case), without charges, of one handedness and with spins $\pm\frac{1}{2}$, respectively, operating on the vacuum state $|\psi_{oc}>= \stackrel{03}{[+i]}\stackrel{12}{[-]}$: $\hat{b}^{h\dagger}_1 =\stackrel{03}{[+i]}\stackrel{12}{(+)}$ and $\hat{b}^{h\dagger}_2 =\stackrel{03}{(-i)}\stackrel{12}{[-]}$, Eq. (9.20, 9.21) [6], with $h = 1$, representing the right handedness. These two "basis vectors" should be compared with the two vectors, one corresponding to the spin $\frac{1}{2}$ and the other to the spin $-\frac{1}{2}$ in the Dirac case.

Since Dirac did not postulate such creation operators on the level of $\hat{b}^{h\dagger}_m$, let us postulate them now on the level of $\hat{b}^{h\dagger}_m$, to be able to compare in this paper presented creation operators for this particular case, $\hat{a}^{h\dagger}_{\uparrow}$ and $\hat{a}^{h\dagger}_{\downarrow}$, of right handedness h and spin up and down $(\uparrow, \downarrow)$ as follows

$$\hat{b}^{h\dagger}_1 =\stackrel{03}{[+i]})\stackrel{12}{(+)} \;\text{ to be related to }\; \hat{a}^{h\dagger}_{\uparrow}\,, \qquad \hat{b}^{h\dagger}_2 =\stackrel{03}{(-i)})\stackrel{12}{[-]} \;\text{ to be related to }\; \hat{a}^{h\dagger}_{\downarrow}\,.$$

One should repeat this also for left handedness $h = -1$. But these creation operators $\hat{a}^{h\dagger}_m$, $m = (1, 2) = (\uparrow, \downarrow)$, still can not be compared with the Dirac's ones.

Let us make the superposition of both creation operators of particular handedness h, $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k) := \alpha^{hs}_{\uparrow}(\vec{p}_k)\,\hat{a}^{h\dagger}_{\uparrow} + \alpha^{hs}_{\downarrow}(\vec{p}_k)\,\hat{a}^{h\dagger}_{\downarrow}$, with the coefficients $\alpha^{hs}_{\uparrow}(\vec{p}_k)$ and $\alpha^{hs}_{\downarrow}(\vec{p}_k)$ chosen so that $\underline{\hat{\mathbf{a}}}^{hs\dagger}_{tot}(\vec{p}_k)\!:= \hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k)\, e^{-i(p^0x^0 - \vec{p}_k\cdot\vec{x})}$ solves the equations of motion, Eq. (9.23) [7], for a plane wave $e^{i\varepsilon\vec{p}_k\cdot\vec{x}}$ for $|\vec{p_k}| = |p^0_k|$, then it follows

$$\underline{\hat{\mathbf{a}}}^{hs\dagger}_{tot}(\vec{p}_k) := (\alpha^{hs}_{\uparrow}(\vec{p}_k)\,\hat{a}^{h\dagger}_{\uparrow} + \alpha^{hs}_{\downarrow}(\vec{p}_k)\,\hat{a}^{h\dagger}_{\downarrow})\, e^{-i(p^0x^0 - \vec{p}_k\cdot\vec{x})} = \sum_m \hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k) v^{hs}_m(\vec{p}_k)\,, \tag{9.42}$$

where the summation runs over m up and down spin m of the chosen handedness h.

Since $v^{hs}_m(\vec{p}_k) = u^{hs}_m(\vec{p}_k)\, e^{-i(p^0x^0 - \vec{p}_k\cdot\vec{x})}$ it follows also that

$$\hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k) = \sum_m u^{hs}_m\,\hat{a}^{h\dagger}_m,$$

and $u^{hs}_m(\vec{p}_k) = \alpha^{hs}_m(\vec{p}_k)$. We conclude that $\underline{\hat{\mathbf{a}}}^{hs\dagger}_{tot}(\vec{p}_k)$ obviously determine

$$\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k) v^{hs}_m(\vec{p}_k) = \hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k)\, u^{hs}_m(\vec{p}_k) e^{-i(p^0x^0 - \vec{p}_k\cdot\vec{x})}.$$

Anticommutation relations of Eq. (9.41), postulated by Dirac, ensure the equivalent anticommutation relations also for $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k)$ and $\hat{\mathbf{a}}^{hs}(\vec{p}_k)$.

---

[6] We choose in the Clifford case the first two members of the third family in Table 9.1, since they manifest in $d = (3 + 1)$ the Clifford odd character.

[7] The equations of motion read in the Dirac case: $\{\hat{p}^0 + (-2iS^{0i}\hat{p}_i)\}(\alpha^s_1(\vec{p}_k)\,\hat{\mathbf{a}}^{\dagger}_1 + \alpha^s_2(\vec{p}_k)\,\hat{\mathbf{a}}^{\dagger}_2)e^{-i(p^0x^0 - \vec{p}_k\cdot\vec{x})} = 0$. To solve them we need to recognize that the matrices in the chiral representation S^{0i}, $i = (1, 2)$, transform $\hat{\mathbf{a}}^{\dagger}_1$ into $\hat{\mathbf{a}}^{\dagger}_2$, and opposite.

Now we are able to relate creation and annihilation operators in both cases, the Dirac case and our case of using the odd Clifford algebra to represent the internal space of fermions.

$$\begin{aligned}
\hat{b}^{h\dagger}_{1} &= \overset{03}{[+i]}\overset{12}{(+)} \quad \text{to be related to} \quad \hat{a}^{h\dagger}_{\uparrow}\,,\\
\hat{b}^{h\dagger}_{2} &= \overset{03}{(-i)}\overset{12}{[-]} \quad \text{to be related to} \quad \hat{a}^{h^{h}\dagger}_{\downarrow}\\
\hat{b}^{h}_{1} &= -\,\overset{03}{[+i]}\overset{12}{(-)} \quad \text{to be related to} \quad \hat{a}^{h}_{\uparrow}\,,\\
\hat{b}^{h}_{2} &= \quad \overset{03}{(+i)}\overset{12}{[-]} \quad \text{to be related to} \quad \hat{a}^{h}_{\downarrow}\,,
\end{aligned} \tag{9.43}$$

both sides representing the creation operators, with $S^{12} = \frac{1}{2}$ and handedness $\Gamma^{(3+1)} = 1$, Eq. (9.48), in the first row, and with $S^{12} = -\frac{1}{2}$ and handedness $\Gamma^{(3+1)} = 1 = h$, in the second row [8].

None of the creation operators, $\hat{a}^{h\dagger}_{m}$, $m = (\uparrow, \downarrow)$ and $\hat{b}^{h\dagger}_{m}$, $m = (1,2)$, depend on momenta, but $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}_k)$ and $\hat{\mathbf{b}}^{sf\dagger}(\vec{p}_k)$ as well as $\hat{\underline{\mathbf{a}}}^{hs\dagger}_{tot}(\vec{p}_k)$ and $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p}_k)$ do depend on momenta.

The creation operators $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ fulfill the anticommutation relations of Eqs. (9.28, 9.38, 9.39), the same as $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ do. We can just replace $\hat{\underline{\mathbf{a}}}^{s\dagger}_{tot}(\vec{p}_k)$ by $\hat{\underline{\mathbf{b}}}^{sf\dagger}_{tot}(\vec{p})$ for any of families (for plane waves solutions with continuous $\vec{p}$).

We can conclude:

$$\begin{aligned}
\hat{\underline{\mathbf{a}}}^{hs\dagger}_{tot}(\vec{p}) \quad &\text{is to be related to} \quad \hat{\underline{\mathbf{b}}}^{hs\dagger}_{tot}(\vec{p})\,,\\
\hat{a}^{h\dagger}_{m}, m = (\uparrow, \downarrow) \quad &\text{is to be related to} \quad \hat{b}^{h\dagger}_{m}\; m = (1,2)\,,
\end{aligned} \tag{9.44}$$

with h representing the handedness. This can be done for any chosen family in the Clifford case. In all the relations with $\hat{\underline{\mathbf{b}}}^{hs\dagger}_{tot}(\vec{p})$ the handedness is not written explicitly and is included in the index m and in the index s, while the index f represents the family quantum number. Only in this chapter we introduce handedness in addition to clarify the relations.h

In the Clifford case the charges origin in spins $d \geq 6$. In $d = (13+1)$ all the charges of quarks and leptons and antiquarks and antileptons can be explained, as well as the families of quarks and leptons and antiquarks and antileptons. In the Dirac case charges come from additional groups and so do families.

Let us add: The odd Clifford algebra influences the algebra of the associated creation and annihilation operators acting on the second quantized Hilbert space $\mathcal{H}$; Due to oddness of the Clifford algebra, which determines internal degrees of freedom of fermions, the creation operators and their Hermitian conjugated annihilation partners, determined on the tensor products of internal and momentum space, make the creation and annihilation operators to anticommute.

We conclude: The by Dirac postulated creation operators, $\hat{\mathbf{a}}^{h\dagger}_{m}(\vec{p})$, and their annihilation partners, $\hat{\mathbf{a}}^{h}_{m}(\vec{p})$, Eqs. (9.40, 9.42), related in Eq. (9.44) to the Clifford

[8] The vacuum state is on the left hand side equal to $\overset{03}{[+i]}\overset{12}{[-]}$, while on the right hand side the corresponding vacuum state can be defined, if we follow our way of defining the vacuum state, to be proportional to $(\hat{a}_{\uparrow}\hat{a}^{\dagger}_{\uparrow} + \hat{a}_{\downarrow}\hat{a}^{\dagger}_{\downarrow})$.

odd creation and annihilation operators, manifest that the odd Clifford algebra offers the explanation for the second quantization postulates of Dirac.

9.4 Creation and annihilation operators in $d = (13 + 1)$-dimensional space

The *standard model* offered an elegant new step in understanding elementary fermion and boson fields by postulating:

i. Massless family members of (coloured) quarks and (colourless) leptons, the left handed fermions as the weak charged doublets and the weak chargeless right hand members, the left handed quarks distinguishing in the hyper charge from the left handed leptons, each right handed member having a different hyper charge. All fermion charges are in the fundamental representation of the corresponding groups. Antifermions carry the corresponding anticharges and opposite handedness. The massless families to each family member exist.

ii. The existence of the massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the corresponding adjoint representations.

iii. The existence of a massive scalar Higgs, gaining at some step of the expanding universe the nonzero vacuum expectation value, responsible for masses of fermions and heavy bosons and for the Yukawa couplings. The Higgs carries a half integer weak charge and hyper charge.

iv. Fermions and bosons are second quantized fields.

The *standard model* assumptions have in the literature several explanations, mostly with many new not explained assumptions. The most successful seem to be the grand unifying theories [22–36, 38–40], if postulating in addition the family group and the corresponding gauge scalar fields.

The *spin-charge-family* theory, the project of one of the authors of this paper (N.S.M.B. [1–3, 10–15, 17]), is offering the explanation for all the assumptions of the *standard model*, unifying in $d = (13 + 1)$-dimensional space not only charges, but also charges and spins and families [2, 7], explaining the appearance of families [8, 10, 15], the appearance of the vector gauge fields [12, 14], of the scalar field and the Yukawa couplings [13]. Theory offers the explanation for the dark matter [4, 5], for the matter-antimatter asymmetry [11], and makes several predictions [4, 6, 11].

The *spin-charge-family* theory is a kind of the Kaluza-Klein like theories [17, 42, 44–50] due to the assumption that in $d \geq 5$ (in the *spin-charge-family* theory $d \geq (13+1)$) fermions interact with the gravity only (vielbeins and two kinds of the spin connection fields). Correspondingly this theory shares with the Kaluza-Klein like theories their weak points, at least:

a. Not yet solved the quantization problem of the gravitational field.

b. The spontaneous break of the starting symmetry, which would at low energies manifest the observed almost massless fermions [44].

c. The appearance of gravitational anomalies, what makes the theory not well defined [53], but in the low energy limit the fields manifest in $d = (3 + 1)$ properties of the observed vector and scalar gauge fields.

d. And other problems.

In the *spin-charge-family* theory fermions interact in $d = (13+1)$ with the gravity only: with the spin connections (the gauge fields of S^{ab} and of $\tilde{S}^{ab}$) and vielbeins (the gauge fields of momenta), with fermions as a condensate present, breaking the symmetry (and with no other gauge fields present), manifesting at low energies in $d = (3+1)$ as the ordinary gravity and all the observed vector gauge fields.

It is proven in Refs. [51, 52], that one can have massless spinors even after breaking the starting symmetry. Ref. [12] proves, that at low enough energies, after breaking the staring symmetry, the two spin connections manifest in $d = (3+1)$ as the observed vector gauge fields, as well as the scalar fields, which offer the explanation for the Higgs and the Yukawa couplings. Ref. [11] offers the explanation for the matter-antimatter asymmetry due to the existence of the scalar fields with the "colour charges" in the fundamental representations. In Ref. [17] the *spin-charge-family* theory explains the *standard model* triangle anomaly cancellation better than the $SO(10)$ theory [23].

The working hypotheses of the authors of this paper (in particular of N.S.M.B.) is, since the higher dimensions used in the *spin-charge-family* theory offer in an elegant (simple) way explanations for the so many observed phenomena, that they should not be excluded by the renormalization and anomaly arguments. At least the low energy behavior of the spin connections and vielbeins as vector and scalar gauge fields manifest as the known and more or less well defined theories.

In this paper we present that using the half of the odd Clifford algebra objects to explain the internal degrees of freedom of fermions (the other half represent the Hermitian conjugated partners), as suggested by the *spin-charge-family* theory, leads to the second quantized fermions without postulates of Dirac [9].

9.5 Conclusions

We present in Part I and Part II of this paper that the description of the internal space of fermions with the odd elements of the anticommuting algebra defines the creation and annihilation operators, which anticommute when applied on the corresponding vacuum state. The internal space, described by the odd Clifford algebra, extends its oddness to creation and annihilation operators generated on the tensor products of the internal basis with finite numbers of elements and the momentum basis with infinite number of elements. The application of these creation and annihilation operators on the Hilbert space, determined by the tensor multiplication of all possible creation operators of any numbers, formally observed

[9] The authors of Ref. [43] let us know that their path integral formulation enabled them to see a great deal of what we present in this paper. We went through their paper noting that they did a lot concerning path integral formulation of quantum mechanics, offering ways to treat anomalies. But we couldn't recognize that they propose some replacement for the Dirac postulates of creation and annihilation operators. We also could not found whether our proposal for explaining the Dirac postulates would bring any new light on path integral formulations and anomalies cancelations. To clarify this topics the discussions with authors would be needed.

in this paper and in [3] in the Clifford algebra, manifests the same anticommutation relations as the creation and annihilation of the second quantized fermions, explaining therefore the Dirac postulates of the second quantized fermion fields.

In the subsection 9.1.2 we clarify the relation between our description of the internal space of fermions with "basis vectors", manifesting oddness and transferring the oddness to the corresponding creation and annihilation operators of second quantized fermions, to the ordinary second quantized creation and annihilation operators from a generalized point of view.

We learn in Part I of this paper, that odd products of superposition of θ^a's, Eqs. (8-11,13,22) in Part I, exist forming the odd algebra "basis vectors" in the internal space of "Grassmann fermions" with integer spin, which together with their Hermitian conjugated partners fulfill on the algebraic level on the vacuum state all the requirements for the anticommutation relations for the Dirac fermions. The creation and annihilation operators, defined on the tensor products of the superposition of the Grassmann odd algebra "basis vectors" and the momentum space basis, and manifesting correspondingly the oddness of the "basis vectors", fulfill the anticommutation relations of the second quantized Dirac's fermions on the vacuum state, as well as on the "Slater determinants" of all possibilities of occupied and empty single particle "Grassmann fermion" states of integer spins of any number. These "Slater deerminants", representing the Hilbert space of second quantized "Grassmann fermions", can be represented as well with the tensor product multiplication of any possible choice of single "Grasmann fermion states" of all possible numbers of states, started with none (that is with the identity), distinguishing at least either in one of the quantum numbers of the "basic vectors" or in momentum basis.

In Part II we learn, that the creation and annihilation operators exist in the Clifford odd algebra, defining the internal space of half integer fermions, which applying on the vacuum state fulfill the anticommutation relations postulated by Dirac. Creation operators, defined on the tensor products of the superposition of the finite "basis vectors" of the internal space described with the Clifford algebra and of the infinite momentum basis, fulfill as well together with their Hermitian conjugated annihilation operators the anticommutation relations postulated by Dirac, on the vacuum state and on the Hilbert space of infinite number of the single particle fermion states, $N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-2}}$, Eqs. (9.33, 9.34), creating "Slater determinants" (Eqs. (9.37, 9.39)), but only after the reduction of the degrees of freedom of the Clifford algebras for a factor of two, Eq. (9.12).

The reduction of the Clifford algebras for the factor of two leaves the anticommutation relations of Eqs. (9.2, 9.3) unchanged, enabling the appearance of family quantum numbers. The Clifford fermions carry half integer spins, families and charges in fundamental representations, Eq. (9.5).

The reduction of Clifford space causes the reduction also in Grassmann space, what leads to the disappearance of integer spin fermions, Eq. (9.19).

The Clifford algebra oddness of the "basis vectors", describing the internal space of fermions, makes odd also the corresponding fermion states defined on the tensor products of the internal and momentum space. Correspondingly any two states fulfill the anticommutation relations and so do any tensor products of odd numbers of fermion states, forming the Hilbert second quantized space.

We present the creation operators, defined on the tensor products of ”basis vectors” and plane waves, solve the equations of motion, in our case for free massless fermions, Eq. (9.23), derived from the action, Eq. (9.22).

Anticommutation relations are not postulated, as it is in the Dirac case, they follow from the oddness of the Clifford objects, and correspondingly explain the second postulates of Dirac (what is stressed in several places in Part I and Part II and in a short way also in Subsect. 9.1.2).

The relation between the Dirac's creation and annihilation operators and the ones offered by the odd Clifford algebra, discussed in In Subsect. 9.3.4 demonstrates that the basic differences between these two descriptions is on the level of the single particle creation operators: While the odd Clifford algebra offers the creation and annihilation operators, which fulfill the anticommutation relations, already on the level of the ”basis vectors” determining the internal space of fermions, Eq. (9.17), when applied on the vacuum state, Dirac postulates the anticommutation relations on the level of second quantized objects, following the procedure of Lagrange and Hamilton.

The final result is in both cases equivalent, leading to the Hilbert space of second quantized fields. However, our way not only explains the Dirac postulates but demonstrates in addition, that also the single particle states in the first quantization do anticommute due to the oddness of the ”basis vectors” defining the internal space. The oddness of the Clifford objects of creation and annihilation operators is transmitted from the ”basis vectors” of internal space to the tensor products of the superposition of the ”basis vectors” and the momentum or coordinate space.

Correspondingly the odd Clifford algebra, equipped with the family quantum numbers, and fulfilling the anticommutation relations already on the level of the single particle creation operators applying on the vacuum state, as well as on the level of the whole Hilbert space, offers the explanation for the anticommutation relations, postulated by Dirac.

The Hilbert space of all ”Slater determinants” with any number of occupied or empty states of an odd character, follows in all three cases, the Dirac one (with postulated creation and annihilation operators and offering no families and no charges), the Grassmann one (offering spins and charges in adjoint representations, and no families) and the Clifford one (offering spins, families and charges), in an equivalent way: due to the anticommuting creation and annihilation operators, representing ”basis vectors” and their Hermitian conjugated partners. One can see this in Sect. 9.3.4.

Let us repeat: Internal space contributes the final number of states, the infinity of number of states is due to momentum/coordinate space [10].

The anticommuting single fermion states manifest correspondingly the oddness already on the level of the first quantization. Correspondingly these odd

[10] Let us add that the single particle vacuum state is the sum of products of annihilation $\times$ creation operators: In the Grassmann case it is just an identity, in the Clifford case is the sum of products of projectors for each family.

fermion states form in the tensor products $*_T$ the Hilbert space H of second quantized states.

9.6 APPENDIX: Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [2] of two functions of the Grassmann coordinates $<\mathbf{B}|\theta><\mathbf{C}|\theta>$, $<\mathbf{B}|\theta>=<\theta|\mathbf{B}>^{\dagger}$,

$$<\mathbf{b}|\theta>=\sum_{k=0}^{d} b_{a_1\dots a_k}\theta^{a_1}\cdots\theta^{a_k},$$

by requiring

$$\begin{aligned}
&\{d\theta^a,\theta^b\}_+ = 0\,, \quad \int d\theta^a = 0\,, \quad \int d\theta^a\theta^a = 1\,,\\
&\int d^d \quad \theta\ \theta^0\theta^1\cdots\theta^d = 1\,,\\
&d^d\theta = d\theta^d\dots d\theta^0\,, \quad \omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k}+\theta^k)\,,
\end{aligned}\tag{9.45}$$

with $\frac{\partial}{\partial\theta_a}\theta^c = \eta^{ac}$. We shall use the weight function [2] $\omega = \Pi^d_{k=0}(\frac{\partial}{\partial\theta_k}+\theta^k)$ to define the scalar product in Grassmann space $<\mathbf{B}|\mathbf{C}>$

$$\begin{aligned}
<\mathbf{B}|\mathbf{C}> &= \int d^d\theta^a\ \ \omega <\mathbf{B}|\theta><\theta|\mathbf{C}>\\
&= \sum_{k=0}^{d} b^*_{b_1\dots b_k} c_{b_1\dots b_k}\,.
\end{aligned}\tag{9.46}$$

To define norms in Clifford space Eq. (9.45) can be used as well.

9.7 APPENDIX: Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} = \alpha\varepsilon_{a_1 a_2\dots a_{d-1}}\,{}_{a_d}\, S^{a_1 a_2}\cdot S^{a_3 a_4}\cdots S^{a_{d-1} a_d}\,,\tag{9.47}$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case S^{ab} is defined in Eq. (9.3), while in the Clifford case Eq. (9.47) simplifies, if we take into account that $S^{ab}|_{a\neq b} = \frac{i}{2}\gamma^a\gamma^b$ and $\tilde{S}^{ab}|_{a\neq b} = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} := (i)^{d/2}\prod_a \quad (\sqrt{\eta^{aa}}\gamma^a), \quad \text{if} \quad d = 2n\,.\tag{9.48}$$

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10 Proposal for Compositeness of String out of Objects — Fake Scattering, Finite String Field Theory Formulation ⋆

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Abstract. We review a bit our earlier novel string field theory [1,4] stressing the interesting property, that it becomes expressed in terms of particle like objects called by us "objects" which in our formalism do not at all develop in time. So in this way there is in our picture, in spite of it being supposed to reproduce string theory with an arbitrary number strings present - in this sense a string field theory -, in fact no time! This strange missing of time in the formalism gives rise to slight speculations about the philosophy of the concept of time. There is course then also no need for a Hamiltonian, but we construct or rather attempt to do so, a fake Hamiltonian or phantasy Hamiltonian,

Povzetek. Avtorja predstavita svojo novo teorijo polja strun, v kateri nastopajo objekti podobni delcem, ki se v uporabljenem formalizmu s časom ne spreminjajo. V tej teoriji tako čas ne nastopa, čeprav bi naj reproducirala teorijo strun s poljubnim številom strun (zato jo avtorja imenujeta teorija polja strun. Avtorja ob tem razpravljata o pojmu časa. V tej teoriji tudi ni potrebe po hamiltonki, vendar jo vseeno poskusita konstruirati.

Novel String Field Theory and Unitarity, although in non-relativistic case (Fake Scattering, Hope for finiteness).

10.1 Introduction

We have long worked on new/novel formulation [1] of the bosonic string theory as a second quantized theory in the sense that we describe states with an arbitrary number of strings, an achievement made earlier by String Field Theory by Kaku and Kikkawa and by Witten and others (for Bosonic string theory [2]). Our approach has some similarity to the work by C. Thorn [3] in as far as we also use a

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splitting of the string into small bits. We, however, did it after the description of the single string has been resolved into right and left moving fields $X_R^\mu(\tau - \sigma)$ and $X_L^\mu(\tau + \sigma)$ (using the often used notation in string theory in so called conformal gauge) and most remarkably we achieve that we do not need any time development for what is the analogue of Charles Thorns small bits, we call ours as said somewhat different small parts for "objects". The "objects " stand, we could say, still [4].

It is this property of our string field theory which seems so interesting, that we indeed shall concentrate on precisely this no time development property in the present article.

We thus seek in the present article to go backwards in the sense that we begin by considering such objects, that do not develop, and then afterwards shall seek by what we call "phantasy" to return to the string theory able to describe several strings. We believe it would be not so terribly hard to make the same "object" description of the various superstrings, presumably by having "objects" being fermions with odd spin, but of course they should still not develop in time. Both because it would complicate the presentation and because we have not developed yet the string field theory of ours with such spin-objects yet - we did not even quite finish the bosonic version - we shall keep to the boson string theory in the present article.

The real motivation is to seek generalizations of the string theory. Indeed the main reason that superstring theory is so popularly speculated to be the "theory of everything" is that it avoids the ultraviolet divergence problems plaguing the point particle quantum field theories. This avoidance of ultraviolet divergences is connected with or a consequence of that the string scattering amplitudes - the Veneziano models - are strongly indeed exponentially cut off for large momentum transfer. In our "object" picture this cut off at large momentum transfer can be traced back to that the distribution of the momenta of the objects is cut off in a similar way.

The scattering in our picture of objects has the character of two composite particles(~ strings) composed from the "objects" exchange some collections of objects with each other. Thus after the scattering the momenta of the final composite particles (~ strings) can only deviate from those in the initial state by the momenta of the exchanged collections of objects. We shall namely remember the for this argument to be valid extremely important point that the single object never can change its momentum, because it does not develop at all, especially it does not scatter by itself. The whole seeming scattering of the composite particles composed from objects is purely "fake" in the sense that they are a result of some objects being transfered from one compositum to another one.

(10.1 - 1)**Fake Scattering Concept**

We are so fascinated by this idea of making a quantum field theory like theory, so that in a **fundamental** sense there is **No Time development**, but when looking at it appropriately, then you can "see" it as e.g. string field theory (a theory of second quantized strings).

This fake-scattering concept is implemented in the "Novel string field theory" , which we have put forward long ago.

(10.1 - 2)**Hamiltonian =0 gives no time-development**

So quantum mechanically the **no time-development theory** is just a Hilbert space of the states, and they never develop - there is basically no time needed -.

In the **"Novel string field theory"** of ours the states in this Hilbert space are described formally by a second quantized theory of particles that can occur in different numbers just like in usual second quantized theory. We call these particles "objects" and they are crudely to be considered small pieces of strings like in the Charles Thorn's string bit theory. But very importantly **we** first split the string into bits, **after we hav gone to the light cone variables on the string**: $\tau - \sigma$ and $\tau + \sigma$.

(10.1 - 3)**Introduction of Fake Degrees of Freedom**

In the philosophy that the true **fundamental theory** has **no time** (or say no time development) means that all development with time has to be fake. That is to say it has to be in some degrees of freedom, that do not really exist in nature, but which we the physicists introduce formally so as to make a theory more in agreement with our usual picture of how physics is.

Basically the idea is that we introduce some extra degrees of freedom that only are there in phantasy, so that we construct formally a system/a world with some extra variables or some extra information on its states. These extra inforations shall however only in some way be adjusted to help describing the original degrees of freedom, which we call the true degrees of freedom. In the case we here hope to realize; the original or true degrees of freedom are the ones for the object. But by the addition of the extra degrees of freedom we have in mind giving an information about how the chains of the objects are glued together in possibly different ways. These different ways we hope to describe by means of the extra phantasy degrees of freedom. It is mainly how the cyclically ordered chains of objects are, one can say the extra degrees of freedom should tell which objects belong to which cyclically ordered chain. Thereby the extra phantasy degrees of freedom also come to tell which objects belong to which string. Thus some strings exchanging objects can be a pure phantasy happening. This is what is called than scattering is a fake.

Abstractly we replace each basis vector in a basis for the second quantized Hilbert space by a series of basis-vectors. So to each "fundamental basis vector" we have a lot of basis-vectors in the extended theory only deviating from each other by invented or fake degrees of freedom.

Then we allow the "fake-development" - the fake Hamiltonian - to only move around the basis-vectors into each other which belong to the same fundamental basisvector.

(10.1 - 4) **A String Field Theory Inspired Example**

To a good enough approximation the readers can imagine that our "objects" (after some technical details of only using the "even" ones among them) are (scalar) **particles** with position and momenta in a 25+1 dimensional world (or if we choose an infinite momentum frame in 24 transverse dimensions), and that there in any single particle state for such a particle can be a number of particles $n = 0, 1, 2, ...$, just as in second quantization.

To avoid the problems with relativity, Dirac sea [7] etc., we like to for pedagogical reasons effectively consider a non-relativistic theory, or almost equivalent an infinite momentum frame formulation.

(10.1 - 5) **The Pedagogical Non-relativistic model with Zero Hamiltonian**

We consider a model with say non-relativistic bosons - so that they can occur in any number in any sigle particle state -. To make the theory not develop in time we want to simplify to make the Hamiltonian zero

$$H = 0, \tag{10.1}$$

which in addition to having no interactions mean that we let the non-relativistic mass

$$m \to \infty, \tag{10.2}$$

so that even the kinetic term $\frac{\vec{p}^2}{2m}$ goes to zero.

We can choose a basis for the single particle states to be e.g. either the momentum eigenstates or the position eigenstates (a priori as we wish).

(10.1 - 5 - 1)**Second Quantizing our $H = 0$ Particle Model:**

As basis to use in single particle Hilbert space we shall here choose the position eigenstates because we like to investigate about a "nearness" concept (we want to say if two particles described by such basis vectors chosen are close or far apart.)

Then the corresponding basis in the second quantization state space is enumerated by a function, that to every position $\vec{x}$ assigns a number $n(\vec{x})$ giving the number of particles with exactly the position $\vec{x}$.

In other words a second quantized basis-vector can be described by the number $n(\vec{x})$ of "objects"(=particles) in each position $\vec{x}$:

$$n : \mathbf{R}^{24} \to \{0, 1, 2, ...\} \tag{10.3}$$

and we cannot require it continuos unless we take it to be only constant, because a continuous function mapping the real number type of space $\mathbf{R}^{24}$ into a discrete space, the positive integers and 0, can only be constant if it is continuous.

However, we shall at this stage not describe about continuity.

(10.1 - 5 - 2)**Second Quantized Basis**

A basis - and this is the one we now have chosen to use - in the second quantized state space consists of vectors like

$$|n> = \prod_{\vec{x}} \frac{a^\dagger(\vec{x})^{n(\vec{x})}}{\sqrt{n(\vec{x})}} |n = 0> \tag{10.4}$$

where $a^\dagger(\vec{x})$ is the creation operator for a particle at the position $\vec{x}$. The symbol $\mathbf{R}$ stands for the set of real numbers.

Remember

$$n : \mathbf{R}^{24} \to \{0, 1, 2, ...\}. \tag{10.5}$$

(10.1 - 5 - 3) **Introduction of the Fake Degree of Freedom "The Successor Function" f**

Our extremely simple $H = 0$ theory just introduced has a priori nothing to do with strings (nor much other sensible physics for that matter), but now we want by just explaining to make it into a string field theory!

For each single one $|n>$ of our basis states in the second quantized space we want to introduce a "sucessor function " f, which is a permutation of the particles present in that state.

In the state $|n>$ there are

$$N(n) = \sum_{\vec{x}} n(\vec{x}) \tag{10.6}$$

particles present. Here we assumed that there were not infinitely many particles present.

The "Successor function" f is a Permutation of the Particles present in the state $|n>$.

Assuming that there are only finitely many particles in a second quantized state vector $|n>$ we can think of these $N(n)$ particles as true particles, and you could define $N(n)!$ permutations f of the $N(n)$ particles present.

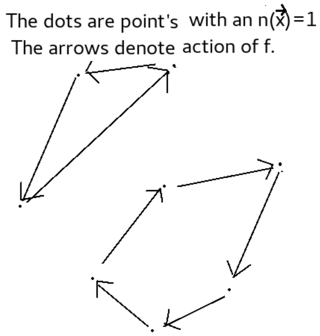

Fig. 10.1. The points denote the objects and the arrows denote the action of the permutation function f. Since a permutation can be resolved into cyclic permutations, the objects will on this drawing get into cyclically ordered chains.

We Think of a Phantasy-space with $|n>$ replaced by $N(n)!$ new phantasy basis vectors representing the same true physics.

So the new basis-vectors in the second quantized space should be denoted

$$|n, f> = (|n>, f) \text{ where } f \in P_{N(n)} \tag{10.7}$$

where again

$$N(n) = \sum_{\vec{x}} n(\vec{x}) \tag{10.8}$$

is the number of particles in the state $|n>$.

Working with Phantasy space Makes Life Easier

Of course it is f which is the phantasy degree of freedom. It was just introduced by us.

"Thus we can decide in the following rule:

We throw away all the choices of permutations f unless it fullfils the following rule (we do so for some reason to be explained later): The position x_{first} of a particle $first$ being mapped by f to $f(fisrt)$ must have a position $x_{f(first)}$ close to x_{first}. That is to say, we require only to include in our phantasy space such combinations that $f$should satisfy f(first) is close to first, i.e. $|x_{first} - x_{f_f(first)}|$ should small.

If f does not obey this restriction,we simply take it out and let there be fewer state vectors in the phantasy Hilbert space."

So we can decide - just we like so for some reason to explained possibly later - to say that we throw away all the choices of the permutation f, for which the position of a particle $\vec{x}_{first}$ and that of the particle into which f maps it $\vec{x}_{f(first)}$ are not close. I.e. we require only to include in our phantasy the f's satisfying

$$f(first) \text{ close to } first \tag{10.9}$$

I.e.

$$|\vec{x}_{first} - \vec{x}_{f(first)}| \qquad \text{small.} \tag{10.10}$$

If f does not obey this restriction, we simply take it out and let there be fewer state vectors in the phantasy Hilbert space.

We can phantasize that f describes successors in long almost connected chains

We can choose the f permutations, we allow, to be such that they describe connected closed loop chains of the particles in the state, so well it is possible.

From our purpose of making theory to be part of a speculated theory for everything we could be allowed to postulate something - if beautiful enough - also **about the state of the universe**, such as that the most likely type of state is one in which the particles sit in long circular chains with rather small distance between the neighbours and even further assumptions involving the momenta.

(10.1 - 6) **About Fundamental Physics We can further only make assumptions about the Initial and /or Final states**

After we settled on no time-development ($\sim$ Hamiltonian being zero) we can **not** as physicists looking for the right theory of nature anymore **speculate about the Hamiltonian**, because that we already took to zero (as operator).

But we may **want to** have a bit of chance to **assume** a little bit to adjust to fit our hoped for model to experimental information etc.

Then we have the chance of speculating about the **initial state** (which is also the final state though, when no development).

Assumptions about Initial and final state

For our purpose with the string theory towards which we are driving in mind we like to make assumptions about initial condition like this:

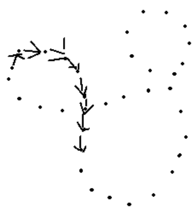

Fig. 10.2. We think of the objects sitting (due to some assumed principle about the liklely configuration of the objects) in cyclically ordered chains, that can be described by giving the permutation function f as here illustrated by the arrows. The reader can imagine contuing marking the arrows to tell how f acts.

- (a) An approximate constraint on the relative state of a couple of particles A and $f(A)$, namely

$$k(\vec{x}_{f(A)} - \vec{x}_A) \approx \vec{p}_{f(A)} + \vec{p}_A, \tag{10.11}$$

  where k is a constant, actually related (as to be seen) to the Regge slope α' so important in string theory.
- (b) The particles shall approximately form cyclic chains.
- (c) And they shall even especially locally along the chains have a certain wave function like they would have in string theory if they were identified with the "objects" of ours (which we have not yet described in detail.)

(10.1 - 7) **Assumptions about (Initial) State Formulated by Density Matrix** ρ

Whatever assumption about a quantum system one might want to make it can in principle be written by means of a **density matrix** ρ.

ρ is a positive operator on the Hilbert space of state vectors for the system normalized to $Tr(\rho) = 1$.

We have one $\rho_{fundamental}$ for the "fundamental degrees of freedom, and we can partly choose one ρ_{full} for the combined system of the fundamental and the phantasy degrees of freedom system. Then you can act

$$\rho_{fundamental}|n>$$

or

$$\rho_{full}(|n>, f)$$

Density Matrix Relation

We shall naturally require for consistency

$$< p|\rho_{fundamental}|n> = \sum_f (< p|, f)\rho_{full}(|n>, f) \tag{10.12}$$

or formulated differently:

$$\rho_{fundamental} = Tr_{w.r.t.\ phantasy}\rho_{full} \tag{10.13}$$

So far we talk about timeless density matrices.

But could we make a purely phantasy time development of only the phantasy or f-degrees of freedom without disturbing the fundamental ($|n>$, $|p>$,...) degrees of freedom ?

Stringy Initial State Assumptions, and Phantasy Notation give String Field Theory The point is we put a fairly large amount of **string theory into assumptions about the initial state**, partly because we cannot do it in the proper Hamiltonian.

The assumption,

"An approximate constraint on the relative state of a couple of particles A and $f(A)$, namely

$$k(\vec{x}_{f(A)} - \vec{x}_A) \approx \vec{p}_{f(A)} + \vec{p}_A, \tag{10.14}$$

where k is a constant, actually related (as to be seen) to the Regge slope α' so important in string theory."

This assumption would if the particles did not have infinite masses mean that the cyclic chain would move along itself.

(10.1 -8) **Yet a complication in relating the trivial static theory to string theory**

The cyclic chains of particles are **not** simply the strings when we identify with string theory - as it would be in Charles Thorns theory -, No.

We have to choose a starting point and go along the cyclical chain from that with two marks in opposite directions along the chain, and then construct for each step an average of the two "people" that started at the start. It is the series of average under this trip of the two "people" that makes up the string.

In this way we get an open string from making this two "people" walk on a cyclically ordered chain.

Main Point: Brought although a bit complicated a correspondence to String Theory

To a set of strings in a known state - e.g. the ground state of their oscillations - one can calculate the state of the corresponding particles (which we usually call "objects") sitting - ordered by the faked f description - in a cyclic chain for each open string (we postpone the closed strings for the moment).

I.e. **We can pretend to see string theory in our game with infinitely heavy particles.**

Most remarkably: When we calculated the overlap between two different sets of strings represented as second quantized states of objects, we got - apart from a wrong sign (a missing i) - the form of the **Veneziano model**.

As the typical example we considered an initial state with two cyclically ordered chains of objects representing two open string in the states of the ground

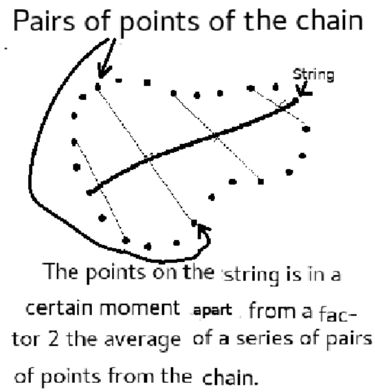

Fig. 10.3. How to construct an open string in terms of objects: You need only one cyclically ordered chain for giving an open string with apart from a factor 2 having the points of the string being the averages of the two object points in the pair. Of course one can pair the objects in different ways even keeping to the continuous type of way illustrated, and that then give the string at a different moment of time. To ensure that the reader identifies the right small spots with the objects one may count that there are 26 objects on this drawing. Corresponding to that there must be 14 points on the open string. The long curved arrows just point to two objects forming a pair, but here are 14 “pairs”,corresponding to 14 points on the string, the objects at the ends of the string being paired with themselves.

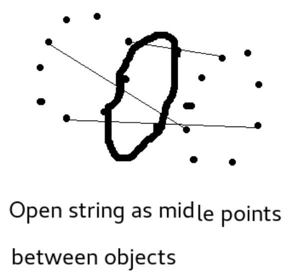

Fig. 10.4. How a closed string is constructed from objects: You need for that **two cyclically ordered chains** of objects (as ordered by f, but f is not drawn on this figure) and except for normalization by a factor 2 you construct the averages for pairs of objects with one object in the one cyclically ordered chain and the other member of the pair in the other cyclically ordered chain.

state of bosonic strings. Then as final state we took a similar state of objects corresponding to two ground state open string. We allowed, however, these open strings to have arbitrary momenta. Then the overlap indeed run out to become the four point Veneziano model for the two incoming and two outgoing particles identified with the strings. We did though get two "small" problems: 1) We had expected to get a Veneziano scattering amplitude being a sum of three Beta-function terms, but we got only one term. 2) If we should expect the overlap we calculated to be identified with the S-matrix element - as the S-matrix in theory without time development would be expected to be 1 - we should have gotten the Veneziano model with an extra factor $i = \sqrt{-1}$ because there is in the expression for the S-matrix in terms of the amplitude which is real in lowest order Veneziano model with an extra i.

But that was what we got at first.

Correspondence with Veneziano Model rather short via thinking on surfaces of string development

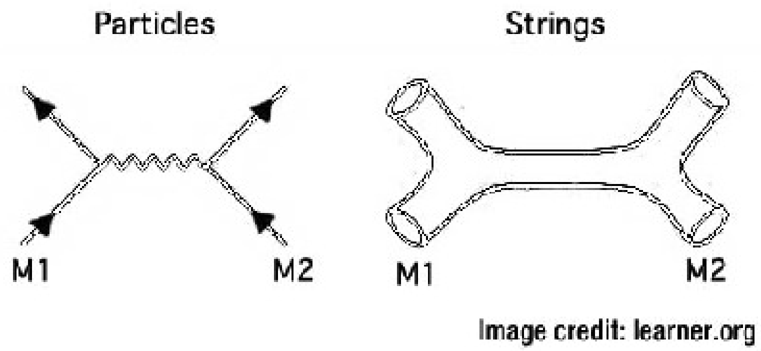

Fig. 10.5.

(10.1 - 9) **Important step in Showing Veneziano Model from Our Novel String field theory**

You think of external ground state strings. They can be produced as in general ground states - by a long imaginary time development with the appropriate Hamiltonian. This development is then written as in complex time development of the string, very reminiscent of what it always used in string theory to compute say Veneziano model.

Very crudely we just give a motivation for this kind of functional integral description of the strings.

Really we do it with a doubled string; i.e. we have a closed string diagram describe the open string. So there are some complications but we did manage to one of the three terms.

Changing Phantasy Degrees of Freedom can Change Number of Cyclic Chains and thus of (Open) Strings

For different "successor functions" f_1 and f_2 you can find different numbers of cyclic chains even for completely the same configuration of the infinitely heavy particles (="objects") and thus in fundamental physics-wise the same situation $|n>$.

Take a fundamental physics situation $|n>$ like this:

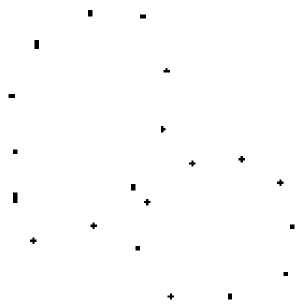

Fig. 10.6.

In choice f_1 of the phantasy you have one open string, in another choice f_2 of the Phantasy gives Two chains, thus Two open strings

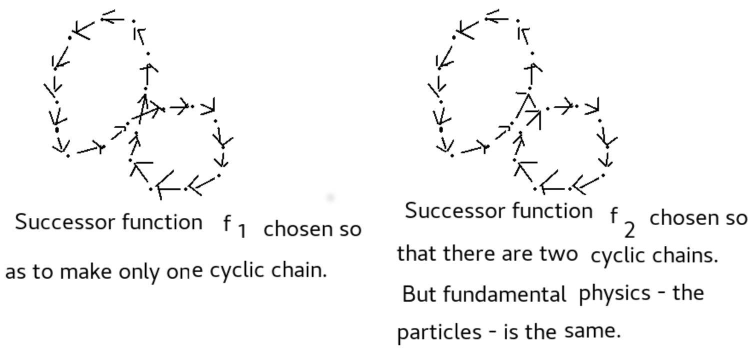

Fig. 10.7. On this figure the reader sees the same points illustrating the "objects" as on foregoing figure, and we hope the reader can see that one can have two separate cyclically closed chains that though come very close to each other at some point. It is such cases that the successor function f can be change a few places and still be in agreement with the approximate requirement that the successor function maps one object to the next in a chain to which it belongs.

Unification of strings can be change of f, thus phantasy, because changing actually actually only a couple of values of the successor function f can cause that e.g. a previously closed cyclically ordered chain of objects get split into two such chains. Since this would correspond to one open string being split into two, we see that splitting can be easily described by changing f. Oppositely of course the opposite change in f would mean a unification of two to become one open string.

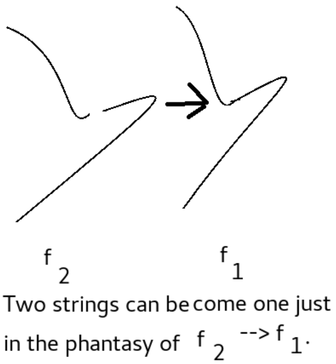

Fig. 10.8.

10.2 Hamiltonian

How to make a purely Phantasy Hamiltonian

The exercise we want to do now is to see what Hamiltonian is allowed working on the extended Hilbert space containing also the phantasy degrees of freedom, so that the basis states are

$$\underline{\text{(Extended)}}\text{ Basis states of the form } (|n>, f) \tag{10.15}$$

$$\text{while}$$

$$\underline{\text{Fundamental}}\text{ basis states of the form } |n> \tag{10.16}$$

Thinking of matrices the extended operator (matrix) consists of a lot of blocks, one block for each matrix element in the (original) fundamental Hamiltonian (which is actually zero).

Attempting to find a phantasy Hamiltonian only moving the Phantsy Degrees of Freedom

For any operator depending only on the physical degrees of freedom O we want the "only phantasy" Hamiltonian $H_{phantasy}$ candidate to commute with it:

$$[O, H_{phantasy}] = 0. \tag{10.17}$$

This condition is, however, too strong, since it would not allow the hamiltonian to depend at all on the "fundamental" degrees of freedom, because if so, the conjugate variable to the one it depended on would be made to vary (and that we wanted to exclude).

We must be satisfied with only having this requirement **approximately.**

Not so good Argument that we can have a wanted $H_{phantasy}$ approximately.

We want the development in the f or phantasy degrees of freedom only to depend on that some cyclic chains come very close / touch; and that is dependent on only very few particles/"objects", so at least it does only involve at first few among an extremely big number of "objects" in the interesting situation.

10.2.1 An Attempt to an $H_{phantasy}$

If one thinks of using position eigen-state say, one could enumerate the particles by the position $\vec{x}$ or we could say that the number I were a function of the position $\vec{x}$ defined for the values of the latter a particle, i.e. defined whenever $n(\vec{x}) = 1$ or bigger. In the case when there are positions with more than one particle the particle number I would in addition have to depend on a small i number enumerating the particles at one special position $\vec{x}$ say. So we would in this more general case write

$$I = I(\vec{x}, i) = \text{The number I of the } i\text{'th particle at the position } \vec{x}. \quad (10.18)$$

where the I is then a name or a number assigned to the ith object at the position $\vec{x}$). In order that at least Hamiltonian $H_{phantasy}$ should commute with those operators O that could be constructed out of the position operators of the "objects" alone, we would have to make the $H_{phantasy}$ operator at least not change the second quantized state in the position basis formulation, when acting on it. As soon as we would make an image of f say $f(I)$ to be change to be an object sitting at a different position than before the action, this could give rise to the change by the action of some only on the positions depending operator, and thus this would not be allowed.

This consideration would leave us with only the possibility that the action with $H_{phantasy}$ to change the image $f(I)$ from what it starts being to the number for an object with the **same position.**. That is to say that if say

$$f(J) = I(\vec{x}, i) \text{before the action with } H_{phantasy}, \quad (10.19)$$

then after the action:

$$f(J) = I(\vec{x}, k) \text{ (where } \vec{x} \text{ is the same, but } k \text{ can be different from } i.) \quad (10.20)$$

This means that under the operation of the phantasy Hamiltonian we can as far as these $\vec{x}$-representation considerations go change the f into another f let us say f' obtained by multiplying it - in the permutation composition way - from the right by a permutation of a subset of objects sitting on the same position. If $P_{\vec{x}}$ denotes the subgroup of the permutations of the objects with position $\vec{x}$, we can say it would be allowed that the sucessor function f changes into f' with $f' = f \circ p$ for some $p \in P_{\vec{x}}$ for some position $\vec{x}$.

That is to say that worrying only about the (fundamental) position dependent operators w.r.t. whether they commute with the phantasy hamiltonian we can allow matrix elements like :

$$< n, f \circ p | H_{phantasy} | n, f > = g \text{ (some nonzero value)} \quad (10.21)$$

where p is a permutation of objects with the same position. (The value g, which we must here introduce, will turn out to be proportional to the coupling constant, also usually called g in the formulation of veneziano models.).

This proposal is, however, although it looks at first o.k. **not good**: The point is that if we concentrate on some successor function f having resulted by multiplication with such a permutation p then you will in that state find that there is an infinite uncertainty in the relative momentum for the two objects that had the same $\vec{x}$ position. It would act much like they had scattered with a pointlike interaction. This would mean indeed that the $H_{phantasy}$ had changed the state of the fundamental degrees of freedom and we wanted to avoid that. Because if our playing or phantasy Hamiltonian truly change the state of the tue fundamental degrees of freedom it is not truly only phantasy.

The occurence of the need for selecting a permutation of the objects sitting on the position $\vec{x}$ say a priori is a little freedom to be specified, but in what we think should be the most important situation:

- (10.2.1.a) That there are in most positions $\vec{x}$ no objects at all, i.e. most $n(\vec{x}) = 0$.
- (10.2.1.b) and the dominant part of the rest of the positions have just one object, i.e. next to $n = 0$ it is the value $n = 1$ that is most common.
- (10.2.1.c) Continuing this way with falling numbers of positions the higher n, the first and dominant value of n for which a non-trivial permutations of the objects at the position is $n = 2$ for the position in question. And in this case there is *only one nontrivial* (i.e. not unity) *permutation of the objects at the position*. So in this most copious non-trivial case the permutation p is not ambiguous.
- (10.2.1.d) Finally we expect higher n-values than 2 to be extremely seldom, and we may ignore approximately this possibility.

Of course it is so to speak the choice of the density matrix ρ which shall give the a priori probability for how to find the system of objects that should be made so that we have this probability for n taking a given value to fall rapidly with the size of this value. It is actually very natural with such a property in the limit of the $\vec{x}$-space going to a continuum.

The reader may check that we also without causing any problem for the conservation under the phantasy Hamiltonian development of the operators depending on the positions for the objects by also allowing

$$< n, f|H_{phantasy}|n, p \circ f > = g^* \text{ (the complex conjugate of } g), \quad (10.22)$$

so that we can achieve that the phantasy Hamiltonian is a Hermitian one in the phantasy Hilbert space constructed as tensor product of the space with the f's as basis vector marks and the fundamental Hilbert space. So we can claim we arrange:

$$H_{phantasy} = H^{\dagger}_{phantasy}. \quad (10.23)$$

10.2.2 The problem of keeping fundamental degrees of freedom fixed

But this proposed $H_{phantasy}$ will not commute with the relative momentum of the typically two objects being permuted by p, because the expression we proposed depends on the relative position and thus will not commute with the conjugate momentum.

Actually as already said it is impossible to solve this problem except at best approximately somehow. If we truly arranged that the phantasy Hamiltonian should commute with all operators from the fundamental degrees of freedom, we would be forced to have a phantasy Hamiltonian only depending on the pure phantasy degrees of freedom, and that would not be so fun.

But we anyhow want to speculate that such a phantasy Hamiltonian can act approximately without disturbing the fundamental degrees of freedom significantly. E.g. one could speculate that as described in the now following subsection, it would approximately commute enough under assumption of the density matrix distribution.

10.2.3 A rather bad example for idea of concrete $H_{phantasy}$

Since we actually have just seen that a fully satisfactory phantasy Hamiltonian is impossible (see the argument below formula (10.17)) , just propose one that has difficulties in the sense of not commuting fully with the fundamental degrees of freedom - meaning operators acting only on the original basis $|n>$ space - would still be of interest. To give the possibility to work on with the idea of constructing a phantasy Hamiltonian that functions approximately let us indeed build the proposal from an operator $N(I, J)$ supposed to act on the space of fundamental states and being effectively zero in all cases when the objects I and J are not close to each other and only significant when these two objects are close to each other. You may take it that it is so to speak a "nearness operator", and that is why we called in by the first letter N in the word "nearness". Such an operator $N(I, J)$ is to be considered an operator of the same kind as an interaction between the two objects I, and J, and thus could be written as a convolutions by some function possibly involving smeared delta functions The operator $N(I, J)$ of course only act on the wave functions for just those two objects I and J, so it could be written acting on the space of all the objects represented by wave function like $\psi(x_1, ..., x_N)$ as

$$N(I, J)\psi(\vec{x}_1, ..., \vec{x}_N) =$$
$$= \int\int d\vec{x}_I' d\vec{x}_J' K(\vec{x}_I, \vec{x}_J; \vec{x}_I', \vec{x}_J')\psi(\vec{x}_1, .., \vec{x}_{I-1}, \vec{x}_I', \vec{x}_{I+1}, ..., \vec{x}_{J-1}, \vec{x}_J', \vec{x}_{J+1}, ..., \vec{x}_N)$$

Of course in order that $N(I, J)$ be a nearness operator the to be chosen function of four spatial vector $K(\vec{x}_I, \vec{x}_J; \vec{x}_I', \vec{x}_J)$ should vanish for any of the four arguments being far away from the other ones.

We should also think of it as being in spite of its locality rather smooth so that it does not change the momenta of the objects i and J too much. Actually the reader should understand that we are hoping for - the impossible - that we have an operator just testing if the two objects numbered I and J are near each other, but preferably without disturbing them. But we know from the discussions of Niels Bohr etc. that in quantum mechanics you cannot measure without disturbing.

Anyway let us go on for pedagogical reasons as if we had arranged an operator $N(I, J)$ that could just observe without disturbing. This would be an only classical intuition that could have that.

If we anyway fall back on classical intuition, we could as well really take it as if the operator also asked for **nearness in momentum space**, i.e. it should be arranged to only be significant in size for the two objects having approximately the same momenta also.

Supposing we now had a for practical purposes such nice operator checking if two objects are in the approximately same point in the phase space $N(I, J)$.

The idea then is that we shall by means of it construct a term in the phantasy Hamiltonian that if $N(I, J)$ is non-zero will permute the actions of the successor function f on the two objects involved. That is to say that with a weight $N(I, J)$ the successor function f shall be changed so that the images of I and J are no longer as at first $f(I)$ and $f(J)$ respectively, but oppositely $f(J)$ and $f(I)$.

This means that we define a term to be put into the phantasy Hamiltonian

$$H_{IJ} = N(I, J)P_{f \to f \circ p_{IJ}}, \tag{10.24}$$

where $P_{f \to f \circ p_{IJ}}$ is an operator only acting on the phantasy-degrees of freedom , i.e. on the f-part, by permuting the two object(numbers) I and J before the action of f. Here the permutation p_{IJ} means the permutation permuting the two objects i and J.

The full proposal for the phantasy Hamiltonian should then be the sum over all pairs of different objects (I, J), with $IneJ$.

That is to say we propose the phantasy Hamiltonian to be of the form:

$$H_{phantasy} = \sum_{(I,J)\ with\ I \neq J} H_{IJ} \tag{10.25}$$

$$= \sum_{(I,J)\ with\ I \neq J} N(I, J)P_{f \to f \circ p_{IJ}}. \tag{10.26}$$

This phantasy Hamiltonian is made so as give some topology change - change in the way the objects are thought to hang together in chains (the cyclically ordered chains) as the phantasy-time goes on, but only provided the chains almost coincide where the change takes place. This will correspond also when translated into strings shifting the topology of how they hang together to only glue the strings in a new way at places where they touch. This is what you expect for physical strings also: they only interfere when they touch.

As already stressed the $H_{phantasy}$ here is at best approximately o.k.. We can argue that it is not so bad again by remarking that if one thinks on strings with infinitely many objects in them and that we can arrange the interaction between the strings to be sufficiently weak - by putting the coupling g above absorbed in $N(I, J)$ sufficiently small - so that one only has about one interaction at a time, meaning that only one out or two of infinitely many objects get disturbed by the operators $N(I, J)$.

10.2.4 Mathematically Formulated Approximate $H_{phantasy}$ Restriction

One idea to make a concrete statement of the sought for purely phantasy hamiltonian, that should preferably only move the phantasy degrees of freedom f but not the fundamental degrees of freedom n, would be to replace the hoped for $[O, H_{phantasy}] = 0$ requirement by the milder

$$Tr(\rho[O, H_{phantasy}]) = 0, \tag{10.27}$$

for all genuinely fundamental operators O and the assumed density matrix ρ expressing our assumption about the state of the system of "objects". We should presumably most wisely only take this relation in the limit of infinitely many objects and then we can hope as just mentioned that a single object being a bit pushed would not count very much, if it stands inside the quantum fluctuations.

It is easy to see by a bit of trivial algebra that if we choose $H_{phantasy}$ to commute with the density matrix ρ we get fulfilled (10.27).

This means that we should look for arranging that our $N(I, J)$'s in the phantasy Hamiltonian commute with the density (matrix) operator ρ.

10.2.5 Unitarity

Once we have settled on a formalism with a constructed phantasy Hamiltonian, we can of course construct corresponding time development operators, say the time-development operator from time t_1 to time t_2 would be

$$U(t_2, t_1) = \exp(-iH_{phantasy}), \tag{10.28}$$

(of course a phantasy development). This time-development - which is also an approximate S-matrix - would of course be a unitary operator acting in the space extended with the phantasy degree of freedoms. Thinking of the development with lowest order perturbation in the parameter g leading to the Veneziano model as we have previously argued, it is essentially obvious that the higher orders will give unitarity corrections to this Veneziano model. So the scheme with the phantasy Hamiltonian should automatically lead to include these Veneziano model unitarity corrections.

(Let us though at this point remind of the problem we had in deriving the Veneziano model: when we did it in the infinite momentum frame - which is very close to the non-relativistic game used in this article - we did not get but one of the three terms we ought to have got. Of course then we shall also miss some of the unitarity correction terms if we just use the here a bit simplified form.)

10.3 On the Concept of Time.

As a little parenthesis at this point let us point out that our picture with the stressing of no time development, really means that in our object-description there is at first *no time*. One can say that the time first comes in when we introduce the phantasy degrees of freedom, and the phantasy Hamiltonian. In this sense the concept of time comes into our scheme as a "phantasy" a fake. The fundamental world has no time. Only by looking at situations in which the various pieces of cyclic chains are screwed together in different combinations as existing at different moments of time a time-concept pops up. That is to say that if one wants to make some ontological model for how a concept of time comes into physics, then we here have the roots for some idea about that:

The time concept could be a phantasy degree of freedom which for some reason could be a reasonable way of describing an a priori timeless physics.

Interestingly enough this attitude of time being a phantasy or fake concept is not actually quite new in as far as we can claim that it is already present in general relativity:

In general relativity all the coordinates and not only (but also) the time coordinate $t = x^0$ are arbitrary and phantasy or fake, in the sense even that the physicist that chooses the coordinates, can decide what these coordinate shall be.

Crudely imposing quantum mechanics and reaching the Wheeler-DeWitt equation one has by this Wheeler-DeWitt equation a restriction on the state, which seemingly tell that the state of the gravity theory is the same at all times. The most close to a Hamiltonian in the gravity theory is namely an integral over the Wheeler-DeWitt equation quantity. This then means that one has got a constraint that the Hamiltonian shall be zero as a constraint. So taking this at face-value one has in gravity a very similar situation as to the one in our scheme: There is no time development, except in some gauge-chosen or fake way.

10.4 Motivations

Purpose of this Faked Scattering String Theory Formulation

Hope you got the idea of **considering a completely trivial** $H = 0$ **quantum field theory** and built up a story of e.g. strings **just by defining some extra "phantasy degrees of freedom"**.

What is the purpose ?:

- (4-a) It is a method to make a second quantized string theory (competing with works by Kaku and Kikkawa and by Witten, ... [2]). You can describe states with several strings.
- (4-b) You may use the idea to look for further models sharing the great property of string theory of **not having the usual divergencies**. Likely this is the only hope for making theories, that make sense, in high dimensions.

Problem of Ultraviolet Divergences Worse the Higher Dimension of Spacetime

Each momentum-formulated loop intergal in a Feynman diagram bring a $\int ... d^d q$ integration and unless there are very many propagators in the loop we cannot avoid divergence for large loop momenta q.

The higher dimension the more different loop integrals lead to divergencies.

To absorb the divergencies into bare coupling constants you need in high dimensions so many that the theory ends up with infinitely many parameters, and is in principle useless.

Direction of Hope for High Dimensional Theories: Formfactors

One needs some factor that can make converge the loops in the high dimensional theory, otherwise you have ultraviolet divergencies and in high dimensions it gets too many different divergencies.

Best hope:
some exponentially falling off factor

$$\text{Factor extra in loop} \propto \exp(-k * q_E^2) \qquad (10.29)$$

much like what one gets from formfactors when one has effective theories for hadrons.

Suggestion:
Replace the particles in the high dimensional theory by **composite (bound) states**, like the hadrons are composite in QCD.
Just Bound States Not Good Enough: Partons [9]
If as we now believe hadrons are bound states [8] but of quarks and gluons called in this connection partons the effective vertices will NOT go down exponentially for very big momentum transfers but will be dominated by the coupling to a single parton and behave at the end more like in the theory of just particles. Thus it will only help a part of the way, but finally at high momenta the divergencies reappear.
Only if there are infinitely many constituents(=partons [9]) in the bound states and they have Bjorken variable $x = 0$, you can postpone parton dominance from popping up, and thus only then we can use the replacement of the original particles in high dimensional theory by bound states.
Hadrons Scatter Crudely by Exchange of Bunches of Constituents
Hadron scattering at energies below where partons collisions become important was described by exchange of other hadrons, pions, vector bosons like ω, again hadrons which again consists of many partons. So it was mainly exchange of lots of partons between one hadron and another one, while the single partons hardly were seen.

Moderate energy Hadron scattering in terms of partons is much like the fake-scattering of just exchanging bunches from one bound state to another one.
(we here ignored the relativity and effects of vacuum)

10.4.1 Bound States Not Perfectly True for Our Fake Model

Let as a side remark call attention to that our model of the string in string theory as "composed" of constituents which we called "objects" should presumably not really be called composite in as far as when we stress that there is no interaction it is not truly a bound state. You could of course think that one could the limit of letting the interaction be weaker and weaker and thus at the end have the non-interacting constituents. One could think of some weakly bound states such as some molecules or atoms and then consider a process in which - for some reason a very fast - exchange of say an electron or some other combination of electrons and nuclei, e.g. some whole atom takes place between a couple of different molecules. If this exchange goes fast compared to the internal quantum mechanical motions of the electrons around the nuclei in two scattering molecules one would make the approximation of taking the scattering or exchange amplitude to be given simply by the overlap of the wave function for the two incoming molecules with that of the two molecules after the scattering. Such an overlap approximation for the scattering or exchange process when it goes fast, would be completely analogous to the type of approximation which we use in our calculations of the Veneziano model amplitude in our fake scattering theory.

In a high energy hadron collision the meeting of the two hadrons goes rather fast compared to the moving around of the constituents / partons in the hadrons. So the usual low transverse momentum type of hadronic collisions are not so far from the described case of a rather fast exchange between the molecules compared to the moving around of the constituents. In this sense one might speculate whether the rather fast passage of the hadrons could be described as being close to being fake in the sense that the genuine interaction between the constituents first shows up before or after the main hitting passage.

When a couple of partons really hit each other there is a fast interaction taking place between the constituents it would not be analogous to the fake process even in the short passage time.

10.5 Unitarity

A Major Achievement of Phantasy Hamiltonian Formulation is Unitarity of Time-development Operator.

If the theory has a formal /phantasy time development given by a Hamiltonian $H_{phantasy}$ then we have automatically that developing during some time interval wil result in a unitary operator development.

Essentially unitary S-matrix.

Perturbation Expansion in Coefficient on the "Phantasy Hamiltonian" $H_{phantasy}$

Really the overall scale of the $H_{phantasy}$ is a matter of the time unit. In fact there is no time in the theory before we introduce the phantasy degrees of freedom and make them move.

Natural to make perturbation theory in the coefficient on $H_{phantasy}$.

Then we get one shift in the topology or way of connection of the cyclic chains for each order in the perturbation. That corresponds to different topologies of string surface diagrams as describing unitarity corrections to the Veneziano model.

10.6 Conclusion

This article was truly inspired by our novel string field theory on which we by now have worked for long, and believe to have formulated a theory in which many strings can be present, so that it is really a string field theory, in terms of what we called objects, which is really pieces of strings taken for right and left movers separately.

The remarkable fact turning out of this our old formalism was that the objects, meaning the bits making up the right and left mover degrees of freedom turned out having zero Hamiltonian, zero time development.

As a pedagogical exercise to study such a system like our objects with zero Hamiltonian we started by considering a second quantized system of infinitely heavy non-relativistic particles they namely have vanishing Hamiltonian if they do not have any interaction:

- We have put forward a very trivial second quantized theory (of infinitely heavy non-relativistic particles identified as our earlier "objects") and assumed for it a Hamiltonian that is zero as operator. So no time development in this "fundamental" theory. (It is this one which is the analogue of the theory of the objects from our Novel String Field Theory.)
- We can only make it more interesting or adjustable by assuming something about the state of it. Say by a density matrix $\rho_{fundamental}$. We use this option to assume that the particles (="objects") sit in (long) closed chains (cyclic chains).
- We interpret each cyclic chain to describe an open string in a string theory.
- We introduced a phantasy system of degrees of freedom by introducing a "successor function" f, which puts all the "objects" (~ particles) into a series of closed chains, thereby making explicit that such chains are assumed to be present by the assumption about the likely state of the trivial second quantized system.
- Mostly we imagine the cyclic ordering is given by the "fundamental" state of the trivial theory, but in some cases it will be ambiguous which chains there are. Then it is we introduce the fake/phantasy/f-variable to distinguish possibilities.
- Then the idea was to make a Hamiltonian supposed to mainly make this fake degree of freedom move, but approximately to avoid varying the "fundamental" degrees of freedom.
  With this we then get a quite phantasy time. We only get time development due to the phantasy degrees of freedom.
  This could be used to realize the philosophy that the very concept of time is indeed a fake concept, so that at the fundamental level there is no time, but only a static state of the universe. Then only by introduction of a fake overbuilding (analogous to our phantasy successor function f we obtain a world seemingly having a time-concept.
  Indeed different moments would corresponds to just different ways of looking at the very same state, whatever the moment in question.

Conclusion on Hopes and Applications

- Really the formulation of ours is a **solution** of second quantized string theory, in the sense that we could say we solved the time development by identifying string theory with several strings with a theory without time development.

- Hope to generalize our "object" picture to different models which have the same great property as string theory of **not having usual divergencies**! This would be absolutely needed in high dimensions, because with point particles high dimensions cause rather hopeless divergencies.
- As a special case we may generalize to p-adic [5,6] Veneziano model [10].

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11 On Dark Stars, Planck Cores and the Nature of Dark Matter

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Abstract. Dark stars are compact massive objects, described by Einstein gravitational field equations with matter. The type we consider possesses no event horizon, instead, there is a deep gravitational well with a very strong redshift factor. Observationally, dark stars can be identified with black holes. Inside dark stars, Planck density of matter is reached, Planck cores are formed, where the equations are modified by quantum gravity. In the paper, several models of dark stars with Planck cores are considered, resulting in the following hypothesis on the composition of dark matter. The galaxies are flooded with low-energetic radiation from the dark stars. The particle type can be photons and gravitons from the Standard Model, can also be a new type of massless particles. The model estimations show that the extremely large redshift factor $z \sim 10^{49}$ and the emission wavelength $\lambda_0 \sim 10^{14}$m can be reached. The particles are not registered directly in the existing dark matter experiments. They come in a density sufficient to explain the observable rotation curves. The emission has a geometric dependence of density on radius $\rho \sim r^{-2}$, producing flat rotation curves. The distribution of sources also describes the deviations from the flat shape. The model provides a good fit of experimental rotation curves. Outbreaks caused by a fall of an external object on a dark star lead to emission wavelength shifted towards smaller values. The model estimations give the outbreak wavelength $\lambda \sim 1$m compatible with fast radio bursts. The paper raises several principal questions. White holes with Planck core appear to be stable. Galactic rotation curves in the considered setup do not depend on the matter type. Inside the galaxy, dark matter can be of hot radial type. At cosmological distances, it can behave like the cold uniform type.

Povzetek. Temne zvezde so masivni kompaktni objekti, ki jih opišejo Einsteinove enačbe za snov v gravitacijskem polju. Avtor obravnava posebno vrsto objektov: nimajo horizonta dogodkov, določa jih globoka gravitacijska potencialna jama z zelo močnim rdečim premikom. Učinkujejo kot črne luknje. Gostota doseže v notranjosti temnih zvezd Planckovo gostoto. V tem Planckovem jedru se enačbe spremenijo zaradi kvantnih efektov gravitacije. Prispevek obravnava vrsto modelov temnih zvezd s Planckovim jedrom ter postavi hipotezo o sestavi temne snovi. Galaksije preplavlja nizkoenergijsko sevanje temnih zvezd. Delci tega sevanja so lahko fotoni in gravitoni Standardnega modela, lahko so pa tudi brezmasni delci nove vrste. Avtor oceni, da lahko sevanje doseže izjemno velik rdeči premik – $z \sim 10^{49}$ – in valovne dolžine $\lambda_0 \sim 10^{14}$m. Obstoječi detektorji temne snovi jih ne zaznajo neposredno. Njihova gostota je dovolj velika, da pojasni izmerjene rotacijske krivulje galaksij. Gostota sevanja se z radijem spreminja, $\rho \sim r^{-2}$, kar povzroči ravne rotacijske krivulje, porazdelitev

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izvorov sevanja pa poskrbi za odstopanja krivulj od ravnih. Model dobro tako dobro opiše izmerjene rotacijske krivulje. Padec zunanjega objekta na temno zvezdo povzroči izbruhe, valovna dolžina sevanja se premakne k manjsšim vrednostim. Model da oceno za valovno dolžino sevanja $\lambda \sim 1$m, kar se ujema s hitrimi radijskimi izbruhi ('fast radio bursts'). Prispevek postavi vrsto bistvenih vprašanj: bele luknje s Planckovim jedrom se zdijo stabilne, rotacijske krivulje galaksij v obravnavanem modelu niso odvisne od vrste snovi — znotraj galaksije je temna snov lahko vroča z radialno odvisnostjo, pri kozmoloških razdaljah pa se lahko obnaša kot hladna snov z enakomerno gostoto.

Keywords: Planck stars, RDM-stars, TOV-stars, dark matter

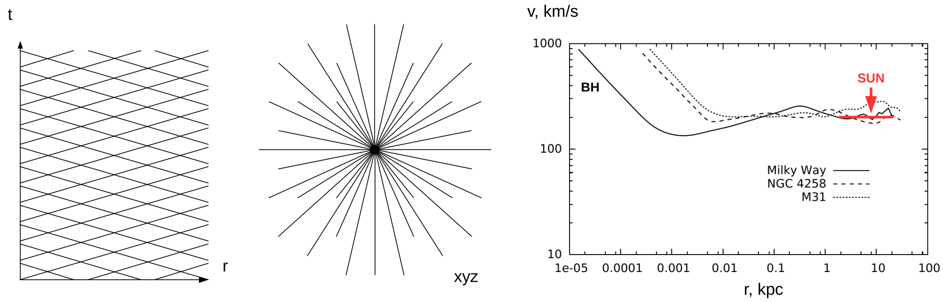

Fig. 11.1. On the left and in the center: an RDM-star – a black hole, coupled to radial flows of dark matter. On the right: experimental rotation curves for three galaxies. Image from [3], data from [4].

11.1 Introduction

Dark stars, also known as quasi black holes, boson stars, gravastars, fuzzballs, are solutions of general theory of relativity, which first follow the Schwarzschild profile and then are modified. Outside they are similar to black holes, inside they are constructed differently, depending on the model of matter used. An overview of these models can be found in the paper by Visser et al. "Small, dark and heavy: but is this a black hole?" [1]. A recent advance has been reported by Holdom and Ren in their paper "Not quite a black hole" [2]. Our contribution to this family are RDM-stars [3], quasi black holes coupled to Radial Dark Matter. A typical configuration of an RDM-star is shown on Fig.11.1 on the left and in the center. It is a stationary solution, including T-symmetric superposition of ingoing and outgoing radially directed flows of dark matter.

An RDM-star can be used as the simplest model of a spiral galaxy. In the limit of weak gravitational fields, the dark matter flows radially converging towards the center of the galaxy produce a typical geometric dependence of mass density on the radius $\rho \sim r^{-2}$, which corresponds to constant orbital velocity $v = Const$, flat rotation curve. It is a qualitatively correct behavior for many experimental rotation curves at large distances, see Fig.11.1 on the right. We will show that a distribution of RDM-stars in the galaxy also allows to describe correctly the deviations of rotation curves from the flat shape. The model of RDM-stars fits very well the experimental rotation curves by Sofue et al. [4–8] and Salucci et al. [9–13].

In strong gravitational fields, RDM-stars behave interestingly. First of all, the event horizon, typical for real black holes, is erased. Instead, a deep gravitational well is formed, where the values of the redshift become enormously large. As a result, for an external observer the star looks black, like a real black hole. Simultaneously, the mass density increases rapidly, reaching and exceeding the Planck value.

This is where Planck stars come into play. This model is based on the calculations in quantum loop gravity, performed for a scalar field cosmology by Ashtekar et al. [14–16]. According to these calculations, the mass density has a quantum correction: $\rho_X = \rho(1 - \rho/\rho_c)$, where the critical density $\rho_c \sim \rho_P$ is of the order of Planck value, ρ is the nominal density before the correction and ρ_X is the effective density participating in Einstein field equations. As a result of this correction, $\rho = \rho_c$ corresponds to $\rho_X = 0$, at the critical density the gravity is effectively switched off, while $\rho > \rho_c$ corresponds to $\rho_X < 0$, in excess of critical density the effective negative mass appears (exotic matter), with gravitational repulsion (quantum bounce phenomenon). In the Planck star model by Rovelli, Vidotto [17], Barceló et al. [18], a collapse of a star leads to the quantum bounce, is replaced by extension, as a result, the black hole turns white.

In this paper we consider a stationary version of a Planck star, stabilized under the pressure of the external matter (Planck core). We will consider two stationary spherically symmetric models with a Planck core in the center. The subject is related to the stability of white holes, earlier investigated in papers by Ori and Poisson [19], Eardley [20], Zel'dovich, Novikov and Starobinskij [21]. It is also related to the origin of fast radio bursts and gives an unusual viewpoint on the nature of dark matter.

The paper is organized as follows. In Section 2 the model of RDM-stars is considered. The main computations have been performed in the author's earlier paper [22], here a short overview of the results is given. In Section 3 the model of TOV-stars with Planck core is presented. In Section 4 the nature of dark matter according to the considered models is discussed. A theoretically interesting question on stability of white holes is considered in the Appendix.

11.2 RDM-stars with Planck core

RDM-stars and rotation curves of galaxies. RDM-star geometry can be used as a simplest model of dark matter distribution in spiral galaxies. Let us consider dark matter flows radially converging towards the center of a galaxy, displayed on Fig.11.1 center, in the limit of weak gravitational fields. The one-line calculation

$$\rho_{dm} \sim r^{-2},\ M_{dm} \sim r,\ v^2 = GM_{dm}/r = Const \tag{11.1}$$

evaluates mass density, enclosed mass function and orbital velocity of stars. Adding a concentrated mass in the center, $v^2 = GM_0/r + Const$, the rotation curve described by the sum of Keplerian and constant terms can be obtained. The real rotation curves, displayed on Fig.11.1 right, possess a similar structure, with Keplerian behavior at small distances and flat shape at large distances. The red line shows a segment 2-20kpc where the rotation curve for the Milky Way can be considered as approximately flat, with the Sun position at 8kpc. These plots show that the real rotation curves deviate from a simple sum of Keplerian and constant terms, revealing additional structures, oscillations. On the other hand, the model with a single RDM-star in the center of the galaxy is also a simplification. Further we consider a model of distributed RDM-stars, able to capture the additional structures. Then we perform a calculation in the limit of strong gravitational fields to analyze the interior structure of an RDM-star.

The detailed description of rotation curves in the RDM-model is based on two assumptions: (1) all black holes are RDM-stars; (2) their density is proportional to the concentration of the luminous matter in the galaxy. As a result, the dark matter mass density can be represented by the integral

$$\rho_{dm}(x) = \int d^3x'\, b(|x-x'|)\, \rho_{lm}(x'),\;\; b(r) = 1/(4\pi L_{KT})/r^2. \qquad (11.2)$$

Here ρ_{lm} is the density of luminous matter, the kernel $b(r)$ represents a contribution of a single RDM-star and L_{KT} is a parameter of length dimension, regulating a coupling between the dark and the luminous matter. This form of coupling has been proposed earlier in a context of a different model in works by Kirillov and Turaev [23,24]. The physical meaning of the L_{KT} parameter is the radius at which the enclosed mass of dark matter equals to the mass of the luminous matter, to which it is coupled: $M_{dm}(L_{KT}) = M_{lm}$.

The detailed rotation curve of Milky Way, known also as Grand Rotation Curve (GRC), has been constructed on the basis of various experimental data by Sofue et al. [4–8]. This curve is presented on Fig.11.2 by data points with errors. Here one can see several structures, including Keplerian contribution of the central black hole (BH), inner and outer bulges (LM1,2), galactic disk (LM3), followed by dark matter contribution (DM) and background outer part (bgr). The red line with the marked Sun position represents the same 2-20kpc approximately flat interval as on the previous figure. This part appears to be relatively small due to a much larger range of distances involved in the analysis.

In paper [8], the distribution of luminous matter in the bulges is described by *exponential spheroid model*, representing the mass density by an exponent $\rho_{lm} \sim \exp(-r/a)$. For the galactic disk *Freeman's model* [25] is used, with the surface mass density described by similar exponent $\rho_{lm} \sim \delta(z)\exp(-r/R_D)$. Taking these distributions, the integral (11.2) and the resulting rotation curve $v(r)$ can be evaluated analytically. The lengthy explicit expressions per every structure are given in [22], also used there as basis functions for the fitting procedure. For stability of the fit, the relative coupling of dark matter to different structures of luminous matter has been fixed as shown in Table 11.1. The λ-constants are used as multiplicative factors to integrals (11.2). Since the different galactic structures may possess a different density and different population of black holes, we can select different coupling constants for them. This procedure is equivalent to a readjustment of the corresponding L_{KT}-parameters, while we prefer to use a single L_{KT}-parameter and adjust the individual couplings by relative λ-factors. Three scenarios have been considered in Table 11.1, the first one assigns all dark matter coupling to the galactic disk, the second one introduces equal coupling among all structures, the third one describes a prevailing coupling for the central structures.

The result of the fit is shown by curves on Fig.11.2. The green line represents the total rotation curve, a quadratic sum over all structures. It has almost the same shape for all three scenarios. Also, the separated contributions of different structures are shown. They depend on the scenario, e.g., the third scenario with prevailing dark matter coupling to central structures also shows a considerable contribution of dark matter in the center. Table 11.2 presents the obtained fitting parameters – the total masses and geometric sizes of the structures. In the considered modeling, the dark matter halo is sharply cut at the radius R_{cut}, further providing Keplerian fall of the outer part of the rotation curve, followed by its linear increase due to the uniform background density. Interestingly, the parameters, characterizing the outer part of the rotation curve, the total mass of dark matter halo $M_{dm}(R_{cut})$ and the background density ρ_{bgr}, appear to be approximately the same for the considered three scenarios.

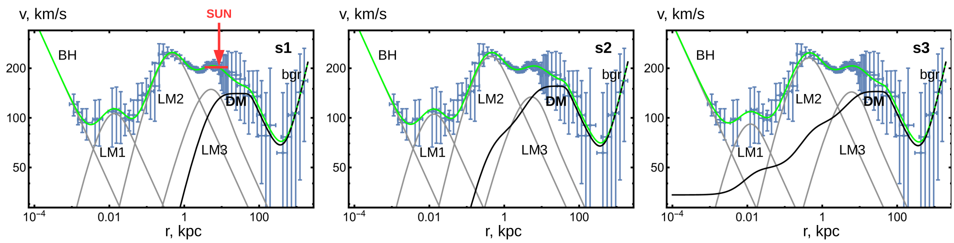

Fig. 11.2. Detailed rotation curve for Milky Way, fitted by RDM-model. Blue points with error bars – data from [8]. Green curve – fit by RDM-model from [22] for three coupling scenarios (s1-3). Contributions of different galactic structures are also shown.

λ_{KT}	s1	s2	s3
λ_{smbh}	0	1	10^3
λ_1	0	1	10^2
λ_2	0	1	2
λ_{disk}	1	1	1

Table 11.1. GRC fit: coupling coefficients for 3 scenarios

Other galaxies can be modeled with a concept of a Universal Rotation Curve (URC) introduced by Salucci et al. [9–13]. It represents averaged experimental rotation

par	s1	s2	s3
M_{smbh}	3.6×10^6	3.6×10^6	3.2×10^6
M_1	5.5×10^7	5.2×10^7	3.6×10^7
a_1	0.0041	0.0039	0.0036
M_2	9.7×10^9	8.6×10^9	8.2×10^9
a_2	0.13	0.13	0.13
M_{disk}	3.2×10^{10}	2.7×10^{10}	3.5×10^{10}
R_D	2.4	2.5	2.8
L_{KT}	7.0	6.3	12.0
R_{cut}	58	45	53
$M_{dm}(R_{cut})$	2.7×10^{11}	2.5×10^{11}	2.6×10^{11}
ρ_{bgr}	646	653	649

Table 11.2. GRC fit: the results*

* masses in $M_\odot$, lengths in kpc, density in $M_\odot/\text{kpc}^3$

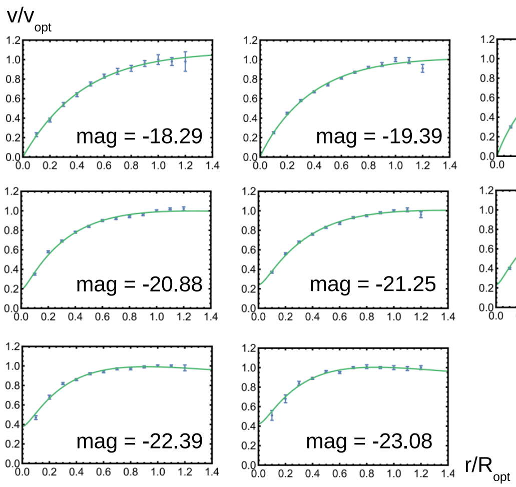

Fig. 11.3. Other galaxies: universal rotation curve, fitted by RDM model. The points with error bars – data from [9]. Green curves – fit by RDM-model from [22]. The data and the fits for different luminosity bins mag are separated.

curves of more than 1000 galaxies. Before averaging, the galaxies are subdivided to bins over the magnitude mag and the curves $v(r, mag)$ are normalized to the values at optical radius: v/v_{opt}, r/R_{opt}. Here, $v_{opt} = v(R_{opt})$ and the optical radius of the galaxy $R_{opt} = 3.2R_D$ is defined as a distance, under which 83% of the luminous mass is located. The averaging smooths the individual features of the curves, their local minima and maxima. The resulting experimental curves appear to be more smooth and are shown by points with errors on Fig.11.3.

On these plots, the radius and velocity are presented in a linear scale, rather than the logarithmic one used in previous plots. As a result, the earlier described central structures are shrinked to a single unresolved central contribution. The modeling is accordingly simplified, preserving only the central and the disk contributions. The basis functions are explicitly written in [22], the result of the fit is presented by green curves on Fig.11.3.

The presented plots show that the model of distributed RDM-stars, based on the Newtonian weak field limit and the proportionality assumptions above, provides a good fit of the experimental rotation curves, for both GRC and URC types.

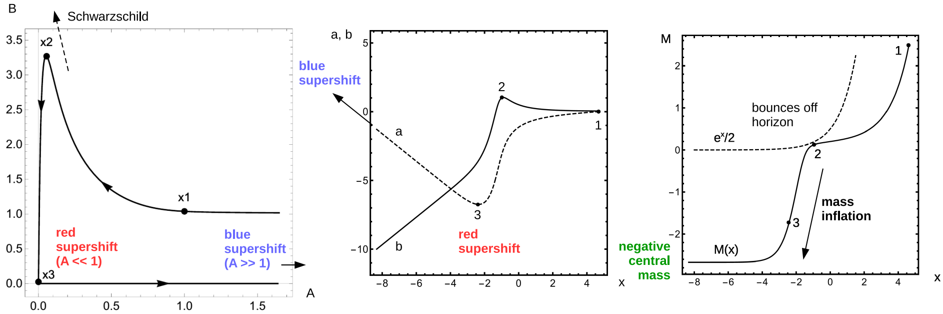

Fig. 11.4. RDM-star model in strong fields. A typical solution in different coordinates (see text).

RDM-star model in strong fields. The system to solve is combined from Einstein gravitational field equations and geodesic equations:

$$G^{\mu\nu} = 8\pi G/c^4 \cdot T^{\mu\nu},\ u^\nu \nabla_\nu u^\mu = 0,\ \nabla_\mu \rho u^\mu = 0. \tag{11.3}$$

We consider the model with T-symmetric non-interacting superposition of ingoing and outgoing flows of dark matter. Therefore, geodesic equations can be applied separately for every flow, described by velocity field u^μ and intrinsic mass density ρ. A static spherically symmetric metric is chosen:

$$ds^2 = -Adt^2 + Bdr^2 + Dr^2(d\theta^2 + \sin^2\theta\ d\phi^2), \tag{11.4}$$

where the profile $A(r) > 0$ describes the redshift and time delay effects, $B(r) > 0$ measures geometric deformation in radial direction. $D = 1$ can be put by convention, so that r is aerial radial coordinate, the area of r-sphere is $4\pi r^2$. Time t is measured by the clock of a distant observer, where $A \to 1$ can be set. Energy-momentum tensor is taken in a form

$$T^{\mu\nu} = \rho(u_+^\mu u_+^\nu + u_-^\mu u_-^\nu),\ u_\pm = (\pm u^t, u^r, 0, 0), \tag{11.5}$$

a sum of T-symmetric radial flows of non-interacting dust matter.

The equations (11.3) have been solved in [3,22]. Geodesic equations possess analytical solution

$$4\pi\rho = c_1/\left(r^2 u^r \sqrt{AB}\right), \tag{11.6}$$
$$u^t = c_2/A,\ u^r = \sqrt{c_2^2 + c_3 A}/\sqrt{AB},$$

in $G = c = 1$ normalization. The integration constants c_{1-3} will be considered in details later. The Einstein equations have a form

$$rA' = -A + AB + 4c_1 B\sqrt{c_2^2 + c_3 A}, \tag{11.7}$$
$$rB' = B/A\left(A - AB + 4c_1 c_2^2 B/\sqrt{c_2^2 + c_3 A}\right),$$

they can be solved numerically. The typical solution is shown on Fig.11.4 left, in (A, B)-coordinates. Initially, near the point x_1, the curve has a hyperbolic form, typical for Schwarzschild solution. The difference starts near the point x_2, where the Schwarzschild solution goes to infinity, the event horizon is formed. In the considered solution, the dark matter acts like a barrier, preventing the formation of the horizon. The solution then goes rapidly towards very small values of A and B, where it exhibits a strong redshift and possesses a small proper length. Then the solution goes to large values of A, a strong blueshift. The same solution is shown in the central part of this figure, in logarithmic coordinates, and on the right part, presenting a Misner-Sharp enclosed mass function:

$$x = \log r,\ a = \log A,\ b = \log B,\ M = (1 - B^{-1})\, r/2. \tag{11.8}$$

In these coordinates the equations obtain the form more convenient for a numerical solution

$$a'_x = -1 + e^b + c_4 e^{b-a}\sqrt{1 + c_5 e^a}, \tag{11.9}$$
$$b'_x = 1 - e^b + c_4 e^{b-a}/\sqrt{1 + c_5 e^a}, \tag{11.10}$$
$$c_4 = 4c_1 c_2,\ c_5 = c_3/c_2^2. \tag{11.11}$$

The convenience follows from the resolution of singularities, typical for polynomial formulation, so that the resulting equations can be easily solved, e.g., by *Mathematica* `NDSolve` algorithm. Also each term in these equations has a clearly defined range of domination, so that normally only one term in the equation is active. This simplifies the asymptotic analysis of the system.

The behavior of the mass function on Fig.11.4 right shows that the solution is bounced off horizon line $M = r/2$, then falls very rapidly. This fall is related to the phenomenon of mass inflation, described in the paper by Hamilton and Pollack [26]. There is a positive feedback loop in black hole solutions with counterstreaming matter flows: (1) increasing energy of the crossing flows leads to (2) increasing pressure, that leads to (3) increasing gravity, that leads again to (1). As a result, an accumulation of very large mass in the counterstreaming region happens. For the considered solutions, the function $M(r)$ decreases with decreasing r. To explain this property, one can imagine spherical shells of positive mass consequentially removed from the star. Finally, the mass arrives to a negative central value, a concentrated negative mass. It corresponds to the well known Schwarzschild singularity of naked type and explains the appearance of a blueshift region in the solution. On the other hand, the singularity is coated in a massive shell appearing due to the mass inflation phenomenon, so that the total mass of the system remains positive. Also, below we will introduce a quantum gravity cutoff in the model, which will remove the naked singularity with most of surrounding structures.

The integration constants $c_{1,2} > 0$, while $c_3 = u_\mu u^\mu = -1, 0, +1$ can take three discrete values, corresponding to the type of dark matter particles: massive, null or (theoretically) tachyonic. Interestingly, the solution in strong fields ($A \ll 1$) depends on the matter type very weakly, since the corresponding term $c_3 A$ in the equations becomes small. Solution in weak fields ($A \sim 1$) depends, at first, on the parameter c_5 that defines asymptotic radial velocity of the dark matter: for $c_5 < -1$, the massive radial flow has a turning point, the matter cannot escape; $c_5 > -1$, possible for all matter types, the matter can escape to large distances, the case further considered:

$$c_6 = c_4\sqrt{1 + c_5},\ c_7 = c_4/\sqrt{1 + c_5},\ \epsilon = (c_6 + c_7)/2. \tag{11.12}$$

The parameter ϵ defines an asymptotic gravitating density $\rho_{grav} = \rho_{eff} + p_{eff}$. The effective density and pressure, produced by counterstreaming dark matter flows, are defined as components of energy-momentum tensor $T_\mu^\nu = \mathrm{diag}(-\rho_{eff}, p_{eff}, 0, 0)$, where

$$\rho_{eff} = c_4/(8\pi r^2 A)/\sqrt{1 + c_5 A},\ p_{eff} = c_4/(8\pi r^2 A) \cdot \sqrt{1 + c_5 A}. \tag{11.13}$$

In the weak field limit $A \sim 1$ we obtain $\rho_{grav} = \epsilon/(4\pi r^2)$, $M_{grav} = \epsilon r$, as in (11.1). This makes ϵ a directly measurable parameter, in physical units $\epsilon = (v/c)^2$, where v is the orbital velocity of stars at large distances from the galaxy center, for Milky Way $v \sim 200\mathrm{km/s}$, $\epsilon \sim 4 \cdot 10^{-7}$.

Quantum gravity cutoff. Further, we omit index *eff* in the formulae, assuming that the effective density and pressure are always considered. Also, for definiteness, we fix the dark matter to null type (NRDM). The resulting model is equivalent to a perfect fluid with the following equation of state (EOS):

$$\rho = p_r,\ p_t = 0, \tag{11.14}$$

there is a relativistic relation between mass density and radial pressure, while the transverse pressure is switched off. The formulae (11.14) become

$$\rho = p_r = \epsilon/(8\pi r^2 A). \tag{11.15}$$

Further, for illustration, we consider the solution for the Milky Way galaxy with a concentrated RDM-star in the center. Fig.11.5 left shows the corresponding metric profiles. The solution starts in point 1 far away from the center, then in point 2 attempts to go to the Schwarzschild regime. We remind that the scale is logarithmic

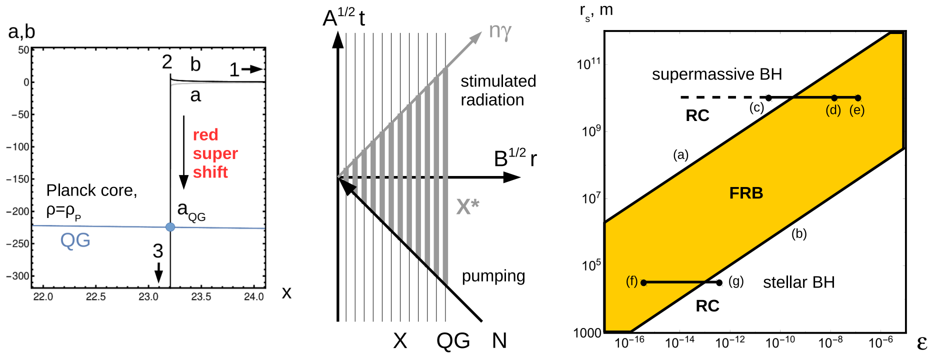

Fig. 11.5. On the left: quantum gravity cutoff in RDM-model for Milky Way scenario. In the center: a mechanism for generating FRB in RDM-model. On the right: simultaneous analysis of rotation curves and FRBs in RDM-model. Images from [22].

and the metric profiles jump many orders of magnitude near the point 2. Then they fall into abyss due to the red supershift phenomenon. Much earlier than the A-profile reaches minimum in point 3, the Planck density is achieved $\rho \sim \rho_P$. At this point we stop the solution and place a Planck core below it. Since the B-profile is also very small at this point and according to the formula (11.8), the Planck core possesses negative total mass, whose repulsive force supports the whole system in equilibrium. In further computations, only the order of the magnitude is important, on necessity, corrections can be applied via phenomenological factors [22]. Taking into account that $\rho_P = l_P^{-2}$ in the units used, where l_P is Planck length, also that redshift factor falls rapidly at almost constant $r \sim r_s$, the value in the cutoff point becomes

$$A_{QG} = \epsilon\,(l_P/r_s)^2/(8\pi). \tag{11.16}$$

For the Milky Way, substituting the estimation of the ϵ-parameter above and the known gravitational radius r_s of the central black hole from Ghez et al. [27], we obtain $A_{QG}^{1/2} = 1.7 \cdot 10^{-49}$. This value will be important for our further calculations.

RDM-stars as sources of Fast Radio Bursts. The common property for all dark star solutions is the presence of high energetic phenomena and strong redshift in their depths. Therefore, high energy photons created in these phenomena on the way out can be shifted to a long wave diapason. This makes dark stars natural candidates for sources of FRBs, the powerful flashes of extragalactic origin, registered in radio band. The lowest FRB frequency of 111 MHz has been reported by Fedorova and Rodin [28], the highest of 8 GHz – by Gajjar et al. [29]. Detailed experimental characteristics of FRBs can be found in *frbcat* catalogue by Petroff et al. [30], there is also a catalogue of existing FRB theories *frbtheorycat* by Platts et al. [31]. At the time of this writing, 118 distinct FRB sources have been registered and 59 FRB theories have been created.

A particular scenario with an RDM-star generating an FRB has been considered in [22]. An object of an asteroid mass falls onto the RDM-star. The gravitational field inside the star acts as an accelerator with super-strong ultrarelativistic factor $\gamma = A_{QG}^{-1/2} \sim 10^{49}$. The nucleons N composing the asteroid enter in the inelastic collisions with particles X forming the Planck core, producing the excited states of a typical energy $E(X^*) \sim \sqrt{2m_X E_N}$. The high-energy photons formed by the decay of X^* with energy $E(\gamma, in) \sim E(X^*)/2$ are subjected to super-strong redshift factor $\gamma^{-1} = A_{QG}^{1/2} \sim 10^{-49}$. The γ-factors do not compensate each other due to the presence of the square root in the formula. Thus, the outgoing energy $E(\gamma, out) \sim \sqrt{m_X m_N/(2\gamma)}$, the wavelength $\lambda_{out} = \sqrt{2\lambda_X \lambda_N \gamma}$. Taking $\lambda_X \sim l_P$, we obtain a formula for FRB wavelength

$$\lambda_{out} = 2(2\pi)^{1/4}\sqrt{\lambda_N r_s}/\epsilon^{1/4}, \tag{11.17}$$

containing only Compton wavelength of nucleon λ_N and (r_s, ϵ)-parameters of the RDM-star. Interestingly, Planck values are canceled out of the formula. Further, taking $\lambda_N = 1.32 \cdot 10^{-15}$m, $r_s = 1.2 \cdot 10^{10}$m, $\epsilon = 4 \cdot 10^{-7}$, the wavelength and frequency of FRB are

$$\lambda_{out} = 0.5\text{m},\ \nu_{out} = 0.6\text{GHz}, \tag{11.18}$$

that falls in the observed range 0.111-8GHz of FRB frequencies.

Further evaluations can be found in [22]. A snowball mechanism is introduced for generating a sequence of excited states, which produces the energy spectrum of photons cut from above by the computed E_{out} value. The spectrum is open towards low energy values, however, the increasing scatter broadening dilutes the signal there. A common mechanism of stimulated emission (aka LASER) can generate a short pulse of coherent radiation, by the scheme displayed on Fig.11.5 center. Other parameters, such as pulse width and pulse delay, spectral and beam efficiency, as well as polarization, repetition and periodicity, observed for some FRBs, have been also discussed in [22]. Most of these parameters are insensitive to the nature of the FRB source, being imparted by local environment and/or interstellar/intergalactic medium on the way of signal propagation. These parameters can be described by the known source independent astrophysical mechanisms, such as scatter broadening and signal dispersion, as well as scenarios with an FRB source passing through a planetary system or an asteroid belt.

The estimation above has been made for a simplified scenario with a concentrated RDM-star in the center of the galaxy with Milky Way alike parameters. Fig.11.5 right shows more possibilities. The coordinates (r_s, ϵ) are the gravitational radius and the parameter defining the contribution of a particular black hole (=RDM-star) to the galactic dark matter halo, $\epsilon = GM\lambda/(c^2 L_{KT} N)$, where M, λ, L_{KT} are parameters from the fit of the galactic structures explained at the beginning of this section, N is the number of black holes in the structure. The band shows detected FRB frequencies between the lines (a) and (b), according to (11.17). Two horizontal lines show two classes of solutions, supermassive and stellar black holes, according to the scenarios considered above: (c) s2, (d) s3, (e) corresponds to the minimal velocity value $v \sim 100$km/s on the plots Fig.11.2, (f) $\epsilon = 4 \cdot 10^{-7}$ divided to $N = 10^9$ stellar black holes, (g) the same with $N = 10^6$, the estimations of the number of stellar black holes are from Wheeler and Johnson [32] and references therein.

The main conclusion from the analysis of the plot on Fig.11.5 right is that the band and the horizontal lines have an intersection, therefore the model of RDM-stars is able to describe simultaneously the rotation curves and FRBs. Moreover, two solution classes exist, stellar and supermassive black holes. The plot is constructed on the basis of Milky Way data, extracted from its highly detailed rotation curve, and is valid for galaxies of similar structure. It would be interesting to populate it with data from other galaxies, that depends on the availability of rotation curves with a comparable detalization.

11.3 TOV-stars with Planck core

In this section we consider Tolman-Oppenheimer-Volkoff (TOV) stars. It is well known system, described by EOS

$$w\rho = p_r = p_t, \tag{11.19}$$

differing from RDM-stars EOS (11.14) by the presence of two components of transverse pressure $T^\mu_\nu = \mathrm{diag}(-\rho, p_r, p_t, p_t)$, equally distributed with the radial one (isotropic pressure). Parameter w regulates the composition and temperature of the star. Small values $w = kT/(mc^2)$ correspond to an ideal gas of massive particles of a given temperature. In this section we will mainly consider an ultrarelativistic plasma or photon gas, corresponding to the value $w = 1/3$. The Einstein equations have a form (see, e.g., Blau [33]):

$$w\rho'_r = -(\rho M/r^2)(1+w)(1+4\pi r^3 w\rho/M)(1-2M/r)^{-1}, \tag{11.20}$$

$$M'_r = 4\pi r^2\rho,\ h'_r = 4\pi r(1-2M/r)^{-1}\rho(1+w), \tag{11.21}$$

where the metric coefficients are chosen as

$$A = e^{2h} f,\ B = f^{-1},\ f = 1 - 2M/r. \tag{11.22}$$

A consequence of this system is so the called hydrostatic equation

$$r(p+\rho)A'_r + 2Arp'_r = 0, \tag{11.23}$$

possessing an analytical solution

$$4\pi w\rho = k_1 A^{k_2} \tag{11.24}$$

with constants

$$k_1 = 4\pi\rho_1 w,\ k_2 = -(1+1/w)/2,\ k_3 = \log k_1. \tag{11.25}$$

The system can be rewritten in logarithmic variables (11.8), to the form convenient for a numerical solution:

$$a'_x = -1 + e^b + 2e^{2x+k_2 a+b+k_3}, \tag{11.26}$$

$$b'_x = 1 - e^b + (2/w)e^{2x+k_2 a+b+k_3}. \tag{11.27}$$

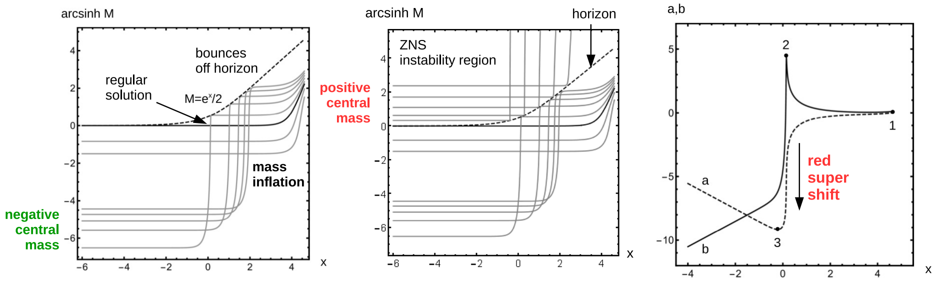

Fig. 11.6. TOV-star solutions. On the left: solutions with negative central mass, in the center: with positive central mass, with phenomenon of Zel'dovich-Novikov-Starobinskij explained, on the right: a typical behavior of metric coefficients.

They are complemented by initial data $a_1 = 0$, $b_1 = -\log(1 - 2M_1/r_1)$, where $w = 1/3$, $k_2 = -2$, $k_3 = \log(4\pi\rho_1/3)$, ρ_1 and M_1 at a large r_1 are given. The typical solution is shown on Fig.11.6 left. The coordinates are $x = \log r$ and $\operatorname{arcsinh} M$, the last one possesses asymptotics $\pm \log |2M|$ at large $|M|$, convenient to display all features of a solution in a single plot.

Usually a regular solution is investigated, satisfying a condition $M = 0$ in the center. This solution is shown by a thick line on the figure. We investigate what happens if this condition is relaxed. If a solution with the same ρ_1 is started with smaller M_1, below the regular line, it simply ends below this line in a negative central value. More interesting, if the solution is started above the regular line, it will not end in a positive central value and will not cross a horizon. Instead, it bounces off the horizon, goes through the mass inflation and ends in an even more negative central value. These solutions are clearly singular in the center, however, they are of interest to us, since the quantum gravity cutoff considered below can remove these singularities, replacing them with a regular Planck core.

For completeness we also consider a case of Fig.11.6 center, when the solution is started above the horizon line, physically under the horizon. It similarly bounces off the horizon from inside and goes to the positive central value. If one reverts the integration, the solution started under the horizon from a positive mass Schwarzschild singularity will stay inside the horizon. This phenomenon was discovered by Zel'dovich, Novikov, Starobinskij [21] investigating the formation of white holes under the influence of matter ejected from the central singularity. The system is described by similar equations and the result is that the ejected matter never leaves the horizon and the white hole under the described circumstances cannot explode. This effect (internal ZNS instability) is one of instability types inherent to white holes, the other one (external Eardley instability) will be considered below in the Appendix. Mainly, in this paper, we consider a dual solution, possessing negative mass Schwarzschild singularity and evolving outside of the instability region.

The plot on Fig.11.6 right shows the typical evolution of metric coefficients, appearing to be very similar to such plots for RDM-stars. An important difference is that the redshift fall and the mass inflation for a TOV-star appear to be much more moderate in comparison with an RDM-star of similar parameters. Table 11.3

shows the scenario with a stellar mass compact object in a cosmic microwave background, described by TOV equations. Although the variation of metric coefficients and enclosed mass in physical units is very large, it is still much smaller than the analogous variations for RDM-stars.

model parameters	$M_1 = 10M_\odot, w = 1/3, \rho_1 = \rho_{cmb} = 4 \cdot 10^{-14}$ J/m^3
starting point of the integration	$r_1 = 10^6$m, $a_1 = 0$, $b_1 = 0.0299773$, $M_1/M_\odot = 10$
supershift begins	$r_2 = 29532.4$m, $a_2 = -54.2719$, $b_2 = 53.7265$, $M_2/M_1 - 1 = -3.64729 \cdot 10^{-23}$, $r_2 - 2GM_2/c^2 = 1.37139 \cdot 10^{-19}$m
supershift ends	$r_3 = 20638.1$m, $a_3 = -107.522$, $b_3 = -104.685$, $\log_{10}(-M_3/M_\odot) = 46.3087$
minimal radius (Planck length), end of the integration	$r_4 = 1.62 \cdot 10^{-35}$m, $a_4 = -17.6594$, $b_4 = -195.278$, $M_4 = 1.728\ M_3$

Table 11.3. TOV-star, scenario with a stellar mass compact object in cosmic microwave background

model parameters	$w = 1/3, \rho_1 = \rho_{cmb} = 4 \cdot 10^{-14}$ J/m^3, $a_{QG} = -146.264$
starting point of the integration	$r_1 = 1.13042 \cdot 10^{-2}$m, $a_1 = 0$, $b_1 = 0.0100503$, $M_1 = 7.61132 \cdot 10^{22}$kg
supershift begins	$r_2 = 1.13042 \cdot 10^{-4}$m, $a_2 = -73.6529$, $b_2 = 73.0876$, $r_2 - 2GM_2/c^2 = 2.04973 \cdot 10^{-36}$m
supershift ends	$r_3 = 7.89967 \cdot 10^{-5}$m, $a_3 = -146.264$, $b_3 = -143.408$, $\log_{10}(-M_3/M_\odot) = 54.7085$
minimal radius (Planck length), end of the integration	$r_4 = 1.62 \cdot 10^{-35}$m, $a_4 = -75.7824$, $b_4 = -214.619$, $M_4 = 1.72898\ M_3$

Table 11.4. micro TOV-star, the critical case

Considering this scenario in more details, we see that $r_2 - r_s \sim 10^{-19}$m, the object comes very close to the gravitational collapse. This is a distance where the matter terms, initially weak, representing cosmic microwave background, are amplified and start to dominate in the equation. Although this is the result of a purely classical model, quantum considerations can change this number.

Further, $\log_{10}(-M_3/M_\odot) \sim 46$, in comparison with the mass of the observable universe: $\log_{10}(M_{uni}/M_\odot) \sim 23$. Thus, the considered compact object contains a core of negative mass, by absolute value much greater than the mass of the universe, compensated by the coat of TOV matter with almost the same positive mass. A similar computation for RDM-model gives an even larger number: $\log_{10}(-M_3/M_\odot) \sim 10^5$.

These enormous numbers could be the result of model idealization. Their origin is the unrestrained phenomenon of mass inflation. It can be changed if a (non-gravitational) interaction between the counterstreaming flows and corresponding corrections to EOS will be taken into account. Also, the considered solutions are stationary and can take enormous amount of time to form. A qualitative interpretation of the obtained solutions is that a permission of negative mass (Planck core) leads to a polarization of the solution to the parts with highly positive and highly negative masses, almost compensating each other in the result.

The other origin of large numbers is Planck density: $\rho_P = 5 \cdot 10^{96}\text{kg/m}^3$. Straightforward estimation for the Planck density core of only $R = 1$mm radius gives the mass $M = (4/3)\pi R^3 \rho_P = 2 \cdot 10^{88}$kg, gravitational radius $R_s = 2GM/c^2 = 3 \cdot 10^{61}$m, much larger than the mass and the radius of the observable universe $M_{uni} = 10^{53}$kg, $R_{uni} = 4 \cdot 10^{26}$m. Such a core will immediately cover the universe by its gravitational radius, with a large margin. To place such objects in our universe, a mechanism for mass compensation is necessary. For instance, the one of this paper, effectively negative masses created by quantum gravity and coated by positive mass shells until the equilibrium with a moderate mass value is reached.

Enormous reserve of energy hiding inside TOV-stars can fuel extremely high-energetic phenomena. Figuratively speaking, if such a bubble bursts somewhere, the consequences can be felt throughout the universe. Thus, it is natural to consider these objects as potential sources of FRBs and we will do this, at first considering the quantum gravity cutoff and formation of Planck core in the center of TOV-star.

Quantum gravity cutoff. Setting $w = 1/3$ in a solution of TOV hydrostatic equation (11.24), obtain $\rho \sim A^{-2}$. Therefore for the considered scenario with cosmic microwave background: $\rho_P/\rho_{cmb} = A_{QG}^{-2}$. Taking $\rho_P = 4.633 \cdot 10^{113}\text{J/m}^3$ and $\rho_{cmb} = 4.19 \cdot 10^{-14}\text{J/m}^3$ from Longair [34], in energetic units, have $A_{QG} = (\rho_{cmb}/\rho_P)^{1/2} = 3 \cdot 10^{-64}$, $a_{QG} = -146$. The question now is whether such value can be reached. For RDM-stars with physically interesting parameters the redshift fall is enormous and the Planck density can be always reached before achieving the minimum in a-dependence. For a TOV-star, the redshift fall and associated density increase in solutions are moderate. The solutions shown in Fig.11.6 right and Table 11.3 pass the minimum a_3 before reaching a_{QG} and the Planck density for these solutions is not reached. The necessary condition for formation of the Planck core is $a_3 < a_{QG}$. To investigate a satisfaction of this condition, we have performed the following numerical experiment. Keeping the outer density fixed to ρ_{cmb}, we changed the solution mass, or associated parameter $x_{10s} = \log_{10} r_s$, where r_s is Schwarzschild radius in meters, in the range $x_{10s} \in [-10, 10]$. After the integration of TOV-equations, we detected the minimum a_3 and found that it is well approximated by linear dependence $a_3 = -128.089 + 4.60519 \cdot x_{10s}$. As a result, $a_3 < a_{QG}$ condition is satisfied at $r_s < r_{s,crit} = 0.11$mm (micro TOV-stars), $M < 7.6 \cdot 10^{22}$kg, approximately Moon's mass. The critical case is shown in Table 11.4. The equality $a_3 = a_{QG}$ and the resulting $r_2 \sim r_s$ confirms this computation.

TOV-stars as sources of Fast Radio Bursts. Let us consider a photon of initially Planck energy, $E_{in} \sim E_P$, $\lambda_{in} \sim l_P$, on the surface of the Planck core. After applying the redshift, the outgoing wavelength $\lambda_{out} = l_P A_{QG}^{-1/2} = 0.9$mm. Experimentally it is $\lambda_{exp} = 37.5$mm, for the highest 8 GHz FRB detection of FRB121102 source [29]. The deviation factor $\lambda_{exp}/\lambda_{out} \sim 40$ can still be considered as a good hit, taking into account 127 orders of difference in the input density parameters ρ_{cmb}/ρ_P. Technically, it can be compensated by an attenuation factor $E_{in} = E_P/N$, the initial photon is $N \sim 40$ times weaker than Planck energy. A part of this factor can be related with $(1+z)$ cosmological redshift of the source, $z \sim 0.2-0.3$, the remaining factor to explain is $N \sim 30$.

The analytical formula for the wavelength is also interesting:

$$\lambda_{out} = l_P(\rho_{cmb}/\rho_P)^{-1/4},$$

or, in Planck units, simply $\lambda_{out} = \rho_{cmb}^{-1/4}$, depending only on the cosmic microwave background density.

Consideration of other FRB parameters proceeds similar to [22]. Most of the parameters depend not on the source, but on its environment and propagation medium of the signal. Here we consider one question: can the bursts repeat? For the critical case $r_s = r_{s,crit}$ and isotropic estimation of the total burst energy from Cao et al. [35], there is an inner reserve of energy for $7.6 \cdot 10^{22}\text{kg} \cdot c^2/(10^{32-34}\text{J}) \sim 10^{6-8}$ bursts. The energy can be also refilled from the environment, e.g., a companion, an asteroid belt, etc. In this refilling, when the threshold $r_s > r_{s,crit}$ is passed, the conditions for Planck core existence disappear. This can trigger the FRB, that will return the system to $r_s < r_{s,crit}$ state.

In summary, TOV-stars can also be the sources of FRB, or may represent a species of these signals. Differently to FRB from RDM-star, triggered by the fall of an asteroid, TOV-star signals can be autogenerated, possessing also a mechanism for autonomous oscillations around the critical state.

Comparison of different models. While both RDM- and TOV-stars can generate FRBs, the asymptotically flat rotation curves are generated only by RDM-stars. Only they possess the necessary $\rho \sim r^{-2}$ dependence, while the considered TOV solutions have $\rho \to Const > 0$ asymptotics.

On the other hand, in Barranco et al. [36] a different solution of the TOV system has been investigated, possessing $\rho \sim r^{-2}$ dependence. It is a well known analytical self-similar solution, whose existence follows from scale-invariance of the system, see, e.g., the work by Visser and Yunes [37]. Due to the appropriate density profile, this solution can be used to describe the rotation curves, a configuration known as isothermal dark matter halo. In addition, the asymptotic velocity on this solution appears to be $(v/c)^2 = 2w/(1+w)$. The experimentally observed velocities are non-relativistic, achievable only for small w. From here a conclusion is drawn, that the dark matter composing the galaxies "must be cold".

If one uses RDM model instead of TOV, a physically different system with the absence of transverse pressure is formed. Here all types of dark matter produce the same density profile $\rho \sim r^{-2}$ and the same asymptotically flat rotation curves. The value of the orbital velocity is defined by the parameter $(v/c)^2 = \epsilon$, while the type of the matter by the other parameter c_5. As a result, the consideration of rotation curves in the RDM model does not impose a restriction on the type of dark matter in the galaxies.

Self-similar solutions of the TOV system form a very special class, different from the ones considered in this section. The regular type solutions we consider look like a ball of almost constant density, with a little bump of density in the center, due to self-gravitation. The singular solutions we consider have the same outer asymptotics, just possess a concentrated negative mass or a regular Planck core in the center. Self-similar solutions possess such a strong self-gravitation, that the whole solution shape is changed, also at large distances. This is possible only at a very large mass of solution. Especially, for the photon gas we consider, the mass should be enormous to make the light condense under its own gravitation. The computation shows $\rho_1^* = \epsilon^*/(4\pi r_1^2)$, $M_1^* = \epsilon^* r_1$, $\epsilon^* = 2w/(1 + 6w + w^2)$, in geometrical units, for self-similar solution. With $w = 1/3$, at $r_1 = 3.1 \cdot 10^{21}$ m, the outer range of the Milky Way galaxy, it is $\rho_1^* = 0.21$ J/m^3, $M_1^* = 4.5 \cdot 10^{17} M_\odot$, being compared with $\rho_{cmb} = 4 \cdot 10^{-14}$ J/m^3, $M_{cmb} = (4\pi/3)\rho_{cmb} r_1^3 = 2.8 \cdot 10^4 M_\odot$. Thus, the mass characteristics of the system we consider are 13 orders of magnitude below the formation of self-similar solutions.

The other question is an ability of Planck stars directly generate FRBs, investigated by Barrau, Rovelli, Vidotto in [38]. The BRV model considers a collapse of primordial matter to a black hole going through the quantum bounce to the eruption of the white hole. The eruption appears at a delayed time due to strong gravitation. The time of recollapse depends on the mass of the star and is estimated to $t = 0.2M^2$, in Planck units. Equating it with Hubble time, the mass and the size of Planck stars are estimated, created at the Big Bang and exploding "today": $M = (5t_H)^{1/2} = 1.2 \cdot 10^{23}$kg, $r_s = 2M = (20t_H)^{1/2} = 0.2$mm. This estimation comes close to the critical size of TOV-stars $r_{s,crit} = 0.11$mm obtained in our model.

The BRV model predicts an observable FRB signal at $\lambda \sim r_s \sim 0.2$mm. The cosmological redshift correction can be also applied. The result is numerically similar to our model ($\lambda \sim 0.9$mm), although obtained in a completely different setup: recollapse of Planck stars vs redshift of photons of initially Planck energy that arise in stationary TOV solutions with the Planck core in thermal equilibrium with CMB. Our prediction $\lambda \sim \rho_{cmb}^{-1/4}$ and the BRV formula $\lambda \sim (20t_H)^{1/2}$ coincide up to a numerical factor ~ 4.7, if one takes into account cosmological constraints $\Omega_{cmb} = \rho_{cmb}/\rho_{crit} = 4.2 \cdot 10^{-5}$, $\rho_{crit} = 3H^2/(8\pi)$, $t_H = 1/H$.

In the original BRV model of Planck stars only non-repeating FRBs are possible. The work by Barceló et al. [18] proposes repeating recollapses and final stabilization of an object due to dissipative effects. Such a stationary object can be equivalent to the RDM- and TOV-stars discussed here. After its formation, it can produce both repeating and non-repeating FRBs depending on the environment.

The further paper by Barceló et al. [39] seems to "close" the topic of Planck stars, referring to Eardley instability of the white hole part. Due to this instability, the white holes under the influence of external radiation would turn into black holes, not having time to emit the FRB. Below, in Appendix, we will bring a contra-argument, showing that Eardley instability can be eliminated if the core of the white hole possesses negative mass. Physically, it can be the Planck core, formed as the result of quantum gravity corrections when the Planck density is reached. Therefore the models from the Planck star family, as well as the FRB estimates based on them, avoid the white hole instabilities in a self-consistent way.

11.4 Discussion: What is dark matter made of?

In this section we consider three hypotheses on the composition of dark matter, based on the considered dark star models.

Hypothesis 1: the galactic dark matter can be cold, can be hot, producing the same rotation curves.

It follows from the solution properties of the RDM model, the orbital velocity depends only on intensity factor ϵ, not on matter constitution (cold/hot, M/N/T cases, controlled by the other constant c_5). It can be a new type of particles, which can be sterile for interactions with the known matter sectors, i.e., enter only in gravitational interactions with them. It can be almost sterile, i.e., other interactions allowed at high energies in Planck cores, while extremely weak at low energies outside.

One more fascinating possibility is that the dark matter is composed of known particles, placed in an unusual condition. Let consider a photon of Planck energy, emitted from the surface of the Planck core of an RDM-star: $E_{in} \sim E_P$, $\lambda_{in} \sim l_P$. Applying the redshift $A_{QG}^{1/2} \sim 10^{-49}$, have $\lambda_{out} = l_P A_{QG}^{-1/2} \sim 10^{14}$m. It is an extremely large wavelength, about 4 light days, 16 times the Sun-Pluto distance. Such longwave photons can not be registered by usual means, e.g., via radio telescopes. Although the energy of every such photon is extremely small, they come in numbers providing the necessary mass density to explain the rotation curves of the galaxies. The detailed consideration shows that at the Planck core one Planck energy particle per Planck area per Planck time is emitted, that corresponds to Planck density and pressure on its surface. After that the factor $A_{QG}^{1/2}$ is applied twice, for redshift and gravitational time dilation, then the geometrical $(r_s/r)^2$ factor corresponds to the measured halo density $\rho = p_r = 1/l_P^2 \cdot A_{QG} \cdot (r_s/r)^2 = \epsilon/(8\pi r^2)$, where for the redshift factor the formula (11.16) is used.

In the considered scenario the particles should be massless. For massive particles the Compton length must be greater than $\lambda_{out} \sim 10^{14}$m, obviously, excluding lightest neutrino species and other massive particles. Those particles do not overcome the gravitational barrier and remain bounded inside the RDM-star. From the Standard Model, the only appropriate particles for this scenario can be photons and gravitons. Scenarios with massive particles should have a larger starting energy to overcome the barrier.

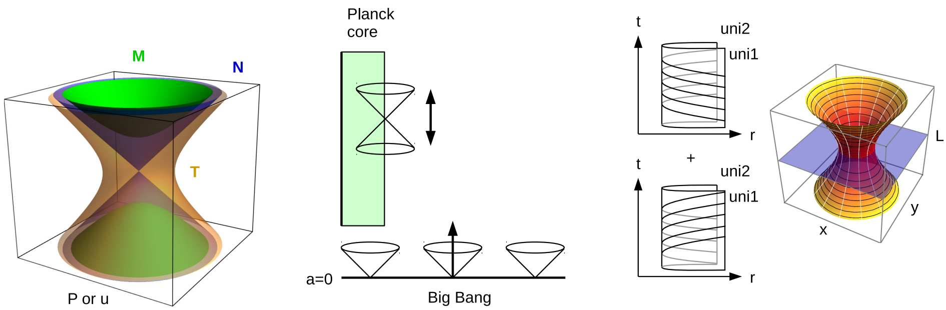

Fig. 11.7. Illustration to hypothesis 2. On the left: the mass shells for massive, null and tachyonic particles. In the center: a difference between Big Bang and Planck core light cones structure. On the right: the world lines of dark matter particles captured by a wormhole.

Hypothesis 2: the emission of galactic dark matter from a Planck core is T-symmetric, in future and in past directions.

We remind that an RDM-star contains two T-symmetric flows, ingoing and outgoing ones. Fig.11.7 left shows the mass shells for momentum P or velocity u vectors. There is a one-sheet tachyonic shell, containing both ingoing and outgoing directions, and two-sheet massive/null shells, where these directions are separated. In any case, we assume that all mass shells become completely occupied at the Planck core. The reason can be an extremely high temperature, in Planck range $T \sim T_P$, the one achievable at Big Bang. It is so hot there, that the vicinity of the Planck core becomes insensitive to the external thermodynamical time arrow and develops an own, T-symmetric thermodynamics. An important difference in this context is that RDM singularity and Planck core are timelike, while Big Bang singularity is spacelike. Different orientation of light cones can lead to the absent time arrow (recovered T-symmetry) near the Planck core and its presence near/after the Big Bang. This difference is shown in Fig.11.7 center, the Big Bang light cones have only the upper part, while the Planck core light cones have both, T-symmetrically occupied parts.

One technical remark about $P_0 < 0$ parts of the mass shells. Although they formally correspond to negative energy and seem to be related with negative mass exotic matter, really they just correspond to T-conjugated flows of the same particles as $P_0 > 0$ counterparts. To verify this, consider T-reflection, that reverts P^μ and u^μ vectors, as well as orientation of the world lines, while preserves the action $A = m\int d\tau|x'_\mu x'^\mu|^{1/2}$ and the energy-momentum tensor $T^{\mu\nu} = \rho u^\mu u^\nu$, which are only physically important.

One more exotic possibility for T-symmetric emission is that the world lines of dark matter are captured by a geometry of wormhole, as shown on Fig.11.7 right. The ingoing flows from one universe become outgoing in the other universe and vice versa. In a stationary scenario, their T-symmetric superposition can be chosen.

Independently on the detailed properties of the considered models, the emission of T-symmetric type is necessary due to simple physical reasons. If only outgoing flows would be present, the total mass of a dark matter halo could not be greater than the mass of (quasi) black holes, from where it originates. Experimentally, the halo mass is much greater than the mass of black holes. In the considered setup, ingoing and outgoing flows compensate each other and allow for arbitrary ratio between halo and black hole masses.

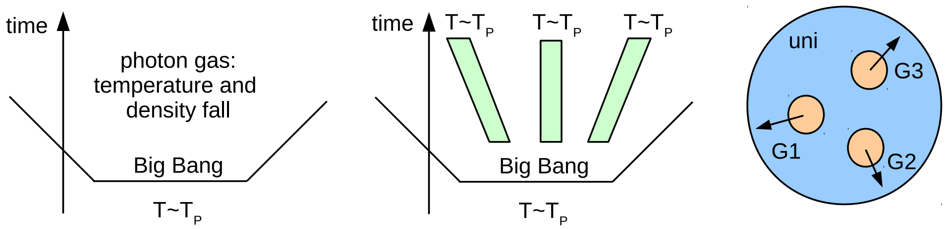

Fig. 11.8. Illustration to hypothesis 3. On the left: evolution of photon gas in standard cosmology. In the center: the same with RDM-stars. On the right: Swiss cheese model with galaxies filled by hot dark matter, surrounded by cold dark matter, in expanding universe.

Hypothesis 3: the cosmological dark matter mimics cold type.

The common opinion is that dark matter both in the galaxies and in between them is cold, i.e., is composed of massive non-relativistic particles. The hot cosmological dark matter would lead to a different expansion rate of the universe. Let us consider an evolution of uniform photon gas in standard cosmology, as shown schematically on Fig.11.8 left. There is an initial flash, then the energy and the number density of photons fall in the expanding universe. For cold dark matter only the density falls. This makes a difference to the evolution of the energy-momentum tensor.

On the other hand, the distribution of dark matter photons in the RDM model is different, see Fig.11.8 center. Their initial energy at Planck core is fixed: $E \sim E_P$, the exit energy is also fixed by the local $A_{QG}^{1/2}$ factor. If the resulting distribution will possess a constant temperature, then in the long-range evolution it will behave like cold dark matter.

The other possibility is that EOS of cosmological dark matter is not identical to the galactic one. There is a class of Swiss cheese models, where the galaxies and their halos do not change their size and structure under cosmological expansion and move as a whole. The cosmological expansion acts only on the level where the matter distribution can be considered as uniform. The galaxies coated in massive halos can behave like macro-particles of cold dark matter, as shown on Fig.11.8 right. Internally they can be filled with hot radiation, externally produce the same gravitational fields as cold massive particles.

As we have mentioned earlier, while fitting the Milky Way rotation curve, the experimental data are compatible with the presence of a cut of dark matter halo at $R_{cut} \sim 50$kpc. While this cut was taken in the model just phenomenologically, the physical mechanisms for it can be constructed. Two-phase distribution, with hot radial dark matter inside the galaxy joined to cold uniform dark matter outside, can be used. It resembles a known phenomenon of termination shock on the border of the Solar system, where the solar wind meets the interstellar medium. This can be modeled similarly in galactic scales. A suitable mechanism for the termination can be any kind of interaction of dark matter particles in the ingoing/outgoing flows and the outer medium. In particular, it can be an absorption or a scatter of longwave dark photons by the intergalactic medium.

11.5 Conclusion

In this paper we have experimented with the insertion of Planck core in several earlier known astrophysical models. What becomes possible as a result of such modification:

(1) RDM solutions can be properly continued to the strong field mode. These are stationary solutions describing black holes, coupled to the radial flows of dark matter. In weak fields, such configuration of dark matter can be used as a model of spiral galaxies possessing realistic rotation curves. In this model, the geometric dependence of the density on the distance $\rho \sim r^{-2}$, typical for the RDM configuration in a single center approximation, gives flat rotation curves, while assuming the coupling of all black holes in the galaxy to RDM, deviations of the rotation curves from the flat shape are also described. In strong fields, a peculiar phenomenon of erasing the event horizon occurs; instead, a spherical region of super-strong redshift is formed. This phenomenon is accompanied by the effect of mass inflation by Hamilton-Pollack, in a thin layer near the gravitational radius a very large positive mass is accumulated, approximately compensated by the negative mass of the Planck core. Outside, such an object, an RDM-star, is perceived as a Schwarzschild black hole of limited mass.

(2) when an external body, for example, an asteroid, falls on an RDM-star, a flash of high-energy photons occurs, then the super-strong redshift of the RDM-star moves the flash frequency to the radio band. This process can be considered as a mechanism for generating fast radio bursts. The calculations lead to the formula for the wavelength $\lambda_{out} = 2(2\pi)^{1/4}(\lambda_N r_s)^{1/2}/\epsilon^{1/4}$, where λ_N is the Compton wavelength of the nucleons that make up the asteroid, r_s is the gravitational radius of the RDM-star, $\epsilon = (v/c)^2$ is the parameter determining the orbital velocity of stars v in the galaxy. Evaluation with the parameters of the Milky Way galaxy gives the wavelength $\lambda_{out} = 0.5$ m and the frequency $\nu_{out} = 0.6$ GHz, in the range 0.111 ... 8 GHz for the observed FRB frequencies.

(3) the Tolman-Oppenheimer-Volkoff system of equations describing the equilibrium of isotropic matter, for the Planck core located in the center, also has an interesting structure of solutions. For the matter, we consider photon gas stitched with cosmic microwave background at infinity. As a result of the calculation, a stationary solution with the parameters of a micro black hole $r_s <$

0.11 mm is obtained. With a smooth change in the external boundary condition (slow accretion of external matter), autonomous oscillations arise in the system, accompanied by self-generation of photonic flashes. In this model, a formula for the outgoing wavelength is $\lambda_{out} = l_P(\rho_{cmb}/\rho_P)^{-1/4}$, numerically $\lambda_{out} = 0.9$ mm, $\nu_{out} = 333$ GHz, that fall close to the observed FRB range.

(4) white holes become stable. We have examined the dynamics of white holes in the Ori-Poisson model and showed that the insertion of negative mass core eliminates both Eardley and Zel'dovich-Novikov-Starobinskij types of instability.

Based on the considered models, in frames of this work, we have proposed three hypotheses about the composition of astrophysical dark matter.

(1) In galaxies, the dark matter can be cold or hot, massive, null or even tachyonic, producing the same rotation curves. Particularly, it can be composed of massless particles with initially Planck energy, finally redshifted to the extremely large wavelength $\lambda_{out} \sim 10^{14}$m. More particularly, it can be composed of low energy photons with such wavelength.

(2) The emission of dark matter particles happens in a T-symmetric way, in future and in past directions. This allows to explain the abundance of dark matter in comparison with the masses of its sources.

(3) At cosmological distances, the dark matter behaves like it is cold. Several mechanisms for this behavior have been proposed.

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APPENDIX: Stability of white holes

The model by Ori-Poisson [19], describing a white hole or, more precisely, its juncture with a black hole, can be considered as a simplification of the stationary models presented in this paper. Instead of continuous superposition of ingoing and outgoing shells of matter, there are precisely two shells, one ingoing, one outgoing. The model is not stationary, it has a white hole as the initial state and a black hole as the final state. The advantage of the model is the existence of an analytical solution. The system is usually considered unstable, but we will now show that it is not.

In addition to the previously described internal ZNS instability, there is external instability found in the work by Eardley [20]. In Ori-Poisson formulation, the instability can be illustrated by the Penrose diagram on Fig.11.9 left. The original white hole (1) explodes, completely releasing its mass into an outgoing null shell (2). There is an ingoing null shell (3) of originally small energy. This shell cannot enter the white hole, in principle (like a shell that cannot exit from the interior of a black hole). Instead, it slows down at the horizon and after a long wait (one can substitute here Hubble time, for instance) receives a super-strong blue shift. It forms a thin super-energetic "blue sheet". Upon its collision with the outgoing shell, an analytically computable rearrangement of energy occurs. As a result, a negligible part of the initial energy comes out (4). The ingoing blue sheet disappears under the horizon of a newly formed black hole (5).

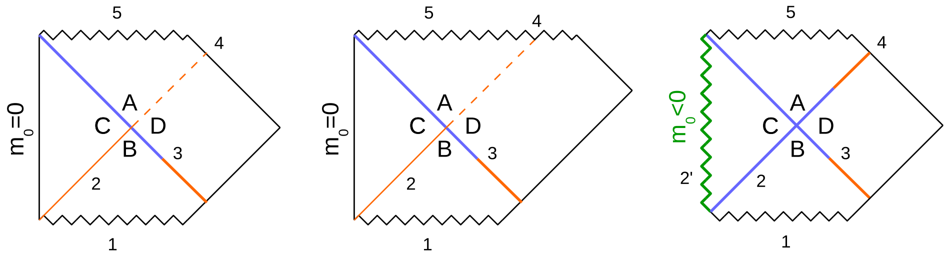

Fig. 11.9. Penrose diagrams for white holes in Ori-Poisson model. On the left: low efficient white hole eruption, in the center: no signal case, on the right: T-symmetric case.

In Barceló et al. [39] an even more asymmetric scenario is considered, depicted on Fig.11.9 center. Here not only the ingoing, but also the outgoing shell disappears in the black hole. Absolutely nothing comes out of this system towards an external observer.

One can immediately ask, how can it be that the black holes are stable, but the white holes are not? What about T-symmetry? As a resolution of this paradox, a T-symmetric scenario can be constructed, according to the principle "for each incoming blue sheet, there is the same outgoing". It is displayed on Fig.11.9 right. Warning: the negative mass is required in the scenario. The original white hole (1) explodes, emitting more energy than its own mass in the form of an outgoing blue sheet (2). It leaves behind the core of negative mass (2'). After the collision between equal blue sheets (2) and (3), a shell of the same energy as the ingoing one comes out (4). The negative mass of the core is compensated by the incoming blue sheet, a black hole of the same mass as the original white hole is formed (5).

The details of the computation are the following. The solution consists of 4 Schwarzschild's patches, marked ABCD on the figures. Their masses are

$$m_A = M - E,\ m_B = M - dm,\ m_C = m_0,\ m_D = M, \tag{11.28}$$

where M is the mass of the white hole with the ingoing shell, dm is the mass of the ingoing shell, E is the mass/energy of the outgoing shell ($G = c = 1$), m_0 is a remainder.

The computation is based on Dray-'t Hooft-Redmount (DTR) relation [40]:

$$f_A f_B = f_C f_D,\ f_i = 1 - 2m_i/R,\ i = A, B, C, D, \tag{11.29}$$

where R is the radius at shells intersection. Relative parameters are introduced:

$$\xi = R/(2M) - 1,\ \alpha = dm/M,\ \beta = m_0/M,\ \eta = E/M, \tag{11.30}$$

where ξ measures the relative distance to the horizon, η is the relative efficiency of white hole eruption. DTR relation results in the expression for the efficiency:

$$\eta = (1 - \alpha - \beta)\, \xi/(\alpha + \xi). \tag{11.31}$$

As a side remark, in [19] a different global structure linking Schwarzschild's patches with a cosmological model was used, however, this does not influence the obtained efficiency formula. Now, let us consider the exponential evolution of the shell

$$\xi = \xi_0 \exp(-\tau/(4M)). \tag{11.32}$$

Here $\tau = 2t$ is the total time for the ingoing shell to reach the point of collision and for the outgoing shell to reach the distant observer, therefore a double factor in the formula. To bring in some values, for $r_s = 1.2 \cdot 10^{10}$m it is an exponential process where the distance is halved every minute and for $\tau = 13.8 \cdot 10^9$years the distance factor takes an enormously small value $\xi \sim \exp(-10^{16})$, practically insensitive to the starting ξ_0.

Considering (11.31) for $\beta = 0$, as in original Ori-Poisson paper, in the limit $0 < \xi \ll \alpha \ll 1$, obtain $\eta \sim \xi/\alpha \sim \exp(-10^{16})$, a vanishingly small efficiency of eruption.

Calculation with $\beta < 0$ reveals a different class of solutions: $\beta \sim -\alpha/\xi$ results in $\eta \sim 1$, 100% efficiency, while for $\eta = \alpha$, $E = dm$, the T-symmetric case is reached selecting

$$\beta = -(\alpha^2 - \xi + 2\alpha\xi)/\xi \sim -\alpha^2/\xi. \tag{11.33}$$

The result of this computation demonstrates that in Ori-Poisson model Eardley instability can be eliminated if the system has a core of negative mass. Being combined with the earlier obtained result, both types of instability, Eardley and ZNS, can be removed from the white hole models by the introduction of a Planck core. A noticeable numerical difference between Ori-Poisson energetic parameters in comparison with those in the other considered models appears mainly due to the under-exponent Hubble time delays for the processes, that run continuously in the RDM and TOV models.

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12 Virtual Institute of Astroparticle Physics as the Online Platform for Studies of BSM Physics and Cosmology

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Abstract. Being a unique multi-functional complex of science and education online, Virtual Institute of Astroparticle Physics (VIA) operating on website **http://viavca.in2p3.fr/site.html**, has provided the platform for completely electronic format of XXIII Bled Workshop ”What comes beyond the Standard models?” in the pandemia conditions, excluding offline meetings. We review VIA experience in presentation online for the most interesting theoretical and experimental results, participation online in conferences and meetings, various forms of collaborative scientific work as well as programs of education at distance, combining online videoconferences with extensive library of records of previous meetings and Discussions on Forum. Since 2014 VIA online lectures combined with individual work on Forum acquired the form of Open Online Courses. Aimed to individual work with students the Course is not Massive, but the account for the number of visits to VIA site converts VIA in a specific tool for MOOC activity. VIA sessions, being a traditional part of Bled Workshops' program, became at XXIII Bled Workshop the only format of the meeting, challenging to preserve the creative nonformal atmosphere of meetings in Bled, Slovenia. We openly discuss the advantages and flaws of VIA platform for online meetings.

Povzetek. Virtual Institute of Astroparticle Physics (VIA, **http://viavca.in2p3.fr/site.html**) je spletišče namenjeno znanosti in izobraževanju. Letos je kot spletna platforma omogočilo XXIII. blejske delavnice, ker nam je pandemija preprečila delavnico v običajni obliki. Organizator in vodja spletišča predstavi izkušnje VIA pri spletni predstavitvi najbolj zanimivih in aktualnih teoretičnih spoznanj, diskusij na daljavo na konferencah, delavnicah, videokonferencah in drugih srečanjih ter izobraževanju preko spleta. Ponuja obsežen arhiv z (video) zapisi preteklih dogodkov in diskusij na forumih spletišča. Od leta 2014 potekajo predavanja preko interneta, ki jim sledijo pogovori, in ki so prerasla v izobraževanje na daljavo. VIA ni namenjena velikemu številu udeležencev, ker pa ima veliko obiskov, je postala orodje tudi za aktivnosti MOOC. Predavanja preko sistema VIA so že tradicionalni del blejskih delavnic, letos pa je delavnica tekla samo preko interneta. Ker so Blejske delavnice namenjene zelo poglobljenim diskusijam, je bilo letošnje vodenje delavnice za organizatorja spletišca poseben izziv. Avtor odkrito obravnava prednosti in slabosti platforme VIA za srečanja preko interneta.

Keywords: astroparticle physics, physics beyond the Standard model, e-learning, e-science, MOOC

12.1 Introduction

Studies in astroparticle physics link astrophysics, cosmology, particle and nuclear physics and involve hundreds of scientific groups linked by regional networks (like ASPERA/ApPEC [1,2]) and national centers. The exciting progress in these studies will have impact on the knowledge on the structure of microworld and Universe in their fundamental relationship and on the basic, still unknown, physical laws of Nature (see e.g. [3,4] for review). The progress of precision cosmology and experimental probes of the new physics at the LHC and in nonaccelerator experiments, as well as the extension of various indirect studies of physics beyond the Standard model involve with necessity their nontrivial links. Virtual Institute of Astroparticle Physics (VIA) [5] was organized with the aim to play the role of an unifying and coordinating platform for such studies.

Starting from the January of 2008 the activity of the Institute takes place on its website [6] in a form of regular weekly videoconferences with VIA lectures, covering all the theoretical and experimental activities in astroparticle physics and related topics. The library of records of these lectures, talks and their presentations was accomplished by multi-lingual Forum. Since 2008 there were **215 VIA online lectures**, VIA has supported distant presentations of **152 speakers at 28 Conferences** and provided transmission of talks at **78 APC Colloquiums**.

In 2008 VIA complex was effectively used for the first time for participation at distance in XI Bled Workshop [7]. Since then VIA videoconferences became a natural part of Bled Workshops' programs, opening the virtual room of discussions to the world-wide audience. Its progress was presented in [8–18].

Here the current state-of-art of VIA complex, integrated since 2009 in the structure of APC Laboratory, is presented in order to clarify the way in which discussion of open questions beyond the standard models of both particle physics and cosmology were presented at the virtual XXIII Bled Workshop on the platform of VIA facility. In the conditions of pandemia, when all the offline meetings were forbidden, VIA videoconferencing became the only possibility to continue in 2020 traditions of open discussions at Bled meetings.

12.2 VIA structure and activity

12.2.1 VIA activity

The structure of the VIA complex is illustrated by the Fig. 12.1. The home page, presented on this figure, contains the information on the coming and records of the latest VIA events. The upper line of menu includes links to directories (from left to right): with general information on VIA (About VIA); entrance to VIA virtual rooms (Rooms); the library of records and presentations (Previous), which contains records of VIA Lectures (Previous $\rightarrow$ Lectures), records of online transmissions of Conferences (Previous $\rightarrow$ Conferences), APC Colloquiums (Previous $\rightarrow$ APC

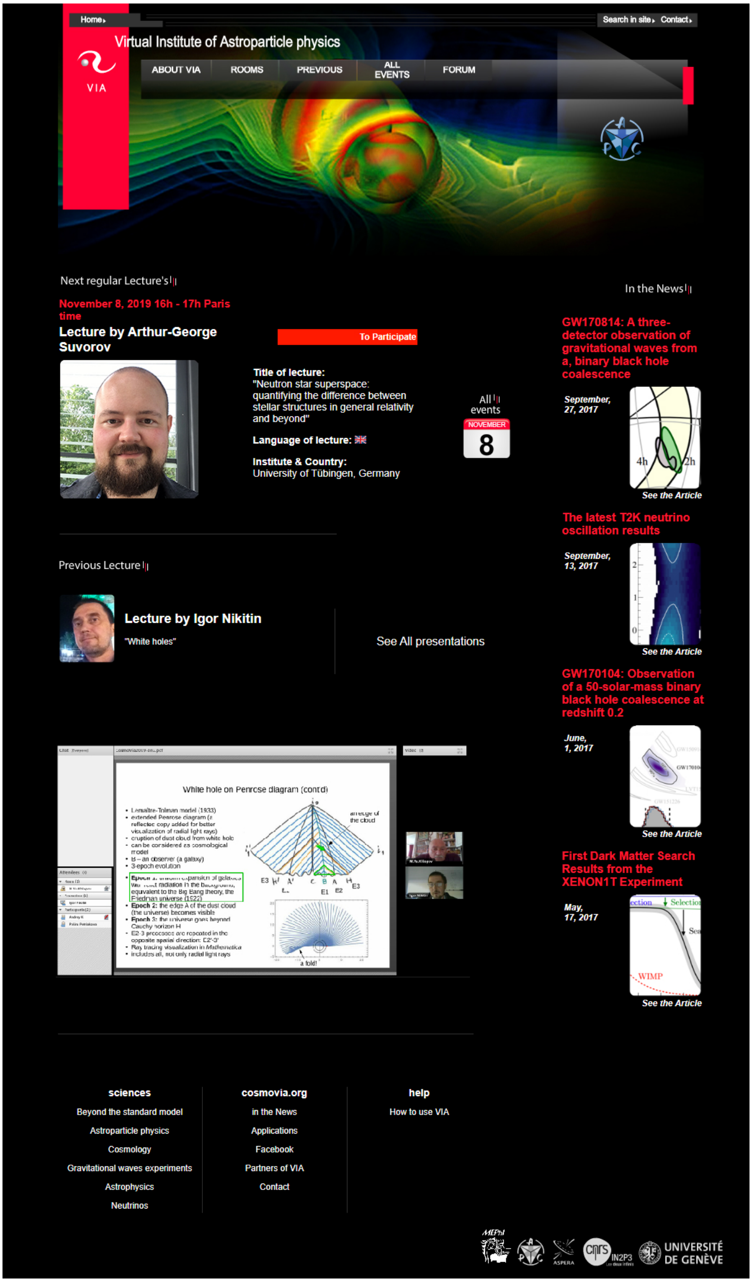

Fig. 12.1. The home page of VIA site

Colloquiums), APC Seminars (Previous → APC Seminars) and Events (Previous → Events); Calendar of the past and future VIA events (All events) and VIA Forum (Forum). In the upper right angle there are links to Google search engine (Search in site) and to contact information (Contacts). The announcement of the next VIA lecture and VIA online transmission of APC Colloquium occupy the main part of the homepage with the record of the most recent VIA events below. In the announced time of the event (VIA lecture or transmitted APC Colloquium) it is sufficient to click on "to participate" on the announcement and to Enter as Guest (printing your name) in the corresponding Virtual room. The Calendar shows the program of future VIA lectures and events. The right column on the VIA homepage lists the announcements of the regularly up-dated hot news of Astroparticle physics and related areas.

In 2010 special COSMOVIA tours were undertaken in Switzerland (Geneva), Belgium (Brussels, Liege) and Italy (Turin, Pisa, Bari, Lecce) in order to test stability of VIA online transmissions from different parts of Europe. Positive results of these tests have proved the stability of VIA system and stimulated this practice at XIII Bled Workshop. The records of the videoconferences at the XIII Bled Workshop are available on VIA site [19].

Since 2011 VIA facility was used for the tasks of the Paris Center of Cosmological Physics (PCCP), chaired by G. Smoot, for the public program "The two infinities" conveyed by J.L.Robert and for effective support a participation at distance at meetings of the Double Chooz collaboration. In the latter case, the experimentalists, being at shift, took part in the collaboration meeting in such a virtual way.

The simplicity of VIA facility for ordinary users was demonstrated at XIV Bled Workshop in 2011. Videoconferences at this Workshop had no special technical support except for WiFi Internet connection and ordinary laptops with their internal webcams and microphones. This test has proved the ability to use VIA facility at any place with at least decent Internet connection. Of course the quality of records is not as good in this case as with the use of special equipment, but still it is sufficient to support fruitful scientific discussion as can be illustrated by the record of VIA presentation "New physics and its experimental probes" given by John Ellis from his office in CERN (see the records in [20]).

In 2012 VIA facility, regularly used for programs of VIA lectures and transmission of APC Colloquiums, has extended its applications to support M.Khlopov's talk at distance at Astrophysics seminar in Moscow, videoconference in PCCP, participation at distance in APC-Hamburg-Oxford network meeting as well as to provide online transmissions from the lectures at Science Festival 2012 in University Paris7. VIA communication has effectively resolved the problem of referee's attendance at the defence of PhD thesis by Mariana Vargas in APC. The referees made their reports and participated in discussion in the regime of VIA videoconference. In 2012 VIA facility was first used for online transmissions from the Science Festival in the University Paris 7. This tradition was continued in 2013, when the transmissions of meetings at Journées nationales du Développement Logiciel (JDEV2013) at Ecole Politechnique (Paris) were organized [22].

In 2013 VIA lecture by Prof. Martin Pohl was one of the first places at which the first hand information on the first results of AMS02 experiment was presented [21].

In 2014 the 100th anniversary of one of the foundators of Cosmoparticle physics, Ya. B. Zeldovich, was celebrated. With the use of VIA M.Khlopov could contribute the programme of the ”Subatomic particles, Nucleons, Atoms, Universe: Processes and Structure International conference in honor of Ya. B. Zeldovich 100th Anniversary” (Minsk, Belarus) by his talk ”Cosmoparticle physics: the Universe as a laboratory of elementary particles” [23] and the programme of ”Conference YaB-100, dedicated to 100 Anniversary of Yakov Borisovich Zeldovich” (Moscow, Russia) by his talk ”Cosmology and particle physics” [24].

In 2015 VIA facility supported the talk at distance at All Moscow Astrophysical seminar ”Cosmoparticle physics of dark matter and structures in the Universe” by Maxim Yu. Khlopov and the work of the Section ”Dark matter” of the International Conference on Particle Physics and Astrophysics (Moscow, 5-10 October 2015). Though the conference room was situated in Milan Hotel in Moscow all the presentations at this Section were given at distance (by Rita Bernabei from Rome, Italy; by Juan Jose Gomez-Cadenas, Paterna, University of Valencia, Spain and by Dmitri Semikoz, Martin Bucher and Maxim Khlopov from Paris) and its work was chaired by M.Khlopov from Paris [29]. In the end of 2015 M. Khlopov gave his distant talk ”Dark atoms of dark matter” at the Conference ”Progress of Russian Astronomy in 2015”, held in Sternberg Astronomical Institute of Moscow State University.

In 2016 distant online talks at St. Petersburg Workshop ”Dark Ages and White Nights (Spectroscopy of the CMB)” by Khatri Rishi (TIFR, India) ”The information hidden in the CMB spectral distortions in Planck data and beyond”, E. Kholupenko (Ioffe Institute, Russia) ”On recombination dynamics of hydrogen and helium”, Jens Chluba (Jodrell Bank Centre for Astrophysics, UK) ”Primordial recombination lines of hydrogen and helium”, M. Yu. Khlopov (APC and MEPHI, France and Russia)”Nonstandard cosmological scenarios” and P. de Bernardis (La Sapiensa University, Italy) ”Balloon techniques for CMB spectrum research” were given with the use of VIA system [30]. At the defense of PhD thesis by F. Gregis VIA facility made possible for his referee in California not only to attend at distance at the presentation of the thesis but also to take part in its successive jury evaluation.

Since 2018 VIA facility is used for collaborative work on studies of various forms of dark matter in the framework of the project of Russian Science Foundation based on Southern Federal University (Rostov on Don). In September 2018 VIA supported online transmission of **17 presentations** at the Commemoration day for Patrick Fleury, held in APC [31].

The discussion of questions that were put forward in the interactive VIA events is continued and extended on VIA Forum. Presently activated in English,French and Russian with trivial extension to other languages, the Forum represents a first step on the way to multi-lingual character of VIA complex and its activity. Discussions in English on Forum are arranged along the following directions: beyond the standard model, astroparticle physics, cosmology, gravitational wave experiments, astrophysics, neutrinos. After each VIA lecture its pdf presentation together with link to its record and information on the discussion during it are put in the corresponding post, which offers a platform to continue discussion in replies to this post.

12.2.2 VIA e-learning, OOC and MOOC

One of the interesting forms of VIA activity is the educational work at distance. For the last eleven years M.Khlopov's course "Introduction to cosmoparticle physics" is given in the form of VIA videoconferences and the records of these lectures and their ppt presentations are put in the corresponding directory of the Forum [25]. Having attended the VIA course of lectures in order to be admitted to exam students should put on Forum a post with their small thesis. In this thesis students are proposed to chose some BSM model and to study the cosmological scenario based on this chosen model. The list of possible topics for such thesis is proposed to students, but they are also invited to chose themselves any topic of their own on possible links between cosmology and particle physics. Professor's comments and proposed corrections are put in a Post reply so that students should continuously present on Forum improved versions of work until it is accepted as admission for student to pass exam. The record of videoconference with the oral exam is also put in the corresponding directory of Forum. Such procedure provides completely transparent way of evaluation of students' knowledge at distance.

In 2018 the test has started for possible application of VIA facility to remote supervision of student's scientific practice. The formulation of task and discussion of progress on work are recorded and put in the corresponding directory on Forum together with the versions of student's report on the work progress.

Since 2014 the second semester of the course on Cosmoparticle physics is given in English and converted in an Open Online Course. It was aimed to develop VIA system as a possible accomplishment for Massive Online Open Courses (MOOC) activity [26]. In 2016 not only students from Moscow, but also from France and Sri Lanka attended this course. In 2017 students from Moscow were accompanied by participants from France, Italy, Sri Lanka and India [27]. The students pretending to evaluation of their knowledge must write their small thesis, present it and, being admitted to exam, pass it in English. The restricted number of online connections to videoconferences with VIA lectures is compensated by the wide-world access to their records on VIA Forum and in the context of MOOC VIA Forum and videoconferencing system can be used for individual online work with advanced participants. Indeed Google Analytics shows that since 2008 VIA site was visited by more than **242 thousand** visitors from **154** countries, covering all the continents by its geography (Fig. 12.2). According to this statistics more than half of these visitors continued to enter VIA site after the first visit. Still the form of individual educational work makes VIA facility most appropriate for PhD courses and it is planned to be involved in the International PhD program on Fundamental Physics, which can be started on the basis of Russian-French collaborative agreement. In 2017 the test for the ability of VIA to support fully distant education and evaluation of students (as well as for work on PhD thesis and its distant defense) was undertaken. Steve Branchu from France, who attended the Open Online Course and presented on Forum his small thesis has passed exam

Fig. 12.2. Geography of VIA site visits according to Google Analytics

at distance. The whole procedure, starting from a stochastic choice of number of examination ticket, answers to ticket questions, discussion by professors in the absence of student and announcement of result of exam to him was recorded and put on VIA Forum [28].

In 2019 in addition to individual supervisory work with students the regular scientific and creative VIA seminar is in operation aimed to discuss the progress and strategy of students scientific work in the field of cosmoparticle physics.

In 2020 the regular course now for M2 students continued, but the problems of adobe Connect, related with the lack of its support for Flash in coming 2021 made necessary to find a solution on the platform of Zoom. This platform is rather easy to use and provides records, while the lack of whiteboard tools for discussions online can be solved by accomplishments of laptops by graphic tabloids.

12.2.3 Organisation of VIA events and meetings

First tests of VIA system, described in [5,7–9], involved various systems of videoconferencing. They included skype, VRVS, EVO, WEBEX, marratech and adobe Connect. In the result of these tests the adobe Connect system was chosen and properly acquired. Its advantages are: relatively easy use for participants, a possibility to make presentation in a video contact between presenter and audience, a possibility to make high quality records, to use a whiteboard tools for discussions, the option to open desktop and to work online with texts in any format. This choice however should be reconsidered in future or at least accomplished by Zoom in view of the lack of support for Flash on which VIA site is based.

Initially the amount of connections to the virtual room at VIA lectures and discussions usually didn't exceed 20. However, the sensational character of the exciting news on superluminal propagation of neutrinos acquired the number of participants, exceeding this allowed upper limit at the talk "OPERA versus Maxwell and Einstein" given by John Ellis from CERN. The complete record of this talk and is available on VIA website [32]. For the first time the problem of necessity in extension of this limit was put forward and it was resolved by creation of a virtual "infinity room", which can host any reasonable amount of participants.

Starting from 2013 this room became the only main virtual VIA room, but for specific events, like Collaboration meetings or transmissions from science festivals, special virtual rooms can be created. This solution strongly reduces the price of the licence for the use of the adobeConnect videoconferencing, retaining a possibility for creation of new rooms with the only limit to one administrating Host for all of them.

The ppt or pdf file of presentation is uploaded in the system in advance and then demonstrated in the central window. Video images of presenter and participants appear in the right window, while in the lower left window the list of all the attendees is given. To protect the quality of sound and record, the participants are required to switch out their microphones during presentation and to use the upper left Chat window for immediate comments and urgent questions. The Chat window can be also used by participants, having no microphone, for questions and comments during Discussion. The interactive form of VIA lectures provides oral discussion, comments and questions during the lecture. Participant should use in this case a "raise hand" option, so that presenter gets signal to switch out his microphone and let the participant to speak. In the end of presentation the central window can be used for a whiteboard utility as well as the whole structure of windows can be changed, e.g. by making full screen the window with the images of participants of discussion.

Regular activity of VIA as a part of APC includes online transmissions of all the APC Colloquiums and of some topical APC Seminars, which may be of interest for a wide audience. Online transmissions are arranged in the manner, most convenient for presenters, prepared to give their talk in the conference room in a normal way, projecting slides from their laptop on the screen. Having uploaded in advance these slides in the VIA system, VIA operator, sitting in the conference room, changes them following presenter, directing simultaneously webcam on the presenter and the audience. If the advanced uploading is not possible, VIA streaming is used - external webcam and microphone are directed to presenter and screen and support online streaming.

12.2.4 VIA activity in the conditions of pandemia

The lack of usual offline connections and meetings in the conditions of pandemia made the use of VIA facility especially timely and important. This facility supports regular weekly meetings of the Laboratory of cosmoparticle studies of the structure and dynamics of Galaxy in Institute of Physics of Southern Federal University (Rostov on Don, Russia) and M.Khlopov's scientific - creative seminar and their announcements occupied their permanent position on VIA homepage (Fig. 12.3), while their records were put in respective place of VIA forum, like [33] for Laboratory meetings.

The platform of VIA facility was used for regular Khlopov's course "Introduction to Cosmoparticle physics" for M2 students of MEPHI (in Russian) and supported regular seminars of Theory group of APC, keeping their records in VIA library [34].

The programme of VIA lectures continued to present hot news of astroparticle physics and cosmology, like talk by Zhen Cao from China on the progress of

Fig. 12.3. Permanent announcements of regular VIA meetings of SFEDU laboratory and Khlopov's seminar

LHAASO experiment [35] or lecture by Sunny Vagnozzi from UK on the problem of consistency of different measurements of the Hubble constant [36].

The results of this activity inspired the decision to hold XXIII Bled Workshop online on the platform of VIA.

12.3 VIA platform for virtual XXIII Bled Workshop

VIA sessions at Bled Workshops continued the tradition coming back to the first experience at XI Bled Workshop [7] and developed at XII, XIII, XIV, XV, XVI, XVII, XVIII, XIX, XX, XXI and XXII Bled Workshops [8–18]. They became a regular but supplementary part of the Bled Workshop's program. It had to be the only form of Workshop activity in 2020.

In the course of XXIII Bled Workshop, the list of open questions was stipulated, which was proposed for wide discussion with the use of VIA facility. The list of these questions was put on VIA Forum (see [37]) and all the participants of VIA sessions were invited to address them during VIA discussions. During the XXIII Bled Workshop the announcement of VIA sessions was put on VIA home page, giving an open access to the videoconferences at the Workshop sessions. The preliminary program as well as the corrected program for each day were continuously put on Forum [37] with the slides and records of all the talks and discussions [37].

Starting from the Opening of the Workshop VIA facility tried to preserve the creative atmosphere of Bled discussions

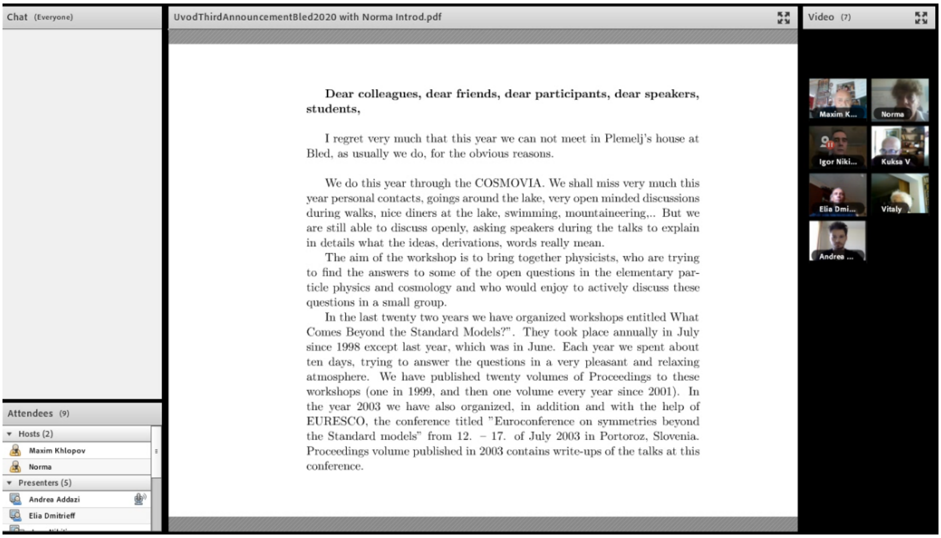

Fig. 12.4. Opening of XXIII Bled Workshop by Norma Mankoc- Borstnik

All the talks in the program of XXIII Bled Workshop were given in the format videoconferences as the talks ”How far has so far the Spin-Charge-Family theory succeeded to offer the explanation for the observed phenomena?” by Norma Mankoc-Borstnik from Ljubljana, Slovenia (Fig. 12.5) or ”Gravitational footprints of massive neutrinos and lepton number breaking” by A. Marciano, (Fig. 12.6), from Rome (see records in [37]).

During the Workshop the VIA virtual room was open, inviting distant participants to join the discussion and extending the creative atmosphere of these discussions to the world-wide audience. The online format of Workshop provided remote presentation of students’ scientific debuts in BSM physics and cosmology as it was, in particular, presented in the interesting talk ”Formation of conserved charges at the de Sitter space” by Valery Nikulin (Fig. 12.7) or in the talk ”Numerical simulation of dark atom interaction with nuclei” by Timur Bikbaev (Fig. 12.8).

The records of all these lectures and discussions can be found on VIA Forum [37].

Though the technical problems didn’t make possible nonformal private discussions of participants, still VIA facility has managed to join scientists from Mexico, USA, France, Russia, Slovenia, Denmark, India, China and many other countries in discussion of open problems of physics and cosmology beyond the Standard models.

12.4 Conclusions

The Scientific-Educational complex of Virtual Institute of Astroparticle physics provides regular communication between different groups and scientists, working

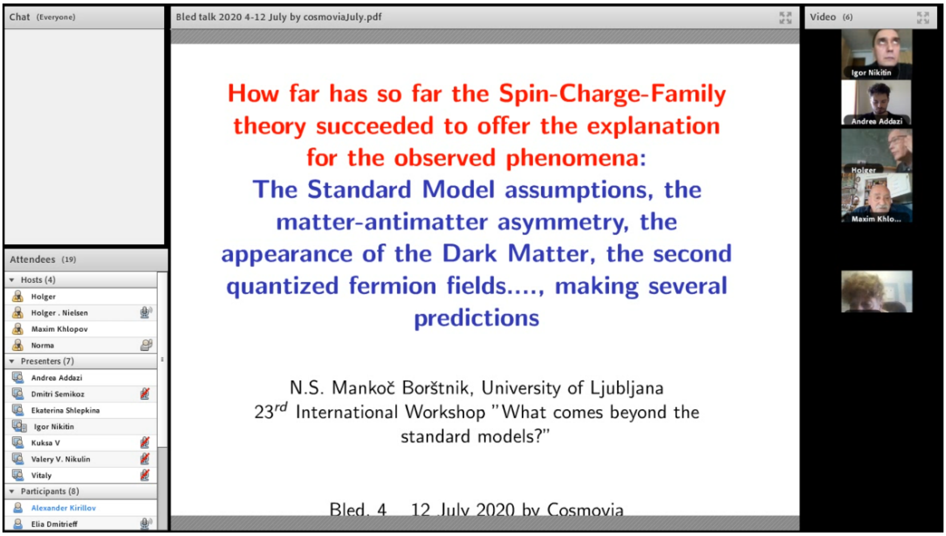

Fig. 12.5. The talk "How far has so far the Spin-Charge-Family theory succeeded to offer the explanation for the observed phenomena?" by Norma Mankoc-Borstnik at XXIII Bled Workshop

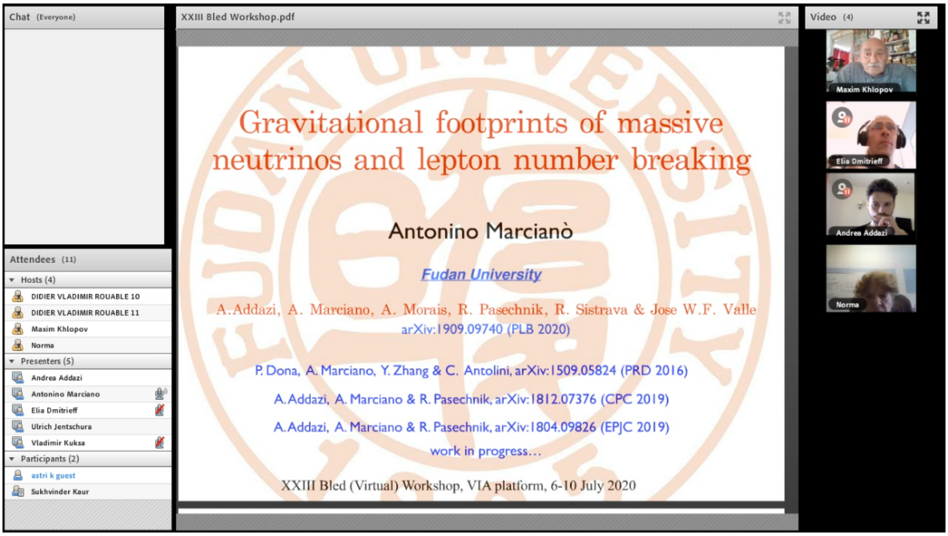

Fig. 12.6. VIA talk "Gravitational footprints of massive neutrinos and lepton number breaking" by A. Marciano from Rome at XXIII Bled Workshop

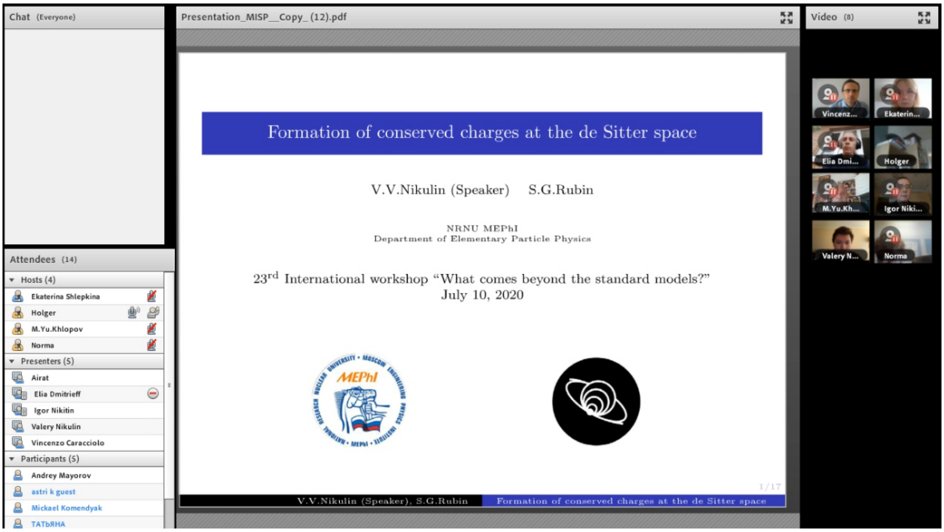

Fig. 12.7. VIA talk "Formation of conserved charges at the de Sitter space" by Valery Nikulin at XXIII Bled Workshop

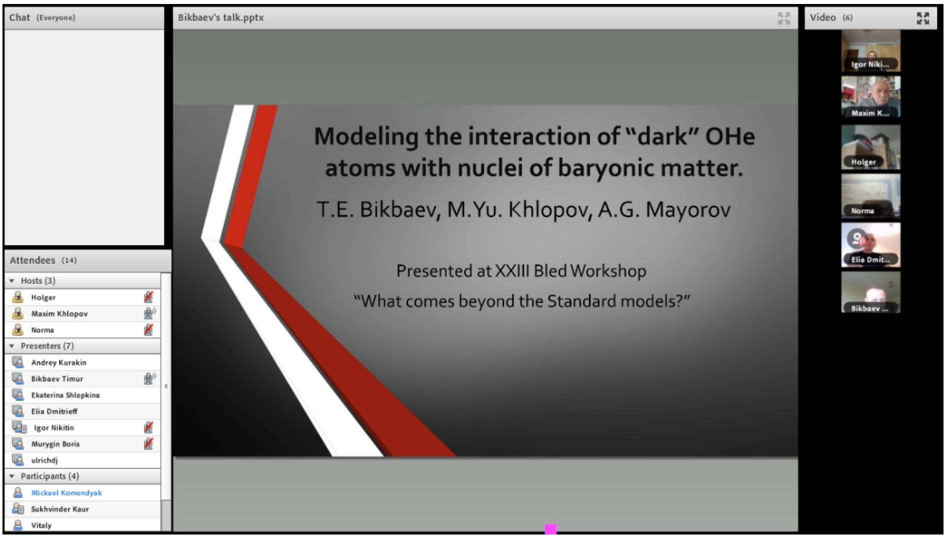

Fig. 12.8. VIA talk "Formation of conserved charges at the de Sitter space" by Timur Bikbaev at XXIII Bled Workshop

in different scientific fields and parts of the world, the first-hand information on the newest scientific results, as well as support for various educational programs at distance. This activity would easily allow finding mutual interest and organizing task forces for different scientific topics of cosmology, particle physics, astroparticle physics and related topics. It can help in the elaboration of strategy of experimental particle, nuclear, astrophysical and cosmological studies as well as in proper analysis of experimental data. It can provide young talented people from all over the world to get the highest level education, come in direct interactive contact with the world known scientists and to find their place in the fundamental research. These educational aspects of VIA activity can evolve in a specific tool for International PhD program for Fundamental physics. Involvement of young scientists in creative discussions was an important aspect of VIA activity at XXIII Bled Workshop. VIA applications can go far beyond the particular tasks of astroparticle physics and give rise to an interactive system of mass media communications.

VIA sessions, which became a natural part of a program of Bled Workshops, maintained in 2020 the platform for online discussions of physics beyond the Standard Model for distant participants from all the world in the lack of possibility of offline meetings. This discussion can continue in posts and post replies on VIA Forum. The experience of VIA applications at Bled Workshops plays important role in the development of VIA facility as an effective tool of e-science and e-learning.

One can summarize the advantages and flaws of online format of Bled Workshop. It makes possible to involve in the discussions scientists from all the world (young scientists, especially) free of the expenses related with meetings in real (voyage, accommodation, ...), but loses the advantage of nonformal discussions at walks along the beautiful surrounding of the Bled lake and other places of interest. The improvement of VIA technical support (e.g. by involvement of Zoom) can provide better platform for nonformal online discussions, but in no case can be the substitute for Bled meetings and its creative atmosphere in real. One can summarize that VIA sessions should remain a useful but still supplementary tool of Bled Workshop meetings in real, provided that such real meetings are possible.

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Poem by Astri Kleppe

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Proceedings to the 23rd [Virtual] Workshop, Volume 1
What Comes Beyond . . . *(p. 267)*
Bled, Slovenia, July 4–12, 2020

13 June

Astri Kleppe

Tonight it all springs into blossom, Lilacs,
Bird Cherry
with strands of willow, weaving
the beginning; How
it all
was meant to be.

Birdsong, sound of running
steps, and soon
the larvae will be heading
for the Bird Cherry, enfolding it
in silver
and cocoons; and soon
the nights of August will bring
darkness
for Orion and the Moon;
But in this fair night, early June
it's all in ecstasy, in vigil
for the blossom;
No one sleeps, the birds, the flies
are all awake.

Who are you then, I asked
the Lilacs
We are strangers here, the answer came,
and we belong to no one.
But tell me who you are, I begged
And flowers sprinkled over me,
a waterfall
of petals,
We just arrived, tonight,
what more is there to say?

II

A loose dream from a corner of the Universe
is driving towards us, our island;
Clouds
awakening of granite and cadavers, blue
and earth;
A boomerang towards The Milky Way, its icy stars
and howling wolves, and tightly curved
around the little heat
from our own speed. We are but animals
of auguries, of hope and salt;
and though the dreams
of Leibniz,
Alan Guth and Hubble
led us, it was other tokens
that the Universe imagined,
of another
kind;
And suddenly this otherness
sticks out: a tree
With roots in galaxies and whispers,
in galactic summer, leaves
that dance in morning breeze
and drizzle, with a scent
of seed and clover, pregnant
visions,
over paths through rain-gray
grass.

Beware of those
who tread on dew and stop
under that tree. So slowly
night is turning, in this space
of the improbable, a darkness
where a tree can grow
from nothing, rise
with flower buds and day.

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Volume 1:Invited Talks

Proceedings to the 23rd workshop 'What Comes Beyond the Standard Models', Bled, July 4.–12., 2020 [Virtual Workshop, July 6.–10., 2020]
Volume 1: Invited Talks

Uredili Norma Susana Mankoč Borštnik, Holger Bech Nielsen in Dragan Lukman

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