Exact Symmetries of the Gravitationally Coupled Dirac Equation and Equivalence Principle for Antimatter: A Lagrangian Approach

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Abstract

In 1928, Dirac developed the equation which bears his name, and which predicted the existence of positrons, i.e., the existence of particles of the same (inertial) mass, but opposite charge, as compared to electrons. The "positive electron" was indeed detected by Anderson in 1933. Apparently, the Dirac equation is able to predict the charge of antiparticles, and their inertial mass. But what does the Dirac equation say about the gravitational mass of antiparticles? In order to answer this question, it is necessary to couple the Dirac equation to gravity, i.e., to describe the quantum dynamics of a relativistic spin-1/2 particle in a curved space-time. This is not completely trivial, because it has to be done covariantly, using the spin-1/2 representation of the gauge group of local Lorentz transformations. Of course, the particle-antiparticle transformation changes in the curved space-time, as compared to flat space. However, once the formalism has been implemented, it is possible to compare the quantum dynamics of particles and antiparticles, in a general combined gravito-electromagnetic curved space-time background. The somewhat disappointing conclusion is that the electromagnetic interaction terms precisely change sign, while the gravitational interactions remain exactly the same for both particles and antiparticles. This means that any speculation about possible interesting deviations of the couplings of antiparticles to gravity can be laid to rest, at least on the level of Dirac theory. We conclude that antimatter gravity experiments test for the existence of exotic, CPT-violating, long-range interactions.

History: Theoretical Predictions before Discovery

First Incidence:
Maxwell equations were discovered before electromagnetic waves were detected in a laboratory:

J. C. Maxwell, "A dynamical theory of the electromagnetic field", Phil. Trans. Roy. Soc. (London) **155**, 459–512 (1865).
$\Rightarrow$ (Theory before Experiment)
H. R. Hertz, "Ueber die Ausbreitungsgeschwindigkeit der electrodynamischen Wirkungen", Ann. Phys. (Lpzg.) **270**, 551–569 (1888).

Second Incidence:
Dirac equation predicts the existence of particles with the same inertial mass as the electron, but opposite charge.

P. A. M. Dirac, "The quantum theory of the electron", Proc. Roy. Soc. (London) A **117**, 610–624 (1928).
$\Rightarrow$ (Theory before Experiment)
C. D. Anderson, "The Positive Electron", Phys. Rev. **43**, 491–409 (1933).

Inspiration

(J. C. Maxwell)

(H. R. Hertz)

(P. A. M. Dirac)

(C. D. Anderson)

We recall the free Dirac equation:

$$\left(\mathrm{i}\gamma^\mu \frac{\partial}{\partial x^\mu} - m_I \right) \psi = 0 \,, \qquad \mu = 0, 1, 2, 3 \qquad \text{(summation convention)}.$$

where:

- γ^μ are 4×4 Dirac matrices.
  $\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$. Yes, they really are 4×4!
- The anticommutators are $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\,\mathrm{diag}(1, -1, -1, -1)$.
- $x^\mu = (t, \vec{r})$ is the space-time-coordinate four-vector.
- $\psi = \psi(t, \vec{r})$ is a four-component wave function.
- m_I is the inertial mass.
- $\hbar = c = \epsilon_0 = 1$.

For the plane-wave *ansatz* $\psi = u \exp(-\mathrm{i}p^\mu x_\mu)$, the Dirac equation has two solutions with positive energy $p^0 = E \geq m_I$ (spin-up and spin-down electron), and two solutions with negative energy $p^0 = E \leq -m_I$ (spin-up and spin-down positron). The negative-energy solutions describe a different kind of particle.

The Dirac equation is a relativistic wave equation. The Dirac wave function ψ is not equal to the second-quantized Dirac field operator $\hat{\psi}$. The prediction of the existence of the positron follows *without* knowing about field quantization, which was invented by H. A. Bethe in 1947. (We recall that the positron was discovered in 1933.)

Dirac Equation with Electromagnetic Coupling

We recall the Dirac equation coupled to an electromagnetic field:

$$\left[\mathrm{i}\gamma^\mu \underbrace{\left(\frac{\partial}{\partial x^\mu} - e\, A_\mu \right)}_{\text{covariant derivative}} - m_I \right] \psi = 0\,.$$

In an atomic system, one has $A^0 = \Phi$ (Coulomb field), and for the spatial components $A^i = 0$ ($\vec{A} = \vec{0}$). So, multiplication from the left with $\gamma^0 = \beta$ under use of the relationship $(\gamma^0)^2 = \mathbf{1}_{4\times 4}$ gives

$$\left[\mathrm{i} \left(\frac{\partial}{\partial t} - e\, A^0 \right) + \mathrm{i}\, \gamma^0\, \gamma^i\, \frac{\partial}{\partial x^i} - m_I \right] \psi = 0\,,$$

or,

$$E\,\psi = \mathrm{i}\frac{\partial}{\partial t}\psi = \underbrace{\left(\vec{\alpha}\cdot\vec{p} + \beta m_I + e\,\Phi\right)}_{\text{Dirac Hamiltonian}} \psi = H\,\psi\,.$$

Here, $\vec{\alpha} = \gamma^0\,\vec{\gamma}$, and $V = e\,\Phi$ is the Coulomb potential (or a different potential, in other contexts). The Dirac Hamiltonian is an differential operator which also constitutes a 4×4 matrix.
The potential $e\,\Phi$ appears in the Dirac Hamiltonian because of the electromagnetic covariant derivative.

Dirac Equation with Gravitational Coupling: A Futile Attempt

Dirac Hamiltonian with electromagnetic coupling:

$$H = \vec{\alpha} \cdot \vec{p} + \beta m_I + e\,\Phi\,.$$

The potential $e\,\Phi$ appears in the Dirac Hamiltonian because of the electromagnetic covariant derivative. Here, $p_\mu = \mathrm{i} D_\mu = \mathrm{i}\partial_\mu - e A_\mu$.

Guess: Just write

$$\boxed{\text{[WRONG!]}} \qquad H \stackrel{?}{=} \vec{\alpha} \cdot \vec{p} + \beta m_I + m_G\,\Phi_G\,,$$

where $m_I = m_G$ is the inertial (equal to the gravitational) mass and Φ_G is the gravitational potential.

Q: Why is this not possible?
A: Because the gravitational coupling has to be formulated as a covariant derivative, and one cannot simply insert the gravitational potential into the Dirac Hamiltonian.

Next question: What is the covariant derivative ∇_μ for gravitational coupling?

Dirac Equation with Gravitational Coupling

Gravitational coupling to curved space-time background: Treat the curved background (metric) as a classical background. Treat the particle as a relativistic particle (either in first or second quantization). No need for space-time quantization! Two replacements:

$$\gamma^\mu \to \overline{\gamma}^\mu \,, \qquad \partial_\mu \to \nabla_\mu = \partial_\mu - \Gamma_\mu \,.$$

Spin-connection matrices Γ_μ. The Dirac equation assumes the form

$$(\mathrm{i}\overline{\gamma}^\mu \, \nabla_\mu - m_I)\, \psi = 0 \,.$$

We recall that the free-space Dirac matrices fulfill

$$\{\gamma^A, \gamma^B\} = 2g^{AB} = 2\,\mathrm{diag}(1, -1, -1, -1) \,.$$

By contrast, the curved-space Dirac matrices fulfill

$$\{\overline{\gamma}^\mu, \overline{\gamma}^\nu\} = 2\,\overline{g}^{\mu\nu}(x) \,, \qquad \overline{\gamma}^\mu = e^\mu_A \, \gamma^A \,.$$

Here, $\overline{g}^{\mu\nu}(x)$ is the metric of curved space-time, and the e^μ_A are the coefficients of the *vielbein* or “tetrad”.

Covariance of the Coupling

Let $S(\Lambda)$ be a representation of a local Lorentz transformation. Then,

$$\nabla'_\mu \psi' = (\nabla_\mu \psi)' \,, \qquad \psi' = S(\Lambda)\,\psi \,, \qquad (\nabla_\mu \psi)' = S(\Lambda)\,\nabla_\mu \psi \,.$$

According to Fock and Ivanenko, and Brill and Wheeler, and others, one sets

$$\Gamma_\mu = \frac{\mathrm{i}}{4}\,\omega_\mu^{AB}\,\sigma_{AB} \,, \qquad \omega_\mu^{AB} = e_\nu^A \nabla_\mu e^{\nu B} \,.$$

The Ricci rotation coefficients are denoted as ω_μ^{AB}, and the spin matrices are $\sigma_{AB} = \frac{\mathrm{i}}{2}\,[\gamma_A, \gamma_B]$. (With the flat-space γ_A.)
Under local Lorentz transformations, one has

$$e'^{\mu A} = \Lambda^A{}_B\, e^{\mu B} \,.$$

Vector and spinor representations:

$$\Lambda^C{}_D = \left(\exp\left[\frac{1}{2}\epsilon^{AB}\,\mathbb{M}_{AB}\right]\right)^C{}_D \approx \delta^C{}_D + \epsilon^C{}_D \,,$$

$$S(\Lambda) = \exp\left(-\frac{\mathrm{i}}{4}\,\sigma^{AB}\,\epsilon_{AB}\right) \approx 1 - \frac{\mathrm{i}}{4}\,\sigma^{AB}\,\epsilon_{AB} \,.$$

Here, the ϵ_{AB} are the local, infinitesimal parameters describing the Lorentz transformations.

Algebra of Generators

Recall that for vector Lorentz transformations, one has

$$(\mathbb{M}_{AB})^{C}{}_{D} = g^{C}{}_{A}\, g_{DB} - g^{C}{}_{B}\, g_{DA}\,.$$

Analogous algebraic relations for spin matrices,

$$[\tfrac{1}{2}\sigma^{CD}, \tfrac{1}{2}\sigma^{EF}] = \mathrm{i}\left(g^{CF}\,\tfrac{1}{2}\sigma^{DE} + g^{DE}\,\tfrac{1}{2}\sigma^{CF} - g^{CE}\,\tfrac{1}{2}\sigma^{DF} - g^{DF}\,\tfrac{1}{2}\sigma^{CE}\right)\,,$$

and generator matrices of local Lorentz transformations:

$$[\mathbb{M}^{CD}, \mathbb{M}^{EF}] = g^{CF}\;\mathbb{M}^{DE} + g^{DE}\;\mathbb{M}^{CF} - g^{CE}\;\mathbb{M}^{DF} - g^{DF}\;\mathbb{M}^{CE}\,.$$

(Generalizations to particles of higher spin for gravitational coupling is thus possible.) Under local Lorentz transformations, one has

$$\omega'^{AB}_{\mu} \approx \omega^{AB}_{\mu} - \partial_\mu\,\epsilon^{AB} + \epsilon^{A}{}_{C}\,\omega^{CB}_{\mu} - \omega^{AD}_{\mu}\,\epsilon_{D}{}^{B}\,,$$

and

$$\Gamma'_\mu = S(\Lambda)\,\Gamma_\mu\,S(\Lambda)^{-1} + [\partial_\mu S(\Lambda)]\;S(\Lambda)^{-1}\,.$$

Here, $\Gamma'_\mu = \frac{\mathrm{i}}{4}\,\omega'^{AB}_{\mu}\,\widetilde{\sigma}_{AB}$.

Dirac Adjoint for Curved Space–Times

The Dirac adjoint is constructed so that it transforms with the inverse Lorentz transformation.

Lorentz transformation of spinor:

$$\psi'(x') = S(\Lambda)\,\psi(x)\,.$$

Lorentz transformation of Dirac adjoint:

$$\overline{\psi}'(x') = \overline{\psi}(x)\,S(\Lambda^{-1}) = \overline{\psi}(x)\,[S(\Lambda)]^{-1}\,.$$

Surprise. Can show that the following definition does the job:

$$\overline{\psi}(x) = \psi^{+}(x)\,\gamma^{0}\,.$$

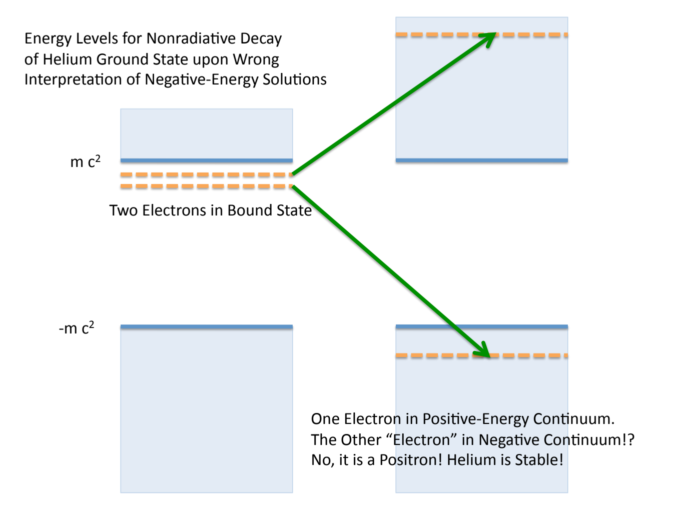
Here, γ^0 is the *flat-space* γ^0 matrix.
Intuitively, this is because the local spinor Lorentz transformations are constructed in an analogous way as compared to flat space-time.
(This is a recent result [arXiv:2003.08733v2].)

Every atomic physicist who has been working with relativistic atomic structure codes knows that the negative-energy solutions of the Dirac equation cannot describe electrons.

In fact, in MCDF (multi-configuration Dirac–Fock) codes, the negative-energy solutions have to be projected out explicitly.

Why? Well, the negative-energy states do not describe electrons. If they did, then the helium atom's ground state would be unstable against a nonradiative simultaneous quantum jump of one electron into the positive-energy continuum, while the other electron jumps into the negative-energy continuum.

Necessity for Re–Interpreting Negative-Energy Solutions



Charge Conjugation of the Dirac Equation

Q: How do we know that the negative-energy solutions of the Dirac equation describe positrons?
A: Consider charge conjugation.
We recall the electromagnetically coupled Dirac equation:

$$\left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - e\,A_\mu \right) - m \right] \psi = 0$$

Charge conjugation swaps positive-energy and negative-energy solutions:

$$\psi \to \psi^{\mathcal{C}} = C\,\overline{\psi}^{\mathrm{T}} = C\,\gamma^0\,\psi^*\,, \qquad \overline{\psi} = \psi^\dagger\,\gamma^0\,,$$

Here, C is the charge conjugation matrix, T denotes the transpose, the dagger (†) denotes the Hermitian adjoint, $C = \mathrm{i}\,\gamma^2\gamma^0$ is the charge conjugation matrix, and the superscript calligraphic-C denotes the charge-conjugated Dirac spinor. *The complex conjugation reverses the sign of the energy.* Voilà:

$$\left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + e\,A_\mu \right) - m \right] \psi^{\mathcal{C}} = 0\,.$$

Charge Conjugation of the Lagrangian

Wait a second. We can do this more elegantly. The Dirac theory allows for a Lagrangian formulation with a density

$$\mathcal{L} = \overline{\psi} \left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - e\, A_\mu \right) - m \right] \psi \,.$$

In terms of the charge-conjugated antiparticle bispinor wave function:

$$\mathcal{L} = \underbrace{-}_{\text{oops!}} \overline{\psi}^{C} \left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + e\, A_\mu \right) - m \right] \psi^{C} \,.$$

Why the overall minus sign? Do we suddenly have to "maximize" the action? Answer: The minus sign goes away when we consider second-quantized (QED) field operators as opposed to first-quantized wave functions:

$$\hat{\mathcal{L}} = \overline{\hat{\psi}} \left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - e\, A_\mu \right) - m \right] \hat{\psi} \,.$$

In terms of the charge-conjugated quantum field operator:

$$\hat{\mathcal{L}} = \underbrace{+}_{\text{looks better!}} \overline{\hat{\psi}}^{C} \left[\mathrm{i}\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + e\, A_\mu \right) - m \right] \hat{\psi}^{C} \,.$$

Charge Conjugation of the Gravitationally Coupled Lagrangian (I)

Start from the Lagrangian

$$\mathcal{L} = \overline{\psi}(x)\left[\overline{\gamma}^{\mu}\left\{\mathrm{i}\left(\partial_{\mu} - \Gamma_{\mu}\right) - e\,A_{\mu}\right\} - m_I\right]\psi(x)\,.$$

Using the relation

$$\Gamma_{\mu}^{+} = -\frac{\mathrm{i}}{4}\,\omega_{\mu}^{AB}\,\sigma_{AB}^{+} = -\frac{\mathrm{i}}{4}\,\omega_{\mu}^{AB}\,\gamma^{0}\,\sigma_{AB}\,\gamma^{0} = -\gamma^{0}\,\Gamma_{\mu}\gamma^{0}\,,$$

one can show that the adjoint of the Lagrangian is

$$\mathcal{L}^{+} = \overline{\psi}(x)\,\left[\overline{\gamma}^{\mu}\left\{-\mathrm{i}\overleftarrow{\partial}_{\mu} - e\,A_{\mu}\right\} - \mathrm{i}\,\Gamma_{\mu}\,\overline{\gamma}^{\mu} - m_I\right]\psi(x)\,.$$

Charge Conjugation of the Gravitationally Coupled Lagrangian (II)

Transposition and insertion of charge conjugation matrix $C = \mathrm{i}\gamma^2\,\gamma^0$ (with the flat-space γ^2 and γ^0) leads to

$$\begin{aligned}\left(\mathcal{L}^+\right)^{\mathrm{T}} = \psi^{\mathrm{T}}(x)\,C^{-1}\,\Big[C\,\left(\overline{\gamma}^{\mu}\right)^{\mathrm{T}}\,C^{-1}\left\{-\mathrm{i}\,\overrightarrow{\partial}_{\mu} - e\,A_{\mu}\right\} \\ -\mathrm{i}\,C\,\left(\overline{\gamma}^{\mu}\right)^{\mathrm{T}}\,C^{-1}\,C\,\Gamma_{\mu}^{\mathrm{T}}\,C^{-1} - m_I\Big]\;C\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,.\end{aligned}$$

On account of the identities

$$\psi^{\mathcal{C}}(x) = C\,\left[\overline{\psi}(x)\right]^{\mathrm{T}}\,,\qquad \overline{\psi^{\mathcal{C}}(x)} = -\psi^{\mathrm{T}}(x)\,C^{-1}\,,$$

one can show that

$$\boxed{\mathcal{L} = \left(\mathcal{L}^+\right)^{\mathrm{T}} = -\overline{\psi^{\mathcal{C}}(x)}\,\left[\overline{\gamma}^{\mu}\,\{\mathrm{i}(\partial_{\mu} - \Gamma_{\mu}) + e\,A_{\mu}\} - m_I\right]\,\psi^{\mathcal{C}}(x)\,.}$$

(i) The electromagnetic coupling term has changed sign, the gravitational term has not. *(ii)* There are no “interference” terms involving combined gravito-electromagnetic effects incurred upon particle–antiparticle interchange (“charge conjugation”) in curved space-time. *(iii)* Result holds for arbitrary *non-static* backgrounds.

Charge Conjugation of the Gravitationally Coupled Lagrangian (III)

Main result:

$$\boxed{\mathcal{L} = \left(\mathcal{L}^{+}\right)^{\mathrm{T}} = -\overline{\psi^{\mathcal{C}}(x)}\,\left[\overline{\gamma}^{\mu}\,\{\mathrm{i}(\partial_\mu - \Gamma_\mu) + e\,A_\mu\} - m_I\right]\,\psi^{\mathcal{C}}(x)\,.}$$

Nice, but disturbing minus sign.

In second quantization:

$$\begin{aligned}\hat{\mathcal{L}} &= \hat{\overline{\psi}}(x)\,\left[\overline{\gamma}^{\mu}\,\{\mathrm{i}\,(\partial_\mu - \Gamma_\mu) - e\,A_\mu\} - m_I\right]\,\hat{\psi}(x)\\ &= +\hat{\overline{\psi}}^{\mathcal{C}}(x)\,\left[\overline{\gamma}^{\mu}\,\{\mathrm{i}(\partial_\mu - \Gamma_\mu) + e\,A_\mu\} - m_I\right]\,\hat{\psi}^{\mathcal{C}}(x)\,.\end{aligned}$$

Get rid of the disturbing minus sign in view of the anti-commutation relations for fermionic creation and annihilation operators.

We thus know that Dirac particles and antiparticles behave exactly the same in gravitational fields.

Charge Conjugation of the Gravitationally Coupled Lagrangian (IV)

However, we still have to match the gravitational and the inertial mass, i.e., show that:

$$m_I = m_G \,.$$

By our previous result, it is sufficient to do this for *particles*. The result for antiparticles then follows per our previous considerations.

The matching can be done by considering a special case, i.e., the case of a central field.

This is described in [Int. J. Mod. Phys. A **34**, 1950180 (2019)] with an affirmative result.

Charge Conjugation of the Gravitationally Coupled Lagrangian (V)

Summary ($m_I = m_G = m$):

$$\mathcal{L} = \overline{\psi} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\underbrace{\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\}}_{\text{covariant–gravitational}} - e\, A_{\mu} \right) - m \right] \psi \,.$$

In terms of the charge-conjugated antiparticle bispinor wave function:

$$\mathcal{L} = \underbrace{-}_{\text{oops!}} \; \overline{\psi}^{C} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\} + e\, A_{\mu} \right) - m \right] \psi^{C} \,.$$

In second quantization:

$$\begin{aligned} \hat{\mathcal{L}} &= \overline{\hat{\psi}} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\} - e\, A_{\mu} \right) - m \right] \hat{\psi} \\ &= \overline{\hat{\psi}}^{C} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\} + e\, A_{\mu} \right) - m \right] \hat{\psi}^{C} \,. \end{aligned}$$

Charge Conjugation of the Gravitationally Coupled Lagrangian (V)

Result:

$$\begin{aligned}\hat{\mathcal{L}} &= \hat{\overline{\psi}} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\} - e\, A_{\mu} \right) - m \right] \hat{\psi} \\ &= \hat{\overline{\psi}}^{\mathcal{C}} \left[\mathrm{i}\overline{\gamma}^{\mu} \left(\left\{ \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu} \right\} + e\, A_{\mu} \right) - m \right] \hat{\psi}^{\mathcal{C}} .\end{aligned}$$

A few remarks are in order:

- Result holds for arbitrary curved space times, also dynamic ones.
- The so-called curved-space metric $\overline{g}^{\mu\nu} = \overline{g}^{\mu\nu}(x)$, which is non-constant, describes the (classical) curved space-time, in which the particle (antiparticle) is moving. Note that $\{\overline{\gamma}^{\mu}, \overline{\gamma}^{\nu}\} = \overline{g}^{\mu\nu}(x)$.
- Why is this interesting? A few very serious scientists (among these, Y. N. Obukhov, J. F. Donoghue and B. Holstein) have studied the gravitationally coupled Dirac equation and found terms which either *(i)* lead to parity-breaking, and particle-antiparticle symmetry breaking spin-gravity coupling terms, or *(ii)* other particle-antiparticle symmetry breaking terms. Corresponding articles have been published in high-profile journals. We show that there is no room for such terms.
- This generalizes previous findings [Phys. Rev. A **88**, 022121 (2013)].

Why was this Symmetry Not Discovered Earlier?

- Possible explanation (over-simplified): "Maybe, because atomic physicists knew that the Dirac equation describes particles and antiparticles simultaneously, but did not know how to couple it to the gravitational field, while gravitational physicists always knew how to couple the Dirac equation to gravity, but failed to notice that it describes particles and antiparticles simultaneously."
- The method of gravitational coupling was invented by Russian and American scientists (Fock, Brill and Wheeler, and others), with mutually consistent results. The spin-connection formalism in curved space-time stands on firm ground.
- Note that, in a central field, $\Gamma_0 \neq \Phi_G$, i.e., one cannot simply write the gravitational potential into the Dirac Hamiltonian just as in the electromagnetic case. The gauge groups of electromagnetism and gravity are completely different, and the spin-connection matrices Γ_μ in a central gravitational field are manifestly matrix-valued. Both space and time are curved around a massive object. The extraction of the leading gravitational coupling terms, in a central field, requires a much more sophisticated formalism, based on a so-called Foldy–Wouthuysen transformation, generalized to gravitational coupling.

Series of Works with Jonathan Noble

Series of Works with Jonathan Noble (Selected)

- Phys. Rev. A **88**, 022121 (2013)
  Nonrelativistic limit of the Dirac-Schwarzschild Hamiltonian: Gravitational zitterbewegung and gravitational spin-orbit coupling
  foreshadows equivalence principle for antiparticles
- J. Phys. A **47**, 045042 (2014)
  Foldy–Wouthuysen transformation, scalar potentials and gravity
- Phys. Rev. A **92**, 012101 (2015)
  Ultrarelativistic decoupling transformation for generalized Dirac equations
- Phys. Rev. A **93**, 032108 (2016)
  Dirac Hamiltonian and Reissner-Nordström metric: Coulomb interaction in curved space-time
  combined electromagnetic and gravitational interactions

Antimatter Gravity Experiments

Q: *Then, what do antimatter gravity experiments test for?*
A: Probably, some very exotic, CPT-violating interaction.

Proposal. [arXiv:2003.08733] Write

$$
\begin{aligned}
\hat{\psi}(x) &= \psi^{(-)}(x) + \psi^{(+)}(x)\,, \\
\hat{\psi}^{(-)}(x) &= \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{m}{E}\,a_s(\vec{p})\,u_s(\vec{p})\,\mathrm{e}^{-\mathrm{i}p\cdot x}\,, \\
\hat{\psi}^{(+)}(x) &= \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\,\frac{m}{E}\,b_s^{+}(\vec{p})\,v_s(\vec{p})\,\mathrm{e}^{\mathrm{i}p\cdot x}\,.
\end{aligned}
$$

Assume "residual" (RS) interaction,

$$
\begin{aligned}
\hat{\mathcal{L}}_{\mathrm{RS}} &= -\frac{\eta}{2}\,e\,\hat{\overline{\psi}}^{(-)}\gamma^\mu\hat{\psi}^{(-)}\,B_\mu + \frac{\eta}{2}\,e\,\hat{\overline{\psi}}^{(+)}\gamma^\mu\hat{\psi}^{(+)}\,B_\mu - \frac{1}{4}G_{\mu\nu}\,G^{\mu\nu}\,, \\
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,,
\end{aligned}
$$

where B_μ is a massless vector field.
Hydrogen: "charge" $\eta\,e$, antihydrogen: "charge" $-\eta\,e$, so hydrogen and antihydrogen behave differently. Can put bounds on η.

Proposal for Experiments(I)

By deliberately making an electric field in a laboratory measuring the acceleration due to gravity of antihydrogen *large* (not small!), one can test for the charge neutrality of antimatter.

(Order-of-magnitude improvement over previous results should be possible.)

Details: [arXiv 2003.08733].

Proposal for Experiments(II)

According to [Phys. Rev. A **98**, 032508 (2018)], one has the following result. Comparing two atomic clocks at different altitudes (labeled 1 and 2 in the gravitational field), which is the essence of relativistic geodesy:

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2} = \frac{\sqrt{1+2\Phi_1} + |\Phi_1|^n \, C_n(M)}{\sqrt{1+2\Phi_2} + |\Phi_2|^n \, C_n(M)} \, . \tag{3}$$

Compatibility with the Einstein equivalence principles would require that $C_n(M) = 0$. Here, n is an exponent, M is the mass of the gravitational center, and C is a coefficient which depends on the physical effect causing the deviation from the Einstein equivalence principle.

There are, indeed, tiny quantum corrections to the Einstein equivalence principle. Reason: A theory which is nondeterministic (quantum mechanics) cannot be fully compatible with fully deterministic general relativity.

- quadrupole gravitational correction [$n = 3$]
- second-order "dipole" gravitational correction [$n = 2$]
- Fokker precession correction [$n = 3$]
- first-order "dipole" gravitational correction [$n = 2$]

(The latter exists only for diatomic molecules.)

Conclusions

- Just as much as the electromagnetically coupled Dirac equation predicts that antiparticles have the opposite charge as compared to particles (but otherwise behave exactly the same under electromagnetic interactions),...
- ...the gravitationally coupled Dirac equation predicts that particles and antiparticles follow exactly the same dynamics in curved space-time, i.e., with respect to gravitational fields (in particular, they have the same gravitational mass, and there is no sign change in the gravitational coupling).

Conclusions

- Our findings have relevance for antimatter gravity experiments.
- We believe that the possibilities for any deviations of the gravitational interactions of antiparticles from those of particles are relatively remote.
- New symmetries have been discovered for the gravitationally coupled Dirac equation and its behavior under particle-antiparticle transformations.
- Antimatter gravity experiments should probably be interpreted in terms of bounds for the coupling parameters of very exotic, CPT-violating interactions which could lead to interactions that differ for particle-antiparticle configurations as opposed to particle-particle configurations.
- For details, see arXiv: 2003.08733v2.
- Interesting experimental possibilities regarding charge neutrality for antimatter.
- Interesting experimental possibilities regarding tiny deviations from Einstein's equivalence principle.