How does Clifford algebra show the way to the second quantized fermions with unified spins, charges and families, and with vector and scalar gauge fields beyond the *standard model*

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Abstract

Fifty years ago the *standard model* offered an elegant new step towards understanding elementary fermion and boson fields, making several assumptions, suggested by experiments. The assumptions are still waiting for an explanation. There are many proposals in the literature for the next step. The *spin-charge-family* theory of one of us (N.S.M.B.) is offering the explanation for not only all by the *standard model* assumed properties of quarks and leptons and antiquarks and antileptons, with the families included, of the vectors gauge fields, of the Higgs's scalar and Yukawa couplings, but also for the second quantization postulates of Dirac and for cosmological observations, like there are the appearance of the *dark matter*, of *matter-antimatter asymmetry*, making several predictions. This theory proposes a simple starting action in $d \geq (13+1)$-dimensional space with fermions interacting with the gravity only (the vielbeins and the two kinds of the spin connection fields), what manifests in $d = (3+1)$ as the vector and scalar gauge fields, and uses the odd Clifford algebra to describe the internal space of fermions, what enables that the creation and annihilation operators for fermions fulfill the anticommutation relations for the second quantized fields without Dirac's postulates: Fermions single particle states already anticommute. We present in this review article a short overview of the *spin-charge-family* theory, illustrating shortly on the toy model the breaks of the starting symmetries in $d = (13+1)$-dimensional space, which are triggered either by scalar fields — the vielbeins with the space index belonging to $d > (3+1)$ — or by the condensate of the two right handed neutrinos, with the family quantum number not belonging to the observed families. We compare properties and predictions of this theory with the properties and predictions of $SO(10)$ unifying theories. .

1 Introduction

Physicists are gaining knowledge about the laws, that apply to the smallest constituents of nature and thus to the whole universe, by observations, experiments, we make mathematical models, predict with models new events, check predictions by experiments, observations, all these lead to new ideas, new models.

Repeating this circle again and again we ask ourselves questions like: **i.** Is the law of nature simple and elegant, manifesting complexity on many fermion and boson states in particular at low energies, or "nature uses mathematics, when just appears to be needed"? **ii.** Is the space-time $(3+1)$-dimensional? Why? Or has it many more dimensions? How many? There are namely only two simple and elegant choices, zero and infinity. **iii.** What are elementary constituents and interactions among constituents in our universe, in any universe? **iv.** How has the space-time of our universe started? **v.** And many others.

Fifty years ago the *standard model* offered an elegant new step towards understanding elementary fermion and boson fields by postulating:
a. The existence of massless fermion family members with the spins and charges in the fundamental representation of the groups, **a.i.** the quarks as colour triplets and colouress leptons, **a.ii** the left handed members as the weak doublets, the right handed weak chargeless members, **a.iii.** the left handed quarks differing from the left handed leptons in the hyper charge, **a.iv.** all the right handed members differing among themselves in hyper charges, **a.v.** antifermions carrying the corresponding anticharges of fermions and opposite handedness, **a.vi.** the families of massless fermions, suggested by experiments and required by the gauge invariance of the boson fields (there is no right handed neutrino postulated, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino allowed in the *standard model*).
b. The existence of massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the adjoint representations of the corresponding charged groups and manifesting the gauge invariance.
c. The existence of the massive weak doublet scalar higgs, **c.i.** carrying the weak charge $\pm\frac{1}{2}$ and the hyper charge $\mp\frac{1}{2}$ (as it would be in the fundamental representation of the two groups), **c.ii.** gaining at some step of the expanding universe the nonzero vacuum expectation value, **c.iii.** breaking the weak and the hyper charge and correspondingly breaking the mass protection, **c.iv.** taking care of the properties of fermions and of the weak bosons masses, **c.v.** as well as the existence of the Yukawa couplings.
d. The presentation of fermions and bosons as second quantized fields.
e. The gravitational field in $d = (3+1)$ as independent gauge field. (The *standard model* is defined without gravity in order that it be renomalizable, but yet the standard model particles are "alowed" to couple to gravity in the "minimal" way.)
The *standard model* assumptions have been experimentaly confirmed without raising any doubts so far, except for some few and possibly statistical fluctuations anomalies [1], but also by offering no explanations for the assumptions. The last among the fields postulated by the *standard model*, the scalar higgs, was detected in June 2012, the gravitational waves were detected in February 2016.

The *standard model* has in the literature several explanations, mostly with many new not explained assumptions. The most popular seem to be the grand unifying theories [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. At least $SO(10)$-unifying theories offer the explanation for the postulates from **a.i.** to **a.iv**, partly to **b.** by assuming that to all the fermion charges there exist the corresponding vector gauge fields — but does not explain the assumptions **a.v.** up to **a.vi.**, **c.** and **d.**, and does not connect gravity with gauge vector and scalar fields.

What questions should we ask ourselves to be able to find a trustworthy next step beyond the *standard models* of elementary particle physics and cosmology, which would offer understanding of not yet understood phenomena?

[1]We think here on the improved *standard model*, in which neutrinos have non-zero masses, and has no ambition to explain severe cosmological problems

i. Where do fermions, quarks and leptons, originate and why do they differ from the boson fields in spins, charges and statistics?
ii. Why are charges of quarks and leptons so different, why have the left handed family members so different charges from the right handed ones and why does the handedness relate charges to anticharges?
iii. Isn't the first quantization an approximation to the second quantization? How should one describe the internal degrees of fermions to explain the Dirac's postulates of the second quantization?
iv. Where do families of quarks and leptons originate and how many families do exist?
v. Why do family members – quarks and leptons — manifest so different masses if they all start as massless?
vi. How is the origin of the scalar field (the Higgs's scalar) and the Yukawa couplings connected with the origin of families and how many scalar fields determine properties of the so far (and others possibly be) observed fermions and masses of weak bosons? (The Yukawa couplings certainly speak for the existence of several scalar fields with the properties of Higgs's scalar.) Why is the Higgs's scalar, or are all scalar fields of similar properties as the higgs, if there are several, doublets with respect to the weak and the hyper charge?
vii. Do possibly exist also scalar fields with the colour charges in the fundamental representation (like the Higgs's scalars are doublets with respect to the weak charge) and where, if they are, do they manifest?
viii. Where do the so far observed (and others possibly non observed) vector gauge fields originate? Do they have anything in common with the scalar fields and the gravitational fields?
ix. Where does the *dark matter* originate?
x. Where does the "ordinary" matter-antimatter asymmetry originate?
xi. Where does the dark energy originate and why is it so small?
xii. What is the dimension of space? $(3+1)$?, $((d-1)+1)$?, ∞?
xiii. And many others.

The assumptions of the *standard model* are still waiting for an explanation.

Motivation for using Clifford algebra objects to describe the internal space of fermions:

In a long series of works we, mainly one of us N.S.M.B. ([7, 8, 12, 13, 15, 16, 17, 19, 20] and the references therein), have found phenomenological success with the model named the *spin-charge-family* theory, with fermions, the internal space of which is described with the Clifford algebra of all linear combinations of odd products of γ^a's in $d = (13+1)$, Sect. 3, interacting with only gravity ([23] and references therein), Sect. 6. The spins of fermions from higher dimensions, $d > (3+1)$, manifest in $d = (3+1)$ as charges of the *standard model*, Sect. 3, gravity in higher dimensions manifest as the *standard model* gauge vector fields as well as the Higgs's scalar and Yukawa couplings Sect. 6 [19, 15].

Let be added that one irreducible representation of $SO(13,1)$ contains, if looked from the point of view of $d = (3+1)$, all the quarks and leptons and antiquarks and antileptons just with the properties, required by the *standard model*, including the relation between quarks and leptons and handedness and antiquarks and antileptons and the opposite handedness, Table 7. All what in the *standard model* have to be assumed (extremely efficiently "read" from experiments), in the *spin-charge-family* theory appear as a possibility from the starting simple action, Eq. (100). This simple starting action offers the explanation for not only the properties of quarks and leptons and antiquarks and antileptons, but also for vector gauge fields, scalar gauge fields, which represent higgs and explain the Yukawa couplings and offers also the scalars, which cause matter/antimatter asymmetry, proton decay, the appearance of the dark matter, Sect. 6.

It has happened so many times in the history of science that the simpler, more elegant, model has shown up as a more "powerful" one. Examples can be found in nonrelativistic classical physics, after

embedded it into relativistic physics, in Newton's laws after embedded them in general theory of gravity, in symmetries of fermion and gauge fields when embedded symmetries into larger groups, etc..
The working hypotheses of at least one of us (N.S.M.B.) is that the laws of nature are simple and correspondingly elegant and that the many body systems around the phase transitions looks to us complicated at least from the point of view of the elementary constituents of fermion and boson fields.

To this working hypotheses belongs also the description of the internal space of fermions with the Clifford algebra.

There appear in the literature two anticommuting kinds of algebras, the Grassmann algebra [8] and the Clifford algebra [1]. We recognized [7, 8, 9, 10, 61, 62, 14, 22, 25] that the Grassmann algebra includes two Clifford algebras, anticommuting with each other, each of them expressible with the Grassmann algebra elements and opposite.

The Grassmann algebra, with elements θ^a, and their Hermitian conjugated partners $\frac{\partial}{\partial\theta^a}$ [20], can be used to describe the internal space of fermions with the integer spins and charges in the adjoint representations, each of the two Clifford algebras, we denote their elements with γ^a and $\tilde{\gamma}^a$ [7, 8, 9, 10, 61, 62], can be used to describe the half integer spins and charges in fundamental representations. The Grassmann algebra is expressible with the two Clifford algebras and opposite, Sects. 3.2.1, 3.2.2.

In both algebras, the Grassmann algebra and the two Clifford algebras, the odd products of objects, arranged to be eigenstates of the Cartan subalgebras of the corresponding Lorentz algebras, form creation and annihilation operators, which fulfill the anticommutation relations postulated by Dirac for the second quantized fermions. These creation operators can be used to describe the internal space of fermions, Sects 3.2.1, 3.2.2.

Steps to the second quantized states with the Clifford or Grassmann algebra:

i. The internal space of a fermion, Sects. 3, 3.2, is described by either Clifford or Grassmann algebra of an odd Clifford character (superposition of an odd number of Clifford "coordinates" — operators — γ^a's or of an odd number of Clifford "coordinates" — operators — $\tilde{\gamma}^a$'s) or of an odd Grassmann character (superposition of an odd number of Grassmann "coordinates" — operators — θ^a's).
ii. The eigenvectors of all the (chosen) Cartan subalgebra members of the corresponding Lorentz algebra are used to define the "basis vectors" of an odd character. (The Cartan subalgebra is in all three cases chosen in the way to be in agreement with the ordinary choice.) The algebraic application of this "basis vectors" on the corresponding vacuum state (either Clifford $|\psi_{oc}>$, defined in Eq. (46), or Grassmann $|\phi_{og}>$, Eq. (22), which is in the Grassmann case just the identity) generates the "basis states", usable for describing the internal space of fermions. The members of the "basis vectors" manifest together with their Hermitian conjugated partners the properties of creation and annihilation operators, which anticommute, Eq. (26) in the Grassmann case and Eq. (36) in the Clifford case, when applying on the corresponding vacuum state, due to the algebraic properties of the odd products of the algebra elements.
iii. The plane wave solutions of the corresponding Weyl equations (either Clifford, Eq. (79) or Grassmann, Eq. (140)) for free massless fermions are the tensor products, $*_T$, of the superposition of the members of the "basis vectors" and of the momentum basis. The coefficients of the superposition correspondingly depend on a chosen momentum $\vec{p}$, with $p^0 = |\vec{p}|$, for any of the continuous values of moments $\vec{p}$.
iv. The creation operators defined on the tensor products, $*_T$, of superposition of the finite number of "basis vectors" defining the internal space and of the infinite (continuous) momentum space, Eq. (55) in the Clifford case and Eq. (141) in the Grassmann case, define the infinite basis.
v. Applied on the vacuum state these creation operators form anticommuting single fermion states of an odd Clifford/Grassmann character.
vi. The second quantized Hilbert space $\mathcal{H}$ consists of all possible tensor products, $*_{T_H}$, of any number

of single fermion states, starting with no single single state, "Slater determinants" with no single particle state occupied (with no creation operators applied on the vacuum state), with one single particle state occupied (with one creation operator applied on the vacuum state), with two single particle states occupied (with two creation operator applied on the vacuum state), and so on.

vii. The creation operators together with their Hermitian conjugated partners annihilation operators fulfill, due to the oddness of the "basis vectors", while the momentum part commutes, the anticommutation relations, postulated by Dirac for second quantized fermion fields, not only when they apply on the vacuum state, but also when they apply on the Hilbert space $\mathcal{H}$, Eq. (93) in the Clifford case and Eq. (146) in the Grassmann case.

In the Clifford case this happens only after "freezing out" half of the Clifford space, what brings besides the correct anticommutation relations also the "family" quantum number to each irreducible representation of the Lorentz group of the remaining internal space. The oddness of the "basis vectors" is transfered to the creation operators forming the single fermion states and further to the creation operators forming the whole Hilbert space of the second quantized fermions in the Clifford case, Sect. 5, Eq. (99), and in the Grassmann case, Sect. A, Eq. (146).

viii. Correspondingly the creation and annihilation operators with the internal space described by either odd Clifford or odd Grassmann algebra, since fulfilling the anticommutation relations required for the second quantized fermions without postulates, explain the Dirac's postulates for the second quantized fermions.

In Sect. 2 we clarify the notation used in the second quantization of fermions in literature and in our description of the internal space of fermions when using either the Grassmann or the Clifford algebras. The comparison of the Dirac and our way of the second quantization of fermions is presented in Sect. 3.3.4.

In Sect. 3 we present in steps our use of the odd Grassmann algebra and the odd Clifford algebras for description of the internal space of fermions — their spins and charges in both algebras, and families in the Clifford algebra, in a tensor product with the external space of coordinates or momenta, relating our way of describing the internal space of fermons in any d, Sect. 3.2, to the usual way using the group theory, Sect. 3.1.

Sect. 3.2.1 presents the Grassmann odd algebra, that is the algebra of odd products of coordinates θ^a's, determining with their Hermitian conjugated partners $\frac{\partial}{\partial\theta_a}$'s ($\theta^{a\dagger} = \eta^{aa}\frac{\partial}{\partial\theta_a}$, $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$) the internal space of second quantized "fermions" with the integer spin. Corresponding creation operators determine in $d = (3+1)$ spins and charges of "fermions" in adjoint representations.

In Sect. 3.2.2 the two anticommuting Clifford odd algebras, γ^a's and $\tilde{\gamma}^a$'s, are discussed, each of them are the superposition of θ^a's and $\frac{\partial}{\partial\theta_a}$'s ($\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$ [8, 61, 62]). They are, up to η^{aa}, Hermitian operators. Each of the two Clifford algebras describes half integer spins, manifesting in $d = (3+1)$ spins, charges and families in the fundamental representations of the corresponding subgroups.

In Sect. 3.2.3 we explain the reduction of the two Clifford algebras to only one, we make a choice of γ^a's, while $\tilde{\gamma}^a$'s determine the "family charges" to each irreducible representation of the generators of the Lorentz algebra in the remaining Clifford space of γ^a's and insure that the corresponding creation and annihilation operators have all the properties of the second quantized fermion fields. The reduction of the Clifford two algebras to only one reduces also the Grassmann algebra space.

In Sect. 3.3 we discuss the tensor products of the internal space of fermions and the space of coordinates or momenta in the ordinary second quantization procedure, as well as in the case when the internal space is described by the Clifford algebra.

In Sect. 3.3.1 the ordinary second quantization procedure is presented.

In Sect. 3.3.2 the second quantization with the Clifford algebra is presented.

Sect. 3.3.3 demonsrates the solutions of equations of motion for free massless chargeless fermions in

$d = (3+1)$ for the case that the internal space is described by the Clifford algebra with γ^a's.
Sect. 3.3.4 relates our way of second quantization and the usual, Dirac's, way of second quantization, Refs. [22, 25].
In Sect. 3.3.5 the discrete symmetry operators for fermions and bosons (vielbeins and spin connections) in $d = ((d-1)+1)$-dimensional space and as observed in $d = (3+1)$ as fermion fields, vector gauge fields and scalars gauge fields in the *spin-charge-family*, Ref. [125], are presented. App. G adds more details about discrete symmetries of vector and scalar gauge fields.
In Sect. 4 the action and the corresponding equations of motion for free massless fermions in the *standard model*, Sect. 4.1 and in the *spin-charge-family* theory, Sect. 4.2, is discussed and solutions expressed with the creation operators in $d = (5+1)$, Sect. 4.3.1, in $d = (9+1)$ and $d = (13+1)$ and in Sect. 4.3.2, with families included. The proposal for the action for free massless "Grassman fermions" can be found in App. A, where also the solutions are presented, again expressed with the creation operators.
In Sect. 5 the Hilbert space determined by all possible numbers of tensor products of all possible odd Clifford creation operators applying on the vacuum state is presented and the anticommutation properties of the application of creation and annihilation operators on the Hilbert space is demonstrated, Sect. 5.1.
In Sect. 6 the simple starting action for massless fermions, the internal space of which is described by the odd Clifford algebra, and which interact with the vielbeins and the two kinds of the spin connection fields only, and for the gravitational fields is presented and discussed.
In Sect. 6.1 the action for massless fermions as seen in $d = (3+1)$ when fermions interact with the dynamical vectors gauge fields, Sect. 6.2.1 and the scalar gauge fields, which for particular scalar space indexes manifest properties of Higgs's scalar, explaining the appearance of the scalar higgs and the Yukawa couplings of the *standard model*, Sect. 6.2.2, while for the rest of the scalar indexes the scalar triplets and antitriplet fields cause the transition of antileptons into quarks and antiquarks into quarks, offering explanation for the matter/antimatter asymmetry in the expanding universe and predict the proton decay.
In Sect. 7 we compare the assumptions of the *standard model*, Sect. 7.1, with in the literature proposed unifying theories, Sect. 7.2, mostly $SO(10)$-unifying theories, Sect. 7.2.1, and the *spin-charge-family* theory, Sect. 7.2.2, discussing besides the assumptions also the predictions of these theories, Sects. 7.3, 7.3.1, searching for the next step beyond the *standard model*.
In Sect. 7.4 the toy model is discussed in $d = (5+1)$ of massless odd Clifford fermions, interacting with the zweibein and spin connection fields in $d = (5,6)$, of particular properties, which cause the break of the $5+1$-dimensional space into $3+1$ times an almost S^2 sphere as an introduction into the study of the *spin-charge-family* theory case in $d = (13+1)$, which is under consideration [2].
In Sect. 7.5 we shortly present other possibilities for which we have our own understanding of the topics.
In Sect. 8 we mention possibilities that the realization and predictions of the ıspin-charge-family theory could influence hadron physics, if at all.
In Sect. 9 we shortly overview what one can learn from this review article.
In App. A the trial to find the action for the massless integer spin "Grassmann fermions" is presented.
In App. B properties of the Clifford even commuting "basis vectors" are discussed, App. evenclifford, and the matrices for γ^a's, $\tilde{\gamma}^a$'s, S^{ab}, $\tilde{S}^{ab}$ are presented and discussed, App. B.2
. In App. C the triangle anomaly cancellation of the *standard model* is explained in the ıspin-charge-family theory and in the $SO(10)$-unifying theories.

Most of the rest of appendixes offer detail explanations or proofs needed in the main text.

[2]The mass of fermions is in this toy model determined either with the dynamics in the 6^{th} and the 7^{th} dimension or, at low energies, by the constant values of the spin connection fields, which break the mass protection.

2 Second quantization in text books and relation to our way of second quantization

This section is to present in a short way how does the main assumption, the decision to describe the internal space of fermions with the "basis vectors" expressed with the superposition of odd products of the anticommuting members of the algebra, as discussed in details in Sect. 3, either the Clifford one or the Grassmann one, acting algebraically, $*_A$, on the internal vacuum state $|\psi_o>$ in each of these algebras, relate to the creation and annihilation anticommuting operators of the second quantized fermion fields, discussed in details in Sect. 3.3, Sect. 3.3.1, Sect. 3.3.2.

Let us first tell that the algebraic product $*_A$ is usually not present in other works, and thus has no well known physical meaning. It is at first a product by which one can multiply two internal "basis vectors", describing internal space of fields, $\hat{b}^{\dagger}_i$ and $\hat{b}^{\dagger}_j$, with each other,

$$\begin{aligned} \hat{c}^{\dagger}_k &= \hat{b}^{\dagger}_i *_A \hat{b}^{\dagger}_j \,, \\ \hat{b}^{\dagger}_i *_A \hat{b}^{\dagger}_j &= \mp \hat{b}^{\dagger}_j *_A \hat{b}^{\dagger}_i \,, \end{aligned} \tag{1}$$

the sign $\mp$ depends on whether $\hat{b}^{\dagger}_i$ and $\hat{b}^{\dagger}_j$ are products of odd or even number of algebra elements: The sign is $-$ if both are (superposition of) odd products of algebra elements, in all other cases the sign is $+$.

Let $\mathbf{R}^{d-1}$ define the external spatial or momentum basis $|\vec{p}>$. Then the tensor product $*_T$ extends the internal "basis vectors" into the creation operator $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ defined in both spaces, Sect. 3.3, App. J

$$\hat{\mathbf{b}}^{\dagger}_{\vec{p},i} = |\vec{p}> *_T \hat{b}^{\dagger}_i \,, \tag{2}$$

where again $\hat{b}^{\dagger}_i$ represent the superposition of products of elements of the anticommuting algebras, in our case either θ^a or γ^a or $\tilde{\gamma}^a$, used in this paper.

We can make a choice of the orthogonal and normalized basis so that $<\hat{\mathbf{b}}_{\vec{p},i}|\hat{\mathbf{b}}^{\dagger}_{\vec{p'},j}>= \delta(\vec{p}-\vec{p'})\,\delta_{ij}$. Let us point out that either $\hat{b}^{\dagger}_i$ or $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ apply algebraically on the corresponding vacuum state, $\hat{b}^{\dagger}_i *_A |\psi_o>$ and $\hat{\mathbf{b}}_{\vec{p},i} *_A |\psi_o>$.

To give to the algebraic product, $*_A$, and to the tensor product, $*_T$, defining the space of single particle wave functions depending on the internal space and external space, the understandable meaning, we postulate the connection between the anticommuting/commuting properties of the "basis vectors", expressed with the odd/even products of the anticommuting algebra elements and the corresponding creation operators, creating second quantized single fermion/boson states.

$$\begin{aligned} \hat{\mathbf{b}}^{\dagger}_{\vec{p},i} *_A |\psi_o> &= |\psi_{\vec{p},i}> \,, \\ \hat{\mathbf{b}}^{\dagger}_{\vec{p},i} *_T \,|\psi_{\vec{p'},j}> &= 0 \,, \\ \text{if}\, \vec{p} &= \vec{p'}\, \text{and}\, i=j \,, \\ \text{in all other cases} & \quad \text{it follows} \\ \hat{\mathbf{b}}^{\dagger}_{\vec{p},i} *_T \,\hat{\mathbf{b}}^{\dagger}_{\vec{p'},j} *_A |\psi_o> &= \mp \hat{\mathbf{b}}^{\dagger}_{\vec{p'},j} *_T \,\hat{\mathbf{b}}^{\dagger}_{\vec{p},i} *_A |\psi_o> \,, \end{aligned} \tag{3}$$

with the sign $\pm$ depending on whether $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ have both an odd character, the sign is $-$, if this is not the case the sign is $+$.

Not necessarily both, odd and even products of anticommuting algebra elements have any physical meaning (any realization in the observed phenomena), if any at all. It will be shown that in Grassmann algebra both, odd and even products of anticommuting algebra elements exist, behaving as either anticommuting or commuting creation and annihilation operators, respectively, Sect. A.2, while in the Clifford algebra only anticommuting odd products of anticommuting elements of the Clifford algebra exist, Sect. B.1 [25].

To each creation operator $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ its Hermitian conjugated partner represents the annihilation operator $\hat{\mathbf{b}}_{\vec{p},i}$

$$\begin{aligned}
\hat{\mathbf{b}}_{\vec{p},i} &= (\hat{\mathbf{b}}^{\dagger}_{\vec{p},i})^{\dagger}\,, \\
\text{with the} & \quad \text{property} \\
\hat{\mathbf{b}}_{\vec{p},i} *_A |\psi_o> &= 0\,, \\
\text{defining the} & \quad \text{vacuum state as} \\
|\psi_o>: &= \sum_i \hat{b}_i *_A \hat{b}^{\dagger}_i \,|\, I > |0_{\vec{p}}>\,,
\end{aligned} \tag{4}$$

where summation i runs over all different products of annihilation operator $\times$ its Hermitian conjugated creation operator, no matter what $\vec{p}$ is chosen, and $|\, I > \ |0_{\vec{p}}>$ represents the identity with the starting momentum state $|0_{\vec{p}}>$, from where all the momentum states $|\vec{p}>$ follow by translation in momentum space, $\hat{b}_i$ represents the Hermitian conjugated annihilation operator to $(\hat{b}^{\dagger}_i)^{\dagger}$, Sect. 3.3.

Let the tensor multiplication $*_{T_H}$ denotes the multiplication of any number of any possible single particle states, and correspondingly of any number of any possible creation operators, Sect. 5.

What further means that to each single particle wave function we define the creation operator $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$, applying in a tensor product from the left hand side on the second quantized Hilbert space — consisting of all possible products of any number of single particle wave functions — adding to the Hilbert space the single particle wave function created by this particular creation operator. In the case of the second quantized fermions, if this particular wave function with the quantum numbers $_i$ and momentum $\vec{p}$ of $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ is already among the single fermion wave functions of a particular product of the fermion wave functions, the action of the creation operator gives zero, otherwise the number of the fermion wave functions increases for one, Sect. 5. (In the boson case the number of boson second quantized wave functions increases always for one.)

If we apply with the annihilation operator $\hat{\mathbf{b}}_{\vec{p},i}$ on the second quantized Hilbert space, then the application gives a nonzero contribution only if the particular products of the single particle wave functions do include the wave function with the quantum number i and $\vec{p}$, Sect. 5.

In a Slater determinant formalism the single particle wave functions define the empty or occupied places of any of infinite numbers of Slater determinants. The creation operator $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ applies on a particular Slater determinant from the left hand side. Jumping over occupied states to the place with its i and $\vec{p}$. If this state is occupied, the application gives in the fermion case zero. (In the boson case the number of particles increase for one.) The particular Slater determinant changes sign in the fermion case if $\hat{\mathbf{b}}^{\dagger}_{\vec{p},i}$ jumps over odd numbers of occupied states. (In the boson case the sign of the Slater determinant does not change.)

When annihilation operator $\hat{\mathbf{b}}_{\vec{p},i}$ applies on particular Slater determinant, is jumping over occupied states to its own place, giving zero, if this place is empty and decreasing the number of occupied states if this place is occupied. The Slater determinant changes sign in the fermion case, if the number of occupied states before its own place is odd. (In the boson case the sign does not change.)

Let us stress again that choosing antisymmetry or symmetry of products of the algebra elements is possible only in the case of the Grassmann algebra — Grassmann odd algebra describes second quantized integer spin "fermions", Sect. 3.2.1, while the commuting Grassmann even algebra can be used to describe second quantized integer spin bosons.

In the Clifford case Clifford odd choice, describing the half integer fermions, is the only choice, since the Clifford even algebra does not offer the description of the second quantized boson states, as explained in App. B.

To describe the second quantized fermion states we use the "basis vectors", which are the superposition of the odd numbers of algebra elements in both algebras, in the two Clifford algebras and in the Grassmann algebra.

The creation operators and their Hermitian conjugated partners annihilation operators therefore in the fermion case anticommute. The single fermion states, which are the application of the creation operators on the vacuum state $|\psi_o>$, manifest correspondingly as well the oddness. The vacuum state, defined as the sum over all different products of annihilation $\times$ the corresponding creation operators, have an even character.

One usually means antisymmetry when talking about Slater-determinants because otherwise one would not get determinants.

In the present paper [7, 8, 12, 61] the choice of the symmetrizing versus antisymmetrizing relates indeed the commutation versus anticommutation with respect to the a priori completely different product $*_A$, of anticommuting members of the Clifford or Grassmann algebra. The oddness or evenness of these products transfer to quantities to which these algebras extend.

Let us add up with the recognition: Oddness and eveness of the "basis vectors" to describe "fermions" and bosons is meaningful only in the case of Grassmann algebra, while the Clifford algebra offers only the description of fermions, offering at the same time family quantum numbers for different irreducible representations of the corresponding Lorentz algebra. This is uniqum of the Clifford algebra that it offers the unification of spins, charges and families. The approaches with the group theory description of the internal space of fermions do not offer this kind of unification.

3 Internal and external space of fermions

Single fermion states are functions of external coordinates and of internal space of fermions. The basis for single fermion states are usually chosen to have some symmetries in both spaces, external and internal.

If M^{ab} denote infinitesimal generators of the Lorentz algebra in both spaces, $M^{ab} = L^{ab} + S^{ab}$, with $L^{ab} = x^a p^b - x^b p^a$, $p^a = i\frac{\partial}{\partial x_a}$, determining operators in ordinary space, while S^{ab} are equivalent operators in internal space of fermions [3]

$$\begin{aligned}
\{M^{ab}, M^{cd}\}_- &= i\{M^{ad}\eta^{bc} + M^{bc}\eta^{ad} - M^{ac}\eta^{bd} - \mathbf{M^{bd}}\eta^{\mathbf{ac}}\}\,, \\
\{M^{ab}, p^c\}_- &= -i\eta^{ac}p^b + i\eta^{cb}p^a\,, \\
\{M^{ab}, S^{cd}\}_- &= i\{S^{ad}\eta^{bc} + S^{ad} - S^{ac}\eta^{bd} - \mathbf{S^{bd}}\eta^{\mathbf{ac}}\}\,,,,
\end{aligned} \tag{5}$$

with $L^{ab\dagger} = L^{ab}$, provided that $p^{a\dagger} = p^a$, since they apply on states which are zero on the boundary [6], while $S^{ab\dagger} = \eta^{aa}\eta^{bb}S^{ab}$. Commutation relation of Eq. (5) are valid for either L^{ab} or S^{ab} in any dimension of space-time, $\{L^{ab}, S^{cd}\}_- = 0$.

We use the metric tensor $\eta^{ab} = diag(1, -1, -1, \dots, -1, -1)$ for $a = (0, 1, 2, 3, 5, \dots, d)$.

The usual choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra is

$$M^{03}, M^{12}, M^{56}, \dots, M^{d-1\,d}\,, \tag{6}$$

defining correspondingly the members of the irreducible representations of the Lorentz group to be the eigenvectors of all the members of the Cartan subalgebra presented in Eqs. (6, 8).

In the literature the single fermion states usually do not anticommute. Dirac postulated anticommuting properties for fermion fields only in the second quantization, by postulating anticomutation

[3] We here mainly think of the algebra and do not for the moment go into the problem that if we take these operators to be represented on the space of states for a particle then x^0 is not the operator, it is instead a moment of time, what we do have in mind.

rules for creation and annihilation operators, which operating on the vacuum state generate the second quantized states. Although all the elementary fermion constituents, quarks and leptons, do carry besides spins also charges and appear in families, and although charges and families are treated as internal degrees of freedom of fermions, yet internal space of fermions is not built from spins, charges and families in equivalent way, unifying spins, charges and families.

One finds in the literature [26, 27] several trials to unify all the charges, but not the spin with the charges. We comment in App. C that the appearance of spins and charges in one irreducible representation of the Lorentz group explains the triangle anomalies cancellation in the *standard model* better than the unification of only charges, since in this first case the handedness and charges are already related, this is happening in the *spin-charge-family* theory, while in the unification of only charges the relation between spins and charges must be postulated like in the *standard model*.

We show in what follows that anticommuting properties of spinor states, brought into the theory by using the anticommuting "basis vectors", bring simplicity, transparency and elegance in the theoretical description of fermion states, if spins and charges appear in the same irreducible representation of the group.

Let us point out that in the literature the anticommuting properties of the second quantized fermion fields are just postulated. In the "Grassmann fermions" and the two "Clifford fermions" the "basis vectors" anticommute due to the oddness of the "basis vectors". The Hermitian conjugated partners of the "basis vectors" belong to independent representations, while in the usual second quantized fields they are just postulated to exist.

We shall use in what follows two kinds of anticommuting algebras, the Grassmann algebra and the two anticommuting kinds of the Clifford algebras, denoting them by θ^a's and $p^{\theta a}$'s $= \frac{\partial}{\partial\theta_a}$'s, for the Grassmann algebra objects, and γ^a's and $\tilde{\gamma}^a$'s for the two kinds of the Clifford algebra objects. The existence of the two kinds of the Clifford algebras is our recognition [7, 8, 10, 61, 62].

We denote the corresponding infinitesimal Lorentz generators as $\mathbf{S}^{ab}$ for the Grassmann algebra, and as S^{ab} and $\tilde{S}^{ab}$ for the two kinds of the Clifford algebras.

$$
\begin{aligned}
\mathbf{S}^{ab} &= i\,(\theta^a \frac{\partial}{\partial\theta_b} - \theta^b \frac{\partial}{\partial\theta_a})\,, \qquad S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,\\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \qquad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,,\\
\{\mathbf{S}^{ab},\, \theta^e\}_- &= -i\,(\eta^{ae}\,\theta^b - \eta^{be}\,\theta^a)\,, \qquad \{\mathbf{S}^{ab},\, p^{\theta e}\}_- = -i\,(\eta^{ae}\,p^{\theta b} - \eta^{be}\,p^{\theta a})\,,\\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,,\\
\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- &= i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,,\\
\{S^{ab}, \tilde{\gamma}^c\}_- &= 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,,
\end{aligned} \tag{7}
$$

the proof for the relation $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ is presented in App. I, Statement 2..

It then follows for the Cartan subalgebra of the commuting operators of the Lorentz algebra for each of the two kinds of the operators of the Clifford algebra, S^{ab} and $\tilde{S}^{ab}$ and for the Grassmann algebra $\mathbf{S}^{ab}$

$$
\begin{aligned}
&\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\,d}\,,\\
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\,d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\,d}\,.
\end{aligned} \tag{8}
$$

We shall see that the Grassmann algebra is uniquely expressible with the two Clifford algebras and opposite.

Table 1: Four spinor states in $d = (3+1)$, two left handed (chiral) and two right handed (chiral), each with spin $\pm\frac{1}{2}$, are the eigenstates of the Cartan subalgebra of the Lorentz algebra in the internal space of fermions.

$h = \Gamma^{(3+1)}$	$\psi_{S^{03},S^{12}}$	S^{03}	S^{12}	Γ^{3+1}
R	$\psi_{+\frac{1}{2},+\frac{1}{2}}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+1$
R	$\psi_{-\frac{1}{2},-\frac{1}{2}}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+1$
L	$\psi_{-\frac{1}{2},+\frac{1}{2}}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1
L	$\psi_{+\frac{1}{2},-\frac{1}{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1

3.1 Internal space of fermions in text books

Fermion states can be represented as a superposition of tensor products of basis in ordinary (coordinate or momentum) space and in internal space. Fermions carry in general as a part of the internal degrees of freedom besides spins also charges.

The spinor part of the internal space of massless fermions (without charges) can be represented by the spinor representation of the Lorentz group S^{mn}, $(m,n) = (0,1,2,3)$. Making the usual choice of the Cartan subalgebra of the Lorentz group, Eq. (6),

$$S^{03}, S^{12}, \tag{9}$$

one usually choses internal part of states to be the eigenstates of the spin, S^{12}, and the handedness, $\Gamma^{(3+1)} = -\frac{i}{3!}\varepsilon_{mnop}S^{mn}S^{op}$, which is in the case of the Cartan subalgebra of Eq. (9) equal to $\Gamma^{(3+1)} = i(-2i)^2 S^{03}S^{12}$.

There are two possibilities for spin, $S^{12} = \pm\frac{1}{2}$, and two for $\Gamma^{(3+1)} = \pm 1$, for right and left handedness, respectively. Due to the relation $\Gamma^{(3+1)} = i(-2i)^2 S^{03}S^{12}$ the spinor part can be represented as well as "eigenstates" of the Cartan subalgebra members.

These four spinor states, which are the eigenstates of the Cartan subalgebra, can be written as presented in Table (1). These two representations, the right and the left one, can be written each in terms of Pauli spin matrices $\vec{\sigma}$ and the unit matrix σ^0.

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Let us write down the 4×4 matrix of handdedness/chirality, $\Gamma^{(3+1)}$, and the corresponding matrices of the infinitesimal generators of the Lorentz transformations in internal space of fermions, $\vec{S}$ ($\vec{S} = \frac{1}{2}\varepsilon_{ijk}S^{jk}$) and $\vec{K}$ ($K^i = S^{0i}$), defined in the space of four spinor states,

$$\mathbf{\Gamma^{(3+1)}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{K} = \frac{i}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}. \tag{11}$$

Each of the two "basis states" represents two decoupled irreducible representations of the Lorentz group. They only can be connected when introducing Dirac matrices $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ with properties

$$\gamma^{\mathbf{0}} = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \gamma^{\mathbf{1}} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \gamma^{\mathbf{2}} = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^{\mathbf{3}} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}. \tag{12}$$

The eigenstates of the Cartan subalgebra, Table 1, are in the literature not assumed to anticommute, they are rather assumed to be commuting objects. In Ref. ([41], page 17) P. Ramond mentions the possibility to give to spinors the Grassmann anticommuting character.

We point out in this paper that the internal space of fermions do have anticommuting character, either with the integer spins, if represented with the odd Grassmann algebra, or with the half integer spins, if represented with any of the two kinds of the odd Clifford algebras. All the "basis vectors" in internal space, either described by the Grassmann algebra or in any of the two Clifford algebras, are taken to be superposition of products of an odd number of algebra elements and to be at the same time eigenstates of all the Cartan subalgebra members, to be comparable with the usual choice of the spin representation.

We show in App. B.2 that using any of the two odd Clifford algebras to describe the internal space of fermions, offers several irreducible representations. However, the left and the right handed representation in $d=(3+1)$ appear in the same irreducible representation (what means that both have the odd handedness) provided that $d>(3+1)$. In the case of higher dimensions than $d=(3+1)$ not only the spin and handedness but also all the charges contribute to the internal space of fermions, determining the anticommuting "basis vectors".

In $d=(13+1)$, for example, in one irreducible representation of the Lorentz group there are all the quarks and all the leptons and the antiquarks and the antileptons with all the properties postulated by the *standard model* for quarks and antiquarks and for leptons and antileptons, all carrying the required charges, handedness and spins.

To the oddness of each member of the irreducible representation the spin part and the charge part contribute. All the irreducible representations are equivalent with respect to the Lorentz group.

3.2 Internal space of fermions described by Grassmann algebra and two kinds of Clifford algebras

We shall pay attention only to even dimensional spaces: $d=2(2n+1)$ and $d=4n$, n is a non negative integer.

In Grassmann d-dimensional space there are d anticommuting operators θ^a, $\{\theta^a,\theta^b\}_+=0$, $a=(0,1,2,3,5,..,d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+=0$, offering together $2\cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\begin{aligned}
&\{\theta^a,\theta^b\}_+=0\,, \qquad \{\frac{\partial}{\partial\theta_a},\frac{\partial}{\partial\theta_b}\}_+=0\,,\\
&\{\theta_a,\frac{\partial}{\partial\theta_b}\}_+ \;=\; \delta_{ab}\,,(a,b)=(0,1,2,3,5,\cdots,d)\,.
\end{aligned}\tag{13}$$

We define [20]

$$\begin{aligned}
(\theta^a)^\dagger &= \eta^{aa}\frac{\partial}{\partial\theta_a}\,,\\
&\text{leading to}\\
(\frac{\partial}{\partial\theta_a})^\dagger &= \eta^{aa}\theta^a\,,
\end{aligned}\tag{14}$$

the identity is the self adjoint member. We make a choice for the complex properties of θ^a, and correspondingly of $\frac{\partial}{\partial\theta_a}$, as follows

$$\begin{aligned}
\{\theta^a\}^* &= (\theta^0,\theta^1,-\theta^2,\theta^3,-\theta^5,\theta^6,...,-\theta^{d-1},\theta^d)\,,\\
\{\frac{\partial}{\partial\theta_a}\}^* &= (\frac{\partial}{\partial\theta_0},\frac{\partial}{\partial\theta_1},-\frac{\partial}{\partial\theta_2},\frac{\partial}{\partial\theta_3},-\frac{\partial}{\partial\theta_5},\frac{\partial}{\partial\theta_6},...,-\frac{\partial}{\partial\theta_{d-1}},\frac{\partial}{\partial\theta_d})\,.
\end{aligned}\tag{15}$$

In d-dimensional space of anticommuting Grassmann coordinates and of their Hermitian conjugated partners derivatives, Eqs. (13, 14), there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — both are expressible in terms of θ^a and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial \theta_a}$ [8].

$$\begin{aligned} \gamma^a &= (\theta^a + \frac{\partial}{\partial \theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})\,, \\ \theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,, \end{aligned} \tag{16}$$

offering together $2\cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$.

Taking into account Eqs. (13, 14) ($\{\theta^a, \theta^b\}_+ = 0$, $\{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0$, $\{\theta_a, \frac{\partial}{\partial \theta_b}\}_+ = \delta_{ab}$, $\theta^{a\dagger} = \eta^{aa}\,\frac{\partial}{\partial \theta_a}$ and $(\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa}\theta^a$) one finds

$$\begin{aligned} \{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,, \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,, \\ (\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,, \\ \gamma^a\gamma^a &= \eta^{aa}\,, \quad \gamma^a(\gamma^a)^\dagger = I\,, \quad \tilde{\gamma}^a\tilde{\gamma}^a = \eta^{aa}\,, \quad \tilde{\gamma}^a(\tilde{\gamma}^a)^\dagger = I\,, \end{aligned} \tag{17}$$

where I represents the unit operator.

Making a choice for the θ^a properties as presented in Eq. (15), it follows for the Clifford objects

$$\begin{aligned} \{\gamma^a\}^* &= (\gamma^0, \gamma^1, -\gamma^2, \gamma^3, -\gamma^5, \gamma^6, ..., -\gamma^{d-1}, \gamma^d)\,, \\ \{\tilde{\gamma}^a\}^* &= (-\tilde{\gamma}^0, -\tilde{\gamma}^1, \tilde{\gamma}^2, -\tilde{\gamma}^3, \tilde{\gamma}^5, -\tilde{\gamma}^6, ..., \tilde{\gamma}^{d-1}, -\tilde{\gamma}^d)\,, \end{aligned} \tag{18}$$

The Cartan subalgebra of the Lorentz algebra is presented in Eqs. (8, 6).

3.2.1 Grassmann algebra and "basis vectors" describing internal space

It appears useful to arrange 2^d products of θ^a into irreducible representations with respect to the Lorentz group with the generators [8], Eq. (7),

$$\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,. \tag{19}$$

2^{d-1} members of the representations have an odd Grassmann character (those which are superposition of odd products of θ^a's). All the members of any particular odd irreducible representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

If we exclude the self adjoint identity there is $(2^{d-1}-1)$ members of an even Grassmann character, they are even products of θ^a's. All the members of any particular even representation follow from any starting member by the application of $\mathbf{S}^{ab}$'s.

The Hermitian conjugated 2^{d-1} odd partners of odd representations of θ^a's and $(2^{d-1}-1)$ even partners of even representations of θ^a's are reachable from odd and even representations, respectively, by the application of Eq. (14).

It appears useful to make the choice of the Cartan subalgebra of the commuting operators of the Lorentz algebra, $\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \cdots, \mathbf{S}^{d-1\,d}$, Eq. (8), and choose the members of the irreducible representations of the Lorentz group to be the eigenvectors of all the members of the Cartan subalgebra of

Eq. (8), $\mathbf{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})$

$$\begin{aligned}
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) &= k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,\\
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) &= 0\,,\\
&\text{or}\\
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,\frac{i}{k}\theta^a\theta^b &= 0\,,
\end{aligned} \tag{20}$$

with $k^2 = \eta^{aa}\eta^{bb}$. The eigenvector $\frac{1}{\sqrt{2}}\,(\theta^0 \mp \theta^3)$ of $\mathbf{S}^{03}$ has the eigenvalue $k = \pm i$, respectively, the eigenvalues of all the other eigenvectors of the rest of the Cartan subalgebra members, Eq. (8), are $k = \pm 1$. In App. I, Statement 2a. the proof for Eq. (20) is presented.

We choose the "basis vectors" to be products of an odd number of nilpotents $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and the rest of the even objects $\frac{i}{k}\theta^a\theta^b$, with eigenvalues $k = \pm i$ and 0, respectively, so that all the Cartan subalgebra eigenvectors appear in the "basis vestor".

Let us check how does $\mathbf{S}^{ac} = i(\theta^a \frac{\partial}{\partial \theta_c} - \theta^c \frac{\partial}{\partial \theta_a})$ transform the product of two "nilpotents" $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ and $\frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d)$. One finds that $\mathbf{S}^{ac}\;\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\;\frac{1}{\sqrt{2}}\,(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d) = -\frac{\eta^{aa}\eta^{cc}}{2k}\;(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)$. $\mathbf{S}^{ac}$ transforms the product of two Grassmann odd eigenvectors of the Cartan subalgebra into the superposition of two Grassmann even eigenvectors.

"Basis vectors" have an odd or an even Grassmann character, if their products contain an odd or an even number of "nilpotents", $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, respectively. "Basis vectors" are normalized, up to a phase, in accordance with Eq. (165) of App. D.

The Hermitian conjugated representations of (either an odd or an even) products of θ^a's can be obtained by taking into account Eq. (14) for each "nilpotent"

$$\begin{aligned}
\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)^\dagger &= \eta^{aa}\,\frac{1}{\sqrt{2}}(\frac{\partial}{\partial \theta_a} + \frac{\eta^{aa}}{-ik}\frac{\partial}{\partial \theta_b})\,,\\
(\frac{i}{k}\,\theta^a\theta^b)^\dagger &= \frac{i}{k}\,\frac{\partial}{\partial \theta_a}\,\frac{\partial}{\partial \theta_b}\,.
\end{aligned} \tag{21}$$

Making a choice of the identity for the vacuum state,

$$|\phi_{og}> \;=\; |\,1>\,, \tag{22}$$

we see that algebraic products — we shall use a dot, $\cdot$, or without a dot for an algebraic product of eigenstates of the Cartan subalgebra forming "basis vectors" and $*_A$ for the algebraic product of "basis vectors" — of different θ^a's, if applied on such a vacuum state, give always nonzero contributions,

$$(\theta^0 \mp \theta^3)\cdot(\theta^1 \pm i\theta^2)\cdots(\theta^{d-1} \mp \theta^d)|\,1 > \neq \text{zero},$$

(this is true also, if we substitute any of nilpotents $\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$ or all of them with the corresponding even operators $(\frac{i}{k}\,\theta^a\theta^b)$; in the case of odd Grassmann irreducible representations at least one nilpotent must remain in the product). The Hermitian conjugated partners, Eq. (21), applied on $|\,1>$, give always zero

$$(\frac{\partial}{\partial \theta_0} \mp \frac{\partial}{\partial \theta_3})\cdot(\frac{\partial}{\partial \theta_1} \pm i\frac{\partial}{\partial \theta_2})\cdots(\frac{\partial}{\partial \theta_{d-1}} \pm i\frac{\partial}{\partial \theta_d})|\,1 >= 0.$$

Let us notice the properties of the odd products of θ^a's and of their Hermitian conjugated partners:
i. Superposition of products of different θ^a's, applied on the vacuum state $|\,1>$, give nonzero contribution. To create on the vacuum state the "fermion" states we make a choice of the "basis vectors" of

the odd number of θ^a's, arranging them to be the eigenvectors of all the Cartan subalgebra elements, Eq. (8).
ii. The Hermitian conjugated partners of the "basis vectors", they are products of derivatives $\frac{\partial}{\partial\theta_a}$'s, give, when applied on the vacuum state $|\,1>$, Eq. (22), zero. Each annihilation operator annihilates the corresponding creation operator, due to : $\frac{\partial}{\partial\theta_a}\theta^b=\eta^{ab}$.
iii. The algebraic product, $*_A$, of a "basis vector" by itself gives zero, the algebraic anticommutator of any two "basis vectors" of an odd Grassmann character (superposition of an odd products of θ^a's) gives zero, due to Eq. (13).
iv. The algebraic application of any annihilation operator on the corresponding Hermitian conjugated "basis vector" gives identity, on all the rest of "basis vectors" gives zero. Correspondingly the algebraic anticommutators of the creation operators and their Hermitian conjugated partners, applied on the vacuum state, give identity, all the rest anticommutators of creation and annihilation operators applied on the vacuum state, give zero.
v. Correspondingly the "basis vectors" and their Hermitian conjugated partners, applied on the vacuum state $|\,1>$, Eq. (22), fulfill the properties of creation and annihilation operator, respectively, for the second quantized "fermions" on the level of one "fermion" state.

From the eigenvectors of the members of the Cartan subalgebra, Eq. (8), we construct 2^{d-1} Grassmann odd "basis vectors" and $2^{d-1}-1$ (we skip self adjoint identity, which we use to describe the vacuum state $|\,1>$) Grassmann even "basis vectors" as superposition of odd and even products of θ^a's, respectively. Their Hermitian conjugated 2^{d-1} odd and $2^{d-1}-1$ even partners are, according to Eqs. (14, 21), determined by the corresponding superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s, respectively [4]. We present in this Sect. 3.2.1 the Grassmann anticommuting odd "basis vectors". In App. A.2 the reader can find illustration of the Grassmann even and correspondigly commuting "basic vectors".

Let us in $d=2(2n+1)$-dimensional space-time, $n\ge 0$, make a choice of the starting Grassmann odd "basis vector" $\hat{b}^{\theta 1\dagger}_1$, which is the eigenvector of the Cartan subalgebra of Eqs. (8, 20) with the egenvalues $(+i,+1,+1,\cdots,+1)$, respectively, and has the Hermitian conjugated partner equal to $(\hat{b}^{\theta 1\dagger}_1)^\dagger=\hat{b}^{\theta 1}_1$,

$$\begin{aligned}
&\hat{b}^{\theta 1\dagger}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\\
&\cdots(\theta^{d-1}+i\theta^d)\,,\\
&\hat{b}^{\theta 1}_1 := (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}}-i\frac{\partial}{\partial\theta^d})\cdots(\frac{\partial}{\partial\theta^0}-\frac{\partial}{\partial\theta^3})\,,\\
d &= 2(2n+1)\,.
\end{aligned}\tag{23}$$

In the case of $d=4n$, $n\ge 0$, the corresponding starting Grassmann odd "basis vector" can be chosen as

$$\begin{aligned}
\hat{b}^{\theta 1\dagger}_1 :&= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1}\,(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)\cdots\\
&\cdots(\theta^{d-3}+i\theta^{d-2})\theta^{d-1}\theta^d\,,\\
d &= 4n\,.
\end{aligned}\tag{24}$$

All the rest of "basis vectors", belonging to the same irreducible representation of the Lorentz group, follow by the application of $\mathbf{S}^{ab}$'s.

[4] Relations among operators and their Hermitian conjugated partners in both kinds of the Clifford algebra objects are more complicated than in the Grassmann case, where the Hermitian conjugated operators follow by taking into account Eq. (14).
In the Clifford case $\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{i\,k}\gamma^b)^\dagger$ is proportional to $\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{i\,(-k)}\gamma^b)$, while $\frac{1}{\sqrt{2}}(1+\frac{i}{k}\gamma^a\gamma^b)$ are self adjoint. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

Let us denote the members m of this starting irreducible representation f by $\hat{b}^{\theta m\dagger}_{f}$, $m=1, f=1$, and their Hermitian conjugated partners by $\hat{b}^{\theta m}_{f}$, with $m=1, f=1$.

"Basis vectors", belonging to different irreducible representations f', will be denoted by $\hat{b}^{\theta m'\dagger}_{f'}$, f' and their Hermitian conjugated partners by $\hat{b}^{\theta m'}_{f'} = (\hat{b}^{\theta m'\dagger}_{f'})^{\dagger}$.

$\mathbf{S}^{ac}$'s, which do not belong to the Cartan subalgebra, transform step by step the two by two "nilpotents", no matter how many "nilpotents" are between the chosen two, up to a constant, as follows: $\mathbf{S}^{ac}\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\cdots\ \frac{1}{\sqrt{2}}(\theta^c + \frac{\eta^{cc}}{ik'}\theta^d) \propto -\frac{\eta^{aa}\eta^{cc}}{2k}(\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)\cdots$, leaving at each step at least one "nilpotent" unchanged, so that the whole irreducible representation remains odd.
The superposition of $\mathbf{S}^{bd}$ and $i\mathbf{S}^{bc}$ transforms $-\frac{\eta^{aa}\eta^{cc}}{2k}\ (\theta^a\theta^b + \frac{k}{k'}\theta^c\theta^d)$ into $\frac{1}{\sqrt{2}}\ (\theta^a - \frac{\eta^{aa}}{ik}\theta^b)\frac{1}{\sqrt{2}}\ (\theta^c - \frac{\eta^{cc}}{ik'}\theta^d)$, and not into $\frac{1}{\sqrt{2}}\ (\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\ \frac{1}{\sqrt{2}}\ (\theta^c - \frac{\eta^{cc}}{ik'}\theta^d)$ or not into $\frac{1}{\sqrt{2}}\ (\theta^a - \frac{\eta^{aa}}{ik}\theta^b)\ \frac{1}{\sqrt{2}}\ (\theta^c + \frac{\eta^{cc}}{ik'}\theta^d)$.

Therefore we can start another odd representation with the "basis vector" $\hat{b}^{\theta m\dagger}_{2}$ as follows

$$\begin{aligned}
\hat{b}^{\theta m=1\dagger}_{2}\!: &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\theta^0 + \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)\cdots(\theta^{d-1} + i\theta^d)\,,\\
(\hat{b}^{\theta m=1\dagger}_{2})^{\dagger} &= \hat{b}^{\theta 1}_{2}\!: = (\frac{1}{\sqrt{2}})^{\frac{d}{2}}\,(\frac{\partial}{\partial\theta^{d-1}} - i\frac{\partial}{\partial\theta^{d}})\cdots(\frac{\partial}{\partial\theta^{0}} -+\frac{\partial}{\partial\theta^{3}})\,. \qquad (25)
\end{aligned}$$

The application of $\mathbf{S}^{ac}$'s determines the whole second irreducible representation $\hat{b}^{\theta 2\dagger}_{j}$.

One finds that each of these two irreducible representations has $\mathbf{\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}}$ members, Ref. [20].

Taking into account Eq. (13), it follows that odd products of θ^a's anticommute and so do the odd products of $\frac{\partial}{\partial\theta_a}$'s. It then follows:

Statement 1. The oddness of the products of θ^a's guarantees the anticommuting properties of all objects which include odd number of θ^a's.

One further sees that $\frac{\partial}{\partial\theta^a}\theta^b = \eta^{ab}$, while $\frac{\partial}{\partial\theta_a}|1>=0$, and $\theta^a|1>=\theta^a|1>$. We can therefore conclude

$$\begin{aligned}
\{\hat{b}^{\theta m}_{f}, \hat{b}^{\theta m'\dagger}_{f'}\}_{*_A+}|1> &= \delta_{ff'}\,\delta^{mm'}\,|1>\,,\\
\{\hat{b}^{\theta m}_{f}, \hat{b}^{\theta m'}_{f'}\}_{*_A+}|1> &= 0\,\cdot|1>\,,\\
\{\hat{b}^{\theta m\dagger}_{f}, \hat{b}^{\theta m'\dagger}_{f'}\}_{*_A+}|1> &= 0\,\cdot|1>\,,\\
\hat{b}^{\theta m}_{f}\ *_A\ |1> &= 0\,\cdot|1>\,,\\
|\phi^m_{of}> &= \hat{b}^{\theta m\dagger}_{f}\,|1>\,, \qquad (26)
\end{aligned}$$

where $\{\hat{b}^{\theta m}_{f}, \hat{b}^{\theta m'\dagger}_{f'}\}_{*_A+} = \hat{b}^{\theta m}_{f} *_A \hat{b}^{\theta m'\dagger}_{f'} + \hat{b}^{\theta m'}_{f'} *_A \hat{b}^{\theta m\dagger}_{f}$.

These anticommutation relations of the "basis vectors" of the odd Grassmann character, manifest on the level of the Grassmann algebra the anticommutation relations required by Dirac [2] for second quantized fermions.

The creation operators, determining the "Grassmann fermions" carrying the integer spins (in adjoint representations, Eq. (20)), are tensor products, $*_T$, of "basis vectors" in internal space and of the (momentum or coordinate) "basis" in ordinary space. Since the oddness of the "basis vectors" transfers to the creation operators, then the creation operators and their Hermitian conjugated partners annihilattion operators, fulfill the anticommutation relations postulated by Dirac for the second quantized "Grassmann fermions" with the integer spin.

The integer spin fermions have not been observed, all the observed fermions carry the half integer spin.

Grassmann integer spin fermions in $d=(5+1)$:

Let us illustrate anticommuting "basis vectors" with spins and charges in adjoint representation in $d=(5+1)$-dimensional space.

Table 2 represents two decuplets, which are the "egenvectors" of the Cartan subalgbra ($\mathbf{S}^{03}$, $\mathbf{S}^{12}$, $\mathbf{S}^{56}$), Eq. (8), of the Lorentz algebra $\mathbf{S}^{ab}$. The two decuplets represent two Grassmann odd irreducible representations of $SO(5,1)$.

One can read on the same table, from the first to the third and from the fourth to the sixth line in each of the two decuplets, two Grassmann even triplet representations of $SO(3,1)$, if paying attention on the eigenvectors of $\mathbf{S}^{03}$ and $\mathbf{S}^{12}$ alone, while the eigenvector of $\mathbf{S}^{56}$ has, as a "spectator", the eigenvalue either $+1$ (the first triplet in both decuplets) or -1 (the second triplet in both decuplets). Each of the two decuplets contains also one "fourplet" with the "charge" $\mathbf{S}^{56}$ equal to zero (($7^{th}, 8^{th}, 9^{th}, 10^{th}$) lines in each of the two decuplets (Table II in Ref. [8])).

Paying attention on the eigenvectors of $\mathbf{S}^{03}$ alone one recognizes as well even and odd representations of $SO(1,1)$: $\theta^0\theta^3$ and $\theta^0 \pm \theta^3$, respectively.

The Hermitian conjugated "basis vectors" follow by using Eq. (21) and is for the first "basis vector" of Table 2 equal to $(-)^2(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$. One correspondingly finds that when $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ applies on $(\frac{1}{\sqrt{2}})^3(\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6)$ the result is identity. Application of $(\frac{1}{\sqrt{2}})^3(\frac{\partial}{\partial\theta_5} - i\frac{\partial}{\partial\theta_6})(\frac{\partial}{\partial\theta_1} - i\frac{\partial}{\partial\theta_2})(\frac{\partial}{\partial\theta_0} + \frac{\partial}{\partial\theta_3})$ on all the rest of "basis vectors" of the decuplet I as well as on all the "basis vectors" of the decuplet II gives zero. "Basis vectors" are orthonormalized with respect to Eq. (165) of App. D.

Let us notice that $\frac{\partial}{\partial\theta_a}$ on a "state" which is just an identity, $|1>$, gives zero, $\frac{\partial}{\partial\theta_a}|1>=0$, while $\theta^a|1>$, or any superposition of products of different θ^a's, applied on $|1>$, gives the "vector" back.

One easily sees that application of products of superposition of θ^a's on $|1>$ gives nonzero contribution, while application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on $|1>$ gives zero.

The two by $\mathbf{S}^{ab}$ decoupled Grassmann decuplets of Table 2 are the largest two irreducible representations of odd products of θ^a's. There are 12 additional Grassmann odd "vectors", arranged into irreducible representations of six singlets and six sixplets

$$
\begin{aligned}
&(\frac{1}{2}(\theta^0 \mp \theta^3), \frac{1}{2}(\theta^1 \pm i\theta^2), \frac{1}{2}(\theta^5 \pm i\theta^6)\,,\\
&\frac{1}{2}(\theta^0 \mp \theta^3)\,\theta^1\theta^2\theta^5\theta^6, \frac{1}{2}(\theta^1 \pm i\theta^2)\,\theta^0\theta^3\theta^5\theta^6\,, \frac{1}{2}(\theta^5 \pm i\theta^6)\,\theta^0\theta^3\theta^1\theta^2)\,.
\end{aligned} \tag{27}
$$

The algebraic application of products of superposition of $\frac{\partial}{\partial\theta^a}$'s on the corresponding Hermitian conjugated partners, which are products of superposition of θ^a's, leads to the identity for either even or odd Grassmann character [5].

Besides 32 Grassmann odd eigenvectors of the Grassmann Cartan subalgebra, Eq. (8), there are $(32-1)$ Grassmann "basis vectors", which we arrange into irreducible representations, which are superposition of even products of θ^a's. The even self adjoint operator identity (which is indeed the normalized product of all the annihilation times, $*_A$, creation operators) is used to represent the vacuum state.

It is not difficult to see that Grassmann "basis vectors" of an odd Grassmann character anticommute among themselves and so do odd products of superposition of $\frac{\partial}{\partial\theta^a}$'s, while equivalent even products commute.

We shall demonstrate the matrix representations of the operators of the Lorentz transformations and of the elements of the algebras for the Clifford algebra elements γ^a's and $\tilde{\gamma}^a$'s when applying on "basis vectors" of the Clifford algebra of γ^a's in App. B.2.

The Grassmann odd algebra (as well as the two odd Clifford algebras) offers, due to the oddness of the internal space giving oddness as well to the elements of the tensor products of the internal space and of the momentum space, the description of the anticommuting second quantized fermion fields, as postulated by Dirac.

[5] We shall see that the vacuum states are in the Clifford case, similarly as in the Grassmann case, for both kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s, normalized sums of products of the annihilation $\times$, $*_A$, its Hermitian conjugated creation operators, and correspondingly self adjoint operators, but they are not the identity.

Table 2: The two decuplets, the odd eigenvectors of the Cartan subalgebra, Eq. (8), ($\mathbf{S}^{03}, \mathbf{S}^{12}$, $\mathbf{S}^{56}$) of the Lorentz algebra $SO(5,1)$ in Grassmann $(5+1)$-dimensional space, forming two irreducible representations, are presented. Table is taken from Refs. [20, 25]. The "basis vectors" within each decuplet are reachable from any member by $\mathbf{S}^{ab}$'s and are decoupled from another decuplet. The two operators of handedness, $\Gamma^{((d-1)+1)}$ for $d=(6,4)$, are invariants of the Lorentz algebra, Eq. (172) of App. F, $\Gamma^{(5+1)}$ for the whole decuplet, $\Gamma^{(3+1)}$ for the "triplets" and "fourplets".

I	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	i	1	1	1	1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	1	1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	$-i$	-1	1	1	1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	i	-1	-1	1	-1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	1	-1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	$-i$	1	-1	1	-1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	i	0	0	1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	$-i$	0	0	1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	1	0
II	i	decuplet of eigenvectors	$\mathbf{S}^{03}$	$\mathbf{S}^{12}$	$\mathbf{S}^{56}$	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1+i\theta^2)(\theta^5+i\theta^6)$	$-i$	1	1	-1	-1
	2	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^1\theta^2)(\theta^5+i\theta^6)$	0	0	1	-1	-1
	3	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1-i\theta^2)(\theta^5+i\theta^6)$	i	-1	1	-1	-1
	4	$(\frac{1}{\sqrt{2}})^3(\theta^0+\theta^3)(\theta^1-i\theta^2)(\theta^5-i\theta^6)$	$-i$	-1	-1	-1	1
	5	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^1\theta^2)(\theta^5-i\theta^6)$	0	0	-1	-1	1
	6	$(\frac{1}{\sqrt{2}})^3(\theta^0-\theta^3)(\theta^1+i\theta^2)(\theta^5-i\theta^6)$	i	1	-1	-1	1
	7	$(\frac{1}{\sqrt{2}})^2(\theta^0+\theta^3)(\theta^1\theta^2+\theta^5\theta^6)$	$-i$	0	0	-1	0
	8	$(\frac{1}{\sqrt{2}})^2(\theta^0-\theta^3)(\theta^1\theta^2-\theta^5\theta^6)$	i	0	0	-1	0
	9	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3-i\theta^5\theta^6)(\theta^1+i\theta^2)$	0	1	0	-1	0
	10	$(\frac{1}{\sqrt{2}})^2(\theta^0\theta^3+i\theta^5\theta^6)(\theta^1-i\theta^2)$	0	-1	0	-1	0

But the Grassmann "fermions" carry the integer spins, while the observed fermions — quarks and leptons — carry half integer spin.

3.2.2 Clifford algebras and "basis vectors" describing internal space

We learn in Sect. 3.2, Eq. (16), that in d-dimensional space of anticommuting Grassmann coordinates (and of their Hermitian conjugated partners — derivatives) there exist two kinds of the Clifford coordinates (operators) — γ^a and $\tilde{\gamma}^a$ — both are expressible in terms of θ^a and their conjugate momenta $p^{\theta a} = i\,\frac{\partial}{\partial\theta_a}$ [8].

$$\begin{aligned}
\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,\\
\theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{28}$$

offering together $2\cdot 2^d$ operators: 2^d of those which are products of γ^a and 2^d of those which are products of $\tilde{\gamma}^a$.

Let us repeat the anticommuting properties of both Clifford algebras, presented already in Eq. (17).

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\
(\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,,
\end{aligned}$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$, leading to the statement that the two Clifford algebras form the independent algebras.

Statement 2. γ^a's and $\tilde{\gamma}^a$'s define two independent Clfford algebras.

The proof of this statement can be found in App. I, Statement 1.

As in the Grassmann case we chose the "basis vectors" in each of the two spaces to beproducts of eigenstates of the Cartan subalgebra members, Eq. (8), of the Lorentz algebras, ($S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a), \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$).

First we look for "eigenstates" of each of the Cartan subalgebra members, Eq. (8), for each of the two kinds of the Clifford algebras separately,

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\,\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, \quad \tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,,
\end{aligned} \tag{29}$$

$k^2 = \eta^{aa}\eta^{bb}$. The proof of Eq. (29) is presented in App. (I), Statement 2a. The Clifford "basis vectors" — nilpotents $\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), (\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b))^2 = 0$ and projectors $\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b), (\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b))^2 = \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)$ — of both algebras are normalized, up to a phase, with respect to Eq. (165) of App. D.

Both, nilpotents and projectors, have half integer spins. The "eigenvalues" of the operator S^{03}, for example, for the "vector" $\frac{1}{2}(\gamma^0 \mp \gamma^3)$ are equal to $\pm\frac{i}{2}$, respectively, for the "vector" $\frac{1}{2}(1 \pm \gamma^0\gamma^3)$ are $\pm\frac{i}{2}$, respectively, while all the rest "vectors" have "eigenvalues" $\pm\frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $\tilde{S}^{03}$: for $\frac{1}{2}\,(\tilde{\gamma^0} \mp \tilde{\gamma}^3)$ the "eigenvalues" $\pm\frac{i}{2}$, respectively, and for the "eigenvectors" $\frac{1}{2}(1 \pm \tilde{\gamma}^0\tilde{\gamma}^3)$ the "eigenvalues" $k = \pm\frac{i}{2}$, respectively, while all the rest "vectors" have $k = \pm\frac{1}{2}$.

To make discussions easier let us introduce the notation for the "eigenvectors" of the two Cartan subalgebras, Eq. (8), Ref. [61, 8].

$$
\begin{array}{rclllll}
\stackrel{ab}{(k)}: &=& \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, & \stackrel{ab}{(k)}^{\dagger} = \eta^{aa}\stackrel{ab}{(-k)}\,, & (\stackrel{ab}{(k)})^2 = 0\,, & \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]} \\
\stackrel{ab}{[k]}: &=& \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, & \stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}\,, & (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, & \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,, \\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &=& 0\,, & \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,, & \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, \\
\stackrel{ab}{(\tilde{k})}: &=& \frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,, & \stackrel{ab}{(\tilde{k})}^{\dagger} = \eta^{aa}\stackrel{ab}{(-\tilde{k})}\,, & (\stackrel{ab}{(\tilde{k})})^2 = 0\,, & \\
\stackrel{ab}{[\tilde{k}]}: &=& \frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,, & \stackrel{ab}{[\tilde{k}]}^{\dagger} = \stackrel{ab}{[\tilde{k}]}\,, & (\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,, & \\
\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} &=& 0\,, & \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, & \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-\tilde{k}]} = \stackrel{ab}{(\tilde{k})}\,, & \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-\tilde{k})} = 0\,,
\end{array}
\tag{30}
$$

with $k^2 = \eta^{aa}\eta^{bb}$ for both algebras. Let us notice that the "eigenvectors" of the Cartan subalgebras and the eigenvalues are equivalent equivalent in both algebras. Both algebras have projectors and nilpotents: $((\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}\,, (\stackrel{ab}{(k)})^2 = 0)$, $((\stackrel{ab}{[\tilde{k}]})^2 = \stackrel{ab}{[\tilde{k}]}\,, (\stackrel{ab}{(\tilde{k})})^2 = 0)$. We pay attention on even dimensional spaces, $d = 2(2n+1)$ or $d = 4n$, $n \ge 0$ only.

The "basis vectors", which are products of $\frac{d}{2}$ either of nilpotents or of projectors or of both, are "eigenstate" of all the members of the Cartan subalgebra, Eq. (8), of the corresponding Lorentz algebra, forming $2^{\frac{d}{2}-1}$ irreducible representations with $2^{\frac{d}{2}-1}$ members in each of the two Clifford algebras cases.

The "basis vectors", which appear below in Eq. (31), are "eigenvectors" of all the Cartan subalgebra members, Eq. (8), in $d = 2(2n+1)$-dimensional space of γ^a's. The first one is the product of nilpotents only and correspondingly a superposition of an odd products of γ^a's. The second one belongs to the same irreducible representation as the first one, since it follows from the first one by the application of S^{01}, for example, the third one follows from the second one by the application of $S^{d-3\,d-1}$.

$$
\begin{array}{l}
\stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, \quad \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, \\
\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,.
\end{array}
\tag{31}
$$

The number of nilpotents in any of the members remains all the time odd, due to the eveness of the generators of the Lorentz transformations. One finds for their Hermitian conjugated partners, up to a sign,

$$
\begin{array}{l}
\stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \quad \stackrel{03}{[-i]}\stackrel{12}{[-i]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,.
\end{array}
$$

The members of the above irreducible representation for the $\tilde{\gamma}^a$'s follow from Eq. (32) by replacing γ^a's with $\tilde{\gamma}^a$'s.

The "basis vectors" form an orthonormal basis within each of the irreducible representations:

$$
\stackrel{d-1\,d}{(-)}\cdots\stackrel{12}{(-)}\stackrel{03}{(-i)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)} = 1\,.
\tag{32}
$$

This is, due to Eq. (30), equal to $\stackrel{03}{[-i]}\stackrel{12}{[-]}\cdots\stackrel{d-1\,d}{[-]}$, what we normalize to one, as we show in Eq. (46).

Usually in the literature, the operators γ^a's are represented as matrices. We use here γ^a's as Clifford objects, which form the "basis vectors". One can calculate, as seen in Ref. [24] and in App. B.2, the matrix representations of γ^a's when apply on "basis vectors" and the generators of the Lorentz transformations S^{ab} when apply on "basis vectors" defined in d-dimensional space. In App. B.2 the matrix representations of γ^a's and S^{ab}'s in $d=(3+1)$-dimensional space-time are presented.

In the Grassmann case the 2^{d-1} odd and $2^{d-1}-1$ (we skip the selfadjoint identity defining the vacuum state, Eq. (22)) even Grassmann "basis vectors" (operators), which are superposition of either odd or even products of θ^a's, respectively, are well distinguishable from their 2^{d-1} odd and $2^{d-1}-1$ even Hermitian conjugated partners, which are superposition of odd and even products of $\frac{\partial}{\partial\theta_a}$'s.

In the Clifford case the relation between "basis vectors" and their Hermitian conjugated partners (made of products of nilpotents ($\stackrel{ab}{(k)}$ or $\stackrel{ab}{(\tilde{k})}$) and projectors ($\stackrel{ab}{[k]}$ or $\stackrel{ab}{[\tilde{k}]}$), Eq. (30), are less transparent (although still easy to be evaluated). This can be noticed in Eq. (30), since for nilpotents we notice that $\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)^\dagger = \eta^{aa}\,\frac{1}{\sqrt{2}}(\gamma^a + \frac{\eta^{aa}}{i\,(-k)}\gamma^b)$, while for projectors we find that they are self adjoint objects, $(\frac{1}{\sqrt{2}}(1+\frac{i}{k}\gamma^a\gamma^b))^\dagger = \frac{1}{\sqrt{2}}(1+\frac{i}{k}\gamma^a\gamma^b)$. This is the case also for representations in the sector of $\tilde{\gamma}^a$'s.

One easily sees that in even dimensional spaces, either in $d=2(2n+1)$ or in $d=4n$, the Clifford odd "basis vectors" (they are products of an odd number of nilpotents and an even number of projectors) have their Hermitian conjugated partners in another irreducible representation, since Hermitian conjugation changes an odd number of nilpotents (changing at the same time the handedness of the "basis vectors"), while the generators of the Lorentz transformations change two nilpotents at the same time (keeping the handedness unchanged).

The Clifford even "basis vectors" have an even number of nilpotents and can have an odd or an even number of projectors. Correspondingly an irreducible representation of even "basis vectors" have among "basis vectors" the product of projectors only, which is therefore selfadjoint.

Taking into account Eq. (30) one finds that the product of Hermitian conjugated partner of a "basis vector" from Eq. (31) and the corresponding "basis vector" is nonzero and normalized to identity, as presented in Eq. (32) $(\stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)})$. This is the case for any irreducible representation in the case of the Clifford algebra of γ^a's as well in the Clifford algebra of $\tilde{\gamma}^a$'s.

Statement 2. The product of any Hermitian conjugated "basis vector" with the corresponding "basic vector" is the same for all the members of an irreducible representation.

The proof is presented in App. I, Statement 4.

Statement 3. Changing a pair of any two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k')}$ to a pair of projectors with the same (k,k') $\stackrel{ab}{[k]}\stackrel{cd}{[k']}$, or of any two projectors $\stackrel{ab}{[k']}\stackrel{cd}{[k'']}$ two a pair of two nilpotents with the same (k',k''), or a pair of any nilpotent (projector) and projector (nilpotent) $\stackrel{ab}{(k)}\stackrel{cd}{[k']}$ to a pair of the projector (nilpotent) and the nilpotent (projektor) with the same (k,k') $\stackrel{ab}{[k]}\stackrel{cd}{(k')}$, the obtained "basis vector" belongs to different irreducible representations., since the Lorentz transformations S^{ce} transform $\stackrel{ab}{(k)}\stackrel{cd}{(k')}$ into $\stackrel{ab}{[-k]}\stackrel{cd}{[-k']}$, never to $\stackrel{ab}{[k]}\stackrel{cd}{[k']}$, and equivalently for other pairs.

This statement is proven in App. I, Statement 2b. Due to Eq. (29) the irreducible representations are equivalent with respect to the eigenvalues of the Cartan subalgebra members and also with respect to the generators of the Lorentz transformations S^{ab}.

Statement is valid also if we replace γ^a with $\tilde{\gamma}^a$.

It can be checked, however, that there are Hermitian conjugated partners of the "basis vectors" of different irreducible representations, which applying on $\stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}$ from the left hand side, give also nonzero contributions, not the identity. Like it is $\stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}$, giving (due to Eq. (30)) $\stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{[-]}$. The "basis vector" of $\stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(-)}$ is $\stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(+)}$, belonging to another irreducible representations.

There are several other choices, like

$$\begin{array}{c}\stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,\\ \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{(-)} *_A \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,,\end{array} \tag{33}$$

which also give nonzero contributions duo to Eq. (30).

Let us recognize:

i. The two Clifford spaces, the one spanned by γ^a's and the second one spanned by $\tilde{\gamma}^a$'s, are independent vector spaces, each with 2^d "basis vectors" (App. I, Statement 1.).

ii. The Clifford odd "basis vectors" (the superposition of products of odd numbers of γ^a's or of $\tilde{\gamma}^a$'s, respectively) can be arranged for each kind of the Clifford algebras into two groups, Hermitian conjugated to each other, of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ irreducible representations of the corresponding Lorentz group.

iii. Different irreducible representations are indistinguishable with respect to the "eigenvalues" of the corresponding Cartan subalgebra members, Statement 4.

iv. The Clifford even part (made of superposition of products of even numbers of γ^a's or $\tilde{\gamma}^a$'s, respectively) splits as well into twice $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ irreducible representations of the Lorentz group. One member of each Clifford even representation, the one which is the product of projectors only, is self adjoint. Members of one irreducible representation are with respect to the Cartan subalgebra indistinguishable from all the other irreducible representations for each of the two algebras.

v. The odd $2^{\frac{d}{2}-1}$ members of each of the $2^{\frac{d}{2}-1}$ irreducible representations are among themselves orthogonal and so are orthogonal $2^{\frac{d}{2}-1}\cdot 2^{\frac{d}{2}-1}$ members of their Hermitian conjugated part. For illustration of the orthogonality one can look at Table 3, and recognizes that any "basis vector" of the first four multiplets (*odd I, II, III, IV*), the third column, gives zero, if it is multiplied from the left hand side or from the right hand side with any other "basis vector" from the third column, according to Eq. (30). The same is true for any "basis vector" from the fourth column, Hermitian conjugated to the third column.

Generalization to any even dimension d is straightforward.

vi. Let us denote the Clifford odd "basis vectors" of the Clifford algebra γ^a by $\hat{b}^{m\dagger}_f$, where f defines different irreducible representations and m a member in the representation f. Then their Hermitian conjugate partners is denoted by $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$, let us start for $d = 2(2n+1)$.

Let us repeat *Statement 2.*: The algebraic product of $\hat{b}^m_{f} {}_{*_A} \hat{b}^{m\dagger}_f$ is the same for all the members of one irreducible representation.

vii. Let us look for the vacuum state $|\psi_{oc}>$ as the sum of the products $\hat{b}^m_{f} {}_{*_A} \hat{b}^{m\dagger}_f$ over all the irreducible representations

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^m_{f} {}_{*_A} \hat{b}^{m\dagger}_f\,|\,1\;>\,, \tag{34}$$

for one of the members m, anyone, of the odd irreducible representation f, with $|\,1\;>$, which is the vacuum without any structure ($\hat{b}^m_f|\,1\;>= 0$).

If we make a choice of the starting member of the starting irreducible representation as

$$\begin{aligned} \hat{b}^{m=1\dagger}_{f=1}: &= \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, \\ (\hat{b}^{m=1\dagger}_{f=1})^{\dagger} = \hat{b}^{m=1}_{f=1}: &= \stackrel{d-1\,d}{(-)}\cdots\stackrel{12}{(-)}\stackrel{03}{(-i)}\,, \end{aligned} \tag{35}$$

we recognize for the "basis vectors" of an odd Clifford character for each of the two Clifford algebras the properties

$$\begin{aligned} \hat{b}^m_{f\;*_A}|\psi_{oc}> &= 0\,|\psi_{oc}>\,, \\ \hat{b}^{m\dagger}_{f\;*_A}|\psi_{oc}> &= |\psi^m_f>\,, \\ \{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\,|\psi_{oc}>\,, \\ \{\hat{b}^{m\dagger}_f, \hat{b}^{m\dagger}_f\}_{*_A+}|\psi_{oc}> &= |\psi_{oc}>\,, \end{aligned} \tag{36}$$

where $*_A$ represents the algebraic multiplication of $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m'}_{f'}$ among themselves and with the vacuum state $|\psi_{oc}>$ of Eq.(34), which takes into account Eq. (17).

All the products of Clifford algebra elements are up to now the algebraic ones and so are also the products in Eq. (36). Since we use here anticommutation relations, we want to point out with $_{*_A}$ this algebraic character of the products, to be later distinguished from the tensor product $_{*_T}$, when the creation and annihilattion operators are defined on an extended basis, which is the tensor product of the superposition of the "basis vectors" of the Clifford space and of the momentum basis, applying on the Hilbert space of "Slater determinants". The tensor product $*_{T_H}$ will be used as the product mapping a pair of the fermion wave functions in to two fermion wave functions and further to many fermion wave functions — that is to the extended algebra of many fermion system.

Obviously, $\hat{b}^{m\dagger}_f$ and $\hat{b}^m_f$ have on the level of the algebraic products, when applying on the vacuum state $|\psi_{oc}>$, *almost* the properties of creation and annihilation operators of the second quantized fermions in the postulates of Dirac, as it is discussed in the next items. We illustrate properties of "basis vectors" and their Hermitian conjugated partners on the example of $d = (5+1)$-dimensional space in Sect. 3.2.3.

viii. There is, namely, the property, which the second quantized fermions should fulfill in addition to the relations of Eq. (36). The anticommutation relations of creation and annihilation operators should include:

$$\{\hat{b}^m_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> \;=\; \delta^{mm'}\delta_{ff'}|\psi_{oc}> \;. \tag{37}$$

For any $\hat{b}^m_f$ and any $\hat{b}^{m'\dagger}_{f'}$ this is not the case, as we demonstrated in Eq. (33) (besides $\hat{b}^{m=1}_{f=1} = \stackrel{d-1\,d}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{03}{(-i)}$, for example, also $\hat{b}^{m'}_{f'} = \stackrel{d-1\,d}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{03}{[+i]}$ and several others give, when applied on $\hat{b}^{m=1\dagger}_{f=1}$, nonzero contributions). There are namely $2^{\frac{d}{2}-1}-1$ too many annihilation operators for each creation operator, which give, applied on the creation operator, nonzero contribution.

viii. a. To use the Clifford algebra objects to describe second quantized fermions, representing the observed quarks and leptons as well as the antiquarks and antileptons [12, 13, 15, 16, 17, 19, 18, 20], *the families should exist.*

viii. b. The operators should exist, which connect one irreducible representation of fermions with all the other irreducible representations.

viii. c. Two independent choices for describing the internal degrees of freedom of the observed quarks and leptons are not in agreement with the observed properties of fermions.

We solve these problems, cited in viii. a., viii. b., viii. c., by reducing the degrees of freedom offered by the two kinds of the Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s, making a choice of one — γ^a's — to describe the internal space of fermions, and using the other one — $\tilde{\gamma}^a$'s — to describe the "family" quantum number of each irreducible representation of S^{ab}'s in space defined by γ^a's.

3.2.3 Reduction of Clifford and Grassmann space and appearance of family quantum number

The creation and annihilation operators of an odd Clifford algebra of both kinds, of either γ^a's or $\tilde{\gamma}^a$'s, would obviously obey the anticommutation relations for the second quantized fermions, postulated by Dirac, at least on the vacuum state, which is a sum of all the products of annihilation times, $*_A$, the corresponding creation operators, provided that each of the irreducible representations would carry a different quantum number.

But we know that a particular member m has for all the irreducible representations the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either γ^a's or $\tilde{\gamma}^a$'s), Eq. (30).

The only possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us say "family" quantum number, is that we "sacrifice" one of the two vector spaces, let us make a choice of $\tilde{\gamma}^a$'s, *and use* $\tilde{\gamma}^a$ *'s to define the "family" quantum number for each irreducible representation of the vector space of* γ^a's, while *keeping the relations of* Eq. (17) *unchanged:* $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, $(\gamma^a)^\dagger = \eta^{aa}\,\gamma^a$, $(\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a$, $(a,b) = (0,1,2,3,5,\cdots,d)$.

We therefore *postulate*:
Let $\tilde{\gamma}^a$'s operate on γ^a's as follows [62, 8, 17, 19, 20]

$$\{\tilde{\gamma}^a B \;=\; (-)^B\, i\, B\gamma^a\}\, |\psi_{oc}>\,, \tag{38}$$

with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [62], $|\psi_{oc}>$ is defined in Eq. (34).

After this postulate the vector space of $\tilde{\gamma}^a$'s is "frozen out". No vector space of $\tilde{\gamma}^a$'s needs to be taken into account any longer, in agreement with the observed properties of fermions. This solves the problems viii.a - viii. c. of Sect. 3.2.2.

Taking into account Eq. (38) we can check that (App. I, Statement 3, 3a, 3b):

a. Relations of Eq. (17) remain unchanged [6].

b. Relations of Eq. (7) remain unchanged [7].

c. The eigenvalues of the operators S^{ab} and $\tilde{S}^{ab}$ on nilpotents and projectors of γ^a's are after the reduction of Clifford space equal to

$$S^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}\,, \qquad \tilde{S}^{ab} \stackrel{ab}{(k)} = \frac{k}{2} \stackrel{ab}{(k)}\,,$$
$$S^{ab} \stackrel{ab}{[k]} = \frac{k}{2} \stackrel{ab}{[k]}\,, \qquad \tilde{S}^{ab} \stackrel{ab}{[k]} = -\frac{k}{2} \stackrel{ab}{[k]}\,, \tag{39}$$

[6] Let us show that the relation $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2\eta^{ab}$ remains valid when applied on B, if B is either an odd or an even product of γ^a's: $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\, \gamma^c = -i\,(\tilde{\gamma}^a\gamma^c\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^a) = -i\,i\,\gamma^c(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c$, while $\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\, \gamma^c\gamma^d = i\,(\tilde{\gamma}^a\gamma^c\gamma^d\gamma^b + \tilde{\gamma}^b\gamma^c\gamma^d\gamma^a) = i(-i)\,\gamma^c\gamma^d(\gamma^b\gamma^a + \gamma^a\gamma^b) = 2\eta^{ab}\gamma^c\gamma^d$. The relation is valid for any γ^c and γ^d, even if $c = d$.

[7] One easily checks that $\tilde{\gamma}^{a\dagger}\gamma^c = -i\gamma^c\gamma^{a\dagger} = -i\eta^{aa}\gamma^c\gamma^a = \eta^{aa}\tilde{\gamma}^a\gamma^c = -i\eta^{aa}\gamma^c\gamma^a$.

demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors of γ^a's differ from the eigenvalues of $\tilde{S}^{ab}$, so that $\tilde{S}^{ab}$ can be used to denote irreducible representations of S^{ab} with the "family" quantum number, what solves the problems viii. a. and viii. b. of Subsubsect. 3.2.2.

d. We further recognize that γ^a transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[-k]}$, never to $\stackrel{ab}{[k]}$, while $\tilde{\gamma}^a$ transform $\stackrel{ab}{(k)}$ into $\stackrel{ab}{[k]}$, never to $\stackrel{ab}{[-k]}$

$$
\begin{array}{llll}
\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}, & \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]}, & \gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)}, & \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}\,, \\
\tilde{\gamma^a} \stackrel{ab}{(k)}= -i\eta^{aa} \stackrel{ab}{[k]}, & \tilde{\gamma^b} \stackrel{ab}{(k)}= -k \stackrel{ab}{[k]}, & \tilde{\gamma^a} \stackrel{ab}{[k]}= \; i \stackrel{ab}{(k)}, & \tilde{\gamma^b} \stackrel{ab}{[k]}= -k\eta^{aa} \stackrel{ab}{(k)}\;.
\end{array}
\tag{40}
$$

e. One finds, using Eq. (38),

$$
\begin{array}{lllll}
\stackrel{ab}{\tilde{(k)}} \stackrel{ab}{(k)} &=& 0\,, & \stackrel{ab}{\tilde{(-k)}} \stackrel{ab}{(k)}= -i\,\eta^{aa} \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{\tilde{(k)}} \stackrel{ab}{[k]}= i\, \stackrel{ab}{(k)}\,, & \stackrel{ab}{\tilde{(k)}} \stackrel{ab}{[-k]}= 0\,, \\
\stackrel{ab}{\tilde{[k]}} \stackrel{ab}{(k)} &=& \stackrel{ab}{(k)}\,, & \stackrel{ab}{\tilde{[-k]}} \stackrel{ab}{(k)}= 0\,, \quad \stackrel{ab}{\tilde{[k]}} \stackrel{ab}{[k]}= 0\,, & \stackrel{ab}{\tilde{[-k]}} \stackrel{ab}{[k]}= \stackrel{ab}{[k]}\;.
\end{array}
\tag{41}
$$

f. From Eq. (40) it follows

$$
\begin{array}{lll}
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &=& -\frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, \qquad S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]}= \frac{i}{2} \stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,, \\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{(k)} &=& \frac{i}{2}\eta^{aa}\eta^{cc} \stackrel{ab}{[k]}\stackrel{cd}{[k]}\,, \qquad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{[k]}= -\frac{i}{2} \stackrel{ab}{(k)}\stackrel{cd}{(k)}\,, \\
S^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &=& -\frac{i}{2}\eta^{aa} \stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, \qquad S^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)}= \frac{i}{2}\eta^{cc} \stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\,, \\
\tilde{S}^{ac} \stackrel{ab}{(k)}\stackrel{cd}{[k]} &=& -\frac{i}{2}\eta^{aa} \stackrel{ab}{[k]}\stackrel{cd}{(k)}\,, \qquad \tilde{S}^{ac} \stackrel{ab}{[k]}\stackrel{cd}{(k)}= \frac{i}{2}\eta^{cc} \stackrel{ab}{(k)}\stackrel{cd}{[k]}\;.
\end{array}
\tag{42}
$$

g. Each irreducible representation has now the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (8). Correspondingly the creation and annihilation operators fulfill algebraically the anticommutation relations, postulated by Dirac for the second quantized fermions, since: **g.a** Different irreducible representations carry different "family" quantum numbers and to each "family" quantum member only one Hermitian conjugated partner with the same "family" quantum number belongs. **g.b** Each summand of the vacuum state, Eq. (34), belongs to a particular "family". This solves the problem viii. a. - viii. c. of Sect. 3.2.2.

The anticommutation relations of Dirac, postulated for fermions, are fulfilled on the vacuum state, Eq. (34), on the algebraic level, without postulating them. They follow by themselves due to the fact that the creation and annihilation operators are superposition of odd products of γ^a's.

Statement 5. The oddness of superpositions of products of γ^a's, representing the "basis vectors", guarantees the anticommuting properties of objects which are tensor, $*_T$, products of these "basis vectors" and the (commuting) basis of the ordinary space.

We show in Sect. 4 that solutions of equations of motion must be the tensor, $*_T$, product of the finite number of the "basis vectors", $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ (determining properties of the internal space of fermions), and the (continuously) infinite number of basis vectors in ordinary space. The corresponding creation and annihilation operators fulfill the Dirac's postulates for the second quantized fermions. We shall see that for positive energy $p^0 = |\vec{p}|$ the solutions of equations of motion for massless fermions of particular handedness and charges the solutions for their antifermions of opposite charges and handedness with $-\vec{p}$ correspond, reducing the (continously infinite) number of creation operators for the factor of 2.

In Sect. 5 we show that the anticommuting property of creation and annihilation operators manifest also on the Hilbert space, formed as the tensor products, $*_{T_H}$, of all possible numbers of all possible fermion and antifermion states solving the equations of motion, the creation operators of which are the tensor products, $*_T$, of the finite number, $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$, of Clifford odd "basis vectors" and of the continuous infinite basis of the ordinary (momentum or coordinate) space, applying on the vacuum state.

Let us write down the anticommutation relations of Clifford odd "basic vectors", representing the creation operators in internal space of fermions with half integer spin represented by γ^a's and of the corresponding Hermitian conjugated partners annihilation operators again. After the reduction of the Clifford algebra any irreducible representation carry the family quantum number, distinguishing families among themselves.

$$
\begin{aligned}
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> &= \delta^{mm'}\,\delta_{ff'}\,|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{*_A+}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_{f}\,{}_{*_A}|\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\hat{b}^{m}_{f}\,{*_A}|\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,
\end{aligned}
\tag{43}
$$

with (m,m') denoting the "family" members and (f,f') denoting "families", $*_A$ represents the algebraic multiplication of $\hat{b}^{\dagger m}_{f}$ and $\hat{b}^{m}_{f}$ with the vacuum state $|\psi_{oc}>$ of Eq. (34) and among themselves, taking into account Eq. (17).

h. Let us make a choice of the starting creation operator $\hat{b}^{m=1\dagger}_{f=1}$ of an odd Clifford character and correspondingly of its Hermitian conjugated partner in $d=2(2n+1)$ and $d=4n$, respectively, as we start in Eq. (35), but now, after reduction of the two kinds of the Clifford algebras into only one, each irreducible representation carries the family quantum number.

$$
\begin{aligned}
\hat{b}^{1\dagger}_{1}: &= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\;\stackrel{d-1\,d}{(+)}\,,\\
(\hat{b}^{1\dagger}_{1})^{\dagger} &= \hat{b}^{1}_{1}:=\stackrel{d-1\,d}{(-)}\;\stackrel{d-3\,d-2}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,,\\
d &= 2(2n+1)\,,\\
\hat{b}^{1\dagger}_{1}: &= \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\;\stackrel{d-1\,d}{[+]}\,,\\
(\hat{b}^{1\dagger}_{1})^{\dagger} &= \hat{b}^{1}_{1}:=\stackrel{d-1\,d}{[+]}\;\stackrel{d-3\,d-2}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{(-)}\stackrel{01}{(-i)}\,,\\
d &= 4n\,.
\end{aligned}
\tag{44}
$$

All the rest creation operators in the internal space described by γ^a's (the "basis vectors"), belonging to the same Lorentz representation, follow by the application of the Lorentz generators S^{ab}'s.

i. The representations with different "family" quantum numbers are reachable by $\tilde{S}^{ab}$, since, according to Eq. (42), we recognize that $\tilde{S}^{ac}$ transforms two nilpotents $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into two projectors $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$, without changing k ($\tilde{S}^{ac}$ transforms $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$ into $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$, as well as $\stackrel{ab}{[k]}\stackrel{cd}{(k)}$ into $\stackrel{ab}{(k)}\stackrel{cd}{[k]}$). All the "family" members are reachable from one member of a new family by the application of S^{ab}'s.

In this way, by starting with the creation operator $\hat{b}^{1\dagger}_{1}$, Eq. (44), $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members follow.

Let us find the starting member of the next "family" to the "family" of Eq. (44) by the application of $\tilde{S}^{01}$

$$
\begin{aligned}
\hat{b}^{1\dagger}_{2}: &= \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{(+)}\;\stackrel{d-1\,d}{(+)}\,,\\
\hat{b}^{1}_{2}: &= \stackrel{d-1\,d}{(-)}\;\stackrel{d-3\,d-2}{(-)}\cdots\stackrel{56}{(-)}\stackrel{12}{[+]}\stackrel{01}{[+i]}\;. \qquad (45)
\end{aligned}
$$

The corresponding annihilation operators, that is the Hermitian conjugated partners of $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ "family" members, following from the starting creation operator $\hat{b}^{1\dagger}_{1}$ by the application of S^{ab}'s — the family members — and the application of $\tilde{S}^{ab}$ — the same family member of another family — can be obtained by Hermitian conjugation.

The creation and annihilation operators of an odd Clifford character, expressed by nilpotents and projectors of γ^a's, obey anticommutation relations of Eq. (43), *without postulating the second quantized anticommutation relations.*

j. The vacuum state $|\psi_{oc}>$ for the vector space determined by γ^a's, which is the sum over family quantum numbers of products of an annihilation operator with its Hermitian conjugated partner creation operator of any family member m, $\sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f} {}_{*A} \hat{b}^{m\dagger}_{f}$.

$$
\begin{aligned}
|\psi_{oc}> &= \frac{1}{\sqrt{2^{\frac{d}{2}-1}}}\Big(\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]} + \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}\\
&+ \stackrel{03}{[+i]}\stackrel{12}{[-]}\stackrel{56}{[+]}\cdots\stackrel{d-1\,d}{[-]} + \cdots\Big)|1>\,,\\
&\text{for } d = 2(2n+1)\,,\\
|\psi_{oc}> &= \frac{1}{\sqrt{2^{\frac{d}{2}-1}}}\Big(\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{35}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[+]}\\
&+ \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\;\stackrel{d-1\,d}{[+]} + \cdots\Big)|1>\,,\\
&\text{for } d = 4n\,,\quad n \text{ is a positive integer}\,. \qquad (46)
\end{aligned}
$$

The Hermitian conjugated part of the space in the Grassmann case is "freezed out" together with the "basis vector" space of $\tilde{\gamma}^a$'s.

The even partners of the Clifford odd creation and annihilation operators follow by either the application of γ^a on the creation operators, leading to $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members, or with the application of $\tilde{\gamma}^a$ on the creation operators, leading to another group of the Clifford even operators, again with the $2^{\frac{d}{2}-1}$ "families", each with $2^{\frac{d}{2}-1}$ members.

It is not difficult to recognize, that each of the Clifford even "families", obtained by the application of γ^a or by $\tilde{\gamma}^a$ on the creation operators, contains one selfadjoint operator, which is the product of projectors only, contributing as a summand to the vacuum state, Eq. (46).

Clifford half integer spin fermions in $d = (5+1)$:

We illustrate properties of the Clifford odd, and correspondingly anticommuting, creation and their Hermitian conjugated partners annihilation operators, belonging to $2^{\frac{6}{2}-1} = 4$ "families", each with $2^{\frac{6}{2}-1} = 4$ members in $d = (5+1)$-dimensional space. The spin in the fifth and the sixth dimension manifests as the charge in $d = (3+1)$.

Half of the Clifford odd "basis vectors" are creation operators $\hat{b}^{m\dagger}_{f}$, appearing in Table 3 in the fourth column. Denoted by *odd I*, *odd II*, *odd III* and *odd IV* they represent four "families", each family having four members. Their Hermitian conjugated partners $\hat{b}^{m}_{f}$ are presented in the fifth column.

All the families have the same eigenvalues of the Cartan subalgebra members (S^{03}, S^{12}, S^{56}), the handedness $\Gamma^{(3+1)} = i(2i)^2 S^{03} S^{12}$, written in the sixth, seventh, eighth and ninth column.

Table 3: The basic creation operators — $\hat{b}^{m=(ch,s)\dagger}_{f}$ (ch (charge), the eigenvalue of S^{56}, and s (spin), the eigenvalues of S^{03} and S^{12}, explain the index m) — and their annihilation partners — $\hat{b}^{m=(ch,s)}_{f}$ — are presented for $d=(5+1)$-dimensional case. Each basic creation operator is a product of projectors and an odd product of nilpotents, and is the "eigenstate" of the Cartan subalgebra members, (S^{03}, S^{12}, S^{56}). The eigenvalues of ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), Eq. (8), presented in the last three columns of the table, determine the family quantum numbers. Operators $\hat{b}^{m=(ch,s)\dagger}_{f}$ and $\hat{b}^{m=(ch,s)}_{f}$ fulfill the anticommutaion relations of Eq. (36).

family f	m	$=(ch,s)$	$\hat{b}^{m=(ch,s)\dagger}_{f}$	$\hat{b}^{m=(ch,s)}_{f}$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
I	1	$(\frac{1}{2},\frac{1}{2})$	$\overset{03}{(+i)}\overset{12}{(+)}\mid\overset{56}{(+)}$	$(-)\overset{56}{(-)}\mid(-)\overset{12}{(-)}\overset{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	2	$(\frac{1}{2},-\frac{1}{2})$	$\overset{03}{[-i]}\overset{12}{[-]}\mid\overset{56}{(+)}$	$(-)\overset{56}{(-)}\mid\overset{12}{[-]}\overset{03}{[-i]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	3	$(-\frac{1}{2},\frac{1}{2})$	$\overset{03}{[-i]}\overset{12}{(+)}\mid\overset{56}{[-]}$	$\overset{56}{[-]}\mid(-)\overset{12}{(-)}\overset{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
I	1	$(-\frac{1}{2},-\frac{1}{2})$	$\overset{03}{(+i)}\overset{12}{[-]}\mid\overset{56}{[-]}$	$\overset{56}{[-]}\mid\overset{12}{[-]}\overset{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
II	1	$(\frac{1}{2},\frac{1}{2})$	$\overset{03}{[+i]}\overset{12}{[+]}\mid\overset{56}{(+)}$	$(-)\overset{56}{(-)}\mid\overset{12}{[+]}\overset{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	2	$(\frac{1}{2},-\frac{1}{2})$	$\overset{03}{(-i)}\overset{12}{(-)}\mid\overset{56}{(+)}$	$(-)\overset{56}{(-)}\mid(-)\overset{12}{(+)}\overset{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	3	$(-\frac{1}{2},\frac{1}{2})$	$\overset{03}{(-i)}\overset{12}{[+]}\mid\overset{56}{[-]}$	$\overset{56}{[-]}\mid\overset{12}{[+]}\overset{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
II	4	$(-\frac{1}{2},-\frac{1}{2})$	$\overset{03}{[+i]}\overset{12}{(-)}\mid\overset{56}{[-]}$	$\overset{56}{[-]}\mid(-)\overset{12}{(+)}\overset{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
III	1	$(\frac{1}{2},\frac{1}{2})$	$\overset{03}{[+i]}\overset{12}{(+)}\mid\overset{56}{[+]}$	$\overset{56}{[+]}\mid(-)\overset{12}{(-)}\overset{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	2	$(\frac{1}{2},-\frac{1}{2})$	$\overset{03}{(-i)}\overset{12}{[-]}\mid\overset{56}{[+]}$	$\overset{56}{[+]}\mid\overset{12}{[-]}\overset{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	3	$(-\frac{1}{2},\frac{1}{2})$	$\overset{03}{(-i)}\overset{12}{(+)}\mid\overset{56}{(-)}$	$(-)\overset{56}{(+)}\mid(-)\overset{12}{(-)}\overset{03}{(+i)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
III	4	$(-\frac{1}{2},-\frac{1}{2})$	$\overset{03}{[+i]}\overset{12}{[-]}\mid\overset{56}{(-)}$	$(-)\overset{56}{(+)}\mid\overset{12}{[-]}\overset{03}{[+i]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
IV	1	$(\frac{1}{2},\frac{1}{2})$	$\overset{03}{(+i)}\overset{12}{[+]}\mid\overset{56}{[+]}$	$\overset{56}{[+]}\mid\overset{12}{[+]}\overset{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	2	$(\frac{1}{2},-\frac{1}{2})$	$\overset{03}{[-i]}\overset{12}{(-)}\mid\overset{56}{[+]}$	$\overset{56}{[+]}\mid(-)\overset{12}{(+)}\overset{03}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	3	$(-\frac{1}{2},\frac{1}{2})$	$\overset{03}{[-i]}\overset{12}{[+]}\mid\overset{56}{(-)}$	$(-)\overset{56}{(+)}\mid\overset{12}{[+]}\overset{03}{[-i]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
IV	4	$(-\frac{1}{2},-\frac{1}{2})$	$\overset{03}{(+i)}\overset{12}{(-)}\mid\overset{56}{(-)}$	$(-)\overset{56}{(+)}\mid\overset{12}{(+)}\overset{03}{(-i)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

The rest half of the Clifford odd "basis vectors" are their Hermitian conjugated partners $\hat{b}^{m}_{f}$, presented in the fifth column of Table 3. The last three columns of this table determine the family quantum numbers, the eigenvalues of ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$), the same for all family members.

The normalized vacuum state is the product of $\hat{b}^{m}_{f}\cdot\,\hat{b}^{m\dagger}_{f}$, summed over four families.

$$
\begin{aligned}
|\psi_{oc}> \;&=\; \frac{1}{\sqrt{2^{\frac{6}{2}-1}}}\,(\overset{03}{[-i]}\overset{12}{[-1]}\overset{56}{[-1]} \;+\overset{03}{[+i]}\overset{12}{[+1]}\overset{56}{[-1]} \\
&+\;\; \overset{03}{[+i]}\overset{12}{[-1]}\overset{56}{[+1]} + \overset{03}{[-i]}\overset{12}{[+1]}\overset{56}{[+1]}). \qquad (47)
\end{aligned}
$$

One easily checks, by taking into account Eq. (41), that the creation operators $\hat{b}^{m\dagger}_{f}$ and the annihilation operators $\hat{b}^{m}_{f}$ fulfill the anticommutation relations of Eq (43) [8].

All the Clifford even "families" with "family" members of Table 3 can be obtained as algebraic products, $*_A$, of the Clifford odd "basis vectors" of Table 3 ([25], Table I).

Let us notice that Clifford algebra "basis vectors" $\hat{b}^{m\dagger}_{f}$ (forming the creation operators and their Hermitian conjugated partners annihilation operators in the internal space of fermions) offer in $d=(3+1)$ the description of only the spin and family degrees of freedom, as can be seen in App. (B.2), while in $d\geq 5$ the Clifford "basis vectors" enables the description of additional families and also of charges, as seen in Table 3 and in Refs. ([94, 12, 13, 19] and the references therein).

[8] If we would choose for creation operators those from the fifth column, then the annihilation operators would be the present creation operators. The vacuum state would be the sum of products of the present creation operators of the fourth column times the present annihilation operators of the fifth column.

3.3 Tensor products of "basis vectors" of internal space and basis in ordinary space

So far we have treated only the internal space of fermions either in the ordinary description of the internal space, Sect. 3.1, or when using the Grassmann algebra, Sect. 3.2.1, or when using the Clifford algebra, Sect. 3.2.2.

The second quantized fermion fields must include beside the "basis vectors", describing the internal space of fermions also the basis in ordinary space. We write the basis in both spaces as tensor products, $*_T$, of the basis in the internal space of fermions, created by $\hat{b}^{s\dagger}_{f}$, which are anticommuting objects (operators), and of the basis in ordinary space, created by $\hat{b}^{\dagger}_{\vec{p}}$, which are commuting objects, since momenta are commuting objects (operators)

$$\{\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) \;\; = \;\; \hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{s\dagger}_{f}\}\,|\psi_{oc}> *_T |0_{\vec{p}}>\,, \tag{48}$$

where $\vec{p}$ determines the momentum in ordinary space and s determines all the rest of quantum numbers. The state written here as $|\psi_{oc}> \; *_T |0_{\vec{p}}>$ is considered as the vacuum for a starting single particle states from which one obtains the other single particle states by the operators, like $\hat{b}_{\vec{p}}$, which pushes the momentum by an amount $\vec{p}$.

The meaning of the operator $\hat{b}^{\dagger}_{\vec{p}}$, which translates (generates) all the momenta from the starting one, is explained and justified in App. J. Although the operator for translation of the momentum or coordinate is well known, we follow to quite a large extent Ref. [135], the operator $\hat{b}^{\dagger}_{\vec{p}}$ can not be found in the literature, at least not to our knowledge. We postulated it in App. J to make easier to present the relationship between the ordinary second quantization procedure and our new one defined in Eq. (48).

The dimension of space-time can be any $d = (d-1)+1$, although usually is taken the observed one $d = (3+1)$. Quantum numbers of fermions are denoted by s. In the usual second quantization procedure for free massless fermions the attention is payed only on the spin and handedness. Since the Euler-Lagrange equations relate p_0 and $\vec{p}$, $(p^0)^2 = (\vec{p})^2$, the continuously infinite basis in the momentum space, created by $\hat{b}^{\dagger}_{\vec{p}}$, depends on only $\vec{p}$, $|0_{\vec{p}}>$ denotes the vacuum state on which $\hat{\mathbf{b}}^{\dagger}(\vec{p})$ applies.

In App. J we discuss the creation and annihilation operators $\hat{b}^{\dagger}_{\vec{p}}$ and $\hat{b}_{\vec{p}}$ in the case of continuous spectra in ordinary momentum and coordinate space in details. Here we only present the most needed definitions.

Let us repeat the relations concerning the momentum part of the single fermion states.

$$\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p>\,, \quad <\vec{p}\,| = <0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') = <0_p\,|\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p}'}|\,0_p>\,,\\
&\text{leading to}\\
\hat{b}_{\vec{p}'}\,\hat{b}^{\dagger}_{\vec{p}} &= \delta(\vec{p}'-\vec{p})\,,
\end{aligned} \tag{49}$$

since we normalize $<0_p\,|\,0_p>=1$ to identity. Let us repeat Eq. (212) from App. J

$$\begin{aligned}
<\vec{p}\,|\,\vec{x}> &= <0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}> = (<0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>)^{\dagger}\\
\{\hat{b}^{\dagger}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p}'}\}_- &= 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p}'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p}'}\}_- = 0\,,\\
\{\hat{b}^{\dagger}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}'}\}_- &= 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x}'}\}_- = 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}'}\}_- = 0\,,\\
&\text{while}\\
\{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{x}}\}_- &= e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{p}}\}_- = e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,,
\end{aligned} \tag{50}$$

If $\hat{b}^{s\dagger}_{f}$ and their Hermitian conjugated partners do fulfill the anticommuting properties of Eq. (43), then also $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and $(\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}))^{\dagger}$, Eq. (48) fulfill the anticommutation relations of Eq. (43) due the commutativity of operators $\hat{b}^{\dagger}_{\vec{p}} = (\hat{b}^{\dagger}_{-\vec{p}})^{\dagger} = \hat{b}_{-\vec{p}}$.

In usual second quantization procedures the internal space does not have the anticommutation properties. Correspondingly the anticommuting creation and annihilation operators must be postulated [133, 3] separately.

The odd Clifford algebra offers the "basis vectors", which together with their Hermitian conjugated partners fulfill the anticommutation relations of Eq. (43), explaining correspondingly the Dirac's postulates for the second quantized fermions, as we shall see in the next subsubsection.

3.3.1 Creation and annihilation operators in internal and ordinary space in usual second quantization procedure of Dirac

As we see in Sect. 3.1, in Table 1, the internal space of fermions is usually presented with states, denoted by their spins and handedness (in the case of free massless fermions, in the case of free massive fermions the states are superposition of both handedness). These states are treated as commuting vectors, and do not have anticommuting character. Correspondingly the anticommuting creation and annihilation operators must be postulated.

Let us follow Ref. [3]. In the Dirac case the second quantized field operators in $d = (3{+}1)$ dimensions, $\mathbf{\Psi}^{hs\dagger}(\vec{x}, x^0)$, assumed to fulfill anticommutation relations

$$\begin{aligned}
\{\mathbf{\Psi}^{hs\dagger}(\vec{x}, x^0),\, \mathbf{\Psi}^{h's'\dagger}(\vec{x'}, x^0)\}_+ &= 0 = \{\mathbf{\Psi}^{hs}(\vec{x}, x^0),\, \mathbf{\Psi}^{h's'}(\vec{x'}, x^0)\}_+ \,,\\
\{\mathbf{\Psi}^{hs}(\vec{x}, x^0),\, \mathbf{\Psi}^{h's'\dagger}(\vec{x'}, x^0)\}_+ &= \delta_{hh'}\delta^{ss'}\delta(\vec{x}-\vec{x'})\,,
\end{aligned} \tag{51}$$

are postulated as follows

$$\mathbf{\Psi}^{hs}(\vec{x}, x^0) = \sum_{m,\vec{p}_k} \hat{\mathbf{a}}^{hs\dagger}_m(\vec{p}_k, x^0)\, v^{hs}_m(\vec{p}_k, \vec{x}, x^0)\,, \quad \mathbf{\Psi}^{hs\dagger}(\vec{x}, x^0) = \sum_{m,\vec{p}_k} \hat{\mathbf{a}}^{hs}_m(\vec{p}_k, x^0)\, v^{hs\dagger}_m(\vec{p}_k, \vec{x}, x^0)\,. \tag{52}$$

Here $v^{hs\dagger}_m(\vec{p}_k, \vec{x}, x^0) = (v^{hs}_m(\vec{p}_k, \vec{x}, x^0))^*$, with * meaning the complex conjugated values, $v^{hs}_m(\vec{p}_k, \vec{x}, x^0) = u^{hs}_m(\vec{p}_k)\; e^{-i(p^0x^0-\varepsilon\vec{p}_k\cdot\vec{x})}$ are the two left handed ($\Gamma^{(3+1)} = -1 = h$) and the two right handed ($\Gamma^{(3+1)} = 1 = h$) two-component column matrices, $m = (1,2)$, representing twice two solutions s of the Weyl equation for free massless fermions of particular momentum $|\vec{p}_k| = |p^0_k|$ (Ref. [3], Eqs. (20-49) - (20-51)), the factor $\varepsilon = \pm 1$ depends on the product of handedness and spin. (Massive fermions are represented by four-vectors, the superposition of two two-vectors of both handedness.)

Vectors $v^{hs}_m(\vec{p}_k, \vec{x}, x^0)$ are in the case of discretized momenta for a fermion in a box orthogonalized as follows

$$\int d\vec{x} \sum_m v^{hs\dagger}_m(\vec{p}_k, \vec{x}, x^0)\, v^{h's'}_m(\vec{p}_{k'}, \vec{x}, x^0) = \delta_{kk'}\,\delta^{ss'}\,\delta^{hh'}\,. \tag{53}$$

Taking into account the anticommutation relations of Eq. (51), the orthogonality of $v^{hs\dagger}_m(\vec{p}_k, \vec{x}, x^0)$, Eq. (53), the orthogonality of continuous spectra, Eq (49), and Eq. (52), the anticommutation relations (Ref. [3], Eqs. (20-49) - (20-51)) follow for by Dirac postulated creation operators $\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k)$ and their Hermitian conjugated partners annihilation operators $\hat{\mathbf{a}}^{h}_m(\vec{p}_k)$

$$\begin{aligned}
\{\hat{\mathbf{a}}^{h\dagger}_m(\vec{p}_k, x^0),\, \hat{\mathbf{a}}^{h'\dagger}_n(\vec{p}_l, x^0)\}_{*_T+} &= 0 = \{\hat{\mathbf{a}}^{h}_m(\vec{p}_k, x^0),\, \hat{\mathbf{a}}^{h'}_n(\vec{p}_l, x^0)\}_{*_T+}\,,\\
\{\hat{\mathbf{a}}^{h}_m(\vec{p}_k, x^0),\, \hat{\mathbf{a}}^{h'\dagger}_n(\vec{p}_l, x^0)\}_{*_T+} &= \delta_{mn}\,\delta^{hh'}\,\delta_{\vec{p}_k\vec{p}_l}\,.
\end{aligned} \tag{54}$$

In Eq. (48) presented creation operators $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})$ are the tensor products of the basis in internal space of fermions, $\hat{b}^{s\dagger}_f$, and the basis in momentum space $\hat{b}^{\dagger}_{\vec{p}}$. The creation operators $\hat{\mathbf{a}}^{hs\dagger}_m(\vec{p}_k, x^0)$ themselves already have the properties of $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})$. They carry all the needed quantum numbers of $u^{hs}_m(\vec{p}_k)$, while $v^{hs}_m(\vec{p}_k, \vec{x}, x^0)$ "support" $\hat{\mathbf{a}}^{hs\dagger}_m(\vec{p}_k, x^0)$ with $e^{-i(p^0x^0-\varepsilon\vec{p}_k\cdot\vec{x})}$ in addition.

We discuss the continuous momentum basis and the continuous coordinate basis in App. J.

We make in Sect. 3.3.4 the comparison between the above second quantization procedure and our way of second quantization, which is presented in the next Sect. 3.3.2. Let us add that the usual second quantized procedure does not take care of charges and families of fermions, although both can be included by assuming additional quantum numbers and relations among them.

3.3.2 Creation and annihilation operators in internal and ordinary space with "basis vectors" described by Clifford algebra

Our Clifford algebra "basis vectors" $\hat{b}^{m\dagger}_{f}$, the superposition of products of odd number of γ^a's, fulfilling the anticommutation relations of Eq. (43), have in d-dimensional spaces $2^{\frac{d}{2}-1}$ families, each with $2^{\frac{d}{2}-1}$ members. We present in Sect. B.2 the matrix representations of γ^a's, S^{ab}'s and $\tilde{S}^{ab}$'s if space-time is $d=(3+1)$.

Since we observe only $d=(3+1)$ space-time, it means that the $(d-4)$-dimensional space must be directly (dynamically) unobservable, although we do observe the charges of quarks and leptons and of antiquarks and antileptons as superposition of spins in $d \ge (13+1)$ and at least some (so far at least three) of families. Quarks and leptons and antiquarks and antileptons, as well as all the compositions of elementary fields (like mesons and baryons) experience at observable energies the momentum space with $p^a = (p^0, p^1, p^2, p^3, 0, 0, \cdots, 0)$.

To describe quarks and leptons and antiquarks and antileptons we need to take into account besides the internal space of fermions also the momentum or coordinate space with continuously infinite number of basis vectors, as suggested in Eq. (48), where $\hat{\mathbf{b}}^{s\dagger}(\vec{p})$ is the tensor product of the basis in internal space of fermions, $\hat{b}^{s\dagger}$, and the basis in momentum space $\hat{b}^{\dagger}_{\vec{p}}$, as presented in Eq. (49).

Statement 6. To derive the anticommutation relations for the Clifford fermions, which would replace (being comparable with) the anticommutation relations of the second quantized fermions, we need to define the tensor product of the Clifford odd "basis vectors" and the momentum basis first for the single fermion states.

We pay attention here to free massless fields. Since the Euler-Lagrange equations for the second quantized free massless field, Sect. 4, relate the energy p_0 and the momentum $\vec{p}$, we let the creation (and correspondingly annihilation) operators to depend only on $\vec{p}$, with $|p^0| = |\vec{p}|$. Let us therefore define

$$\{\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|_{p^0=|\vec{p}|} \stackrel{\text{def}}{=} \sum_{m} c^{ms}{}_{f}\; (\vec{p}, |p^0|=|\vec{p}|)\; \hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{m\dagger}_{f}\}|\psi_{oc}>|0_{\vec{p}}>\,, \qquad (55)$$

with the coefficients $c^{ms}{}_{f}\; (\vec{p}, |p^0|=|\vec{p}|)$ chosen so, that $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|_{p^0=|\vec{p}|} \cdot e^{-ip_a x^a}$ solve the Weyl equations for free massless fermions, presented in Eq. (79) in Sect. 4 for a chosen momentum $\vec{p}$. The creation operator in momentum space is defined in Eq. (49). The tensor product of Eq. (48) is in Eq. (55) rewrittwn in a more explicit way. In Eq. (213) the relation $<\vec{p}|\psi_{\vec{p}}>= f(\vec{p})$ concerns only the momentum space, while in Eq. (55) both basis are related, the internal "basis vectors" $\hat{b}^{m\dagger}_{f}$ are multiplied by the continuous momentum basis, $*_T \hat{b}_{\vec{p}}|0_{\vec{p}}>$, so that the superposition of both spaces is made, $\sum_m c^{ms}{}_{f}\; (\vec{p}, |p^0| = |\vec{p}|)\hat{b}_{\vec{p}} *_T \hat{b}^{m\dagger}_{f}\}|\psi_{oc>}|0_{\vec{p}}>$. In such superposition the function $f(\vec{p})$ of Eq. (213) in App. J carries here the quantum numbers of the internal space and is replaced by $\sum_m c^{ms}{}_{f}\; (\vec{p}, |p^0| = |\vec{p}|)$.

The new *basis vectors* are the tensor product, $*_T$, of the "basis vectors" in the internal space of fermions, described by the Clifford algebra of odd products of γ^a's, Eq. (43), with creation operators $\hat{b}^{m\dagger}_{f}$ (m denoting the family member quantum number, f the family quantum number and s different orthonormalized solutions of the equations of motion) and the "basis" in the ordinary space of momenta, while $|\psi_{oc}>$ and $|0_{\vec{p}}>$ form the "vacuum state", the first defined in Eq. (34), and the second "vacuum state" defining the starting momentum from which one obtains other single particle states with the same internal part by the operators $\hat{b}^{\dagger}_{\vec{p}}$. Then it follows for single fermion states of particular $\vec{p}$ and $|p^0| = |\vec{p}|$ and any family quantum numbers f and for any solution s

$$|\psi^{\mathbf{s}}{}_{\mathbf{f}}(\tilde{\mathbf{p}}, \mathbf{p^0}) >|_{\mathbf{p^0}=|\tilde{\mathbf{p}}|} = \int dp^0 \delta(p^0 - |\vec{p}|)\, \hat{\mathbf{b}}^{s\dagger}_f(\vec{p}) \;*_T |\psi_{oc}> \;*_T |0_{\vec{p}}> = \hat{\mathbf{b}}^{s\dagger}_f(\vec{p}) \;*_T \;|\psi_{oc}> \;*_T |0_{\vec{p}}>\,, \tag{56}$$

while the corresponding state in the coordinate representation is

$$|\psi^{\mathbf{s}}{}_{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{x^0}) > = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \hat{\mathbf{b}}^{s\dagger}_f(\vec{p})\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|} \;*_T \;|\psi_{oc}> \;*_T |0_{\vec{p}}> = \hat{\mathbf{b}}^{s\dagger}_f(\vec{x}) \;*_T \;|\psi_{oc}> \;*_T |0_{\vec{x}}>\,, \tag{57}$$

where we take into account Eqs. (49, 50). Since the "basis vectors" in internal space of fermions are orthogonal and normatized according to Eq. (36),

$$\{\hat{b}^m_f \;*_A\,,\, \hat{b}^{m'\dagger}_{f'} \;*_A\}_+ |\psi_{oc}> = \hat{b}^m_f \;*_A\, \hat{b}^{m'\dagger}_{f'} \;*_A |\psi_{oc}> = \delta^{mm'} \delta_{ff'}\, |\psi_{oc}>\,, \tag{58}$$

it follows by taking into account Eq. (213) of App. J [9] and App. I, Statement 8. (there the below relation is derived), the expression

$$\begin{aligned}
&< \psi^{\mathbf{s'}}{}_{\mathbf{f'}}(\tilde{\mathbf{p'}}) | \psi^{\mathbf{s}}{}_{\mathbf{f}}(\tilde{\mathbf{p}}) > = < \mathbf{0}_{\tilde{\mathbf{p}}}| < \psi_{\mathbf{oc}}|\hat{\mathbf{b}}^{\mathbf{s'}}_{\mathbf{f'}}(\tilde{\mathbf{p}}') *_{\mathbf{T}}\, \hat{\mathbf{b}}^{\mathbf{s}\dagger}_{\mathbf{f}}(\tilde{\mathbf{p}})\, |\psi_{\mathbf{oc}}> |\mathbf{0}_{\tilde{\mathbf{p}}}> \\
&= < 0_{\vec{p}}| < \psi_{oc}| \sum_{m,m'} c^{*s'm'}{}_{f'}(\vec{p'})\, \hat{b}_{\vec{p'}}\, \hat{b}^{m'}_{f'}\, c^{sm}{}_f(\vec{p})\, \hat{b}^{m\dagger}_f\, \hat{b}^{\dagger}_{\vec{p}} |\psi_{oc}> |0_{\vec{p}}> \\
&= \delta^{ss'} \delta_{ff'}\, \delta(\vec{p'} - \vec{p})\,,
\end{aligned} \tag{59}$$

due to $< 0_{\vec{p}}|\hat{b}_{\vec{p'}}\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}> = \delta(\vec{p'} - \vec{p})$, App. J, while

$$\sum_m c^{ms*}{}_f(\vec{p}, |p^0| = |\vec{p}|)\;\; c^{ms'}{}_{f'}(\vec{p}, |p^0| = |\vec{p}|) = \delta^{ss'} \delta_{ff'}\;. \tag{60}$$

We shall from now on leave out the algebraic product, $*_A$, among the "basis vectors" in internal space of fermions and the tensor product, $*_T$, when creating the single fermion states out of the "basis vectors" and the momentum (or coordinate) basis. We shall use the tensor product, $*_T$, again for generating the Hilbert space out of single fermion states.

It remains to evaluate the scalar product $< \psi^{\mathbf{s'}}{}_{f'}(\vec{x'}, x^0)\, |\, \psi^{\mathbf{s}}{}_f(\vec{x}, x^0) >$, taking into account that the scalar product is evaluated at a time x^0

$$\begin{aligned}
&< \psi^{\mathbf{s'}}{}_{f'}(\vec{x'}, x^0)|\, \psi^{\mathbf{s}}{}_f(\vec{x}, x^0) > = < 0_{\vec{x}}|\;\; < \psi_{oc}|\hat{\mathbf{b}}^{s'}_{f'}(\vec{x'}, x^0)\, |\hat{\mathbf{b}}^{s\dagger}_f(\vec{x}, x^0)\, |\psi_{oc}> |0_{\vec{x}}> = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int \frac{dp'^0}{\sqrt{2\pi}} \delta(p^0 - p'^0) \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} \delta(p^0 - |\vec{p}|)\, \delta(p'^0 - |\vec{p'}|) \\
&< 0_{\vec{x}}| < \psi_{oc}|(\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'}, p'^0)\, \hat{\mathbf{b}}^{s\dagger}_f(\vec{p}, p^0))_{*_A} |\psi_{oc}> |0_{\vec{p}}> e^{ip'_a x'^a}\, e^{-ip_a x^a} = \\
&\int \frac{dp^0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d^{d-1}p'}{(\sqrt{2\pi})^{d-1}}\, \delta(p^0 - |\vec{p'}|) \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \delta(p^0 - |\vec{p}|) \\
&< 0_{\vec{x}}| < \psi_{oc}|(\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}, p^0)\, \hat{\mathbf{b}}^{s\dagger}_f(\vec{p'}, p^0))_{*_A} |\psi_{oc}> |0_{\vec{x}}> e^{i(p^0x^0 - \vec{p}\cdot\vec{x})}\, e^{-i(p^0x'^0 - \vec{p'}\cdot\vec{x})} = \\
&\delta^{ss'}\, \delta_{ff'} \delta(\vec{x'} - \vec{x})\,,
\end{aligned} \tag{61}$$

[9] In Eq. (213) one finds the useful relation $< \vec{p}|f^*(\hat{\vec{p}})\, f(\hat{\vec{p}})|\vec{p'}> = f^*(\hat{\vec{p}})\, f(\hat{\vec{p}})\delta(\vec{p} - \vec{p'})$.

with $< 0_{\vec{x}}| < \psi_{oc}|\psi_{oc} > |0_{\vec{x}} >= 1$.

The scalar product $< \psi^{\mathbf{s}'}{}_{f'}(\vec{x'}, x^0) \,|\, \psi^{\mathbf{s}}{}_{f}(\vec{x}, x^0) >$ has obviously the desired properties of the second quantized states.

Let us now evaluate the commutation relation for $\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}, p^0)\}_{+}|\psi_{oc} > |0_{\vec{p}} >$, by taking into account that $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) = \sum_m c^{ms}{}_{f}\, \hat{b}^{m\dagger}_{f} \hat{b}^{\dagger}_{\vec{p}}$, while $\hat{b}^{m}_{f}|\psi_{oc} >= 0$, Eq. (43), $\hat{b}_{\vec{p'}} \hat{b}^{\dagger}_{\vec{p}} = \delta(\vec{p'} - \vec{p})$, Eq. (49), and $\sum_m c^{ms*}{}_{f}(\vec{p'}) c^{ms'}{}_{f'}(\vec{p}) \delta(\vec{p'} - \vec{p}) = \delta^{ss'} \delta_{ff'} \delta(\vec{p'} - \vec{p})$, Eq. (60). One obtains

$$\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+}|\psi_{oc} > |0_{\vec{p}} >= \{\sum_{m'} c^{m's'*}{}_{f'}(\vec{p'})\, \hat{b}^{m'}_{f'}\, \hat{b}_{\vec{p}}\,,\, \sum_{m} c^{ms}{}_{f}(\vec{p})\, \hat{b}^{m\dagger}_{f}\, \hat{b}^{\dagger}_{\vec{p}}\}_{+}|\psi_{oc} > |0_{\vec{p}} >=$$
$$\sum_{m,m'} \hat{\mathbf{b}}^{m'}_{f'}\, \hat{b}^{m\dagger}_{f}\, \hat{b}_{\vec{p'}}\, \hat{b}^{\dagger}_{\vec{p}}\, c^{m's'*}{}_{f'}(\vec{p'})\, c^{ms}{}_{f}(\vec{p})\, |\psi_{oc} > |0_{\vec{p}} >= \delta^{ss'}\, \delta_{ff'}\, \delta(\vec{p'} - \vec{p})\,. \qquad (62)$$

Let us now write down all the commutation relations for $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ by taking into account Eq. (43)

$$\begin{aligned}
\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+}\, |\psi_{oc} > |0_{\vec{p}} > &= \delta^{ss'} \delta_{ff'}\, \delta(\vec{p'} - \vec{p})\, |\psi_{oc} > |0_{\vec{p}} >\,,\\
\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})\,,\, \hat{\mathbf{b}}^{s}_{f}(\vec{p})\}_{+}\, |\psi_{oc} > |0_{\vec{p}} > &= 0\, |\psi_{oc} > |0_{\vec{p}} >\,,\\
\{\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+}\, |\psi_{oc} > |0_{\vec{p}} > &= 0\, |\psi_{oc} > |0_{\vec{p}} >\,,\\
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\, |\psi_{oc} > |0_{\vec{p}} > &= |\psi^{s}_{f}(\vec{p}) >\\
\hat{\mathbf{b}}^{s}_{f}(\vec{p})\, |\psi_{oc} > |0_{\vec{p}} > &= 0\, |\psi_{oc} > |0_{\vec{p}} >\\
|p^0| &= |\vec{p}|\,.
\end{aligned} \qquad (63)$$

The creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}, p^0))$ and their Hermitian conjugated partners annihilation operators $\hat{\mathbf{b}}^{s}_{f}(\vec{p}, p^0))$, both creating the single fermion state, fulfill when applying on the vacuum state, $|\psi_{oc} > |0_{\vec{p}} >$ the anticommutation relations for the second quantized fermions Eq. (54).

Let us use Eqs. (55, 57) to write the creation operators in the coordinate representation in a more explicit form:

$$\begin{aligned}
|\psi^{\mathbf{s}}{}_{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{x^0}) > &= \sum_{m} \hat{b}^{m\dagger}_{f}\, |\psi_{oc} > \int^{+\infty}_{-\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, c^{ms}{}_{f}\, (\vec{p})\, \hat{b}^{\dagger}_{\vec{p}}\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})} |0_{\vec{p}} >\\
&= \sum_{m} \hat{b}^{m\dagger}_{f}\, |\psi_{oc} > c^{ms}{}_{f}\, (\vec{x})\, \hat{b}^{\dagger}_{\vec{x}}\, |0_{\vec{x}} >= \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x})\, |\psi_{oc} > |0_{\vec{x}} > \;.
\end{aligned} \qquad (64)$$

As expected, the "basis vectors" in the internal space remain the same. there are the momentum parts $\hat{b}^{\dagger}_{\vec{p}}$, which Fourier transform, $\int^{+\infty}_{-\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, c^{ms}{}_{f}\, (\vec{p})\, \hat{b}^{\dagger}_{\vec{p}}\, e^{-i(p^0 x^0 - \varepsilon \vec{p}\cdot\vec{x})}$, into the corresponding coordinate operators, $\hat{b}^{\dagger}_{\vec{x}}$, and so do transform the coefficients which now depend on the coordinates.

We can write down the commutation relations in the coordinate representation in the equivqlent way, just by replacing in Eq. (63) $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}))$ with $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x}, x^0))$, $\delta(\vec{p'} - \vec{p})$ with $\delta(\vec{x}' - \vec{x})$, $|0_{\vec{p}} >$ with $|0_{\vec{x}} >$ and drop $|p^0| = |\vec{p}|$.

$$\begin{aligned}
\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{x'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x})\}_{+}\, |\psi_{oc} > |0_{\vec{x}} > &= \delta^{ss'} \delta_{ff'}\, \delta(\vec{x'} - \vec{x})\, |\psi_{oc} > |0_{\vec{x}} >\,,\\
\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{x'})\,,\, \hat{\mathbf{b}}^{s}_{f}(\vec{x})\}_{+}\, |\psi_{oc} > |0_{\vec{x}} > &= 0\, |\psi_{oc} > |0_{\vec{p}} >\,,\\
\{\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{x'})\,,\, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x})\}_{+}\, |\psi_{oc} > |0_{\vec{x}} > &= 0\, |\psi_{oc} > |0_{\vec{x}} >\,,\\
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x})\, |\psi_{oc} > |0_{\vec{x}} > &= |\psi^{s}_{f}(\vec{x}) >\\
\hat{\mathbf{b}}^{s}_{f}(\vec{x})\, |\psi_{oc} > |0_{\vec{x}} > &= 0\, |\psi_{oc} > |0_{\vec{x}} > \;.
\end{aligned} \qquad (65)$$

$\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ creates on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$ in Eqs. (62, 63) the single fermion state. We can multiply, using this time the tensor product $*_T$ for multiplying single particle states, an arbitrary number of such single particle states, what means that we multiply an arbitrary number of creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) *_T \hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p'}) *_T \cdots *_T \hat{\mathbf{b}}^{s''\dagger}_{f''}(\vec{p''})$, applying on $|\psi_{oc}> |0_{\vec{p}}>$, which gives nonzero contributions, provided that they distinguish among themselves in at least one of the properties $(s, f, \vec{p})$, in the internal space quantum numbers (s, f) or in momentum part $\vec{p}$, due to the orthonormal property continuous spectrum.

All new creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ are the superposition of tensor products of the "basis vectors", described by the odd products of the Clifford γ^a's, and correspondingly anticommuting, and of the commuting "basis" in momentum (or coordinate in the case of $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x})$ space, and correspondingly fulfilling the anticommutation relations of the second quantized fermions, due to anticommuting properties of the "basis vectors" of the internal space and the eveness (commuting properties) of the "basis" in coordinate space.

We can replace the "basis vectors" described by γ^a's with the "basis vectors" described by θ^a's, thus replacing $\hat{b}^{m\dagger}_{f}$ with $\hat{b}^{\theta m\dagger}_{f}$. But after the reduction of the space of the two odd Clifford algebras, γ^a's and $\tilde{\gamma}^a$'s to only the odd Clifford γ^a's, with $\tilde{S}^{ab}$'s used to find all the irreducible representations and to equip them by the family quantum numbers, also the Grassmann algebra of θ^a's and $\frac{\partial}{\partial \theta_a}$'s reduces to the Clifford odd algebra of γ^a's, as we learn in Sect. 3.2.3.

3.3.3 Example for solutions of equation of motion for free massless fields using Clifford algebra

Let us discuss the simplest case, the case which is the closest to the case discussed in Sect. 3.1. This is the case of free massless Clifford fermions in $d = (3+1)$, where $\hat{\mathbf{b}}^{s\dagger}_{f}$, applying on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$, determine the s^{th} solution of the Eq. (79) of the f^{th} family. Fermions carry in this case no charge, since there are in our Clifford algebra representation of the internal space of fermions superposition of spins in $d > (3+1)$ which manifest in $d = (3+1)$ the charges. There are $2^{\frac{4}{2}-1} = 2$ Clifford odd "basis vectors" appearing in $2^{\frac{4}{2}-1} = 2$ families: $\hat{b}^{m=1\dagger}_{f=1} = \overset{03}{[+i]}\,\overset{12}{(+)}$, $\hat{b}^{m=2\dagger}_{f=1} = \overset{03}{(-i)}\,\overset{12}{[-]}$, $\hat{b}^{m=1\dagger}_{f=2} = \overset{03}{(+i)}\,\overset{12}{[+]}$, $\hat{b}^{m=2\dagger}_{f=2} = \overset{03}{[-i]}\,\overset{12}{(-)}$, represented in Table 4 together with their Hermitian conjugated partners. (These two by two "basis vectors" can be "recognized" in Table 3, representing the case of $d = (5+1)$, in the thirteen, fourteen, seventeen and eighteen line if we "drop the charges" represented with $\overset{56}{(+)}$ and pay attention on only the odd Clifford parts in $d = (3+1)$.)

There are $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|_{p^0=|\vec{p}|} e^{-ip_a x^a} = \sum_m c^{ms}{}_f\,(\vec{p})\, \hat{b}^{m\dagger}_{f} e^{-ip_a x^a}$, with the appropriate chosen coefficients $c^{ms}{}_f\,(\vec{p})$ and with $p^0 = |\vec{p}|$, which solve the Weyl equations for free massless fermions, Eq. (79) in Sect. 4, for a chosen momentum $\vec{p}$.

Let us write down two solutions $s = (1, 2)$ for each of the two families, distinguishing from each

Table 4: The "basic vectors" — the creation operators $\hat{b}^{m\dagger}_f$, which on the normalized vacuum state $|\psi_{oc}>= \frac{1}{\sqrt{2}}(\stackrel{03}{[+i]}\stackrel{12}{[-1]} + \stackrel{03}{[-i]}\stackrel{12}{[+1]})$ defines the internal part of states, and their annihilation partners $\hat{b}^{3}_f$ — are presented for $d=(3+1)$-dimensional case. The basic creation operators are the odd products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra members, $(S^{03},\, S^{12})$. The eigenvalues of $(\tilde{S}^{03},\, \tilde{S}^{12},\, \tilde{\Gamma}^{3+1})$, Eqs. (8, 39), presented in the last three columns of the table, determine the family quantum numbers of the "basis vectors", any two of the three values, since $\tilde{\Gamma}^{3+1} = -4i\tilde{S}^{03}\,\tilde{S}^{12}$. Operators $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m}_f$, appearing in two families, fulfill the anticommutaion relations of Eq. (36). They do not carry any charges.

family f	m	$\hat{b}^{m\dagger}_f$	$\hat{b}^{m}_f$	S^{03}	S^{12}	Γ^{3+1}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{\Gamma}^{3+1}$
1	1	$\stackrel{03}{[+i]}\,\stackrel{12}{(+)}$	$(-)\,\stackrel{12}{(-)}\stackrel{03}{[+i]}$	$\frac{i}{2}$	$\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	-1
1	2	$\stackrel{03}{(-i)}\,\stackrel{12}{[-]}$	$\stackrel{12}{[-]}\stackrel{03}{(+i)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	$-\frac{i}{2}$	$\frac{1}{2}$	-1
2	1	$\stackrel{03}{(+i)}\,\stackrel{12}{[+]}$	$\stackrel{12}{[+]}\stackrel{03}{(-i)}$	$\frac{i}{2}$	$\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	-1
2	2	$\stackrel{03}{[-i]}\,\stackrel{12}{(-)}$	$(-)\,\stackrel{12}{(+)}\stackrel{03}{[-]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	$\frac{i}{2}$	$-\frac{1}{2}$	-1

other in the family quantum numbers $(\tilde{S}^{03},\, \tilde{S}^{12})$

$$
\begin{aligned}
p^0 &= |p^0|\,, \quad f \,\text{is determined by}\, \tilde{S}^{03} = -\frac{i}{2}, \tilde{S}^{12} = \frac{1}{2}\,,\\
\hat{\mathbf{b}}^{s=1\dagger}_{f=1}(\vec{p}) &= \beta\left(\stackrel{03}{[+i]}\,\stackrel{12}{(+)} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\stackrel{03}{(-i)}\,\stackrel{12}{[-]}\right) e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\mathbf{b}}^{s=2\dagger}_{f=1}(\vec{p}) &= \beta^*\left(\stackrel{03}{(-i)}\,\stackrel{12}{[-]} - \frac{p^1-ip^2}{|p^0|+|p^3|}\,\stackrel{03}{[+i]}\,\stackrel{12}{(+)}\right) e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,\\
p^0 &= |p^0|\,, \quad f \,\text{is determined by}\, \tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{1}{2}\,,\\
\hat{\mathbf{b}}^{s=1\dagger}_{f=2}(\vec{p}) &= \beta\left(\stackrel{03}{(+i)}\,\stackrel{12}{[+]} + \frac{p^1+ip^2}{|p^0|+|p^3|}\,\stackrel{03}{[-i]}\,\stackrel{12}{(-)}\right) e^{-i(|p^0|x^0-\vec{p}\cdot\vec{x})}\,,\\
\hat{\mathbf{b}}^{s=2\dagger}_{f=2}(\vec{p}) &= \beta^*\left(\stackrel{03}{[-i]}\,\stackrel{12}{(-)} - \frac{p^1-ip^2}{|p^0|+|p^3|}\,\stackrel{03}{(+i)}\,\stackrel{12}{[+]}\right) e^{-i(|p^0|x^0+\vec{p}\cdot\vec{x})}\,,
\end{aligned}
\tag{66}
$$

with $\beta^*\beta = \frac{|p^0|+|p^3|}{2|p^0|}$ taking care that the corresponding states are normalized. We emphasize again, that using the odd Clifford algebra (after the reduction of the two Clifford algebra to only one, Sect. 3.2.3) for the description of the internal space of fermions, there appear families already in $d=(3+1)$, with the family quantum numbers. Both families have with respect to the Lorentz algebra, S^{ab}, identical quantum numbers.

We shall latter discuss the appearing of charges in more details, as well as the appearance of scalar fields, which are responsible for masses of fermions.

3.3.4 Comparison between creation and annihilation operators of Dirac and of those offered by the odd Clifford algebra

Let us compare the usual, the Dirac way of the second quantization and our way using the odd Clifford algebra objects.

In Table 1 the basis states for single fermions in $d=(3+1)$-dimensional space are presented, two left handed and two right handed basis states. We treat free massless fermions, looking for solutions of the Weyl equation, Eq. 79, then we need either the left or the right handed basis states.

Let us make a choice of the right handed solutions. Then we have the the solution with the superposion of the first two states in Table 1 with spin $S^{12} = \pm\frac{1}{2}$.

We can write down the creation operator which would, applying on a vacuum state, generate such a single fermion state. Eq. (52) suggests that the corresponding creation operator of particular momentum $\vec{p}$ can be written as $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}) \stackrel{def}{=} \sum_m \hat{\mathbf{a}}^{hs\dagger}_m(\vec{p})\, u^{hs}_m(\vec{p})$, which is our definition.

Table 4 of Sect. 3.3.3 represents the "basis vectors", that is the creation operators in $d = (3+1)$ for two families, offered by the Clifford algebra. Both are right handed and both give the solution of the Weyl equations, Eq. 79. (If we start from the right handed "basis vectors", we can not generate the left handed ones either by the application of S^{ab} or $\tilde{S}^{ab}$ on the "basis vectors". But "basis vectors" in $d = (5+1)$, for example, do have with respect to $d = (3+1)$ both kinds of "basis vectors", those with $\Gamma^{3+1} = 1$ as well as those with $\Gamma^{3+1} = -1$, as we can see in Table 1 in the thirteenth and fourteenth lines and in the seventeenth and eighteenth lines.) We choose one of these two families, they both have the same properties with respect to the solutions of the Weyl equations, Eq. 79, and with respect to S^{ab}. They only differ in the family quantum numbers, $\tilde{S^{03}}$ and $\tilde{S^{12}}$. The creation operators, which on the vacuum state generate the particular solution s of the Weyl equation for a chosen momentum $\vec{p}$ in the odd Clifford algebra case, are $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p}) = \sum_m c^{ms}{}_f(\vec{p})\, \hat{b}^{m\dagger}_f\, \hat{b}_{\vec{p}}$.

Let us write down both kinds of creation operators, the Dirac one and ours, both for the right handed case, leaving out therefore the index describing handedness h in the Dirac case and f, describing family, in our case

$$\hat{\mathbf{a}}^{s\dagger}(\vec{p}) \stackrel{def}{=} \sum_m \hat{\mathbf{a}}^{s\dagger}_m(\vec{p})\, u^s_m(\vec{p})\,, \quad \hat{\mathbf{b}}^{s\dagger}(\vec{p}) = \sum_m c^{ms}(\vec{p})\, \hat{b}^{m\dagger}\, \hat{b}_{\vec{p}}\,, \tag{67}$$

where we introduced a new creation operator, related to the one proposed by Dirac. Comparing both sides of Eq. (67) one recognizes that the creation operator of Dirac $\hat{\mathbf{a}}^{sm\dagger}(\vec{p})$ (where $v^{sm}(\vec{p},\vec{x}) = u^{sm}(\vec{p})\, e^{i\vec{p}\cdot\vec{x}}$ solve the Dirac equation for two choices of s), multiplied by $u^{sm}(\vec{p})$ and summed over m — $\hat{\mathbf{a}}^{hs\dagger}(\vec{p}) = \sum_m u^{sm}(\vec{p})\hat{a}^{m\dagger}(\vec{p})$ — has to be related to $\hat{\mathbf{b}}^{s\dagger}_f = \sum_m c^{ms}{}_f(\vec{p})\, \hat{b}^m_f\, \hat{b}_{\vec{p}}$, for any of the two families $(f = (\tilde{S}^{03} = -\frac{i}{2}, \tilde{S}^{12} = \frac{i}{2})$ and $f = (\tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{i}{2}))$,

$$\hat{\mathbf{a}}^{s\dagger}(\vec{p}) = \sum_m \hat{\mathbf{a}}^{m\dagger}(\vec{p})\, u^s_m(\vec{p}) \qquad \text{to be related to} \quad \hat{\mathbf{b}}^{s\dagger}(\vec{p}) = \sum_m c^{ms}{}_f(\vec{p})\, \hat{b}^m_f\, \hat{b}_{\vec{p}}\,. \tag{68}$$

Both creation operators, $\hat{\mathbf{a}}^{s\dagger}(\vec{p})$ and $\hat{\mathbf{b}}^{s\dagger}(\vec{p})$, fulfill the anticommutation relations of Eq. (63). $\hat{\mathbf{a}}^{s\dagger}(\vec{p})$ fulfill also the anticommutation relations of Eq. (54), due to the orthogonality relations of solutions of the Weyl equations, Eq. (53). An example of the coefficients $c^{ms}{}_f(\vec{p})$ can be read in Eq. (66) in Sect. 3.3.3.

Dirac equipped the creation operators (and correspondingly also the annihilation operators) with the quantum numbers (s, m) and with $\vec{p}$. He postulated for such creation and annihilation operators anticommutation relations. Our creation and annihilation operators, $\hat{\mathbf{b}}^{s\dagger}(\vec{p})$ and $\hat{\mathbf{b}}^{s}(\vec{p})$, have anticommuting properties due to the anticommutativity of $\hat{\mathbf{b}}^{m\dagger}_f$ and $\hat{\mathbf{b}}^{m}_f$, which are Clifford odd objects.

We conclude: The by Dirac postulated creation operators, $\hat{\mathbf{a}}^{h\dagger}_m(\vec{p})$, and their annihilation partners, $\hat{\mathbf{a}}^{h}_m(\vec{p})$, Eqs. (51, 54), related in Eq. (68) to the Clifford odd creation and annihilation operators, manifest that the odd Clifford algebra offers the explanation for the second quantization postulates of Dirac.

3.3.5 Discrete symmetries in $d = ((d-1)+1)$ with the observed properties in $d = (3+1)$

When treating representations of particle and antiparticle states the discrete symmetry operators in the second quantized picture are needed. We follow here Refs. [125, 126].

We have treated so far free massless fermions with the internal space described by Grassmann and Clifford algebras. After the reduction of the Clifford algebras to only the algebra generated by γ^a's, Sect. 3.2.3, we pay attention to only the "Clifford fermions", since besides the Clifford algebras space also the Grassmann algebra space is reduced and has no more desired properties for the description of the internal space of the integer "fermion states". In Refs. [21, 22, 25] the discrete symmetry operators for the Grassmann integer spin "fermions" and the Clifford half integer spin fermions are presented. The fermion anticommuting "nature" of creation and annihilation operators originates in both cases, the Grassmann case and the Clifford case, in the anticommuting properties of the "basis vectors" describing the internal spaces, Eq. (26) in the Grassmann case and Eq. (43, 63) in the Clifford case. (Fermions in both spaces are in superposition of eigenstates of the Cartan subalgebra operators of $\mathbf{S}^{ab}$ in the Grassmann case, in the Clifford case they are in superposition of the Cartan subalgebra operators of S^{ab} as well as of $\tilde{S}^{ab}$.)

In theories of the Kaluza-Klein kind, as also the *spin-charge-family* theory is, there are spins or total angular moments in higher dimensions $d > (3+1)$ which manifest as charges in the observable $d = (3+1)$ space. The charge conjugation requirement, if following the prescription in $d = (3+1)$, is in contradiction with what we observe for the anti-particles. In Ref. [125] we redefined the discrete symmetries so that we stay within the subgroups of the starting group of symmetries, while we require that the angular moments in higher dimensions manifest as charges in $d = (3+1)$. In the low energy regime the angular momenta in higher dimensions manifest only spins. As in the whole paper also in this part we pay attention on spaces with even d only.

A theory, which would in the low energy regime explain all the observed phenomena, must have the concept of the discrete symmetries (C, P, T) well understood, manifesting the Lorentz invariance, causality and CPT symmetry in $d = (3+1)$ [10].

We start with the definition of the discrete symmetries as they follow from the prescription in $d = (3+1)$, defining the $\mathcal{C}_{\mathcal{H}} = \prod_{\gamma^a \in \Im} \gamma^a \, K$, which transforms any single particle state $\psi^{s\dagger}_f(\vec{p})$ created by $\hat{\mathbf{b}}^{s\dagger}_f$, Eq. (3.3.2), on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$, Eq. (56), solving the Weyl equation, Eq. (79), for a free massless spinor with a positive energy into the charge conjugated one with the negative energy. We define the new operator $\mathbb{C}_{\mathcal{H}}$ as the operator which creates the antiparticle state with the positive energy and all the properties of $\psi^{s\dagger}_f(\vec{p})$. We do the same for the other two discrete operators, the time reversal $\mathcal{T}_{\mathcal{H}}$ and parity $\mathcal{P}^{(d-1)}_{\mathcal{H}}$ operators, again following the $d = (3+1)$ prescription

$$
\begin{aligned}
\mathbb{C}_{\mathcal{H}} : &= (-)^{\frac{d}{2}+1} \prod_{\Im \gamma^a} \gamma^a \, \Gamma^{(d)} K \prod_{\gamma^a \in \Im} \gamma^a \, K \\
\mathcal{T}_{\mathcal{H}} : &= \gamma^0 \prod_{\gamma^a \in \Re} \gamma^a \; K \, I_{x^0} \,, \\
\mathcal{P}^{(d-1)}_{\mathcal{H}} : &= \gamma^0 \, I_{\vec{x}} \,, \\
I_x x^a &= -x^a \,, \quad I_{x^0} x^a = (-x^0, \vec{x}) \,, \quad I_{\vec{x}} \vec{x} = -\vec{x} \,, \\
I_{\vec{x}_3} x^a &= (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d) \,.
\end{aligned} \tag{69}
$$

The product $\prod \gamma^a$ is meant in the ascending order in γ^a.

These discrete symmetry operators do not lead to the desired properties of the observed anti-particle states.

Ref. [125] proposes new operators by transforming the above defined discrete symmetries so that, while remaining within the same groups of symmetries, the redefined discrete symmetries manifest the experimentally acceptable properties in $d = (3+1)$, which is of the essential importance for all

[10] The concept of what the symmetries C , P and T separately mean is in effective theories somewhat a matter of definition partly arranged so as to make them conserved if possible.

the Kaluza-Klein theories [43, 49, 44, 50] in which fermions carry only the spin and family quantum numbers [9, 19]

$$
\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \mathcal{C}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, e^{i\pi J_{11\,13}}, \dots, e^{i\pi J_{(d-3)(d-1)}}\,,\\
\mathcal{T}_{\mathcal{N}} &= \mathcal{T}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{36}}\, e^{i\pi J_{8\,10}}\, e^{i\pi J_{12\,14}}, \dots, e^{i\pi J_{(d-2)d}}\,,\\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{56}}\, e^{i\pi J_{78}}\, e^{i\pi J_{9\,10}}\, e^{i\pi J_{11\,12}}\, e^{i\pi J_{13\,14}}, \dots, e^{i\pi J_{(d-1)d}}\,,\\
\mathbb{C}_{\mathcal{N}} &= \mathbb{C}_{\mathcal{H}}\, \mathcal{P}^{(d-1)}_{\mathcal{H}}\, e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, e^{i\pi J_{11\,13}}, \dots, e^{i\pi J_{(d-3)(d-1)}}\,,
\end{aligned}
\tag{70}
$$

The rotations $(e^{i\pi J_{12}}\, e^{i\pi J_{35}}\, e^{i\pi J_{79}}\, \dots, e^{i\pi J_{(d-3)(d-1)}})$ together with (multiplied by) $\mathcal{P}^{(d-1)}_{\mathcal{H}}$, which are included in $\mathbb{C}_{\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}}$, keep p^i for $i = (1,2,3)$ unchanged, while they transform a state so that all the eigenvalues of the Cartan subalgebra except S^{03} and J^{12} (or at the low energy regime S^{12}) change sign [11]. Correspondingly a second quantized state transforms into the anti-particle state with the same four momentum as the starting state but with the opposite values of the total angular momentum (or at the low energy regime rather the spin) determined by the Cartan subalgebra eigenvalues, except for S^{03} and J^{12} (or at the low energy regime S^{12}). The parity operator $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ changes p^i into $-p^i$ only for $i = (1,2,3)$, while the time reversal operator corrects all the properties of the new $\mathbb{C}_{\mathcal{N}}$) and $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ so that

$$
\mathcal{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}} \mathcal{T}_{\mathcal{N}} \;=\; \mathcal{C}_{\mathcal{H}} \mathcal{P}^{(d-1)}_{\mathcal{H}} \mathcal{T}_{\mathcal{H}} \rightarrow \Gamma^{(d)}\, I_x\,. \tag{71}
$$

The above for the Clifford case proposed discrete symmetry operators [125], commute among themselves as also the old ones do. If manifesting dynamics only in $d = (3+1)$ space they can be written also, up to a phase, as

$$
\begin{aligned}
\mathcal{C}_{\mathcal{N}} &= \prod_{\Im\gamma^m, m=0}^{3} \gamma^m\;\; \Gamma^{(3+1)}\, K\, I_{x^6,x^8,\dots,x^d}\,,\\
\mathcal{T}_{\mathcal{N}} &= \prod_{\Re\gamma^m, m=1}^{3} \gamma^m\;\; \Gamma^{(3+1)}\, K\, I_{x^0}\, I_{x^5,x^7,\dots,x^{d-1}}\,,\\
\mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0\, \Gamma^{(3+1)}\, \Gamma^{(d)}\, I_{\vec{x}_3}\,,\\
\mathbb{C}_{\mathcal{N}} &= \prod_{\Re\gamma^a, a=0}^{d} \gamma^a\;\; K \prod_{\Im\gamma^m, m=0}^{3} \gamma^m\;\; \Gamma^{(3+1)}\, K\, I_{x^6,x^8,\dots,x^d} = \prod_{\Re\gamma^s, s=5}^{d} \gamma^s\;\; I_{x^6,x^8,\dots,x^d}\,,\\
\mathbb{C}_{\mathcal{N}} \mathcal{P}^{(d-1)}_{\mathcal{N}} &= \gamma^0 \prod_{\Im\gamma^a, a=5}^{d} \gamma^a\, I_{\vec{x}_3}\, I_{x^6,x^8,\dots,x^d}
\end{aligned}
\tag{72}
$$

Operators I operates as follows in $d = 2n$: $I_{\vec{x}_3} x^a = (x^0, -x^1, -x^2, -x^3, x^5, x^6, \dots, x^d)$,
$I_{x^5,x^7,\dots,x^{d-1}}\, (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \dots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, -x^5, x^6, -x^7, \dots, -x^{d-1}, x^d)$,
$I_{x^6,x^8,\dots,x^d}\, (x^0, x^1, x^2, x^3, x^5, x^6, x^7, x^8, \dots, x^{d-1}, x^d) = (x^0, x^1, x^2, x^3, x^5, -x^6, x^7, -x^8, \dots, x^{d-1}, -x^d)$.

Let us take a simple case that $p^a = (p^0, 0, 0, p^3, 0, \dots, d)$ and $d = (13+1)$. The solution of the Weyl equation, Eq. (79), for free massless right handed u-quark with the spin $\frac{1}{2}$ and the colour charge $C^1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$, presented in Table 7 in the first line with the creation operator $\hat{u}^{c1\dagger}_R$, applying on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$, is: $\hat{u}^{c1\dagger}_R = \beta \stackrel{03}{(+i)} \stackrel{12}{[+]} | \stackrel{56}{[+]} \stackrel{78}{(+)} || \stackrel{9\,10}{(+)} \;\; \stackrel{11\,12}{[-]} \;\; \stackrel{13\,14}{[-]}$.

[11] Since in our extra dimension picture J_{35} is no longer a symmetry (for the metric taken as a background field) in coordinate space, the operation $e^{i\pi J_{35}}$ looks suspicious as being not a symmetry, but it is. Indeed, the operation $e^{i\pi J_{35}}$ is in the coordinate part composed just of a mirror reflection around the $x^3 = 0$ plane in usual space and reflection in the extra dimension space around the surface $x^5 = 0$.

The application of the Clifford odd operator as it is $\mathbb{C}_{\mathcal{N}}$ would not belong to the same irreducible representation. Clifford even operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$ transforms $\hat{u}^{c1\dagger}_{R}$ as follows

$$\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\,\hat{u}^{c1\dagger}_{R}\,\cdot e^{-ip^0+ip^3x^3} = \beta\,\stackrel{03}{[-i]}\,\stackrel{12}{[+]}\,|\,\stackrel{56}{(-i)}\,\stackrel{78}{[-]}\,||\,\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{(+)}\,\cdot e^{-ip^0-ip^3x^3}\,,$$

which is the positive energy solution for the left handed u-antiquark, the creation operator of which is presented in the 35^{th} line of Table 7.

The operators $\mathbb{C}_{\mathcal{H}}, \mathcal{P}^{(d-1)}_{\mathcal{H}}$ and $\mathcal{T}_{\mathcal{H}}$, Eq. (69), indexed by $\mathcal{H}$, are good symmetries only when also boson fields, in the Kaluza-Klein theories the gravitational fields, in higher than $(3+1)$ dimensions are correspondingly transformed and not considered as *background* fields. However, the operators $\mathbb{C}_{\mathcal{N}}, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ with index $\mathcal{N}$ will be good symmetries even if we take it that there is a *background* field depending only on the extra dimension coordinates - independent of whether the extra dimension space is compactified or not - so that they are not transformed [12].

The properties of the gravitational field, described by vielbeins and spin connections, under the discrete symmetry operators $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathbb{T}_{\mathcal{N}}$, Eq. (72), observed in $d=(3+1)$ as vector and scalar gauge fields, are discussed in App. G.

4 From action to equations of motion

Let us start with the real action functional of local fields Ψ (their Hermitian conjugated fields $\Psi^{\dagger}$) and their first derivative with respect to coordinates $\frac{\partial \Psi}{\partial x_a}$ (and $\frac{\partial \Psi^{\dagger}}{\partial x_a}$) in $d=(d-1)+1$-dimensional space-time, which is invariant under Poincare transformations,

$$\mathcal{A} = \int d^d x\, \mathcal{L}(\Psi, \frac{\partial \Psi}{\partial x_a})\,, \tag{73}$$

the integration is limited to the space of interest. (In the case that we pay attention to free massless fermions, as we have done so far, the integration goes over the whole d-dimensional space.) The arbitrary change of Ψ to $\Psi+\delta\Psi$ causes the corresponding change of the action $\mathcal{A}$

$$\delta\mathcal{A} \;=\; \frac{1}{2}\int d^dx[\{\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\Psi^{\dagger}} - \hat{p}_a\,\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi^{\dagger}}\}\delta\Psi^{\dagger} + \{\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\Psi} - \hat{p}_a\,\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi}\}\delta\Psi]\,. \tag{74}$$

We assume that the two surface terms are equal to zero since the fields are zero on the boundary: $\int d^dx\hat{p_a}[\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi^{\dagger}}\delta\Psi^{\dagger}]=0$ and $\int d^dx\hat{p_a}[\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi}\delta\Psi]=0$.

We pay in the above Eq. (74) attention on the possibility that the fields can be fermion fields, and according to our recognition that the fermion fields, described by either the odd Grassmann or the odd Clifford algebra, anticommute, the left derivative is not the same as the right one.

Requiring that the action is stationary with respect to an arbitrary change of fields, with the boundary terms equal to zero, the Euler-Lagrange equations follow

$$\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\Psi^{\dagger}} - \hat{p}_a\,\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi^{\dagger}} \;=\; 0\,,\quad \frac{\overleftarrow{\partial}\mathcal{L}}{\partial\Psi} - \hat{p}_a\,\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi} = 0\,. \tag{75}$$

[12] One can namely easily see that the transformations of the coordinates of the extra dimensions are *cancelled* between the π-rotations and the actions of e.g. $P_{\mathcal{H}}$ on the extra dimensional coordinates. Thus it can be easily seen that even if we consider a background gravitational field for the extra dimensions - but the $(3+1)$ dimensional space is either flat or their gravitational field is considered dynamical so as to be also transformed - these operators with index $\mathcal{N}$, $\mathbb{C}_{\mathcal{N}}, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$, are good symmetries with respect to the space-time transformations.They are indeed good symmetries according to their action on the Weyl field. The crucial point really is that the $\mathcal{N}$-indexed operators $\mathbb{C}_{\mathcal{N}}, \mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathbb{T}_{\mathcal{N}}$ with their associated x-transformations do *not transform the extra* $(d-4)$*coordinates* so that background fields depending on these extra dimension coordinates do not matter.

Let us write down the momentum canonically conjugate to the fields, $\Pi_{\Psi} = \frac{\delta \mathcal{A}}{\delta \frac{\partial \Psi}{\partial x_0}} = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi}{\partial x_0}} = i \frac{\overrightarrow{\partial \mathcal{L}}}{\partial \hat{p}_0 \Psi}$, $\Pi_{\Psi^\dagger} = \frac{\delta \mathcal{A}}{\delta \frac{\partial \Psi^\dagger}{\partial x_0}} = \frac{\partial \mathcal{L}}{\partial \frac{\partial \Psi^\dagger}{\partial x_0}} = i \frac{\overleftarrow{\partial \mathcal{L}}}{\partial \hat{p}_0 \Psi^\dagger}$, and the Hamilton density $\mathcal{H} = \frac{1}{2}\{\Pi_{\Psi} \hat{p}_0 \Psi + \hat{p}_0 \Psi^\dagger \, \Pi_{\Psi^\dagger}\} - \mathcal{L}$.

$$\begin{aligned}
\Pi_{\Psi} &= i \frac{\overrightarrow{\partial \mathcal{L}}}{\partial \hat{p}_0 \Psi}\,, \quad \Pi_{\Psi^\dagger} = i \frac{\overleftarrow{\partial \mathcal{L}}}{\partial \hat{p}_0 \Psi^\dagger}\,, \\
\mathcal{H} &= -i \frac{1}{2} \{\Pi_{\Psi}\, \hat{p}_0 \Psi + \hat{p}_0 \Psi^\dagger \, \Pi_{\Psi^\dagger}\} - \mathcal{L}\,,
\end{aligned} \tag{76}$$

in the last line we took again into account that the fields can have an odd Clifford (or an odd Grassmann) character.

4.1 Action for free massless fermion fields in standard model and beyond

In this paper we treat so far only free massless fermions. The Lagrange action is for general cases, therefore in $d = (d-1)+1$ spaces, equal to

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\Psi^\dagger \gamma^0 \, \gamma^a \hat{p}_a \Psi) + h.c. = \int d^d x \, \frac{1}{2} \, (\Psi^\dagger \gamma^0 \, \gamma^a \hat{p}_a \Psi - \hat{p}_a \Psi^\dagger \gamma^0 \, \gamma^a \Psi)\,, \tag{77}$$

up to a surface term, which is assume to contribute nothing.

The action is invariant under the Lorentz transformations and translations and it is real [13].

The solutions for free massless fermions are in usual theories, in the *standard model* and in all the theories, which introduce charges and families with the additional groups, or even the unifying groups for charges, like $SO(10)$ and $SU(5)$, while the space is $d = (3+1)$-dimensional, the superposition of the states, presented in Table 1, for each handedness separately. The left and the right handed solutions can not be obtained by the Lorentz transformations in the internal space of fermions. Families need to be introduced by a separate, additional, group.

The right handed solutions of Table 1 are for the two families, existing in $d = (3+1)$ in the case that the internal space is described by the odd Clifford algebra (they are reachable from each other by the application of $\tilde{S}^{ab}$'s), presented in Eq. (66). They are superposition of the "basis vectors", presented in Table 4.

We present in the next Sect. 4.2 the solutions of the equations of motion for $d = (d-1)+1$, $d = (5+1)$ and $d = (13+1)$ in the case that only $p^a = (p^0, p^1, p^2, p^3, 0, 0, 0, 0, \dots, 0)$ is different from zero — this is expected in the low energy, the observable, regime.

4.2 Action for free massless Clifford fermions and solutions of Weyl equations

We take the simplest action, the one from Eq. (77),

$$\mathcal{A} = \int d^d x \, \frac{1}{2} \, (\psi^\dagger \gamma^0 \, \gamma^a \hat{p}_a \psi - \hat{p}_a \psi^\dagger \gamma^0 \, \gamma^a \psi)\,, \tag{78}$$

$\hat{p}_a = i \frac{\partial}{\partial x^a}$, leading to the equations of motion

$$\gamma^a \hat{p}_a |\psi> = 0\,, \tag{79}$$

[13] γ^0 appears in the action to pay attention that the action is Lorentz invariant: $S^{ab\dagger}\gamma^0 = \gamma^0 S^{ab}\,, S^\dagger \gamma^0 = \gamma^0 S^{-1}$, $S = e^{-\frac{i}{2}\omega_{ab}(S^{ab}+L^{ab})}$.

and fulfilling the Klein-Gordon equation

$$\gamma^a p_a \gamma^b p_b |\psi> \;=\; p^a p_a |\psi>= 0\,, \tag{80}$$

for each of the $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})|_{p^0=|\vec{p}|} e^{-ip_a x^a}|\psi_{0c}>$, with $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})|_{p^0=|\vec{p}|}$ from Eqs. (55,56), fulfilling the anticommutation relations of Eqs. (63) on the vacuum state $|\psi_{oc}>|0_{\vec{p}}>$, Eq. (46).

$$\{\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})|_{p^0=|\vec{p}|} \;=\; \sum_m c^{ms}{}_f\;(\vec{p}, |p^0|=|\vec{p}|)\; \hat{b}^\dagger_{\vec{p}}\, \hat{b}^{m\dagger}_f\}|\psi_{oc}>|0_{\vec{p}}>\,,$$

The Euler-Lagrange equations, Eq. (75), $\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\Psi^\dagger} - \hat{p}_a \frac{\overrightarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi^\dagger} = 0$ and $\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\Psi} - \hat{p}_a \frac{\overleftarrow{\partial}\mathcal{L}}{\partial\hat{p}_a\Psi} = 0$, lead for the Lagrange density,

$$\mathcal{L}_C \;=\; \frac{1}{2}\{\psi^\dagger\,\gamma^0\,\gamma^a\,\hat{p}_a\psi - \hat{p}_a\psi^\dagger\,\gamma^0\,\gamma^a\,\psi\,\}\,, \tag{81}$$

to

$$\gamma^0\gamma^a\,\hat{p}_a\,\psi \;=\; 0\,,\quad -\hat{p}_a\,\psi^\dagger\gamma^0\,\gamma^a = 0\,. \tag{82}$$

Taking into account Eq. (76) one finds the momenta and the Hamilton density

$$\begin{aligned} \Pi_\psi &= i\frac{\overrightarrow{\partial}\mathcal{L}}{\partial\hat{p}_0\psi} = \frac{i}{2}\psi^\dagger\,, \quad \Pi_{\psi^\dagger} = i\frac{\overleftarrow{\partial}\mathcal{L}}{\partial\hat{p}_0\psi^\dagger} = -\frac{i}{2}\psi\,,\\ \mathcal{H} &= \psi^\dagger\,\gamma^0\gamma^i\hat{p}_i\,\psi\,, \qquad\quad i = 1,2,3,4,\ldots,d\,. \end{aligned} \tag{83}$$

The solutions $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})|_{p^0=|\vec{p}|} e^{-ip_a x^a}$, with $\hat{\mathbf{b}}^{s\dagger}_f(\vec{p})|_{p^0=|\vec{p}|} = \sum_m c^{ms}{}_f\;(\vec{p}, |p^0|=|\vec{p}|)\;\hat{b}^\dagger_{\vec{p}}\,\hat{b}^{m\dagger}_f$, applying on the vacuum states $|\psi_{oc}>|0_{\vec{p}}>$ for $d=((d-2)+1)$, for $d=(5+1), (9+1)$ and $(13+1)$, will be discussed in Sect. 4.3.

Statement 7. All the states, belonging to different values of the Cartan subalgebra — they differ at least in one value of either the set of S^{ab} or the set of $\tilde{S}^{ab}$, Eq. (8) — are orthogonal according to the scalar product, defined as the integral over the Grassmann coordinates, Eq. (165), for a chosen vacuum state $|1\;>$.

Statement 8. Spaces with $d=(d-1)+1$, $d=2(2n+1)$, $n\geq 1$, have the special property: Each family contains fermions and antifermions — with the properties in $d=(3+1)$ as observed for quarks and antiquarks and leptons and antileptons — what is not the case in $d=4n$.

This in not difficult to understand: Each antifermion carries the same spin, determined by the member S^{12} of the Cartan subalgebra, as does the fermion, the opposite handedness in $d=(3+1)$ as the fermion and the opposite charges, determined by the spins in $d>3+1$ as the fermion. To achieve this in the Clifford description of the internal space of fermions $\frac{d}{2}$ must be odd in order that S^{ab} change an even number of eigenvalues of Cartan subalgebra members. In Table 3 the "basis states" for $d=(5+1)$-dimensional space, with $2^{\frac{d}{2}-1}=4$ family members in $2^{\frac{d}{2}-1}=4$ families are presented. One can see that each family includes charged fermions and their antifermions of the opposite charge and opposite handedness in $d=(3+1)$. We shall comment this case and two more cases in the next subsection.

4.3 Examples for solutions of equation of motion for free massless fields using Clifford odd algebra

In any d-dimensional space each family contains $2^{\frac{d}{2}-1}$ members. We are in particular interested in spaces with $d = 2(2n+1)$, since any family contains fermions and antifermions with the property of the observed quarks and leptons and antiquarks and antileptons. The fermions and antifermions carry: **i.** the same spin (the same eigenvalue of the Cartan subalgebra member S^{12}), **ii.** the opposite handedness in $d = (3+1)$, **iii.** the opposite charges (which in our case origin in spins in $d > (3+1)$). In ordinary theories the charges of fermions and antifermions must be related by "hand", while in the *spin-charge-family* theory, using the odd Clifford algebra, the relation between handedness in $d = (3+1)$ and the charges appear by itself. We discuss in the previous subsection the case of $d = (5+1)$. Here we discuss the three cases: The case with $d = (13+1)$, the case with $d = (9+1)$ and the simplest one, the case with $d = (5+1)$.

The first case, $d = (13+1)$, is of the particular interest since one family contains all the creation operators for all the quarks and leptons and antiquarks and antileptons, with the weak, colour and hypercharges required by the *standard model* for massless quarks and leptons before the electroweak transition. There appear after the break of the starting symmetry, caused by the condensate of neutrino and antineutrino as explained in Table. 6, two (at low energies almost decoupled) groups of four families, the fourth of the lower four families belonging to the observed three, the upper four families contributing to the dark matter in the universe[14].

The second case, $d = (9+1)$ is only for demonstration, as the case which is the subgroup of $SO(13+1)$.

The simplest one, $d = (5+1)$, is also only for illustration.

We analyse properties of creation operators (of the states when creation operators apply on the vacuum state $|\psi_{oc}> \, |0_{\vec{p}}>$) from the point of view of $d = (3+1)$-dimensional space, with the momentum in ordinary space $p^a = (p^0, p^1, p^2, p^3, 0, 0, \cdots, 0)$, so that the charges "seen" in $d = (3+1)$ are determined by the Cartan subalgebra of the Lorentz algebra in the internal space — $S^{st}, (s,t) = (5, 6, 7, \cdots, d)$.

We discuss one family in details (let be reminded that the generators S^{ab} connect all the members belonging to one family, while $\tilde{S}^{ab}$ transform a particular member of one family into the same member of another family), and comment also on the appearance of families (all the families are reachable by $\tilde{S}^{ab}$) and present them briefly.

The discrete symmetry operators are in the Clifford case presented in Eq. (72).

We start with examples in $d = (5+1)$-dimensional space, with charges determined by $S^{st}, (s,t) = (5, 6)$.

4.3.1 Clifford fermions and antifermions in $d = (5+1)$

We look for solutions of the Weyl equation, Eq. (79), by taking into account four basis creation operators of the first family, $f = 1$, in Table 3. Assuming that moments in the fifth and the sixth dimensions are zero, $p^a = (p^0, p^1, p^2, p^3, 0, 0)$, the following four plane wave solutions for $p^0 = |\vec{p}|$, can be found, two with the positive charge $\frac{1}{2}$ and with spin S^{12} either equal to $\frac{1}{2}$ or to $-\frac{1}{2}$, and two with the negative charge $-\frac{1}{2}$ and again with S^{12} either $\frac{1}{2}$ or $-\frac{1}{2}$. Since we assume that the nonzero components of the momenta are only those in $d = (3+1)$, and we treat only free fermions, the dynamics (kinematic) seems similar as the one from Eq. (66), but in that case the space has $d = (3+1)$ and the Clifford algebra offers two families and no charge. In $d = (5+1)$ case the spin S^{56} determines the charge in $d = (3+1)$. We treat now only one family out of four families from Table 3, and each family has four members.

[14]In Ref. [64, 127, 128, 129] ($d = 5+1$)-dimensional space is studied as a toy model to manifest that the break of symmetry from the higher dimensional space to the $(3+1)$-dimensional space *can* lead to massless fermions. Fermions were described in Clifford space. Here we briefly follow these references, and Refs. [125, 126], adding new observations.

Half of them represent fermions and half antifermions, each fermion has the same S^{12} and the opposite handedness than its antifermion, as expected.

$$
\begin{aligned}
&& &\text{Clifford odd creation operators in d} = (5+1)\\
p^0 &=& &|p^0|\,,\;\; S^{56} = \frac{1}{2}\,,\;\; \Gamma^{(3+1)} = 1\,,\\
\left(\hat{\mathbf{b}}^{s=1\dagger}_{f=1}(\vec{p}) \right. &=& &\beta\left(\stackrel{03}{(+i)}\stackrel{12}{(+)}|\stackrel{56}{(+)} + \frac{p^1+ip^2}{p^0+p^3}\stackrel{03}{[-i]}\stackrel{12}{[-]}|\stackrel{56}{(+)}\right)\Bigg)\cdot\\
&& &e^{-i|p^0x^0-\vec{p}\cdot\vec{x})}\,,\\
\left(\hat{\mathbf{b}}^{s=2\dagger}_{f=1}(\vec{p}) \right. &=& &\beta^*\left(\stackrel{03}{[-i]}\stackrel{12}{[-]}|\stackrel{56}{(+)} - \frac{p^1-ip^2}{p^0+p^3}\stackrel{03}{(+i)}\stackrel{12}{(+)}|\stackrel{56}{(+)}\right)\Bigg)\cdot\\
&& &e^{-i(p^0x^0+\vec{p}\cdot\vec{x})}\,,\\
&& &\text{Clifford odd creation operators in d} = (5+1)\\
p^0 &=& &|p^0|\,,\;\; S^{56} = -\frac{1}{2}\,,\;\; \Gamma^{(3+1)} = -1\,,\\
\left(\hat{\mathbf{b}}^{s=3\dagger}_{f=1}(\vec{p}) \right. &=& &-\beta\left(\stackrel{03}{[-i]}\stackrel{12}{(+)}|\stackrel{56}{[-]} + \frac{p^1+ip^2}{p^0+p^3}\stackrel{03}{(+i)}\stackrel{12}{[-]}|\stackrel{56}{[-]}\right)\Bigg)\cdot\\
&& &e^{-i(p^0x^0+\vec{p}\cdot\vec{x})}\,,\\
\left(\hat{\mathbf{b}}^{s=4\dagger}_{f=1}(\vec{p}) \right. &=& &-\beta^*\left(\stackrel{03}{(+i)}\stackrel{12}{[-]}|\stackrel{56}{[-]} - \frac{p^1-ip^2}{p^0+p^3}\stackrel{03}{[-i]}\stackrel{12}{(+)}|\stackrel{56}{[-]}\right)\Bigg)\cdot\\
&& &e^{-i(p^0x^0-\vec{p}\cdot\vec{x})}\,, \qquad (84)
\end{aligned}
$$

Index $^{s=(1,2,3,4)}$ counts different solutions of the Weyl equations, index $_{f=1}$ denotes the family quantum number, all solutions belong to the same family, while $\beta^*\beta = \frac{p^0+p^3}{2p^0}$ takes care that the corresponding states are normalized.

All four superposition of $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|_{p^0=|\vec{p}|} = \sum_m c^{ms}{}_f\; (\vec{p}, p^0 = |\vec{p}|)\; \hat{b}^\dagger_{\vec{p}}\, \hat{b}^{m\dagger}_f e^{-i(p^0x^0-\varepsilon\vec{p}\cdot\vec{x})}$, Eq. (55), with $m=(1,2)$ for the first two states, and with $m=(3,4)$ for the second two states, Table 3, $s=(1,2,3,4)$, are orthogonal and correspondingly normalized.

The vacuum state $|\psi_{oc}>$, Eq. (46), is the sum of selfajoint operators ($\stackrel{03}{[-i]}\stackrel{12}{[-]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[+]}|\stackrel{56}{[-]}$, $\stackrel{03}{[+i]}\stackrel{12}{[-]}|\stackrel{56}{[+]}$, and $\stackrel{03}{[-i]}\stackrel{12}{[+]}|\stackrel{56}{[+]}$), needed that the first, second, third and fourth family creation operators, respectively, applying on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$, give nonzero value, while their Hermitian conjugated partners $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|_{p^0=|\vec{p}|}$, applying on $|\psi_{oc}> |0_{\vec{p}}>$, give zero.

The discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$, Eq. (72), which is in this particular case equal to $\gamma^0\gamma^5 I_{\vec{x}_3}$, transforms $\hat{\mathbf{b}}^{s=1\dagger}_{f=1}(\vec{p})$ into $\hat{\mathbf{b}}^{s=3\dagger}_{f=1}(\vec{p})$ and $\hat{\mathbf{b}}^{s=2\dagger}_{f=1}(\vec{p})$ into $\hat{\mathbf{b}}^{s=1\dagger}_{f=1}(\vec{p})$.

4.3.2 Clifford fermions and antifermions in $d=(9+1)$ and $d=(13+1)$

We look for in both cases, in the case of $d=(9+1)$ and in the case of $d=(13+1)$, the solutions of the Weyl equation, Eq. (79), under the assumption that moments in all higher dimensions, $d\geq 5$, are in the low energy regime equal to zero, $p^a=(p^0,p^1,p^2,p^3,0,0,\ldots,0)$. We make a choice in both cases of only one family out of eight families presented in Table 5.

Let us first look at the properties of one family (one irreducible representation) of $SO(13,1)$, with $2^{\frac{d}{2}-1}=64$ members, presented in Table 7. This family contains also $SO(9,1)$ family with $2^{\frac{d}{2}-1}=16$ members, which can be recognized in Table 7.

In Table 7 the "basis vectors" — the creation operators in internal space of fermions with the quantum numbers of the observed quarks and leptons and antiquarks and antileptons, $\hat{b}^{m\dagger}_f$, $m=$

$(1, 2, \ldots, 2^{\frac{d}{2}-1} = 64,$ — are presented. The whole table follows from the starting $\hat{b}^{m=1\dagger}_{f=1} = \hat{u}^{c1\dagger}_{R}$, representing the right handed u-quark with $S^{12} = \frac{1}{2}$, the colour charge ($\tau^{33} = \frac{1}{2}, \tau^{38} = \frac{1}{2\sqrt{3}}$), the weak charge $\tau^{13} = 0$, the hypercharge $Y = \frac{2}{3}$, the electromagnetic charge $Y = \frac{2}{3}$ and the "fermion charge" $\tau^4 = \frac{1}{6}$.

The reader can calculate all the quantum numbers of Table 7 and Table 5, if taking into account the generators of the two $SU(2)$ ($\subset SO(3,1) \subset SO(7,1) \subset SO(13,1)$) groups, describing spins of fermions and the corresponding family quantum numbers

$$\vec{N}_{\pm}(=\vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm i S^{01}, S^{31} \pm i S^{02}, S^{12} \pm i S^{03})\,, \quad \vec{\tilde{N}}_{\pm}(=\vec{\tilde{N}}_{(L,R)}) := \frac{1}{2}(\tilde{S}^{23} \pm i \tilde{S}^{01}\,, \tag{85}$$

the generators of the two $SU(2)$ ($SU(2) \subset SO(4) \subset SO(7,1) \subset SO(13,1)$) groups, describing the weak charge, $\vec{\tau}^1$, and the second kind of the weak charge, $\vec{\tau}^2$, of fermions and the corresponding family quantum numbers

$$\begin{aligned}
\vec{\tau}^1 : &= \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})\,, \quad \vec{\tau}^2 := \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78})\,,\\
\vec{\tilde{\tau}}^1 : &= \frac{1}{2}(\tilde{S}^{58} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78})\,, \quad \vec{\tilde{\tau}}^2 := \frac{1}{2}(\tilde{S}^{58} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78}),
\end{aligned} \tag{86}$$

and the generators of $SU(3)$ and $U(1)$ subgroups of $SO(6) \subset SO(13,1)$, describing the colour charge and the "fermion" charge of fermions as well as the corresponding family quantum number $\tilde{\tau}^4$

$$\begin{aligned}
\vec{\tau}^3 := &\quad \frac{1}{2}\{S^{9\,12} - S^{10\,11}\,, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}\,, S^{9\,14} - S^{10\,13},\\
&\quad S^{9\,13} + S^{10\,14}\,, S^{11\,14} - S^{12\,13}\,, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})\}\,,\\
\tau^4 := &\quad -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})\,,\\
\tilde{\tau}^4 := &\quad -\frac{1}{3}(\tilde{S}^{9\,10} + \tilde{S}^{11\,12} + \tilde{S}^{13\,14})\,.
\end{aligned} \tag{87}$$

The (chosen) Cartan subalgebra operators, determining the commuting operators in the above equations, is presented in Eq. (8).

The hypercharge Y and the electromagnetic charge Q and the corresponding family quantum numbers then follows as

$$\begin{aligned}
&Y := \tau^4 + \tau^{23}\,, \;\; Q := \tau^{13} + Y\,, \;\; Y' := -\tau^4 \tan^2\vartheta_2 + \tau^{23}\,, \;\; Q' := -Y \tan^2\vartheta_1 + \tau^{13}\,, \quad ,\\
&\tilde{Y} := \tilde{\tau}^4 + \tilde{\tau}^{23}\,, \;\; \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}\,, \;\; \tilde{Y'} := -\tilde{\tau}^4 \tan^2\vartheta_2 + \tilde{\tau}^{23}\,, \;\; \tilde{Q}' = -\tilde{Y} \tan^2\vartheta_1 + \tilde{\tau}^{13}\,. \quad .
\end{aligned} \tag{88}$$

Below are some of the above expressions written in terms of nilpotents and projectors

$$\begin{aligned}
N^{\pm}_{+} &= N^1_+ \pm i\, N^2_+ = -\overset{03}{(\mp i)}\overset{12}{(\pm)}\,, \quad N^{\pm}_{-} = N^1_- \pm i\, N^2_- = \overset{03}{(\pm i)}\overset{12}{(\pm)}\,,\\
\tilde{N}^{\pm}_{+} &= -\overset{03}{\tilde{(\mp i)}}\overset{12}{\tilde{(\pm)}}\,, \quad \tilde{N}^{\pm}_{-} = \overset{03}{\tilde{(\pm i)}}\overset{12}{\tilde{(\pm)}}\,,\\
\tau^{1\pm} &= (\mp)\ \overset{56}{(\pm)}\overset{78}{(\mp)}\,, \quad \tau^{2\mp} = (\mp)\ \overset{56}{(\mp)}\overset{78}{(\mp)}\,,\\
\tilde{\tau}^{1\pm} &= (\mp)\ \overset{56}{\tilde{(\pm)}}\overset{78}{\tilde{(\mp)}}\,, \quad \tilde{\tau}^{2\mp} = (\mp)\ \overset{56}{\tilde{(\mp)}}\overset{78}{\tilde{(\mp)}}\ .
\end{aligned} \tag{89}$$

The corresponding annihilation operators in internal space of fermions $\hat{b}^m_f$ are the Hermitian conjugated partners of the creation operators, $\hat{b}^{m\dagger}_f$, of Table 7 and Table 5.

Table 5: Eight families of the "basis vectors", the creation operators $\hat{b}^m_f$, of $\hat{u}^{c1\dagger}_R$ — the right handed u-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table 7 — and of the colourless right handed neutrino $\hat{\nu}^\dagger_R$ of spin $\frac{1}{2}$, appearing in the 25^{th} line of Table 7 — are presented in the left and in the right part of this table, respectively. Table is taken from [19]. Families belong to two groups of four families, one (I) is a doublet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a singlet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eqs. (85, 86), the other (II) is a singlet with respect to ($\vec{\tilde{N}}_L$ and $\vec{\tilde{\tau}}^{(1)}$) and a doublet with respect to ($\vec{\tilde{N}}_R$ and $\vec{\tilde{\tau}}^{(2)}$), Eqs. (85, 86). All the families follow from the starting one by the application of the operators ($\tilde{N}^{\pm}_{R,L}$, $\tilde{\tau}^{(2,1)\pm}$), Eq. (89). The generators ($N^{\pm}_{R,L}$, $\tau^{(2,1)\pm}$), Eq. (89), transform $\hat{u}^\dagger_{1R}$ to all the members of one family of the same colour charge. The same generators transform equivalently the right handed neutrino $\hat{\nu}^\dagger_{1R}$ to all the colourless members of the same family.

					$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$	$\tilde{\tau}^4$
I	$\hat{u}^{c1\dagger}_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,2}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
I	$\hat{u}^{c1\dagger}_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,4}$	$\stackrel{03}{[+i]}\stackrel{12}{(+)}\vert\stackrel{56}{(+)}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,5}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,6}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\vert\stackrel{56}{[+]}\stackrel{78}{[+]}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,7}$	$\stackrel{03}{[+i]}\stackrel{12}{[+]}\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
II	$\hat{u}^{c1\dagger}_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}$	$\hat{\nu}^\dagger_{R\,8}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)}\vert\stackrel{56}{(+)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\;\;\stackrel{11\,12}{(+)}\;\;\stackrel{13\,14}{(+)}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{3}$

One sees that quarks of all three colours as well as the colourless leptons have the same $SO(7,1)$ part and the same is true for the antiquarks and antilepons. The operator S^{09}, for example, transforms $\hat{u}^{c1\dagger}_R$ into $\hat{\bar{e}}^\dagger_L$, presented on 57 line of Table 7, while the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^0\,\gamma^5\,\cdots\gamma^{d-1}\,I_{\vec{x}_3}\,I_{x^6,x^8,\ldots,x^d}$, presented in Eq. (72) of Sect. 3.3.5, transforms $\hat{u}^{c1\dagger}_R$ into $\bar{u}^{\bar{c1}\dagger}_L$, which is the left handed anti u-quark of the same spin S^{12} and all the charges of opposite values, appearing in this table on the line 35.

Table 5 presents "basis vectors" ($\hat{b}^{m\dagger}_f$, Eq. (43)) for eight families of the right handed u-quark of the colour ($\frac{1}{2}$, $\frac{1}{2\sqrt{3}}$) and the right handed colourless ν — the SO(7,1) content of the $SO(13,1)$ group are in both cases identical, they distinguish only in the $SU(3)$ and $U(1)$ subgroups of $SO(6)$. All the members of any of these eight families of Table 5 follows from either the u-quark or the ν by the application of S^{ab}. Each family carries the family quantum numbers, determined by the Cartan subalgebra of $\tilde{S}^{ab}$ in Eq. (8) and presented in Table 5. When we treat the $d = (9+1)$ case, the families can be assumed, determined in this case with the Cartan subalgebra members of $\tilde{\tau}^{13}, \tilde{\tau}^{23}, \tilde{N}^3_L, \tilde{N}^3_R$ and $\tilde{S}^{9\,10}$. This is not the case, which would be realized in nature, at least it is not yet observed.

The Weyl equations for free massless fermions, Eq. (79), offer the same solutions for all the families. Only when fermions interact with the scalar fields, which "see" the family quantum numbers, the solutions depend on the family quantum numbers.

It is the assumption that the eight families from Table 5 remain massless after the break of symmetry from $SO(13,1)$ to $SO(7,1)\times SO(6)$. We proved for the toy model [127, 128] that the break of symmetry can leave some families of fermions massless, while the rest become massive. But we have not yet proven the masslessness of the $2^{\frac{7+1}{2}-1}$ families after the break from $SO(13,1)$ to $SO(7,1)\times SO(6)$. The assumed break from the starting symmetry $SO(13,1)$ to $SO(7,1)\times SU(3)\times U(1)$is supposed to be caused by the appearance of the condensate of two right handed neutrinos with the family quantum numbers of the upper four families, that is of the four families, which do not contain the three so far observed families, at the energy of $\geq 10^{16}$ GeV. This condensate is presented in Table 6.

Table 6: The condensate of the two right handed neutrinos ν_R, with the quantum numbers of the $VIII^{th}$ family, coupled to spin zero and belonging to a triplet with respect to the generators τ^{2i}, is presented, together with its two partners. The condensate carries $\vec{\tilde{\tau}}^1 = 0$, $\tau^{23} = 1$, $\tau^4 = -1$ and $Q = 0 = Y$. The triplet carries $\tilde{\tau}^4 = -1$, $\tilde{\tau}^{23} = 1$ and $\tilde{N}^3_R = 1$, $\tilde{N}^3_L = 0$, $\tilde{Y} = 0$, $\tilde{Q} = 0$. The family quantum numbers of quarks and leptons are presented in Table 5.

state	S^{03}	S^{12}	τ^{13}	τ^{23}	τ^4	Y	Q	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{\tau}^4$	$\tilde{Y}$	$\tilde{Q}$	$\tilde{N}^3_L$	$\tilde{N}^3_R$
$(\lvert\nu^{\mathbf{VIII}}_{\mathbf{1R}}>_{\mathbf{1}}\ \lvert\nu^{\mathbf{VIII}}_{\mathbf{2R}}>_{\mathbf{2}})$	0	0	0	1	−1	0	0	0	1	−1	0	0	0	1
$(\lvert\nu^{VIII}_{1R}>_1\ \lvert e^{VIII}_{2R}>_2)$	0	0	0	0	−1	−1	−1	0	1	−1	0	0	0	1
$(\lvert e^{VIII}_{1R}>_1\ \lvert e^{VIII}_{2R}>_2)$	0	0	0	−1	−1	−2	−2	0	1	−1	0	0	0	1

In the case that the kinematics of quarks and leptons are determined by moments in $d = (3+1)$, the Weyl equation, Eq. (79), connects the creation operators with spin up and down, that is the first two lines in Table 7, when we treat the right handed weak chargeless u-quark of the colour charge $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, with $\tau^4 = \frac{1}{6}$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$. When we treat their antiparticles, then the 35^{th} and the 36^{th} lines of Table 7 contribute to the solutions of the Weyl equation. They are the left handed weak chargeless u-antiquark of the colour charge $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the "fermion" charge $\tau^4 = -\frac{1}{6}$, the hyper charge $Y = -\frac{2}{3}$ and the electromagnetic charge $Q = -\frac{2}{3}$. The solutions then look as follows

$$
\begin{aligned}
&\text{Clifford odd creation operators in d} = (13+1)\\
p^0 &= |p^0|\,,\ \ c_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})\,,\ \ \Gamma^{(3+1)} = 1\,,\\
\Big(\hat{\mathbf{u}}^{c1 s=1\dagger}_{Rf=1}(\vec{p}) &= \beta \Big(\overset{03}{(+i)}\overset{12}{[+]} + \frac{p^1 + ip^2}{p^0 + p^3}\, \overset{03}{[-i]}\overset{12}{(-)} \Big) \cdot \overset{56}{[+]}\overset{78}{(+)} \,||\, \overset{9\,10}{(+)}\ \overset{11\,12}{[-]}\ \overset{13\,14}{[-]} \Big) \cdot\\
&\quad e^{-i(p^0 x^0 - \vec{p}\cdot\vec{x})}\,,\\
\Big(\hat{\mathbf{u}}^{c1 s=2\dagger}_{Rf=1}(\vec{p}) &= \beta^* \Big(\overset{03}{[-i]}\overset{12}{(-)} - \frac{p^1 - ip^2}{p^0 + p^3}\, \overset{03}{(+i)}\overset{12}{[+]} \Big) \cdot \overset{56}{[+]}\overset{78}{(+)} \,||\, \overset{9\,10}{(+)}\ \overset{11\,12}{[-]}\ \overset{13\,14}{[-]} \Big) \cdot\\
&\quad e^{-i(p^0 x^0 + \vec{p}\cdot\vec{x})}\,,\\
&\text{Clifford odd creation operators in d} = (13+1)\\
p^0 &= |p^0|\,,\ \ \bar{c}_1 = (-\frac{1}{2}, -\frac{1}{2\sqrt{3}})\,,\ \ \Gamma^{(3+1)} = -1\,,\\
\Big(\hat{\bar{\mathbf{u}}}^{\bar{c}1 s=35\dagger}_{Lf=1}(\vec{p}) &= -\beta \Big(\overset{03}{[-i]}\overset{12}{[+]} + \frac{p^1 + ip^2}{p^0 + p^3}\, \overset{03}{(+i)}\overset{12}{(-)} \Big) \cdot \overset{56}{(-)}\overset{78}{[-]} \,||\, \overset{9\,10}{[-]}\ \overset{11\,12}{(+)}\ \overset{13\,14}{(+)} \Big) \cdot\\
&\quad e^{-i(p^0 x^0 + \vec{p}\cdot\vec{x})}\,,\\
\Big(\hat{\bar{\mathbf{u}}}^{\bar{c}1 s=36\dagger}_{Lf=1}(\vec{p}) &= -\beta^* \Big(\overset{03}{(+i)}\overset{12}{(-)} - \frac{p^1 - ip^2}{p^0 + p^3}\, \overset{03}{[-i]}\overset{12}{[+]} \Big) \cdot \overset{56}{(-)}\overset{78}{[-]} \,||\, \overset{9\,10}{[-]}\ \overset{11\,12}{(+)}\ \overset{13\,14}{(+)} \Big) \cdot\\
&\quad e^{-i(p^0 x^0 - \vec{p}\cdot\vec{x})}\,.
\end{aligned}
\tag{90}
$$

Let us recognize that the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$, Eq. (72), which is in this case equal to $\gamma^0\gamma^5\gamma^7\gamma^9\gamma^{11}\gamma^{13} I_{\vec{x}_3}$, transforms $\hat{\mathbf{u}}^{c1s=1\dagger}_{Rf=1}(\vec{p})$ into $\hat{\bar{\mathbf{u}}}^{\bar{c}1s=35\dagger}_{Lf=1}(\vec{p})$ and $\hat{\mathbf{u}}^{c1s=2\dagger}_{Rf=1}(\vec{p})$ into $\hat{\bar{\mathbf{u}}}^{\bar{c}1s=36\dagger}_{Lf=1}(\vec{p})$.

We can look for the solutions for leptons in the same way. The left handed neutrino, for example, with momentum $p^a = (p^0, \vec{p}, 0, 0, \ldots, 0)$, with $\vec{p} = (p^1, p^2, p^3)$, would be the superposition of the two creation operators appearing in 31^{st} and 32^{nd} lines of the Tabele 7, with the same coefficients as presented in Eq. (90), but the handedness and the weak charge would be changed and its colour chargeless would be represented by $\overset{9\,10}{(+)}\ \overset{11\,12}{(+)}\ \overset{13\,14}{(+)}$. The righ handed antineutrino would be superposition of the lines 61 and 62.

To treat the $d = (9+1)$-dimensional case the same table, Table 7, can be used. The first eight lines and the lines from $33-40$ belong to one irreducible representation, sharing the quantum numbers with respect to $SO(7,1)$ with quarks and leptons, and antiquarks and antileptons. This $d = (9+1)$ case distinguishes from the $d = (13+1)$ case in the wealth of the colour charges: There is only one possibility for the "colour" charge. The solutions of the Weyl equation under the assumption that the momentum $p^a = (p^0, \vec{p}, 0, 0, \dots, 0)$, with $\vec{p} = (p^1, p^2, p^3)$ are changed with respect to Eq. (90) only in the colour part.

Table 7 represents in the *spin-charge-family* theory the "basis states" (the creation operators) for internal space of the observed *quarks and leptons and antiquarks and antileptons* for the first of the eight families of Table 5. Hermitian conjugation of the creation operators of Table 7 generates the corresponding annihilation operators, fulfilling together with the creation operators, if applying on the vacuum state $|\psi_{oc}>$, Eq. (46), the anticommutation relations of Eq. (43).

Creation and annihilation operators for quarks and leptons and antiquarks and antileptons are the tensor products of the finite "basis vectors" in internal space and the continuously infinite basis in ordinary space, as presented in Eq. (55). Since the "basis vectors" in internal space transfer their oddness to creation and annihilation operators, the creation and annihilation operators for quark and leptons and antiquarks and antileptons obey on the vacuum state $|\psi_{oc}> |0_{\vec{p}}>$ the anticommutation relations of Eq. (63), as discussed in Sect. 3.3.2.

All the creation operators and their Hermitean conjugated partners, presented in Eqs. (84, 90), fulfill the anticommutation relations for the second quantized fermion fields of Eq. (63). Tables 7 and 5 represent 64 "basic states" in the internal space of fermions, appearing in eight families. If applying on the vacuum state $|\psi_{oc}>$, these creation operators describe the internal space of quarks and lepons and antiquarks and antileptons, which carry one of the two handedness, one of the two S^{12} spins, one of the two weak charges τ^{13}, one of the two $SU(2)_{II}$ charges τ^{23}, either one of the three colour charges (τ^{33}, τ^{38}) or one of the three colour anticharges or they carry the colourless or anticolourless quantum numbers of leptons. Quarks carry the "fermion" quantum number $\tau^4 = \frac{1}{6}$, antiquarks carry the "fermion" quantum number $-\frac{1}{6}$, leptons carry the "fermion" quantum number $-\frac{1}{2}$, antileptons carry the "fermion" quantum number $\frac{1}{2}$. The reader can clearly see in Table 7 the strong correlation among the "fermion" quantum number, charges and handedness. Each of the members carries in addition the family quantum numbers, the same for all the members of one irreducible representation Of S^{ab}. The eight possibilities, presented in Table 5, are clustered into two groups. Each of the two groups of four families manifests its own $SU(2) \times SU(2)$ structure as seen in the last fifth columns of Table 5. The first four families are doublets with respect to $\vec{\tilde{\tau}}^1$ and $\vec{\tilde{N}}_L$ and singlets with respect $\vec{\tilde{\tau}}^2$ and $\vec{\tilde{N}}_R$, the last four families are doublets with respect to $\vec{\tilde{\tau}}^2$ and $\vec{\tilde{N}}_R$ and singlets with respect$\vec{\tilde{\tau}}^1$ and $\vec{\tilde{N}}_L$. They all have the same $\tilde{\tau}^4$.

One can take subgroups $SO(3,1)$, $SO(4)$, $SU(3)$ and $U(1)$ of $SO(13,1)$ and add also the family groups. But the correlation among spin, handedness, charges and families for fermions and antifermions must in such a case be postulated and also the symmetry of families must be chosen. In this case oddness or evenness of creation operators must be postulated as well, in the same way as Dirac did, Eqs. (51, 52).

All the creation operators, creating quarks and leptons and antiquarks and antileptons, have an odd Clifford character. Together with their Hermitian conjugated partners they take care of the anticommutation properties of creation operators, determining the states in the internal and momentum (or coordinate) space, as presented in Eqs. (84, 90).

Let us at the end point out again that the description of the internal space of fermions in $d = 2(2n+1)$-dimensional space (the one describing spin and handedness in $d = (3+1)$ and the one representing in $d = (3+1)$ the fermion charges, originating in $d \geq 5$) with odd Clifford algebra objects makes handedness related to charges, takes care of the families, as well as of the anticommuting properties of fermions. In the *standard model*, as well as in the unified theories, the relations among spins, handedness, charges and families of fermions and antifermions must be postulated, as well as the anticommutation relations among creation and annihilation operators, the last one in the way Dirac did [15].

Table 7: The left handed ($\Gamma^{(13,1)} = -1$), multiplet of creation operators of fermions — the members of the first family in Table 5 (each family represents one fundamental representation of the $SO(13,1)$ group), manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented in the massless basis as odd products of nilpotents and the rest of projectors (together are $\frac{d}{2} = 7$ nilpotents and projectors). The multiplet contains the left handed ($\Gamma^{(3+1)} = -1$ weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$, ($\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})$) and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$, $\vec{\tau}^2 = \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78})$) quarks and leptons and the right handed ($\Gamma^{(3+1)} = 1$), weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($(\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$, ($\vec{\tau}^3 = \frac{1}{2}(S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}))$, carrying the "fermion charge" ($\tau^4 = \frac{1}{6}, = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$. The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). To the same multiplet of creation operators the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons belong. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Anti-quarks are antitriplets, carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anti-colourless antileptons carry the "fermion" charge ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. The creation operators of opposite charges (antifermion creation operators) are reachable from the fermion ones besides by S^{ab} also by the application of the discrete symmetry operator $\mathbb{C}_{\mathcal{N}}\, \mathcal{P}_{\mathcal{N}}$, presented in Refs. [125, 126]. The reader can find this Weyl representation also in Refs. [13, 130, 131, 19] and in the references therein.

i		$^a\hat{b}_i^\dagger$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1$, $\Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
1	$\hat{u}_R^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$\hat{u}_R^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$\hat{d}_R^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\hat{d}_R^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$\hat{d}_L^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$\hat{d}_L^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$\hat{u}_L^{c1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	$\hat{u}_L^{c1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$\hat{u}_R^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$\hat{u}_R^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$\hat{d}_R^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$\hat{d}_R^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$\hat{d}_L^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$\hat{d}_L^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$\hat{u}_L^{c2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\;\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

Continued on next page

[15] In App. B.1 the properties of even Clifford algebra objects are discussed, as well as the relation in this case between creation operators and their Hermitian conjugated partners.

i		${}^{a}\hat{b}_{i}^{\dagger}$	$\Gamma^{(3+1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
		(Anti)octet, $\Gamma^{(7+1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\ -1$ of (anti)quarks and (anti)leptons									
16	$\hat{u}_{L}^{c2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$\hat{u}_{R}^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$\hat{u}_{R}^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$\hat{d}_{R}^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$\hat{d}_{R}^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	$\hat{d}_{L}^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$\hat{d}_{L}^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$\hat{u}_{L}^{c3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$\hat{u}_{L}^{c3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	$\hat{\nu}_{R}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	$\hat{\nu}_{R}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	$\hat{e}_{R}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	$\hat{e}_{R}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	$\hat{e}_{L}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	$\hat{e}_{L}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	$\hat{\nu}_{L}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	$\hat{\nu}_{L}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\hat{\bar{d}}_{L}^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\hat{\bar{d}}_{L}^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\hat{\bar{u}}_{L}^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\hat{\bar{u}}_{L}^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\hat{\bar{d}}_{R}^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\hat{\bar{d}}_{R}^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\hat{\bar{u}}_{R}^{\bar{c}1\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\hat{\bar{u}}_{R}^{\bar{c}1\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\hat{\bar{d}}_{L}^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\hat{\bar{d}}_{L}^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\hat{\bar{u}}_{L}^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\hat{\bar{u}}_{L}^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\hat{\bar{d}}_{R}^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\hat{\bar{d}}_{R}^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\hat{\bar{u}}_{R}^{\bar{c}2\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\hat{\bar{u}}_{R}^{\bar{c}2\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\hat{\bar{d}}_{L}^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\hat{\bar{d}}_{L}^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\hat{\bar{u}}_{L}^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\hat{\bar{u}}_{L}^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\hat{\bar{d}}_{R}^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\hat{\bar{d}}_{R}^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\hat{\bar{u}}_{R}^{\bar{c}3\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\hat{\bar{u}}_{R}^{\bar{c}3\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\hat{\bar{e}}_{L}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\hat{\bar{e}}_{L}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\hat{\bar{\nu}}_{L}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\hat{\bar{\nu}}_{L}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\hat{\bar{\nu}}_{R}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\hat{\bar{\nu}}_{R}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\hat{\bar{e}}_{R}^{\dagger}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\hat{\bar{e}}_{R}^{\dagger}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

5 Hilbert space of Clifford fermions

The Clifford odd creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$, with $p^0 = |\vec{p}|$, are defined in Eq. (55) on the tensor products, $*_T$, of the $(2^{\frac{d}{2}-1})^2$ "basis vectors" $\hat{b}^{m\dagger}_{f}$ (describing the internal space of fermion fields) and of the (continuously) infinite basis in the momentum space, $\hat{b}^{\dagger}_{\vec{p}}$, applying on the vacuum state $|\psi_{oc}>|0_{\vec{p}}>$. The solutions of the Weyl equation, Eq. (79), are plane waves of particular momentum $\vec{p}$ and with the energy related to the momentum, $p^0 = |\vec{p}|$.

The creation operator $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ defines, when applying on the vacuum state $|\psi_{oc}>|0_{\vec{p}}>$, the s^{th} of the $2^{\frac{d}{2}-1}$ plane wave solutions of a particular momentum $\vec{p}$ belonging to the f^{th} of the $2^{\frac{d}{2}-1}$ "families". They fulfill together with the Hermitian conjugated partners annihilation operators $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ the anticommutation relations of Eq. (63).

The Hilbert space of the second quantized fermions consists of any number of tensor products, $*_{T_H}$, of all possible $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$, with the finite number of different (s,f) — $(2^{\frac{d}{2}-1})^2$ — and continuous many momentum $\vec{p}$.

The tensor products, $*_{T_H}$, of single fermion states offer any number of possibilities, starting with the empty case (empty "Slater determinant", with none of possible single fermion states occupied) up to

$$N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}} \tag{91}$$

possibilities for a chosen momentum $\vec{p}$, since any of the fermion states of particular $\vec{p}$ can be or is not among the chosen states (creating $2^{2^{d-2}}$ "Slater determinants" with none single fermion state occupied up to all the 2^{d-2} single fermion states of particular $\vec{p}$ occupied), up to infinite many possibilities when any momentum $\vec{p}$ can be empty or occupied

$$N_{\mathcal{H}} = \prod_{\vec{p}}^{\infty} 2^{2^{d-2}}\,. \tag{92}$$

Statement 9. The Hilbert space of Clifford fermions is generated by the tensor product multiplication, $*_{T_H}$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ of any $(s,f,\vec{p})$).

The Hilbert space of a particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, contains the finite number of "Slater determinants", $2^{2^{d-2}}$.

The total Hilbert space of anticommuting Clifford fermions is the product $\otimes_N$ of the Hilbert spaces of particular $\vec{p}$

$$\mathcal{H} = \prod_{\vec{p}}^{\infty} \otimes_N \mathcal{H}_{\vec{p}}\,. \tag{93}$$

The total Hilbert space $\mathcal{H}$ is correspondingly infinite.

Before starting to comment the application of the creation operators $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ on the Hilbert space $\mathcal{H}$ (determined by all possible "Slater determinants" of all possible occupied and empty fermion states of all possible $(s,f,\vec{p})$, what means the tensor products $*_{T_H}$ of all possible single fermion states of all possible $(s,f,\vec{p})$, with the identity — the empty "Slater deteminant" included) let us discuss properties of creation and annihilation operators, the anticommutation relations of which are presented in Eq. (63).

The creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and the annihilation operators $\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})$, having an odd Clifford character, anticommute, manifesting the properties as follows

$$\begin{aligned}
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p}') &= -\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p}') *_{T_H} \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\,, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s'}_{f'}(\vec{p}') &= -\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}') *_{T_H} \hat{\mathbf{b}}^{s}_{f}(\vec{p})\,, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p}') &= -\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p}') *_{T_H} \hat{\mathbf{b}}^{s}_{f}(\vec{p})\,, \\
\text{if at least one of } (s,f,\vec{p}) & \quad \text{is different from } (s',f',\vec{p}')\,, \\
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) &= 0\,, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s}_{f}(\vec{p}) &= 0\,, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) &= 1\,(\text{identity})\,, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p})|\psi_{oc}>|0_{\vec{p}}> &= 0\,.
\end{aligned} \tag{94}$$

The above relations following from the anticommutation relations of Eq. (63), determine the rules of the application of creation and annihilation operators on "Slater determinants":

i. The creation operator $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ jumps over the creation operators determining the occupied state of another kind (that is over the occupied state distinguishing from the jumping creation operator one in any of the internal quantum numbers (s,f) or in $\vec{p}$) up to the last step when it comes to its own empty state with the quantum numbers (f,s) and $\vec{p}$, occupying this empty state, or, if this state is already occupied, gives zero. Whenever $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ jumps over an occupied state, changes the sign of the "Slater determinant".

ii. The annihilation operator changes the sign whenever jumping over the occupied state carrying different internal quantum numbers (s,f) or different $\vec{p}$, unless it comes to the occupied state with its own internal quantum numbers (s,f) and its own $\vec{p}$, emptying this state, or, if this state is empty, gives zero.

Let us point out again that the Clifford odd creation operators, $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$, and annihilation operators, $\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})$, fulfill the anticommutation relations of Eq. (63) for any $\vec{p}$ and any (s,f) due to the anticommuting character (the Clifford oddness) of the "basis vectors", $\hat{b}^{m\dagger}_{f}$ and their Hermitian conjugated partners $\hat{b}^{m}_{f}$, Eqs. (43, 44, 45), what means that the anticommuting character of creation and annihilation operators is not postulated, it origins in the Clifford oddness describing the internal space of fermions.

The total Hilbert space $\mathcal{H}$ has infinite number of degrees of freedom (of "Slater determinants") due to the infinite number of Hilbert spaces $\mathcal{H}_{\vec{p}}$ of particular $\vec{p}$, $\mathcal{H}=\prod^{\infty}_{\vec{p}} \otimes_{N} \mathcal{H}_{\vec{p}}$, while the Hilbert space $\mathcal{H}_{\vec{p}}$ of particular momentum $\vec{p}$ has the finite dimension $2^{2^{d-2}}$.

Let us write down the number operator, counting the number of fermions with particular choice of quantum numbers (s,f) and particular $\vec{p}$, $\hat{N}^{sf}_{\vec{p}}$,

$$\hat{N}^{sf}_{\vec{p}} = \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\,\hat{\mathbf{b}}^{s}_{f}(\vec{p})\,. \tag{95}$$

$\hat{N}^{sf}_{\vec{p}}$ is obviously the Clifford even operator, and when jumping over the occupied states of with (s',f') and $\vec{p'}$ not equal to (s,f) and $\vec{p}$ at least in one of the three properties, no sign is changed.

It follows that the application of the number operator $\hat{N}^{sf}_{\vec{p}}$ on the vacuum state $|\psi_{oc}>|0_{\vec{p}}>$ and on all the "Slater determinants" with the particular fermion state, defined by the creation operator, $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$, unoccupied (denoted in Eq. (96) as $\mathbf{0^{sf}}_{\vec{p}}$), gives zero contribution, while the application of the number operator $\hat{N}^{sf}_{\vec{p}}$ on all the rest "Slater determinants", with this particular state occupied (denoted in Eq. (96) as $\mathbf{1^{sf}}_{\vec{p}}$), gives identity

$$\begin{aligned}
\hat{N}^{sf}_{\vec{p}}\,|\psi_{oc}>|0_{\vec{p}}> &=& 0\cdot|\psi_{oc}>\,|0_{\vec{p}}>\,, \qquad \hat{N}^{sf}_{\vec{p}}\,\mathbf{0^{sf}}_{\vec{p}}=0\,,\\
\hat{N}^{sf}_{\vec{p}}\,\mathbf{1^{sf}}_{\vec{p}} &=& 1\,\cdot\mathbf{1^{sf}}_{\vec{p}}\,,\quad \hat{N}^{sf}_{\vec{p}}\,\hat{N}^{sf}_{\vec{p}}\,\mathbf{1^{sf}}_{\vec{p}}=1\cdot\mathbf{1^{sf}}_{\vec{p}}\,.
\end{aligned} \tag{96}$$

One can check the above relations on the example of $d=(5+1)$, with the "basis vectors" for $f=(1,2,3,4)$ presented in Table 3 and with the solution for Weyl equation, Eq. (79), presented in Eq. (84).

In Sect. 5.1 the properties of Hilbert spaces are discussed in more details and also illustrated.

5.1 Application of $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ on Hilbert space of Clifford fermions

Let us write down first the Hilbert space of second quantized fermions $\mathcal{H}_{\vec{p}}$, of particular $\vec{p}$. The generalization to the total Hilbert space, $\mathcal{H}$, is presented in Eq. (93). Let us use the simplified notation by denoting for $f=1$ empty states as $\mathbf{0_{rp}}$, and occupied states as $\mathbf{1_{rp}}$, with $r=(1,\dots,2^{\frac{d}{2}-1})$, for $f=2$ we count $r=2^{\frac{d}{2}-1}+1,\cdots,2\cdot 2^{\frac{d}{2}-1}$, anding up with $r=2^{d-2}$. Correspondingly we can represent $\mathcal{H}_{\vec{p}}$ as follows

$$\begin{array}{c}
|\mathbf{0_{1p}},\mathbf{0_{2p}},\mathbf{0_{3p}},\dots,\mathbf{0_{2^{d-2}p}}>|_1 \quad ,\\
|\mathbf{1_{1p}},\mathbf{0_{2p}},\mathbf{0_{3p}},\dots,\mathbf{0_{2^{d-2}p}}>|_2 \quad ,\\
|\mathbf{0_{1p}},\mathbf{1_{2p}},\mathbf{0_{3p}},\dots,\mathbf{0_{2^{d-2}p}}>|_3 \quad ,\\
|\mathbf{0_{1p}},\mathbf{0_{2p}},\mathbf{1_{3p}},\dots,\mathbf{0_{2^{d-2}p}}>|_4 \quad ,\\
\vdots\\
|\mathbf{1_{1p}},\mathbf{1_{2p}},\mathbf{0_{3p}},\dots,\mathbf{0_{2^{d-2}p}}>|_{2^{d-2}+2} \quad ,\\
\vdots\\
|\mathbf{1_{1p}},\mathbf{1_{2p}},\mathbf{1_{3p}},\dots,\mathbf{1_{2^{d-2}p}}>|_{2^{2^{d-2}}} \quad ,
\end{array} \tag{97}$$

with a part with none of states occupied ($N_{rp}=0$ for all $r=1,\dots,2^{d-2}$), with a part with only one of states occupied ($N_{rp}=1$ for a particular $r=(1,\dots,2^{d-2})$, while $N_{r'p}=0$ for all the others $r'\neq r$), with a part with only two of states occupied ($N_{rp}=1$ and $N_{r'p}=1$, where r and r' run from $(1,\dots,2^{d-2})$, and so on. The last part has all the states occupied.

It is not difficult to see that the creation and annihilation operators, when applied on this Hilbert space $\mathcal{H}_{\vec{p}}$, fulfill the anticommutation relations for the second quantized Clifford fermions.

$$\begin{aligned}
\{\hat{\mathbf{b}}^{s}_{f}(\vec{p})\,,\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p})\}_{+}*_{T_H}\mathcal{H}_{\vec{p}} &=& \delta^{ss'}\,\delta^{ff'}\mathcal{H}_{\vec{p}}\,,\\
\{\hat{\mathbf{b}}^{s}_{f}(\vec{p})\,,\hat{\mathbf{b}}^{s'}_{f'}(\vec{p})\}_{+}*_{T_H}\mathcal{H}_{\vec{p}} &=& 0\,\cdot\mathcal{H}_{\vec{p}}\,,\\
\{\hat{\mathbf{b}}^{s}_{f}(\vec{p})\,,\hat{\mathbf{b}}^{s'}_{f'}(\vec{p})\}_{+}*_{T_H}\mathcal{H}_{\vec{p}} &=& 0\,\cdot\mathcal{H}_{\vec{p}}\,.
\end{aligned} \tag{98}$$

The proof for the above relations easily follows if one takes into account that whenever the creation or annihilation operator jumps over an odd products of occupied states the sign of the "Slater determinant" changes due to the oddness of each of the occupied states. The contribution of the application of $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})*_T\ \hat{\mathbf{b}}^{s'}_{f'}(\vec{p})\ *_T$, $(s,f)\neq(s',f')$, on $\mathcal{H}_{\vec{p}}$ has the opposite sign than the contribution of $\hat{\mathbf{b}}^{s'}_{f'}(\vec{p})$ $*_T\,\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\ *_T$ on $\mathcal{H}_{\vec{p}}$, due to exchanged places of both operators. The sum of both contributions gives therefore zero. If the creation and annihilation operators are Hermitian conjugated to each other, the result follows

$$(\hat{\mathbf{b}}^{s}_{f}(\vec{p})\ *_T\ \hat{\mathbf{b}}^{sf\dagger}_{f}(\vec{p})+\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\ *_T\ \hat{\mathbf{b}}^{s}_{f}(\vec{p})\,)\ *_T\ \mathcal{H}_{\vec{p}}=\mathcal{H}_{\vec{p}}\,,$$

manifesting that this application on $\mathcal{H}_{\vec{p}}$ gives the whole $\mathcal{H}_{\vec{p}}$ back. Each of the two summands operates on their own half of $\mathcal{H}_{\vec{p}}$. Jumping together over an even number of occupied states, $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{sf\,\dagger}_{f}(\vec{p})$ do not change the sign of the particular "Slater determinant". (Let us add that $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ reduces for the particular s and f the Hilbert space $\mathcal{H}_{\vec{p}}$ for the factor $\frac{1}{2}$, and so does $\hat{\mathbf{b}}^{sf\dagger}_{tot}(\vec{p})$. The sum of both, applied on $\mathcal{H}_{\vec{p}}$, reproduces the whole $\mathcal{H}_{\vec{p}}$.)

The generalization of the commutation relations to all possible $\vec{p}$ is due to the orthogonality of momentum basis, Eq. (49), straightforward.

Let us repeat that the number of "Slater determinants" in the Hilbert space of particular momentum $\vec{p}$, $\mathcal{H}_{\vec{p}}$, in d-dimensional space is finite and equal to $N_{\mathcal{H}_{\vec{p}}} = 2^{2^{d-2}}$.

The total Hilbert space of anticommuting fermions is the continuously infinite product of the Hilbert spaces of particular $\vec{p}$, Eq. (93), $\mathcal{H} = \prod^{\infty}_{\vec{p}} \otimes_N \mathcal{H}_{\vec{p}}$, with the empty "Slater determinant" counted.

Due to the Clifford odd character of creation and annihilation operators, Eq. (63), and the orthogonality of the solutions of different momenta $\vec{p}$, App. J, it follows that $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}\,') *_{T_H} \mathcal{H} \neq 0$, $\vec{p} \neq \vec{p}\,'$, while $\{\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}) *_{T_H} \hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}\,') + \hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}\,') *_{T_H} \hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})\} *_{T_H} \mathcal{H} = 0$, $\vec{p} \neq \vec{p}\,'$. This can be proven if taking into account Eq. (94). The number of "Slater determinants" in the Hilbert space $\mathcal{H}$ in d-dimensional space is infinite $N_{\mathcal{H}} = \prod^{\infty}_{\vec{p}} 2^{2^{d-2}}$.

Since the creation operators $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ and the annihilation operators $\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}\,')$ fulfill for particular $\vec{p}$ the anticommutation relations on $\mathcal{H}_{\vec{p}}$, Eq. (98), and since the momentum states, the plane wave solutions, are orthogonal, and correspondingly the creation and annihilation operators defined on the tensor products of the internal basis and the momentum basis, representing fermions, anticommute, Eq. (63) (the Clifford odd objects $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ demonstrate their oddness also with respect to $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}\,')$), the anticommutation relations follow also for the application of $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$ on $\mathcal{H}$

$$
\begin{aligned}
\{\hat{\mathbf{b}}^{s}_{f}(\vec{p}) \,, \hat{\mathbf{b}}^{s'\,\dagger}_{f'}(\vec{p}\,')\}_{*_{T_H}+} \mathcal{H} &= \delta^{ss'}\, \delta_{ff'}\, \delta(\vec{p} - \vec{p}\,')\, \mathcal{H} \,, \\
\{\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}), \hat{\mathbf{b}}^{s'\,\dagger}_{f'}(\vec{p}\,')\}_{*_{T_H}+} \mathcal{H} &= 0 \,\cdot \mathcal{H} \,, \\
\{\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p}), \hat{\mathbf{b}}^{s'\,\dagger}_{f'}(\vec{p}\,')\}_{*_{T_H}+} \mathcal{H} &= 0 \,\cdot \mathcal{H} \,.
\end{aligned} \tag{99}
$$

Let us illustrate the properties of $\mathcal{H}$ and the application of the creation operators on $\mathcal{H}$ with a simple case of $d = (1+1)$ dimensional space in a toy model with two discrete momenta (p^1_1, p^1_2). Generalization to many momenta is straightforward.

The internal space of fermions contains only one creation operator, one "basis vector" $\hat{b}^{1\dagger}_{1} \overset{01}{=(+i)}$, one family member $m = 1$ of the only family $f = 1$. Correspondingly the creation operators multiplied by the plane waves (what solve the equations of motion), $\hat{\mathbf{b}}^{1\,\dagger}_{1}(\vec{p}^{1}_{i})|_{p^0 = |p^1_i|}\, e^{-i(p^0 x^0 - p^1_i x^1)}: \overset{01}{=(+i)}\; e^{-i(p^0 x^0 - p^1_i x^1)}|_{p^0_i = |p^1_i|}$ differ only in momentum space. Their Hermitian conjugated annihilation operators are $\hat{\mathbf{b}}^{1}_{1}(\vec{p}^{1}_{i})_{p^0 = |p^1_i|}$, while the vacuum state is $|\psi_{oc}> \; |0_{p^1_i} \overset{01}{=(-i)} \cdot \overset{01}{(+i)} \overset{01}{=[-i]}$.

The whole Hilbert space for this toy model has correspondingly four members, four "Slater determinants", numerated by $| \quad >_i, i = (1, 2, 3, 4)$

$$
(|\mathbf{0_{p_1} 0_{p_2}} > |_1 \,, |\mathbf{1_{p_1} 0_{p_2}} > |_2 \,, |\mathbf{0_{p_1} 1_{p_2}} > |_3 \,, |\mathbf{1_{p_1} 1_{p_2}} > |_4) \,,
$$

$\mathbf{0_{p^1_i}}$ represents an empty state and $\mathbf{1_{p^1_i}}$ the corresponding occupied state. Let us evaluate the application of

$\{\hat{\mathbf{b}}^{1}_{1}(\vec{p}^{1}_{1})\,,\,\hat{\mathbf{b}}^{1\dagger}_{1}(\vec{p}^{1}_{2})\}_{*_{T_H}+}$ on the Hilbert space $\mathcal{H}$. It follows

$$\begin{aligned}
&\{\hat{\mathbf{b}}^{1}_{1}(\vec{p}^{1}_{1})\,,\hat{\mathbf{b}}^{1\dagger}_{1}(\vec{p}^{1}_{2})\}_{*_{T_H}+}\mathcal{H} = \\
&\hat{\mathbf{b}}^{1}_{1}(\vec{p}^{1}_{1})\;*_{T_H}\;(|\mathbf{0_{p_1}1_{p_2}}>|_{1\to 3}\,,\,-|\mathbf{1_{p_1}1_{p_2}}>|_{2\to 4})+ \\
&\hat{\mathbf{b}}^{1\dagger}_{1}(\vec{p}^{1}_{2})\;*_{T_H}\;(|\mathbf{0_{p_1}0_{p_2}}>_{2\to 1}\,,\,+|\mathbf{0_{p_1}1_{p_2}}>_{4\to 3}) = \\
&(-|\mathbf{0_{p_1}1_{p_2}}>_{2\to 4\to 3}+|\mathbf{0_{p_1}1_{p_2}}>_{2\to 1\to 3})\;=0\,.
\end{aligned}$$

6 Simple action for interacting fields in $d=(13+1)$ in *spin-charge-family* theory

In Sect. 4 we discussed actions for free massless fermions in d-dimensional spaces, Eq. (78), using odd Clifford algebra, represented by γ^a's, to describe internal space of fermions. We found that d-dimensional space, $d>(3+1)$, offers the description of spins, families and charges of fermions. Of particular interest is $d=(13+1)$-dimensional space, offering spins, families and charges postulated by the *standard model* before the electroweak break. In the internal space quarks and leptons and antiquarks and antileptons appear together in the same irreducible representation of the generators of the Lorentz group S^{ab}, with spins and handedness of quarks and leptons related to their charges, and spins and handedness of antiquarks and antileptons related to charges as required by the *standard model*, each irreducible representation carrying the family quantum number.

In order that fermions manifest the observed charges of quarks and leptons in $d=(3+1)$ the symmetry must be broken from $SO(13+1)$ first to $SO(7,1)\times SU(3+1)\times U(1)$. This break is (assumed to be) caused by the condensate of the two right handed neutrinos, presented in Table 6. The description of the internal space of fermions with the Clifford odd algebra of one of the two kinds — γ^a's determine internal space while the second kind of the Clifford algebra objects, $\tilde{\gamma}^a$'s, determines the family quantum number of fermions — offers that the corresponding creation operators and their Hermitian conjugated partners annihilation operators fulfill the anticommutation relations on the vacuum state and on the whole Hilbert space of the second quantized fermions without postulates. There are the anticommuting "basis vectors" of the odd Clifford algebra which take care of the anticommuting properties of the second quantized fermions.

Let us now generalize the action for free massless fermions to massless interacting fermions under the requirement that the theory remains simple and therefore elegant.

The *spin-charge-family* theory of one of us, N.S.M.B., ([7, 8, 12, 13, 15, 16, 17, 19, 18], and the references therein) offers a simple (and accordingly elegant) starting action for fermions, coupled in $d=(13+1)$-dimensional space to only gravitational field through the vielbeins $f^{\alpha}{}_{a}$, the gauge fields of momenta, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, the gauge fields of the two kinds of the generators of the Lorentz transformations of the two Clifford algebras, S^{ab} and $\tilde{S}^{ab}$, respectively, Eq. (7) [16].

Let the action for interacting second quantized massless fermions and the corresponding gauge fields

[16] Let us remind the reader that after the reduction of the Clifford space to the part generated only by γ^a's, Sect. 3.2.3, the generators $\tilde{S}^{ab}$'s determine the family properties of fermions, and $\tilde{S}^{ab}$'s transform a family member of particular family into the same family member of the rest of families.

be in $d = (13+1)$-dimensional space as proposed by the *spin-charge-family* theory

$$
\begin{aligned}
\mathcal{A} &= \int d^d x \; E \; \frac{1}{2} (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. + \\
& \int d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \\
p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \{ p_{\alpha}, E f^{\alpha}{}_a \}_- \,, \\
p_{0\alpha} &= p_{\alpha} - \frac{1}{2} S^{ab} \omega_{ab\alpha} - \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab\alpha} \,, \\
R &= \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta}) \} + h.c. \,, \\
\tilde{R} &= \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta}) \} + h.c. \,.
\end{aligned} \tag{100}
$$

Here [17] $f^{\alpha [a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$.

It is shown in Ref. [15, 23] that the spin connection gauge fields manifest in $d = (3+1)$ as the ordinary gravity, the known vector gauge fields and the scalar gauge fields, offering the (simple) explanation for the origin of higgs assumed by the *standard model*, explaining as well the Yukawa couplings.

The theory predicts new vector and scalar gauge fields, Sect. 6.2, what offers explanation for the *dark matter* [94, 12], Sect. 7.3.1 and for the *matter-antimatter asymmetry* [13] in the universe, Sect. 6.2.2.

The appearance of the scalar condensate (so far just assumed, not yet proven that it appears spontaneously) of the two right handed neutrinos with the family quantum numbers of the group of four families, which does not include the observed three families (Table 5), Sect. 6.2, brings masses of the scale $\propto 10^{16}$ GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [13], Sect. 4.

Since the left handed spinors couple differently (with respect to $M^{(7+1)}$) to scalar fields than the right handed ones, the break can leave massless and mass protected $2^{((7+1)/2-1)} (= 8)$ families [127], Sect.7.4, Eq. (120). The rest of families get heavy masses [18].

The manifold $M^{(7+1)}$ breaks further by the scalar fields, presented in Sect. 6.2.2, to $M^{(3+1)} \times SU(2) \times SU(2)$ at the electroweak break. This happens since the scalar fields with the space index $(7,8)$, Subsubsect. 6.2.2, they are the part of a simple starting action Eq.(100), gain the constant values, independent of the coordinates in $d = (3+1)$ (or as interpreted usually gain the nonzero vacuum expectation values). These scalar fields carry with respect to the space index the weak charge $\pm\frac{1}{2}$ and the hyper charge $\mp\frac{1}{2}$ [13, 12], Sect. 6.2.2, just as required by the *standard model*, manifesting with respect to $\tilde{S}^{ab}$ and S^{ab} additional quantum numbers.

Let us point out that all the fermion fields (with the families of fermions and the neutrinos forming the condensate included), the vector and the scalar gauge fields, offering explanation for by the *standard model* postulated ones, origin in the simple starting action.

The starting action, Eq. (100), has only a few parameters. It is assumed that the coupling of fermions to $\omega^{ab}{}_c$'s can differ from the coupling of fermions to $\tilde{\omega}^{ab}{}_c$'s, The reduction of the Clifford space,

[17] $f^{\alpha}{}_a$ are inverted vielbeins to $e^{a}{}_{\alpha}$ with the properties $e^{a}{}_{\alpha} f^{\alpha}{}_b = \delta^{a}{}_b$, $e^{a}{}_{\alpha} f^{\beta}{}_a = \delta^{\beta}_{\alpha}$, $E = \det(e^{a}{}_{\alpha})$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[18] A toy model [127, 128, 132] was studied in $d = (5+1)$-dimensional space with the action presented in Eq. (100), Sect.7.4, Eq. (120). The break from $d = (5+1)$ to $d = (3+1)\times$ an almost S^2 was studied. For a particular choice of vielbeins and for a class of spin connection fields the manifold $M^{(5+1)}$ breaks into $M^{(3+1)}$ times an almost S^2, while $2^{((3+1)/2-1)}$ families remain massless and mass protected. Equivalent assumption, although not yet proved how does it really work, is made also for the $d = (13+1)$ case. This study is in progress quite some time.

Sect. 3.2.3, causes this difference. The additional breaks of symmetries influence the coupling constants in addition. This is under consideration for quite a long time and has not yet been finished.

All the observed properties of fermions, of vector gauge fields and scalar gauge fields are the part of the simple starting action, while the breaks of symmetries dictate the properties of fermions and boson fields after these breaks.

In next subsections the properties of fermions, vector gauge fields and scalar gauge fields will be discussed and achievements and predictions shortly presented.

6.1 Properties of massless interacting fermions as manifesting in $d = (3+1)$ before electroweak break

Let the fermion part of the action, Eq. (100), be rewritten in the way that the fermion action manifests in $d = (3+1)$, that is in the low energy regime before the electroweak break, by the *standard model* postulated properties of: **i.** Fermions, their spins, handedness, charges and family quantum numbers, Eqs. (85, 86, 87), determined by the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$, while the internal space of fermions is described by the Clifford "basis vectors". **ii.** Couplings of fermions to the vector gauge fields, which are the superposition of gauge fields $\omega^{st}{}_m$, Sect. 6.2.1, with the space index $m = (0,1,2,3)$ and with charges determined by the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$ ($S^{ab}\omega^{cd}{}_e = i(\omega^{ad}{}_e\eta^{bc} - \omega^{bd}{}_e\eta^{ac})$ and equivalently for the other two indexes of $\omega^{cd}{}_e$ gauge fields, manifesting the symmetry of space $(d-4)$), and to the scalar gauge fields [12, 14, 17, 19, 130, 131, 7, 8, 93, 94, 96, 23] with the space index $s \geq 5$ and the charges determined by the Cartan subalgebra of S^{ab} and $\tilde{S}^{ab}$ (as explained in the case of the vector gauge fields), and which are superposition of either $\omega^{st}{}_s$ or $\tilde{\omega}^{abt}{}_s$, Sect. 6.2.2

$$\begin{aligned}
\mathcal{L}_f &= \bar{\psi}\gamma^m(p_m - \sum_{A,i} g^{Ai}\tau^{Ai}A^{Ai}_m)\psi + \\
&\quad \{\sum_{s=7,8} \bar{\psi}\gamma^s p_{0s}\,\psi\} + \\
&\quad \{\sum_{t=5,6,9,\dots,14} \bar{\psi}\gamma^t p_{0t}\,\psi\}\,, \qquad (101)
\end{aligned}$$

where $p_{0s} = p_s - \frac{1}{2}S^{s's"}\omega_{s's"s} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $p_{0t} = p_t - \frac{1}{2}S^{t't"}\omega_{t't"t} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abt}$, with $m \in (0,1,2,3)$, $s \in (7,8)$, $(s', s") \in (5,6,7,8)$, (a,b) (appearing in $\tilde{S}^{ab}$) run within either $(0,1,2,3)$ or $(5,6,7,8)$, t runs $\in (5,\dots,14)$, $(t',t")$ run either $\in (5,6,7,8)$ or $\in (9,10,\dots,14)$. The spinor function ψ represents all family members of all the $2^{\frac{7+1}{2}-1} = 8$ families.

The first line of Eq. (101) determines in $d = (3+1)$ the kinematics and dynamics of fermion fields, coupled to the vector gauge fields [15, 19, 12]. The vector gauge fields are the superposition of the spin connection fields ω_{stm}, $m = (0,1,2,3)$, $(s,t) = (5,6,\cdots,13,14)$, and are the gauge fields of S^{st}, Sect. 6.2.1.

The operators τ^{Ai} ($\tau^{Ai} = \sum_{a,b} c^{Ai}{}_{ab}\,S^{ab}$, S^{ab} are the generators of the Lorentz transformations in the Clifford space of γ^a's) are presented in Eqs. (86, 87) of Sect. 4.3. They represent the colour charge, $\vec{\tau}^3$, the weak charge, $\vec{\tau}^1$, and the hyper charge, $Y = \tau^4 + \tau^{23}$, τ^4 is the fermion charge, originating in $SO(6) \subset SO(13,1)$, τ^{23} belongs together with $\vec{\tau}^1$ of $SU(2)_{weak}$ to $SO(4)$ ($\subset SO(13+1)$).

One fermion irreducible representation of the Lorentz group contains, as seen in Table 7, *quarks and leptons and antiquarks and antileptons*, belonging to the first family in Table 5. One notices that the $SO(7,1)$ subgroup content of the $SO(13,1)$ group is the same for the quarks and leptons and the same for the antiquarks and antileptons. Quarks distinguish from leptons, and antiquarks from antileptons, only in the $SO(6) \subset SO(13,1)$ part, that is in the colour (τ^{33}, τ^{38}) part and in the "fermion" quantum number τ^4. The quarks distinguish from antiquarks, and leptons from antileptons, in the handedness, in the colour part and in the τ^4 part, explaining the relation between handedness and charges of fermions and antifermions, postulated in the *standard model*, App. C [19].

All the vector gauge fields, which interact with the condensate, presented in Table 6, become massive, Sect. 6.2.1. The *vector gauge fields not interacting with the condensate — the weak, colour, hyper charge and electromagnetic vector gauge fields — remain massless*, in agreement with by the *standard model* assumed gauge fields before the electroweak break of the mass protection [20].

After the electroweak break, caused by the scalar fields, the only conserved charges are the colour and the electromagnetic charge $Q = \tau^{13} + Y$, $Y = \tau^4 + \tau^{23}$.

The second line of Eq. (101) is the mass term, responsible in $d = (3+1)$ for the masses of fermions and of the weak gauge field (originating in spin connection fields $\omega^{st}{}_m$). The interaction of fermions with the scalar fields with the space index $s = (7, 8)$ (to these scalar fields particular superposition of the spin connection fields $\omega^{ab}{}_s$ and all the superposition of $\tilde{\omega}^{ab}{}_s$ with the space index $s = (7, 8)$ and $(a, b) = (0, 1, 2, 3)$ or $(a, b) = (5, 6, 7, 8)$), which gain the constant values in $d = (3 + 1)$, makes fermions and antifermions massive. *The scalar fields, presented in the second line of Eq. (101), are in the s̃tandard model interpreted as the higgs and the Yukawa couplings*, Sect. 6.2.2, predicting in the *spin-charge-family* theory that there must exist several scalar fields [21].

These scalar gauge fields split into two groups of scalar fields, one group of two triplets and three singlets manifesting the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),L)} \times \widetilde{SU}(2)_{(\widetilde{SO}(4),L)} \times U(1)$ — and the other group of another two triplets and the same three singlets manifesting the symmetry — $\widetilde{SU}(2)_{(\widetilde{SO}(3,1),R)}$ $\times \widetilde{SU}(2)_{(\widetilde{SO}(4),R)} \times U(1)$.

The three $U(1)$ singlet scalar gauge fields are superposition of $\omega_{s't's}$, $s = (7, 8)$, $(s', t') = (5, 6, 7, 8, 9, \cdots, 14)$, with the sum of $S^{s't'}$ arranged into superposition of τ^{13}, τ^{23} and τ^4. The three triplets interact with both groups of quarks and leptons and antiquarks and antileptons [91, 92, 93, 94, 97, 118, 119].

Families of fermions from Table 5, interacting with these scalar fields, split as well into two groups of four families, each of these two groups are coupled to one of the two groups of scalar triplets while all eight families couple to the same three singlets. The scalar gauge fields, manifesting $\widetilde{SU}(2)_{L,R} \times \widetilde{SU}(2)_{L,R}$, are the superposition of the gauge fields $\tilde{\omega}_{abs}$, $s = (7, 8)$, $(a, b) =$ either $(0, 1, 2, 3)$ or $(5, 6, 7, 8)$, manifesting as twice two triplets.

The third line of Eq. (101) represents the scalar fields, which cause transitions from antileptons and antiquarks into quarks and leptons and back, offering the explanation for the matter/antimatter asymmetry in the expanding universe at non equilibrium conditions [13] and for the proton decay. These scalar fields are colour triplets with respect to the space index equal to $(9, 10, 11, 12, 13, 14)$, while they carry the quantum numbers with respect to the superposition of $\mathcal{S}^{ab}$ in adjoint representations, as can be seen in Table 9 and in Fig. 1 of Sect. 6.2.2.

6.2 Vector and scalar gauge fields before electroweak break

This subsection partly follows Ref. ([23] and references therein).

The second line of the starting action, Eq. (100), represents the action for gauge fields in $d = (13+1)$-dimensional space, vector and scalar ones, written explicitly in the fifth and the sixth line, and here

[19] Ref. [18] points out that the connection between handedness and charges for fermions and antifermions, both appearing in the same irreducible representation, explains the triangle anomalies in the *standard model* with no need to connect "by hand" the handedness and charges of fermions and antifermions.

[20] The superposition of the scalar gauge fields $\tilde{\omega}^{st}{}_7$ and $\tilde{\omega}^{st}{}_8$, which at the electroweak break gain constant values in $d = (3 + 1)$, bring masses to all the vector gauge fields, which couple to these scalar fields.

[21] The requirement of the *standard model* that there exist the Yukawa couplings, speaks by itself that there must exist several scalar fields explaining the Yukawa couplings.

repeated as $\mathcal{A}_{gf}$, with the index $_{gf}$ denoting gauge fields

$$
\begin{aligned}
\mathcal{A}_{gf} &= \int d^d x \, E \, (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \,, \\
R &= \frac{1}{2} \, \{ f^{\alpha[a} f^{\beta b]} \; (\omega_{ab\alpha,\beta} - \omega_{ca\alpha} \, \omega^{c}{}_{b\beta}) \} + h.c. \,, \\
\tilde{R} &= \frac{1}{2} \, \{ f^{\alpha[a} f^{\beta b]} \; (\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha} \, \tilde{\omega}^{c}{}_{b\beta}) \} + h.c. \,,
\end{aligned} \tag{102}
$$

the notation $f^{\alpha[a} f^{\beta b]}$ means that the two indexes a and b must be exchanged and the exchanged value taken with the negative sign. Whenever in Eq. (102) two indexes are equal the summation over these two is meant.

In the *spin-charge-family* theory, as in all the Kaluza-Klein theories, the vector gauge fields and the scalar gauge fields — the gauge fields of the charges originating in higher $(d-4)$-dimensional spaces — are represented through the vielbeins $f^{\sigma}{}_{m}$, $m = (0,1,2,3)$ and $f^{\sigma}{}_{s}$, $s \geq 5$, respectively. We proved in Ref. [15] that the vector and the scalar gauge fields manifest in $d = (3+1)$, after the break of the starting symmetry, as the superposition of spin connection fields, when the space $(d-4)$ manifest the assumed symmetry.

$f^{\beta}{}_{a}$ and $e^{a}{}_{\alpha}$ are vielbeins and inverted vielbeins respectively, $e^{a}{}_{\alpha} f^{\beta}{}_{a} = \delta^{\beta}_{\alpha}$, $e^{a}{}_{\alpha} f^{\alpha}{}_{b} = \delta^{a}_{b}$, $E = det(e^{a}{}_{\alpha})$.

Varying the action of Eq. (102) with respect to the spin connection fields, the expression for the spin connection fields $\omega_{ab}{}^{e}$ follows

$$
\begin{aligned}
\omega_{ab}{}^{e} &= \frac{1}{2E} \{ e^{e}{}_{\alpha} \, \partial_\beta (E f^{\alpha}{}_{[a} f^{\beta}{}_{b]}) - e_{a\alpha} \, \partial_\beta (E f^{\alpha}{}_{[b} f^{\beta e]}) - e_{b\alpha} \partial_\beta (E f^{\alpha[e} f^{\beta}{}_{a]}) \} \\
&+ \frac{1}{4} \{ \bar{\Psi} (\gamma^{e} \, S_{ab} - \gamma_{[a} S_{b]}{}^{e}) \Psi \} \\
&- \frac{1}{d-2} \{ \delta^{e}_{a} [\frac{1}{E} \, e^{d}{}_{\alpha} \partial_\beta (E f^{\alpha}{}_{[d} f^{\beta}{}_{b]}) + \bar{\Psi} \gamma_d S^{d}{}_{b} \, \Psi] - \delta^{e}_{b} [\frac{1}{E} e^{d}{}_{\alpha} \partial_\beta (E f^{\alpha}{}_{[d} f^{\beta}{}_{a]}) + \bar{\Psi} \gamma_d S^{d}{}_{a} \, \Psi] \} \,.
\end{aligned} \tag{103}
$$

If replacing S^{ab} in Eq. (103) with $\tilde{S}^{ab}$, the expression for the spin connection fields $\tilde{\omega}_{ab}{}^{e}$ follows.

In Ref. [15] it is proven [22] that in spaces with the desired symmetry the vielbein can be expressed with the gauge fields, $f^{\sigma}{}_{m} = \sum_{A} \vec{\tau}^{A\sigma} \, \vec{A}^{A}_{m}$, and $\tau^{Ai\sigma} = \sum_{st} c^{Ai}{}_{st} \, (e_{s\tau} \, f^{\sigma}{}_{t} - e_{t\tau} \, f^{\sigma}{}_{s}) x^{\tau}$, and

$$
\begin{aligned}
A^{Ai}_{m} &= \sum_{st} c^{Ai}{}_{st} \, \omega^{st}{}_{m} \,, \\
\tau^{Ai} &= \sum_{st} c^{Ai}{}_{st} \, S^{st} \,, \\
\{ \tau^{Ai}, \tau^{Bj} \}_{-} &= i \delta^{AB} f^{Aijk} \tau^{Ak} \,,
\end{aligned} \tag{104}
$$

$m = (0,1,2,3)$ denotes the vector gauge fields. If m is replaced by s, $s = (5,6,\ldots,14)$ the equivalent expression follows for the scalar gauge fields.

If fermions are not present then spin connections of both kinds are uniquely determined by vielbeins, as can be noticed from Eq. (103). If fermions are present, carrying both — family members and family — quantum numbers, then vielbeins and both kinds of spin connections are influenced by the presence of fermions.

[22] We presented in Ref. [15] the proof, that the vielbeins $f^{\sigma}{}_{m}$ (Einstein index $\sigma \geq 5$, $m = 0,1,2,3$) lead in $d = (3+1)$ to the vector gauge fields, which are the superposition of the spin connection fields ω_{stm}: $f^{\sigma}{}_{m} = \sum_{A} \vec{A}^{A}_{m} \, \vec{\tau}^{A\sigma}{}_{\tau} \, x^{\tau}$, with $A^{Ai}_{m} = \sum_{s,t} c^{Ai}{}_{st} \, \omega^{st}{}_{m}$, when the metric in $(d-4)$, $g_{\sigma\tau}$, is invariant under the coordinate transformations $x^{\sigma'} = x^{\sigma} + \sum_{A,i,s,t} \varepsilon^{Ai} (x^{m}) \, c^{Ai}{}_{st} \; E^{\sigma st}(x^{\tau})$ and $\sum_{s,t} c^{Ai}{}_{st} \, E^{\sigma st} = \tau^{Ai\sigma}$, while $\tau^{Ai\sigma}$ solves the Killing equation: $D_\sigma \, \tau^{Ai}_{\tau} + D_\tau \tau^{Ai}_{\sigma} = 0$ ($D_\sigma \, \tau^{Ai}_{\tau} = \partial_\sigma \, \tau^{Ai}_{\tau} - \Gamma^{\tau'}_{\tau\sigma} \tau^{Ai}_{\tau'}$). And similarly also for the scalar gauge fields.

The vector gauge fields A^{Ai}_m of τ^{Ai} represent in the *spin-charge-family* theory all the observed gauge fields, as well as the additional non observed vector gauge fields, which interacting with the condensate gain heavy masses.

The scalar (gauge) fields, carrying the space index $s = (5, 6, \ldots, d)$, offer in the *spin-charge-family* for $s = (7, 8)$ the explanation for the origin of the Higgs's scalar and the Yukawa couplings of the *standard model*, while scalars with the space index $s = (9, 10, \ldots, 14)$ offer the explanation for the proton decay, as well as for the matter/antimatter asymmetry in the universe.

The explicit expressions for $c^{Ai}{}_{ab}$, and correspondingly for τ^{Ai}, and A^{Ai}_a, are written in Sects. 6.2.1 and 6.2.2.

6.2.1 Vector gauge fields

In the *spin-charge-family* theory the simple starting action (in which fermions, the internal space of which is described by the odd Clifford algebra, interact with gravity only — vielbeins and two kinds of spin connection fields), Eq. (100) [23], manifests in $d = (3 + 1)$ before the electroweak break besides as massless gravity, colour, weak $SU(2)_I$ and hyper charge vector gauge fields, also as massive second $SU(2)_{II}$ weak and $U(1)_{\tau^4}$, "fermion" vector gauge fields. The gauge fields $\vec{A}^2_m$ and A^4_m obtain masses in interaction with the condensate of two right handed neutrinos, Table 6. All the vector and scalar gauge fields of S^{ab} and of $\tilde{S}^{ab}$, which interact with the condensate, become massive.

The $U(1)_4$ vector gauge field is the vector gauge field of $\tau^4 (= -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}))$, with τ^4 denoting the "fermion" charge. The hyper charge vector gauge field of the *standard model* is the superposition of the third component of the second $SU(2)_{II}$ vector gauge fields and the $U(1)_{\tau^4}$ vector gauge field ($A^Y_m = \cos\theta_2 A^{\tau^4}_m + \sin\theta_2 A^{23}_m$, θ_2 is the angle of the break of the $SU(2)_{II} \times U(1)_{\tau^4}$ symmetry to $U(1)_Y$ at the scale $\geq 10^{16}$ GeV ([19] and references therein), caused by the condensate, Table 6). Also the two components of the second $SU(2)_{II}$ vector gauge fields and the superposition $A^{Y'}_m = -\sin\theta_2 A^4_m + \cos\theta_2 A^{23}_m$, which is the gauge field of $Y' (= -\tan^2\theta_2 \tau^4 + \tau^{23})$ gain as well masses due to the interaction with the condensate.

If there are no fermions present (like it is the condensate) the spin connection fields are expressible with only vielbeins and opposite [15].

At the electroweak break all the scalar gauge fields, $\vec{\tilde{A}}^{Ai}_s$, $s = (7, 8)$, together with A^Q_s and A^Y_s, gain constant values (nonzero vacuum expectation values), making massive (or contributing to the masses of) all the rest of $\vec{\tilde{A}}^{Ai}_m$, which interact with these scalar fields, leaving massless only gravity, $\vec{A}^3_m$ and A^Q_m.

All the vector gauge fields are expressible with the spin connection fields ω_{stm} as

$$A^{Ai}_m = \sum_{s,t} c^{Ai}{}_{st}\, \omega^{st}{}_m \,, \tag{105}$$

with $\sum_{A,i} \tau^{Ai} A^{Ai}_m = \sum^{*}_{a,b} S^{ab} \omega^{ab}{}_m$,* means that summation runs over (a, b) respecting the symmetry $SO(7, 1) \times SU(3) \times U(1)$.

Let us present expressions for the two $SU(2)$ vector gauge fields, $SU(2)_I$ and $SU(2)_{II}$, before the break of symmetries

$$\begin{aligned} \vec{\mathcal{A}}^1_m &= \vec{A}^1_m = (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m}) \,, \\ \vec{\mathcal{A}}^2_m &= \vec{A}^2_m = (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m}) \,. \end{aligned} \tag{106}$$

[23] In $SO(13+1)$ there are seven members of Cartan subalgebra, Eq. (8), which determine in Table 7 the spin, handedness, weak charges of two kinds, colour charge and "fermion" charge τ^4 of quarks and leptons before the electroweak break. As discussed in Sect. 6.2 the superposition of spin connection fields $\omega^{st}{}_m$, $m = (0, 1, 2, 3)$, manifest in $d = (3 + 1)$ as the vector gauge fields of τ^{Ai}, that is $\vec{A}^3_m$, $\vec{A}^1_m$, A^Q_m and A^Y_m, with the coefficients $c^{Ai}{}_{st}$ determined in Eq. (104) (and as ordinary gravity for (a, b) of $\omega^{ab}{}_m$ belonging to index $m = (0, 1, 2, 3)$).

It is demonstrated in Ref. [15] for the case when $SO(7,1)$ breaks into $SO(3,1)\times SU(2)\times SU(2)$ that $\sum_{A,i}\tau^{Ai}\,A^{Ai}_{m}=\sum_{s,t}S^{st}\,\omega_{stm}$ and that the effective action in flat $(3+1)$ space (with no gravitational field) for the vector gauge fields is $\int\, d^4x\,\{-\frac{1}{4}\,F^{Ai}{}_{mn}\,F^{Aimn}\,\}$, where $F^{Ai}{}_{mn}=\partial_m A^{Ai}_{n}-\partial_n A^{Ai}_{m}-i f^{Aijk}\,A^{Aj}_{m}\,A^{Ak}_{n}$, and f^{Aijk} are the structure constants of the corresponding gauge groups.

The reader can similarly construct all the other vector gauge fields from the coefficients for the corresponding charges, or find the expressions in Refs. [23, 17, 13, 19] or references therein.

The generalization of the break of $SO(13,1)$ into $SO(3,1)\times SU(2)\times SU(2)\times SU(3)\times U(1)$, used in the *spin-charge-family* theory, goes equivalently. In a general case one has $\sum_{A,i}\tau^{Ai}\,A^{Ai}_{m}=\sum^{*}_{s,t}S^{st}\,\omega_{stm}$, where * means that the summation concerns only those (s,t), which appear in $\tau^{Ai}=\sum_{s,t}c^{Ai}{}_{st}\,S^{st}$. These vector gauge fields A^{Ai}_{m}, expressible with the spin connection fields, $A^{Ai}_{m}=\sum_{s,t}c^{Aist}\,\omega_{stm}$, offer an elegant explanation for the appearance of the vector gauge fields in the observed $(3+1)$ space.

The effective action for all the massless vector gauge fields before the electroweak break is in the general case of all the vector gauge fields, which do not interact with the condensate and remain therefore massless, equal to $\int\, d^4x\,\{-\frac{1}{4}\,F^{Ai}{}_{mn}\,F^{Aimn}\,\}$, with the structure constants f^{Aijk} concerning the colour $SU(3)$, weak SU(2) and hyper charge $U(1)$ groups.

All these relations are valid as long as spinors and vector gauge fields are weak fields in comparison with the fields which force $(d-4)$ space to be (almost) curled, Sect. 7.4. When all these fields, with the scalar gauge fields included, start to be comparable with the fields (spinors or scalars), which determine the symmetry of $(d-4)$ space, the symmetry of the whole space changes.

The electroweak break, caused by the constant (non zero vacuum expectation) values of the scalar gauge fields, carrying the space index $s=(7,8)$, makes the weak and the hyper charge gauge fields massive. The only vector gauge fields which remain massless are, besides the gravity, the electromagnetic and the colour vector gauge fields — the observed three massless gauge fields.

6.2.2 Scalar gauge fields in $d=(3+1)$

Scalar fields, taking care of the masses of quarks and leptons in the *spin-charge-family* theory, have the space index $s=(7,8)$ and carry with respect to this space index the weak charge $\tau^{13}=\pm\frac{1}{2}$ and the hyper charge $Y=\mp\frac{1}{2}$, Table 8, Eq. (110). With respect to the index determined by $\tau^{Ai}=\sum_{ab}c^{Ai}{}_{ab}S^{ab}$ and $\tilde{\tau}^{Ai}=\sum_{ab}c^{Ai}{}_{ab}\tilde{S}^{ab}$, that is with respect to S^{ab} and $\tilde{S}^{ab}$, they carry charges and family charges in adjoint representations, Eq. (107).

There are in the starting action of the *spin-charge-family* theory, Eq. (100), scalar fields, which transform antileptons and antiquarks into quarks and leptons and back. They carry space index $s=(9,10,\dots,14)$, They are with respect to the space index colour triplets and antitriplets, while they carry charges τ^{Ai} and $\tilde{\tau}^{Ai}$ in adjoint representations.

Following partly Refs. [19, 23] we shall review both kinds of scalar fields.

Let us demonstrate how do the infinitesimal generators $\mathcal{S}^{ab}$ apply on the spin connections fields ω_{bde} $(=f^{\alpha}{}_{e}\;\omega_{bd\alpha})$ and $\tilde{\omega}_{\tilde{b}\tilde{d}e}$ $(=f^{\alpha}{}_{e}\;\tilde{\omega}_{\tilde{b}\tilde{d}\alpha})$, on either the space index e or any of the indices $(b,d,\tilde{b},\tilde{d})$

$$\mathcal{S}^{ab}\,A^{d\dots e\dots g}\;=\;i\,(\eta^{ae}\,A^{d\dots b\dots g}-\eta^{be}\,A^{d\dots a\dots g})\,, \tag{107}$$

(Section IV. and Appendix B in Ref. [19]).

The scalar fields low energy action [15] is proportional to $\int E\,d^{d-4}x\;\;R$, where

$$\begin{aligned} R &= \{\Gamma^{\sigma}{}_{\tau[\tau',\sigma]}+\Gamma^{\sigma}{}_{\tau''[\sigma}\,\Gamma^{\tau''}{}_{\tau\tau']}\,\}\,g^{\tau\tau'}\\ &= \frac{1}{2}\,\{f^{\sigma[s}f^{\tau t]}\;(\omega^{st}{}_{\tau,\sigma}+\omega_{st'\sigma}\,\omega^{t'}{}_{t\tau})\}+h.c.\,, \end{aligned}$$

and $\Gamma^{\sigma}{}_{\tau\sigma'}=\frac{1}{2}\,g^{\sigma\tau'}\,(g_{\sigma'\tau'},_{\tau}+g_{\tau\tau'},_{\sigma'}-g_{\tau\sigma'},_{\tau'})$. Similar relation follows also for the superposition of the spin connection fields.

If $\omega_{st'\sigma}$ depend on x^m (x^m are coordinates in $(3+1)$ space), the scalar fields are the dynamical fields in $(3+1)$, explaining, for example, after the break of the starting symmetry, the appearance of the Higgs's scalars and the Yukawa couplings [12, 13, 19, 17, 16], as well as the proton decay and the appearance of the matter-antimatter asymmetry in our universe.

a. Scalar gauge fields determining scalar higgs and Yukawa couplings

To be in agreement with the experiments (and with the *standard model* assumptions) the spin-charge-family theory chooses the space index for scalars, gaining constant values and causing correspondingly the electroweak phase transition, equal to $s=(7,8)$ (the choice of $(s=5,6)$ would also work). All the family quantum numbers of eight families, Table 5, that is all the superposition of $\tilde{\omega}_{\tilde{a}\tilde{b}s}$ are allowed, while with respect to ω_{abs} only the superposition representing the scalar gauge fields A^Q_s, A^Y_s and A^4_s, $s=(7,8)$ (or any three superposition of these three scalar fields) may contribute.

It is convenient to use the common notation A^{Ai}_s for all the scalar gauge fields with $s=(7,8)$, independently of whether they originate in ω_{abs} — in this case $Ai=(Q,Y,\tau^4)$ — or in $\tilde{\omega}_{\tilde{a}\tilde{b}s}$. All these gauge fields contribute to the masses of quarks and leptons and antiquarks and antileptons after gaining constant values (nonzero vacuum expectation values).

$$\begin{aligned}
A^{Ai}_s &\quad \text{represents} \quad (\, A^Q_s\,, A^Y_s\,, A^4_s\,, \vec{\tilde{A}}^{\tilde{1}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s\,, \vec{\tilde{A}}^{\tilde{2}}_s\,, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s\,)\,,\\
\tau^{Ai} &\quad \text{represents} \quad (Q,\, Y,\, \tau^4,\, \vec{\tilde{\tau}}^1,\, \vec{\tilde{N}}_L,\, \vec{\tilde{\tau}}^2,\, \vec{\tilde{N}}_R)\,.
\end{aligned} \tag{108}$$

Here τ^{Ai} represent all the operators, which apply on fermions. These scalars with the space index $s=(7,8)$, they are scalar gauge fields of the generators τ^{Ai} and $\tilde{\tau}^{Ai}$, are expressible in terms of the spin connection fields (Ref. [19], Eqs. (10, 22, A8, A9)).

Let us demonstrate [19] that all the scalar fields with the space index $(7,8)$ carry with respect to this space index the weak and the hyper charge $(\mp\frac{1}{2},\, \pm\frac{1}{2})$, respectively. This means that all these scalars have properties as required for the higgs in the *standard model*.

To compare the properties of the scalar fields with those of the Higgs's scalar of the *standard model* we make the scalar fields the eigenstates of $\tau^{13}=\frac{1}{2}(\mathcal{S}^{56}-\mathcal{S}^{78})$.

For this purpose we need to apply the operators τ^{13} $(=\frac{1}{2}(\mathcal{S}^{56}-\mathcal{S}^{78})$, Y $(=\tau^4+\tau^{23})$ and Q $(=\tau^{13}+Y)$, Eqs. (86, 87, 88) on the scalar fields with the space index $s=(7,8)$, taking into account Eq. (107).

Let us rewrite the second line of Eq. (101), ignoring the momentum p_s, $s=(5,6,\ldots,d)$, since it is expected that solutions with nonzero momenta in higher dimensions do not contribute to the masses of fermion fields at low energies in $d=(3+1)$. We pay correspondingly no attention to the momentum $p_s\,, s\in(5,\ldots,8)$, when having in mind the lowest energy solutions, manifesting at low energies.

$$\begin{aligned}
&\sum_{s=(7,8),A,i} \bar{\psi}\,\gamma^s\,(-\tau^{Ai}\,A^{Ai}_s\,)\,\psi =\\
&-\sum_{A,i} \bar{\psi}\,\{\,\overset{78}{(+)}\;\tau^{Ai}\,(A^{Ai}_7 - i\,A^{Ai}_8) + \overset{78}{(-)}\,(\tau^{Ai}\,(A^{Ai}_7 + i\,A^{Ai}_8)\,\}\,\psi\,,\\
&\overset{78}{(\pm)} = \frac{1}{2}\,(\gamma^7 \pm i\,\gamma^8\,)\,, \quad A^{Ai}_{\overset{78}{(\pm)}} := (A^{Ai}_7 \mp i\,A^{Ai}_8)\,,
\end{aligned} \tag{109}$$

with the summation over A and i performed, since A^{Ai}_s represent the scalar fields $(A^Q_s,\, A^Y_s,\, A^4_s)$ determined by $\omega_{s',s'',s}$, as well as $(\vec{\tilde{A}}^{\tilde{4}}_s,\, \vec{\tilde{A}}^{\tilde{1}}_s,\, \vec{\tilde{A}}^{\tilde{2}}_s,\, \vec{\tilde{A}}^{\tilde{N}_R}_s$ and $\vec{\tilde{A}}^{\tilde{N}_L}_s)$, determined by $\tilde{\omega}_{a,b,s}\,, s=(7,8)$.

The application of the operators τ^{13}, Y $(Y=\frac{1}{2}(\mathcal{S}^{56}+\mathcal{S}^{78})-\frac{1}{3}(\mathcal{S}^{9\,10}+\mathcal{S}^{11\,12}+\mathcal{S}^{13\,14}))$ and Q on the scalar fields $(A^{Ai}_7 \mp i\,A^{Ai}_8)$ with respect to the space index $s=(7,8)$, by taking into account Eq. (107)

Table 8: The two scalar weak doublets, one with $\tau^{23} = -\frac{1}{2}$ and the other with $\tau^{23} = +\frac{1}{2}$, both with the "fermion" quantum number $\tau^4 = 0$, are presented. In this table all the scalar fields carry besides the quantum numbers determined by the space index also the quantum numbers A and i from Eq. (108). The table is taken from Ref. [19].

name	superposition	τ^{13}	τ^{23}	spin	τ^4	Q
$A^{Ai}_{\underset{(-)}{78}}$	$A^{Ai}_7 + i A^{Ai}_8$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
$A^{Ai}_{\underset{(-)}{56}}$	$A^{Ai}_5 + i A^{Ai}_6$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1
$A^{Ai}_{\underset{(+)}{78}}$	$A^{Ai}_7 - i A^{Ai}_8$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0
$A^{Ai}_{\underset{(+)}{56}}$	$A^{Ai}_5 - i A^{Ai}_6$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	+1

to make the application of the generators $\mathcal{S}^{ab}$ on the space indexes, gives

$$\begin{aligned}
\tau^{13}\,(A^{Ai}_7 \mp i\, A^{Ai}_8) &= \pm \frac{1}{2}\,(A^{Ai}_7 \mp i\, A^{Ai}_8)\,,\\
Y\,(A^{Ai}_7 \mp i\, A^{Ai}_8) &= \mp \frac{1}{2}\,(A^{Ai}_7 \mp i\, A^{Ai}_8)\,,\\
Q\,(A^{Ai}_7 \mp i\, A^{Ai}_8) &= 0\,.
\end{aligned} \tag{110}$$

Since τ^4, Y, τ^{13} and τ^{1+}, τ^{1-} give zero if applied on (A^Q_s, A^Y_s and A^4_s) with respect to the quantum numbers (Q, Y, τ^4), and since Y, Q, τ^4 and τ^{13} commute with the family quantum numbers, one sees that the scalar fields A^{Ai}_s ($=(A^Q_s$, A^Y_s, $A^{Y'}_s$, $\tilde{A}^{\tilde{4}}_s$, $\tilde{A}^{\tilde{Q}}_s$, $\vec{\tilde{A}}^{\tilde{1}}_s$, $\vec{\tilde{A}}^{\tilde{2}}_s$, $\vec{\tilde{A}}^{\tilde{N}_R}_s$, $\vec{\tilde{A}}^{\tilde{N}_L}_s))$, $s = (7, 8)$, rewritten as $A^{Ai}_{\underset{(\pm)}{78}}$ $= (A^{Ai}_7 \mp i\, A^{Ai}_8)$, are eigenstates of τ^{13} and Y, having the quantum numbers of the *standard model* Higgs's scalar.

These superposition of $A^{Ai}_{\underset{(\pm)}{78}}$ are presented in Table 8 as two doublets with respect to the weak charge τ^{13}, with the eigenvalue of τ^{23} (the second $SU(2)_{II}$ charge) equal to either $-\frac{1}{2}$ or $+\frac{1}{2}$, respectively. The operators $\tau^{1\boxed{\pm}} = \tau^{11} \pm i\tau^{12}$, $\tau^{1\boxed{\pm}} = \frac{1}{2}[(\mathcal{S}^{58} - \mathcal{S}^{67}) \boxed{\mp}\, i\,(\mathcal{S}^{57} + \mathcal{S}^{68})]$, transform one member of a doublet from Table 8 into another member of the same doublet, keeping τ^{23} $(= \frac{1}{2}\,(\mathcal{S}^{56} + \mathcal{S}^{78}))$ unchanged, clarifying the above statement.

It is not difficult to show that the scalar fields $A^{Ai}_{\underset{(\pm)}{78}}$ are *triplets* as the gauge fields of the family quantum numbers ($\vec{\tilde{N}}_R$, $\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^2$, $\vec{\tilde{\tau}}^1$; Eqs. (85, 86, 107)) or singlets as the gauge fields of $Q = \tau^{13} + Y$, $Q' = -\tan^2\vartheta_1 Y + \tau^{13}$ and $Y' = -\tan^2\vartheta_2 \tau^4 + \tau^{23}$.

We show this in App. E, Eq. (111), concluding

$$\begin{aligned}
\tilde{N}^3_L\,\tilde{A}^{\tilde{N}_L\boxed{\pm}}_{\underset{(\pm)}{78}} &= \boxed{\pm}\,\tilde{A}^{\tilde{N}_L\boxed{\pm}}_{\underset{(\pm)}{78}}\,, \quad \tilde{N}^3_L\,\tilde{A}^{\tilde{N}_L 3}_{\underset{(\pm)}{78}} = 0\,,\\
Q\,A^{Q}_{\underset{(\pm)}{78}} &= 0\,,
\end{aligned}$$

taking into account $Q = \mathcal{S}^{56} + \tau^4 = \mathcal{S}^{56} - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14})$, and with τ^4 defined in Eq. (87), if replacing S^{ab} by $\mathcal{S}^{ab}$ from Eq. (107). Similarly one finds properties with respect to the Ai quantum numbers for all the scalar fields $A^{Ai}_{\underset{(\pm)}{78}}$.

We demonstrated that these scalar fields, three singlets — A^Q_s, A^Y_s and A^4_s, the scalar gauge fields of $(Q,\, Y,\, \tau^4)$ — and two groups of two triplets — ($\vec{\tilde{A}}^{\tilde{1}}_s$, $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s$) and ($\vec{\tilde{A}}^{\tilde{2}}_s$, $\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s$), the scalar gauge fields of ($\vec{\tilde{\tau}}^1$, $\vec{\tilde{N}}_L$) and ($\vec{\tilde{\tau}}^2$, $\vec{\tilde{N}}_R$)), respectively — all with the space index $s = (7, 8)$ do behave as Higgs's scalars.

In Ref. [15] it is proven that either vielbeins or spin connection fields with the scalar index with respect to $d = (3+1)$ do manifest in $d = (3+1)$ as several scalar fields.

The necessity to postulate the Yukawa couplings into the standard model is the strong signal that there must be several scalar fields, manifesting as scalar higgs and Yukawa couplings. Although the severe warnings against more than one scalar higgs can be found in the literature and also in our Sect. 7.5.2, the appearance of several scalar fields in the *spin-charge-family* theory, origining in the simple starting action, Eq. (100), and offering the explanations for all the assumptions of the *standard model*, supports (together with the estimations made) the optimism that the theory does explain how and why the flavour changing neutral currents and the proton decay have not been observed.

Table 5 represents two groups of four families. Let us check how do the scalar fields $\vec{\tilde{\tau}}^{1} \cdot \vec{\tilde{A}}^{\tilde{1}}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}$, $\vec{\tilde{N}}_{L} \cdot \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}$, $\vec{\tilde{\tau}}^{2} \cdot \vec{\tilde{A}}^{\tilde{2}}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}$, $\vec{\tilde{N}}_{R} \cdot \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}$ influence the two groups of four families, presented in this table.

Taking into account Eqs. (89, 41) one finds that $\tilde{N}^{\pm}_{+}$ and $\tilde{\tau}^{1\pm}$ transform the first four families among themselves, leaving the second group of four families untouched, while $\tilde{N}^{\pm}_{-}$ and $\tilde{\tau}^{2\pm}$ do not influence the first four families and transform the second four families among themselves. The operators $(Q \cdot A^{Q}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}, Y \cdot A^{Q]Y}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}, \tau^{4} \cdot A^{4}_{\genfrac{}{}{0pt}{}{78}{(\pm)}})$ are diagonal, their application on the family members depend on family members quantum numbers and on properties of scalar fields. All the scalar fields with $s = (7,8)$ "dress" the right handed quarks and leptons with the hyper charge and the weak charge so that they manifest charges of the left handed partners.

How strong is the influence of scalar fields on the masses of quarks and leptons, depends on the coupling constants and the masses of the scalar fields. But we see that in both groups of four families, the mass matrices 4×4 have the symmetry $SU(2)\times SU(2)\times U(1)$ of the form [24]

$$\mathcal{M}^{\alpha} = \begin{pmatrix} -a_1 - a & e & d & b \\ e* & -a_2 - a & b & d \\ d* & b* & a_2 - a & e \\ b* & d* & e* & a_1 - a \end{pmatrix}^{\alpha}, \tag{111}$$

with α representing family members — quarks and leptons [91, 92, 93, 96, 97, 119]. In App. H the symmetry of mass matrices of Eq. (111) is discussed.

The *spin-charge-family* theory treats quarks and leptons in equivalent way. The difference among family members occur due to the scalar fields $(Q \cdot A^{Q}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}, Y \cdot A^{Q]Y}_{\genfrac{}{}{0pt}{}{78}{(\pm)}}, \tau^{4} \cdot A^{4}_{\genfrac{}{}{0pt}{}{78}{(\pm)}})$ [97, 119].

Since we measure the coupling constants of the vector gauge fields and since the break of symmetries influences both coupling constants, the ones of the vector gauge fields and the one of the scalar gauge fields, we should be able to demonstrate how do the coupling constants among scalar fields at low energies manifest in the estimated, not yet derived, Lagrange density after the electroweak break

$$\begin{aligned} \mathcal{L}_{sg} &= E \sum_{A,i} \{(p_m\, A^{Ai}_{s})^{\dagger}\, (p^m\, A^{Ai}_{s}) - (\lambda^{Ai} - (m'_{Ai})^2))\, A^{Ai\dagger}_{s} A^{Ai}_{s} \\ &+ \sum_{B,j} \Lambda^{AiBj}\, A^{Ai\dagger}_{s} A^{Ai}_{s}\; A^{Bj\dagger}_{s} A^{Bj}_{s}\}\,. \end{aligned} \tag{112}$$

This work is under consideration for already few years.

Let us add that at the electroweak break, causing by constant values of the scalar fields with the space index $s = (7,8)$, the mass matrices of the two groups of four families manifest either $\widetilde{SU}(2)_{\widetilde{SO}(3,1)L} \times \widetilde{SU}(2)_{\widetilde{SU}(4)L} \times U(1)$ symmetry, this is the case for the lower four families of the eight families, presented in Table 5, or $\widetilde{SU}(2)_{\widetilde{SO}(3,1)R} \times \widetilde{SU}(2)_{\widetilde{SU}(4)R} \times U(1)$ symmetry, this is the case for the higher four families,

[24] The symmetry $SU(2)\times SU(2)\times U(1)$ of the mass matrices, Eq. (111), is expected to remain in all loop corrections [118].

presented in Table 5. The two $SU(2)$ triplet fields are for each of the two groups different, although manifesting the same symmetries. The same three $U(1)$ singlet fields contribute to the masses of both groups.

The mass matrix of a family member — of quarks and leptons — are 4×4 matrices. The observed three families of quarks and leptons form the 3×3 submatrices of the 4×4 matrices. The symmetry of the mass matrices, manifesting in all orders [118], limits the number of free parameters.

The mass matrices of the upper four families have the same symmetry as the mass matrices of the lower four families, but the scalar fields determining the masses of the upper four families have different properties (masses and coupling constants) than those of the lower four, giving to quarks and leptons of the upper four families much higher masses in comparison with the lower four families of quarks and leptons, what offers the explanation for the appearance of the *dark matter*, studied at Refs. [94, 122].

Let us conclude: Twice four families of Table 5, with the two groups of two triplets applying each on one of the two groups of four families, and one group of three singlets applying on all eight families, **i.** offer the explanation for the appearance of the Higgs's scalar and Yukawa couplings of the observed three families, predicting the fourth family to the observed three families and several scalar fields, **ii.** predicts that the stable of the additional four families with much higher masses that the lower four families contributes to the *dark matter*.

b. Scalar gauge fields causing transitions from antileptons and antiquarks into quarks and leptons [13]

This part follows to a great deal the similar part from Ref. [12].

Besides the scalar fields with the space index $s = (7,8)$, which manifest in $d = (3+1)$ as scalar gauge fields with the weak and hyper charge $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively, and which gaining at low energies constant values cause masses of families of quarks and leptons and of the weak gauge field, there are in the starting action, Eqs. (100, 101), additional scalar gauge fields with the space index $t = (9,10,11,12,13,14)$. They are with respect to the space index t either triplets or antitriplets causing transitions from antileptons into quarks and from antiquarks into quarks and back. These scalar fields are in Eq. (101) presented in the third line.

Like in the case of scalar gauge fields with the properties of Higgs's scalars included in the second line of Eq. (101) we can also here rearrange the third line of $\mathcal{L}_f$ in Eq. (101) so that $\omega_{stt'}$ and $\tilde{\omega}_{stt'}$ are eigenvectors of the Cartan subalgebra operators, Eq. (6), with respect to the space index t', where the application of $\mathcal{S}^{tt'}$ on $\omega_{stt'}$ and $\tilde{\omega}_{stt'}$ is determined by Eq. (107) ($\mathcal{S}^{ab}\, A^{d\ldots e\ldots g} = i\,(\eta^{ae}\, A^{d\ldots b\ldots g} - \eta^{be}\, A^{d\ldots a\ldots g})$). One finds that $(\mathcal{S}^{910}(\omega_{st9} \mp i\omega_{st10} = \pm(\omega_{st9} \mp i\omega_{st10}$. Taking this into account we can rewrite the third line of the fermion action Eq. (101) $\mathcal{L}_{f'} = \psi^\dagger\,\gamma^0\,\gamma^t\,\{\sum_{t=(9,10,\ldots 14)}\,[p_t - (\,\frac{1}{2}\,S^{s's"}\,\omega_{s's"t} + \,\frac{1}{2}\,S^{t't''}\,\omega_{t't"t} + \frac{1}{2}\,\tilde{S}^{ab}\,\tilde{\omega}_{abt}\,)]\}\;\psi$, after taking into account that in the low energy regime $p_t, t \geq 5$, can be neglected, as follows

$$\begin{aligned}
\mathcal{L}_{f"} &= \psi^\dagger\,\gamma^0(-)\,\{\sum_{+,-}\sum_{(t\,t')}\,(\overset{tt'}{\oplus})\;\cdot \\
& \quad [\,\tau^{2+}\,A^{2+}_{\underset{(\oplus)}{tt'}} + \tau^{2-}\,A^{2-}_{\underset{(\oplus)}{tt'}} + \tau^{23}\,A^{23}_{\underset{(\oplus)}{tt'}} + \tau^{1+}\,A^{1+}_{\underset{(\oplus)}{tt'}} + \tau^{1-}\,A^{1-}_{\underset{(\oplus)}{tt'}} + \tau^{13}\,A^{13}_{\underset{(\oplus)}{tt'}} \\
&+ \tilde{\tau}^{2+}\,\tilde{A}^{2+}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{2-}\,\tilde{A}^{2-}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{23}\,\tilde{A}^{23}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{1+}\,\tilde{A}^{1+}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{1-}\,\tilde{A}^{1-}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{13}\,\tilde{A}^{13}_{\underset{(\oplus)}{tt'}} \\
&+ \tilde{N}^{+}_{R}\,\tilde{A}^{N_R+}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{-}_{R}\,\tilde{A}^{N_R-}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{3}_{R}\,\tilde{A}^{N_R3}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{+}_{L}\,\tilde{A}^{N_L+}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{-}_{L}\,\tilde{A}^{N_L-}_{\underset{(\oplus)}{tt'}} + \tilde{N}^{3}_{L}\,\tilde{A}^{N_L3}_{\underset{(\oplus)}{tt'}} \\
&+ \sum_i \tau^{3i}\,A^{3i}_{\underset{(\oplus)}{tt'}} + \tau^{4}\,A^{4}_{\underset{(\oplus)}{tt'}} + \sum_i \tilde{\tau}^{3i}\,\tilde{A}^{3i}_{\underset{(\oplus)}{tt'}} + \tilde{\tau}^{4}\,\tilde{A}^{4}_{\underset{(\oplus)}{tt'}}\;]\,\}\,\psi\,, \qquad (113)
\end{aligned}$$

where (t,t') run in pairs over $[(9,10),(11,12),(13,14)]$ and the summation must go also over $+$ and $-$ of

$\stackrel{tt'}{(\oplus)}$, so that we take into account that $\gamma^9 \omega_{st9} + \gamma^1 0 \omega_{st10} = \frac{1}{2}\{(\gamma^9 + i\gamma^{10})(\omega_{st9} - i\omega_{st10}) + (\gamma^9 - i\gamma^{10})(\omega_{st9} + i\omega_{st10})\}$ and similarly for all the other terms.

In Eq. (113) the relations below are used

$$
\begin{aligned}
\sum_{t,s',s''} \gamma^t \, \frac{1}{2}\, S^{s's"}\, \omega_{s's"t} &= \sum_{+,-} \sum_{(tt')} \stackrel{tt'}{(\oplus)} \, \frac{1}{2}\, S^{s's"}\, \omega_{s"s"\stackrel{tt'}{(\oplus)}}\,, \\
\omega_{s"s"\stackrel{tt'}{(\oplus)}} : &= \omega_{s"s"\stackrel{tt'}{(\pm)}} = (\omega_{s's"t} \mp i\, \omega_{s's"t'})\,, \\
\stackrel{tt'}{(\oplus)}: &= = \frac{1}{2}\,(\gamma^t \pm \gamma^{t'})\,, \\
\sum_{+,-} \sum_{(tt')} \stackrel{tt'}{(\oplus)} \, \frac{1}{2}\, S^{s's"}\, \omega_{s"s"\stackrel{tt'}{(\oplus)}} &= \stackrel{tt'}{(\oplus)} \, \{ \tau^{2+}\, A^{2+}_{\stackrel{tt'}{(\oplus)}} + \tau^{2-}\, A^{2-}_{\stackrel{tt'}{(\oplus)}} + \tau^{23}\, A^{23}_{\stackrel{tt'}{(\oplus)}} \\
&+ \tau^{1+}\, A^{1+}_{\stackrel{tt'}{(\oplus)}} + \tau^{1-}\, A^{1-}_{\stackrel{tt'}{(\oplus)}} + \tau^{13}\, A^{13}_{\stackrel{tt'}{(\oplus)}} \}\,, \\
A^{2\boxminus}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{58\stackrel{tt'}{(\oplus)}} + \omega_{67\stackrel{tt'}{(\oplus)}}) \boxplus i(\omega_{57\stackrel{tt'}{(\oplus)}} - \omega_{68\stackrel{tt'}{(\oplus)}})\,, \quad A^{23}_{\stackrel{tt'}{(\oplus)}} = (\omega_{56\stackrel{tt'}{(\oplus)}} + \omega_{78\stackrel{tt'}{(\oplus)}})\,, \\
A^{1\boxminus}_{\stackrel{tt'}{(\oplus)}} &= (\omega_{58\stackrel{tt'}{(\oplus)}} - \omega_{67\stackrel{tt'}{(\oplus)}}) \boxplus i(\omega_{57\stackrel{tt'}{(\oplus)}} + \omega_{68\stackrel{tt'}{(\oplus)}})\,, \quad A^{13}_{\stackrel{tt'}{(\oplus)}} = (\omega_{56\stackrel{tt'}{(\oplus)}} - \omega_{78\stackrel{tt'}{(\oplus)}})\,, \\
(t\,t') &\in ((9\,10), (11\,12), (13\,14))\,. \qquad (114)
\end{aligned}
$$

The rest of expressions in Eq. (114) are obtained in a similar way. They are presented in Eq. (170) of App. E.2.

The scalar fields with the space index $s = (9, 10, \cdots, 14)$, presented in Table 9, carry one of the triplet colour charges and the "fermion" charge equal to twice the quark "fermion" charge, or the antitriplet colour charges and the "antifermion" charge. They carry in addition the quantum numbers of the adjoint representations originating in S^{ab} or in $\tilde{S}^{ab}$ [25].

If the antiquark $\bar{u}^{\bar{c2}}_L$, from the line 43 presented in Table 7, with the "fermion" charge $\tau^4 = -\frac{1}{6}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the hyper charge $Y (= \tau^4 + \tau^{23} =) -\frac{2}{3}$ and the electromagnetic charge $Q (= Y + \tau^{13} =) -\frac{2}{3}$ submits the $A^{2\boxminus}_{\stackrel{9\,10}{(\oplus)}}$ scalar field, it transforms into u^{c3}_R from the line 17 of Table 7, carrying the quantum numbers $\tau^4 = \frac{1}{6}$, $\tau^{13} = 0$, $\tau^{23} = \frac{1}{2}$, $(\tau^{33}, \tau^{38}) = (0, -\frac{1}{\sqrt{3}})$, $Y = \frac{2}{3}$ and $Q = \frac{2}{3}$. These two quarks, d^{c1}_R and u^{c3}_R can bind together with u^{c2}_R from the 9^{th} line of the same table (at low enough energy, after the electroweak transition, and if they belong to a superposition with the left handed partners to the first family) -into the colour chargeless baryon - a proton. This transition is presented in Fig. 1.

The opposite transition at low energies would make the proton decay.

Let us conclude this section with the recognition that these scalar triplet and antitriplet gauge fields, since transforming quarks into leptons and back, have the properties appearing in the literature as leptoquarks.

---

[25] Although carrying the colour charge in one of the triplet or antitriplet states, these fields can not be interpreted as superpartners of the quarks since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.

Table 9: Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in Eq. (101), are presented. The space degrees of freedom contribute one of the triplets values to the colour charge of all these scalar fields. These scalars are with respect to the two $SU(2)$ charges, (τ^{13} and $\vec{\tau}^2$), and the two $\widetilde{SU}(2)$ charges, ($\vec{\tilde{\tau}}^1$ and $\vec{\tilde{\tau}}^2$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "fermion" number (τ^4) of the quarks. The quantum numbers of the two vector gauge fields, the colour and the $U(1)_{II}$ ones, are added.

field	prop.	τ^4	τ^{13}	τ^{23}	(τ^{33}, τ^{38})	Y	Q	$\tilde{\tau}^4$	$\tilde{\tau}^{13}$	$\tilde{\tau}^{23}$	$\tilde{N}_L^3$	$\tilde{N}_R^3$
$A^{1\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{11\,12\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{11\,12\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{1\boxplus}_{13\,14\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	$\boxplus 1$	0	$(0, \oplus\frac{1}{\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{13}_{13\,14\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(0, \oplus\frac{1}{\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{2\boxplus}_{9\,10\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	$\boxplus 1$	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3} + \boxplus 1$	$\oplus\frac{1}{3} + \boxplus 1$	0	0	0	0	0
$A^{23}_{9\,10\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{1\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	$\boxplus 1$	0	0	0
$\tilde{A}^{13}_{910\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{2\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	$\boxplus 1$	0	0
$\tilde{A}^{23}_{910\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{N_L\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	$\boxplus 1$	0
$\tilde{A}^{N_L 3}_{910\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\tilde{A}^{N_R\boxplus}_{910\,(\oplus)}$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	$\boxplus 1$
$\tilde{A}^{N_R 3}_{910\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{3i}_{9\,10\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\boxplus 1 + \oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$A^{4}_{9\,10\,(\oplus)}$ $\cdots$	scalar	$\oplus\frac{1}{3}$	0	0	$(\oplus\frac{1}{2}, \oplus\frac{1}{2\sqrt{3}})$	$\oplus\frac{1}{3}$	$\oplus\frac{1}{3}$	0	0	0	0	0
$\vec{A}^3_m$	vector	0	0	0	octet	0	0	0	0	0	0	0
A^4_m	vector	0	0	0	0	0	0	0	0	0	0	0

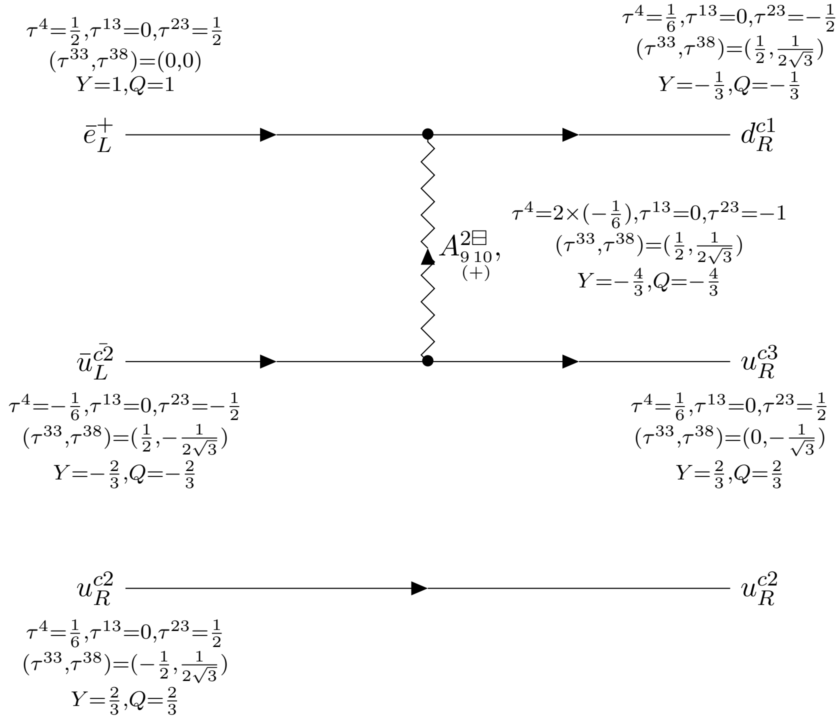

Figure 1: The birth of a "right handed proton" out of an positron $\bar{e}^+_L$, antiquark $\bar{u}^{\bar{c2}}_L$ and quark (spectator) u^{c2}_R. The family quantum number can be any out of the first four families, presented in Table 5.

7 Fermions and bosons at observable energies

We confront in this section the *standard model* with its rather simple, elegant — and waiting to be explained — assumptions about the properties of fermions coupled to the corresponding vector gauge fields and to the Higgs's scalar field with in the literature proposed unifying theories, mainly those built on $SO(10)$ [27], which unifies the charge groups of quarks and leptons separately and of antiquarks and antileptons separately, unifying correspondingly also the vector gauge fields of the charge groups of quarks and leptons, and the *spin-charge-family* theory of one of us (N.S.M.B.). The unifying theories built on $SU(4)\times SU(2)\times SU(2)$ group, Ref. [28], and on $SU(5)$, Ref. [29], which are subgroups of the group $SO(10)$, are counted under the $SO(10)$-unifying theories.

All the unifying theories wait to be confirmed by experiments.

Although the $SO(10)$-unifying theories made an important step in understanding the origin of charges of quarks and leptons and antiquarks and antileptons, as well as the origin of vector gauge fields, they do not offer the explanation either for the correspondence between the handedness and the charges of quarks and leptons and the handedness and the charges of antiquarks and antileptons, or for the appearance of families of quarks and leptons and do not unify all the vector and scalar fields with gravity. Correspondingly the $SO(10)$-unifying theories do not offer explanation for the appearance of the Higgs's scalar and the Yukawa couplings, except by assuming also the existence of scalar fields with the quantum numbers of the vector gauge fields.

One can, namely, just repeat the requirement that not only the vector gauge fields but also the scalar gauge fields should exist with the same charges in adjoint representations and with the hyper and weak charge of the Higgs's scalar.

We compare the offer of the $SO(10)$-unifying theories with the offer of the *spin-charge-family* theory, presented and discussed in this article (and the articles cited in this article), to explain the assumptions of the *standard model*.

We do not discuss supersymmetric models and only mention shortly the string theories.

All the assumptions of the *standard model*, cleverly "read" from the experimental data, are still waiting to be explained. Although there are no evidences either in experiments or in observations, which would help to show the clear step towards a simple and elegant theory, the *standard model* itself requires the next step — a simple and elegant theory, which would explain all its assumptions and would manifest at low energy in (the observable) $d = (3+1)$ dimensions what observations and experiments suggest.

We namely learn from the scientific experiences so far that each next step to simpler and more elegant explanation of the observed phenomena has helped to understand better the low of nature.

We hope that the *spin-charge-family* theory, which we presented shortly in this review article, is this new appropriate step beyond the *standard model*.

7.1 Standard model of fermion fields, vector gauge fields, and scalar fields

More than 50 years ago the "electroweak and colour" *standard model* offered an elegant new step in understanding the origin of fermion and boson fields by postulating properties of massless fermions and vector gauge fields and of massive scalar field before the electroweak break:

i. The existence of several families of massless quarks and leptons of the left or right handedness, with the charges in the fundamental representations of the weak $SU(2)$ charge, colour $SU(3)$ charge, and hyper $U(1)$ charge groups, and of massless antiquarks and antileptons, of the same spin S^{12} and of the opposite handedness and the opposite charges as quarks and leptons, like it is presented in Table 10 for one family, are postulated. Since the ν_R lepton and $\bar{\nu}_L$ antilepton carry in the *standard model* no charge at all, as seen in Table 10, the *standard model* does not include these two members among regular members of a family. The hyper charge Y is chosen so that the electromagnetic charges of quarks and leptons are in accordance with the measured onces.

Since each quark or lepton carries either spin $S^{12} = \frac{1}{2}$ or $S^{12} = -\frac{1}{2}$, there are 30 quarks and leptons in each family and the same number of antiquarks and antileptons, if one counts states with the same $p^0 = |\vec{p}|$ [26]. Counting states with a chosen $\vec{p}$ only, as usually is done, there are correspondingly 15 quarks and leptons in each family and the same number of antiquarks and antileptons. Counting ν_R and $\bar{\nu}_L$ as regular members of a family, there would be for $p^0 = |\vec{p}|$ 32 members of quarks and leptons and 32 members of antiquarks and antileptons in each family. Requirement that $\vec{p}$ and $-\vec{p}$ are treated separately, the number of quarks and leptons and antiquarks and antileptons in one family is then 32. The relation between handedness and charges of quarks and leptons and between handedness and charges of antiquarks and antileptons is in the *standard model* postulated.

The sum of any of these gauge charges over the family members quarks and leptons is equal to zero, and the same is true also for the gauge charges of the antiquarks and antileptons. The family gauge charges are just assumed to exist, they are not explained.

The reader can notice the two additional gauge charges in Table 10, named τ^{23} and τ^4, respectively, separated from the rest of data by vertical lines. These two gauge charges do not appear in the *standard model* at all. Both appear, however, in the $SO(10)$-unifying theories and in the *spin-charge-family* theory — τ^{23} as the third component of the second $SU(2)_{II}$ charge, and τ^4 as the "fermion" gauge charge. We shall comment them when discussing these two suggested steps beyond the *standard model*. In these two kinds of theories the hyper charge Y is equal to $Y = \tau^4 + \tau^{23}$, and the electromagnetic charge $Q = Y + \tau^{13} = \tau^4 + \tau^{23} + \tau^{13}$.

[26]For a chosen momentum $\vec{p}$ fermions with spin $S^{12} = -\frac{1}{2}$ carry opposite momentum $-\vec{p}$, as one can see in Eqs. (66, 84, 90). In Table 7 the "basis vectors" of one irreducible representation of the Lorentz generator S^{ab} in $d = 13+1$, that is of one family, is presented. Each family includes all "basis vectors", those with spin $S^{12} = \frac{1}{2}$ and those with spin $S^{12} = -\frac{1}{2}$. The requirement that the superposition of the tensor product, $*_T$, of the "basis vectors" and the basis in momentum or coordinate space solve the equation of motion reduces the number of states for the factor of 2.

Table 10: Members of each, $i = (1, 2, 3)$, of the so far observed families, before the electroweak break, are presented. Each family contains: **i.** The left handed weak charged quarks and leptons and the right handed weak chargeless quarks and leptons ($\tau^{13} = 0$), quarks carrying one of the triplet colour charges $(1/2, 1/(2\sqrt{3}))$, $(-1/2, 1/(2\sqrt{3}))$, $(0, -1/(\sqrt{3}))$, leptons are colourless, with the hyper charge Y (and with Y also the electromagnetic charge $Q = (\tau^{13} + Y)$) depending on handedness of fermions and on whether fermions carry the colour charge (quarks) or they are colourless (leptons). **ii.** The corresponding antiquarks and antileptons carry the same spin S^{12} and opposite handedness and charges as quarks and leptons. The right handed ν^i_R and the left handed $\bar{\nu}^i_L$, which would carry all the charges equal to zero, are not accepted as the regular members of the antifamily. They are therefore presented in Table 10 as ν^i_R and $\bar{\nu}^i_L$. The hyper charges Y are determined so that the electromagnetic charges agree with the observable values. The two columns within the vertical lines, τ^{23} and τ^4 have no meaning in the *standard model*. They are presented here in purpose for later comparison with the $SO(10)$ theories and with the *spin-charge-family* theory.

name	handedness $-4iS^{03}S^{12}$	weak charge τ^{13}	τ^{23}	colour charge (τ^{33}, τ^{38})	τ^4	hyper charge Y	elm. charge $Q = \tau^{13} + Y$
u^i_L	-1	$\frac{1}{2}$	0	colour triplet	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
d^i_L	-1	$-\frac{1}{2}$	0	colour triplet	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
ν^i_L	-1	$\frac{1}{2}$	0	colourless	$-\frac{1}{2}$	$-\frac{1}{2}$	0
e^i_L	-1	$-\frac{1}{2}$	0	colourless	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
u^i_R	1	0	$\frac{1}{2}$	colour triplet	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
d^i_R	1	0	$-\frac{1}{2}$	colour triplet	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
ν^i_R	1	0	$\frac{1}{2}$	*colourless*	$-\frac{1}{2}$	0	0
e^i_R	1	0	$-\frac{1}{2}$	colourless	$-\frac{1}{2}$	-1	-1
$\bar{u}^i_R$	1	$-\frac{1}{2}$	0	colour antitriplet	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
$\bar{d}^i_R$	1	$\frac{1}{2}$	0	colour antitriplet	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
$\bar{\nu}^i_R$	1	$-\frac{1}{2}$	0	colourless	$\frac{1}{2}$	$\frac{1}{2}$	0
$\bar{e}^i_R$	1	$\frac{1}{2}$	0	colourless	$\frac{1}{2}$	$\frac{1}{2}$	1
$\bar{u}^i_L$	-1	0	$-\frac{1}{2}$	colour antitriplet	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\bar{d}^i_L$	-1	0	$\frac{1}{2}$	colour antitriplet	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
$\bar{\nu}^i_L$	-1	0	$-\frac{1}{2}$	*colourless*	$\frac{1}{2}$	0	0
$\bar{e}^i_L$	-1	0	$\frac{1}{2}$	colourless	$\frac{1}{2}$	1	1

The gauge invariance requirement leads to masslessness of quarks and leptons and all vector gauge fields before the electroweak transition. It is obviously an elegant requirement, since zero and infinity do not need to be justified, as both are natural possibility. The massless elementary fermions and gauge fields indicate that we are looking at a low energy manifestation of the elegant and simple theory, due to which at some scale all the bosons and fermions are massless fields, with the Higgs's field included.

The fact that the left and the right handed quarks and leptons differ in the gauge charges takes care of their mass protection up to the point when Higgs's scalar gains a constant value, "dressing" the right handed quarks and leptons with the appropriate weak and hyper charge ($\tau^{13} = \frac{1}{2}, Y = -\frac{1}{2}$ for u_R and ν_R, and $\tau^{13} = -\frac{1}{2}, Y = \frac{1}{2}$ for d_R and e_R), while the left handed antiquarks and antileptons are dressed with the appropriate weak and hyper charge ($\tau^{13} = -\frac{1}{2}, Y = \frac{1}{2}$ for u_L and ν_L, and $\tau^{13} = \frac{1}{2}, Y = -\frac{1}{2}$ for d_L and e_{RL}) as well.

ii. The existence of massless vector fields before the electroweak break — the gauge fields of the observed charges of the family members quarks and leptons and antiquarks and antileptons — carrying charges in the adjoint representations of the charge groups are postulated: **a.** The hyper photon carrying no charge, yet interacting through the hyper charge with quarks and leptons. **b.** The weak charge triplet boson, carrying the triplet electromagnetic charge (through relation $Q = Y + \tau^{13}$, with $Y = 0$). **c.** The colour octet gauge field. These vector gauge fields are presented in Table 11.

Table 11: Massless vector gauge fields in $d = (3+1)$ before the electroweak break are presented. They are the gauge fields of the hyper, weak and colour charges, carrying all the charges in the adjoint representations. None of them carries the second weak charge and "fermion" charge, presented in the fourth and the last column, respectively, added because of the two additional vector fields, presented below the first three lines and separated from the above three by the horizontal line. Both vector gauge fields appear in $SO(10)$-unifying theories and the *spin-charge-family* theory. One of these two vector gauge fields, the triplet gauge field of the (second $SU(2)_{II}$) $\vec{\tau}^2$ charge, is with respect to the *standard model* the new field. The second one, the singlet vector gauge field of the "fermion" charge τ^4, explains the vector gauge field of the hyper charge Y of the *standard model*, since it appears as the superposition of the τ^4 vector gauge field and the gauge vector field of τ^{23}, the third component of the new $\vec{\tau}^2$ triplet gauge field. We present these two gauge fields here for later comments on the *standard model* assumptions from the point of view the $SO(10)$-unifying theories and the *spin-charge-family* theory, Sect. 6.2.1.

name	hand-edness	weak charge	*weak* *charge*$_{II}$	hyper charge	colour charge	elm charge	*"fermion" charge*
hyper photon	0	0	0	0	colourless	0	0
weak bosons	0	triplet	0	0	colourless	triplet	0
gluons	0	0	0	0	colour octet	0	0
weak bosons$_I$	0	0	triplet	0	colourless	triplet	0
hyper photon$_I$	0	0	0	0	colourless	0	0

For future discussions there appear in the same table two additional vector gauge fields, suggested by the $SO(10)$-unifying theories as well by the *spin-charge-family* theory: The additional $SU(2)$ triplet vector gauge field and the $U(1)$ singlet vector gauge fields. These vector gauge fields are discussed in Sect. 6.2.1 from the point of view of the s̑pin-charge-family theory.

iii. The existence of the scalar gauge field, with the weak and hyper charges $\pm\frac{1}{2}$ and $\mp\frac{1}{2}$, respectively, coupled to different families of quarks and leptons according to Yukawa couplings is postulated. The Higgs's scalar field, after gaining constant values at the electroweak break, gives masses to quarks and leptons and weak charge gauge fields, due to the interactions of the weak gauge fields and quarks and leptons and antiquarks and antileptons with the weak and hyper charge of the scalar field with the nonzero constant value [27]. The scalar higgs is presented in Table 12. There are only components of two doublets presented in the Table 12, which gain constant values at the electroweak break, presented in the table in bold phase. In the same table the two triplets and three singlets, discussed in Sect. 6.2.2, are presented for future discussions on the next step beyond the *standard model*, offered by the *spin-charge-family* theory. There are three singlets and two triplets, which explain the origin of the Higgs's scalar and the Yukawa couplings, Eq. (108).

The Lagrange densities for fermions (ψ includes all the quarks and leptons and antiquarks and antileptons of all the observed families) and for the vector gauge fields A^{Ai}_m of the *standard model* before

[27]The interaction of quarks and leptons with the constant scalar field resembles the interaction of electrons with the ions of the crystal during tunneling of the electrons through it, what causes the change of the mass of electrons.

Table 12: The Higgs's scalar is a massive scalar field in $d = (3+1)$, with the charges like there would be in the fundamental representation of the weak and hyper charge groups. Only $<\mathbf{Higgs_u}>$ and $<\mathbf{Higgs_d}>$ gain (nonzero vacuum expectation) constant values, the two other components do not obtain the constant values. The explanation for the appearance of the scalar gauge fields is offered by the *spin-charge-family* theory in Sect. 6.2.2. The two triplets, $\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$, $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$, and the three singlets, $A^{Q}_{\substack{78\\(\pm)}}, A^{Y}_{\substack{78\\(\pm)}}, A^{4}_{\substack{78\\(\pm)}}$, Eq. (109), presented below the double line, replace and explain the higgses and Yukawa couplings of the *standard model*.

name	handedness	weak charge	hyper charge	colour charge	elm charge
$Higgs_u$	0	$\frac{1}{2}$	$\frac{1}{2}$	*colourless*	1
$<\mathbf{Higgs_d}>$	**0**	$-\frac{1}{2}$	$\frac{1}{2}$	**colourless**	**0**
$<\mathbf{Higgs_u}>$	**0**	$\frac{1}{2}$	$-\frac{1}{2}$	**colourless**	**0**
$Higgs_d$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	*colourless*	-1
$A^{Q}_{\substack{78\\(\pm)}}, A^{Y}_{\substack{78\\(\pm)}}, A^{4}_{\substack{78\\(\pm)}}$	0	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	colourless	0
$\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$	0	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	colourless	0
$\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$	0	$\mp\frac{1}{2}$	$\pm\frac{1}{2}$	colourless	0

the electroweak break is

$$\begin{aligned}
\mathcal{L}_f &= \bar{\psi}\gamma^m (p_m - \sum_{A,i} g^{Ai}\tau^{Ai} A^{Ai}_m)\psi\,, \\
\{\tau^{Ai},\, \tau^{Bj}\}_- &= \delta^{Ab}\, i\, f^{Aijk}\tau^k\,, \\
\mathcal{L}_{vg} &= -\sum_{Ai} \frac{\varepsilon^A}{4}\, F^{Ai}{}_{mn}\, F^{Aimn}\,, \\
F^{Ai}{}_{mn} &= -i(p_m A^{Ai}_n - p_n A^{Ai}_m) - f^{Aijk} A^{Aj}_m A^{Ak}_n\,,
\end{aligned} \tag{115}$$

with the structure constants f^{Aijk} concerning the colour $SU(3), A = 3$, weak $SU(2), A = 1$ and hyper charge $U(1), A = 2$ groups.

The Lagrange density of the Higgs's scalar field is before the electroweak phase transition equal to

$$\mathcal{L}_{sH} = \{(p_m\,\phi)^\dagger\,(p^m\,\phi) - (m^2_\phi\,\phi^\dagger\phi + \lambda\,(\phi^\dagger\phi)^2\,)\,. \tag{116}$$

After the electroweak phase transition $p_m\phi$ become $p_{0m}\phi = (p_m + g^1\vec{\tau}^1\,\vec{A}^1_m + g^Y Y A^Y_m)\,\phi$, since the constant higgs field, $<\mathbf{Higgs_{(d,u)}}>$, causes coupling to the weak vector gauge field. The postulated Yukawa couplings together with the higgs ϕ, gaining the constant value, require coupling of quarks and leptons to the higgs as well

$$\mathcal{L}_{YH} = -\sum_{ff'i} Y^{ff'}_i\,\psi^{i\dagger}_{Lf}\gamma^0 \,< Higgs_i >\, \psi^i_{Rf'}\,, \tag{117}$$

with $Y^{ff'}_i$ which is for each family member $i = (u, d, \nu, e)$ 3×3 matrix, (f, f') represent one of the three families of the family members, and $< Higgs_i > = < Higgs_u >$ for $i = (u, \nu)$ and $< Higgs_i > = < Higgs_d >$ for $i = (d, e)$. At the electroweak break $(m^2_\phi\,\phi^\dagger\phi + \lambda\,(\phi^\dagger\phi)^2\,)$ (is assumed to) changes to $(-m^2_\phi\,\phi^\dagger\phi + \lambda\,(\phi^\dagger\phi)^2\,)$, with the higgs obviously gaining the imaginary mass.

7.2 Discussions on $SO(10)$ unifying theories and *spin-charge-family* theory

In both theories, the $SO(10)$-unifying theories and the *spin-charge-family* theory, the right handed neutrino is the regular member of quarks and leptons and the left handed antineutrino is the regular member of antiquarks and antileptons, with the difference, that in the *spin-charge-family* theory one irreducible representation of the Lorentz group $SO(13+1)$ contains both quarks and leptons and antiquarks and antileptons with the handedness and charges uniquely related. (The Lorentz group $SO(13+1)$ contains as the subgroups $SO(3+1)$ and $SO(10)$.) This is not the case for the $SO(10)$-unifying theories, in which the relation between handedness and charges must be assumed, like in the *standard model*.

Right handed neutrino and left handed antineutrino namely carry the τ^{23} charge of the additional $SU(2)_{II}$ charge group, ν_R has $\tau^{23} = \frac{1}{2}$ and $\bar{\nu}_L$ has $\tau^{23} = -\frac{1}{2}$, as presented in Table 10 (the same can be seen also in Table 7, representing in the *spin-charge-family* theory the "basis vectors" of all the family members of one particular family out of two times two groups of four families, presented in Table 5).

In both theories fermions carry also the "fermion" gauge charge τ^4, quarks have $\tau^4 = \frac{1}{6}$, antiquarks $\tau^4 = -\frac{1}{6}$, leptons have $\tau^4 = -\frac{1}{2}$, antileptons have $\tau^4 = \frac{1}{2}$. The hyper charge is in both cases equal to $Y = \tau^4 + \tau^{23}$.

Both theories predict the existence of the corresponding triplet, $\vec{A}^2_m$, vector gauge field and the singlet, A^4_m, vector gauge fields, the superposition of which is observable at low energies as the hyper charge $Y = (\tau^{23} + \tau^4)$ vector gauge field, A^Y_m, assumed by the *standard model*.

The main differences between these two searches — $SO(10)$-unifying theories [27] (with $SU(5)$ [29] and $SU(4) \times SU(2) \times SU(2)$ [28] as the subgroups of the $SO(10)$ group included) and the *spin-charge-family* theory [8, 9, 11, 12, 21, 22] — for steps beyond the *standard model*, which could explain the *standard model* assumptions, are:

i. The $SO(10)$-unifying theories use, as all the literature does, the Dirac's second quantized theory for the description of massless fermions in $d = (3+1)$-dimensional space-time, as discussed in Sects. 3.1 and 3.3, and Sects. 3.3.1, 3.3.4 of this paper.

The *spin-charge-family* theory describes the internal space of massless fermions with the Clifford algebra in $d = (13+1)$-dimensional space-time, Sects. 3, 3.2, of this paper, what not only offers the explanation for the second quantization postulates of Dirac, Sect. 3.3.4, but offers at the same time the explanation for the appearance of families of fermions, as we explain step by step in Sect. 3, and in particular in 3.2.3, 3.3.2 and 3.3.4.

ii. The $SO(10)$-unifying theories unify all the charges by analyzing the subgroups of the $SO(10)$ group from the point of view of the *standard model* groups, assuming the existence of vector gauge fields to the corresponding charge groups, unifying consequently also all the vector gauge fields, but they do not explain either the appearance of families of fermions or the appearance of the Higgs's scalar and Yukawa couplings, or they suggest the common origin of all the forces with gravity included.

The *spin-charge-family* theory assumes the simple starting action, Eq. (100), in $d = (13+1)$-dimensional space, with fermions interacting with gravity only — with the vielbeins and the two kinds of the spin connection fields [28], unifying gravity and vector and scalar gauge fields. Fermions with the spins defined in $SO(13+1)$ manifest in $d = (3+1)$ the ordinary spin and handedness, charges and families unified, Sect. 3. The starting action, Eq. (100), manifests in $d = (3+1)$ all the vector gauge fields, Sect. 6.2.1, assumed by the *standard model*, as well as the scalar gauge fields, Sect. 6.2.2, with the properties of the Higgs's scalar, what explains the appearance of the Higgs's scalar and the Yukawa couplings of the *standard model*, predicting the number of families and the symmetry of mass matrices, discussed in Sect. 6.2.2, Eq. (111).

iii. In both theories, the $SO(10)$-unifying theories and the *spin-charge-family* theory, the breaks

[28]The spin connection fields are expressible uniquely with vielbeins if there are no fermion condensates present, Eq. (103)), Sect. 6.

of the starting symmetry lead at low energies to the *standard model* content of the charges and correspondingly also of the vector gauge fields of charges [29].

The $SO(10)$ group can break into $SU(3) \times U(1) \times SU(2) \times SU(2)$ either over $SO(6) \times SO(4)$ or over $SU(4) \times SO(4)$ or over $SU(5) \times U(1)$.

In order to keep the handedness and charges in relations as presented in Table 7 and what the *standard model* assumes, the $SO(13+1)$ group must break first to $SO(7,1) \times SU(3) \times U(1)$ (in both sectors, in $SO(13+1)$ sector determining the spins, handedness and charges of family members of different irreducible representations (families), and in $\widetilde{SO}(13+1)$ sector, which connects particular member of a particular family with the same family member of all the families, equipping different irreducible representations of $SO(13+1)$ with the family quantum numbers), and then further to $SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$ [30]. Sect. 7.4 explains in the case of a toy model how the breaks of symmetries can spontaneously appear in the *spin-charge-family* theory.

iv. The way of breaking symmetries determines in both cases the coupling constants of fermions to the corresponding vector gauge fields, manifesting at the low energy regime. In the *spin-charge-family* theory the way of breaking of symmetries determines also the coupling constants of fermions to the scalar gauge fields.

v. The "miraculously" cancellation of the triangle anomalies of the *standard model* both, the $SO(10)$-unifying theories and the *spin-charge-family* theory, explain, App. C, the first one after relating charges and handedness by assumption, in the second one the handedness and charges are related due to the unification of spins and charges.

7.2.1 Short overview of $SO(10)$ *unifying theories* [27, 28, 29]

.

The $SO(10)$-unifying theories offer the explanation for some of the postulates of the *standard model*. Namely, assuming the $SO(10)$ unifying charge group and the existence of the corresponding vector gauge fields with the charges in the adjoint representations, the theories do treat all the charges in an unique way and correspondingly also all the vector gauge fields in an unique way, postulating the existence of the charges and vector gauge fields only once. The theories could repeat this game also for the scalar gauge fields, but the scalar index of the *spin-charge-family* theory, which naturally appears from the simple starting action, Sect. 6.2, since the gravity in $d = (13+1)$ manifests in $d = (3+1)$ as ordinary gravity and all the vector and scalar gauge fields appearing in the starting action, Eq. (100), has to be in the $SO(10)$-unifying theories postulated like in the *standard model*.

i. In the Dirac's second quantization procedure there are in $d = (3+1)$ massless fermions with the half integer spins described by the fundamental representation of the Lorentz group $SO(3,1)$, with the spins, S^{12} and S^{03}, which are the members of the (chosen) Cartan subalgebra of the $SO(3,1)$ algebra, defining as well the handedness, $-4i\, S^{03} S^{12}$, as presented in Sect. 3.3.4 of this paper.

In the $SO(10)$ unifying theories the second quantized fermions carry spins and handedness as in the *standard model*, charges are in the fundamental representation of the subgroups of the group $SO(10)$, carrying in addition to the charges of the *standard model*, presented in Table 10 with the eigenvalues

[29]The break of the starting symmetry should appear spontaneously during the expansion of the universe. To evaluate this from the simple starting action assumed in the *spin-charge-family* theory, without knowing the boundary conditions, is impossible, but yet we can do a lot, since we know the low energy manifestation of quarks and leptons, antiquarks and antileptons, of the vector gauge fields and of the scalar gauge fields.

[30]If we would make the approximation that all the $(14-4)$ extra dimensions are curved with almost spheres, Sect. 7.4, of the same radius, or even with different radii, we could have a kind of $SO(10)$ theory, but in the way of the *spin-charge-family* theory, that is with the gravity as the only gauge field manifesting in $d = (3+1)$ as the ordinary gravity, the vector and the scalar gauge fieds and with fermions manifesting families, the massess of which are determined by the scalar gauge fields, Sect 7.2.2, paragraph **v.**.

of the operators τ^{13} (the weak charge), (τ^{33}, τ^{38}) (the colour charge) and the hyper charge Y, also the charges $\vec{\tau}^2$ and τ^4, presented as well in the same table 10 [31].

ii. To all the charges of fermions the corresponding massless vector gauge fields correspond, carrying charges in adjoint representations: the weak $SU(2)$ is the triplet, the second $SU(2)_{II}$ is the triplet, the colour $SU(3)$ is the octet and the new one $U(1)_I$, which is the singlet, representing the *standard model* hyper charge photon singlet as the superposition of the A^{23}_m vector gauge field (of τ^{23} charge) and this second $U(1)$ singlet vector gauge field. In Table 11 the triplet vector gauge field of the second $SU(2)_{II}$ charge is presented as *weak bosons*$_I$ and the singlet $U(1)_I$ as the *hyper photon*$_I$.

iii. The $SO(10)$ unifying theories can postulate the appearance of families, similarly as the *standard model* does, that is by introducing additional, this time the family, group and correspondingly do not explain the appearance of the Higgs's scalar fields and Yukawa couplings of the *standard model*, Table 12.

iv. The relations among the coupling constants and correspondingly the unification scale depend on the way of breaking the starting $SO(10)$ to the *standard model* $SU(3) \times U(1) \times SU(2)$ symmetry groups. These theories must postulate the scalar fields to break these symmetries as well.

v. In usual $SO(10)$-unifying theories one gets in the first approximation the same requirement for the gauge couplings being unified in the same way as in SU(5) (which is a subgroup of the $SO(10)$) after with the renormalization group extrapolated coupling constants to the unification scale [32].

7.2.2 Short overview of the *spin-charge-family* theory [8, 9, 11, 130, 131, 12, 21, 22]

The *spin-charge-family* theory of one of us (N.S.M.B.) [8, 9, 11, 130, 131, 12, 21, 22] assumes in $d = (13+1)$-dimensional space simple action, Eq. (100), for massless fermions and for massless vielbeins and two kinds of the spin connection fields, with which fermions interact. Description of the internal degrees of fermions by the odd Clifford algebra objects offers the unique explanation of spins, charges and families, Sect. 3, explaining the second quantization postulates of Dirac, Sect. 3.3.

i. *The odd Clifford algebra describes the whole internal space of fermions, Sect. 3.*

The *spin-charge-family* theory assumes massless fermions in $d = (13+1)$-dimensional space, describing the internal space of fermions by the odd Clifford algebra, which offers besides the description of spins and with respect to $d = (3+1)$ spins and charges also the appearance of families of fermions, Sects. 3, 3.2.3, predicting the number of families and the symmetry of their mass matrices, Sect. 6.2.2.

i.a. *The "basis vectors"* of the odd Clifford algebra presentations of the Lorentz group *anticommute*, Sects. 3.2, 5.1. Anticommutativity of the "basis vectors" dictates the anticommuting properties to the creation operators and their Hermitian conjugated partners annihilation operators, Sects. 3.3.2, 4.2, what explains the Dirac's second quantization postulates, Sect. 3.3.4, Eq. (68).

i.b. *Each irreducible representation of the Lorentz group*, analyzed with respect to the *standard model* groups, Table 7, includes quarks and leptons and antiquarks and antileptons related to handedness as required by the *standard model*. (The $SO(10)$ unifying theories must relate charges and handedness "by hand", following the *standard model*.)

i.c. *The spins in* $d > (3+1)$, if analyzed with respect to the *standard model* groups, Eqs. (85, 86, 87), manifest at low energies charges and spins and handedness of quarks and leptons and antiquarks and antileptons assumed by the *standard model*.

[31] The group $SO(10)$ has 5 commuting operators, 5 members of the Cartan subalgebra, presented in Table 10: τ^{13} (the weak $SU(2)$ group of the $\vec{\tau}^1$ charge), τ^{23} (the second $SU(2)$ group of the $\vec{\tau}^2$ charge, the weak $SU(2)$ group and the second $SU(2)$ group are the subgroups of the $SO(4)$ group), (τ^{33}, τ^{38}) (the colour SU(3) group of $\vec{\tau}^3$ charge) and τ^4 (the $U(1)$ group, called in this paper the "fermion" charge, the $U(1)$ group and $SU(3)$ group are the subgroups of the $SO(6)$ group [27] (their expressions with the infinitesimal generators of the Lorentz group are written in Eqs. (86, 87), their relations to the hyper charge and the electromagnetic charge are written in Eq. (88) of this paper), or the subgroup of $SU(4)$ [28]. The hyper charge is in both cases equal to $Y = \tau^{23} + \tau^4$.

[32] Let us comment that either in the $SO(10)$-unifying theories or in the *spin-charge-family* theory there is no need that all three coupling constants should meet at the same point, due to several breaks of symmetries [63].

Since $SO(10)$ group is together with the $SO(3,1)$ the subgroup of $SO(13,1)$, in both groups, $SO(10)$ and $SO(13,1)$, the $SU(2)_{II}$ charge group appears together with the weak $SU(2)_I$ charge group as the subgroups of the $SO(4)$ group. The $U(1)$ charge group appears together with the $SU(3)$ colour charge group as subgroups of $SO(6)$ group (while in Ref. [28] the charge group $U(1)$ belongs together with the $SU(3)$ colour charge group to the $SU(4)$ charge group). The infinitesimal generators of the groups $SU(2)_{II}$ and $U(1)$ are denoted in Table 10 and in Table 7 as τ^{23} and τ^4 charges, respectively[33].

ii. *Fermions interact with the gravity only, manifesting in* $d = (3+1)$ *all the observed fermion fields coupled to the observed vector and scalar gauge fields) as well as gravity*, Ref. [15].

Gravity is represented by the vielbeins (the gauge fields of momenta) and the two kinds of the spin connection fields (the gauge fields of S^{ab} and $\tilde{S}^{ab}$), discussed in Sect. 6. In Eq. (100) the simple starting action in $d = (13+1)$ is presented, manifesting in $d = (3+1)$ all the observed fermion fields coupled to the observed vector and scalar gauge fields) as well as gravity.

ii.a. In Sect. 6.1, in Eq. (101) *the fermion Lagrange density is presented, manifesting the couplings to the vector gauge fields and the scalar gauge fields.* As seen in Table 5 there remain after the (assumed spontaneous) break of the starting symmetry $SO(13,1)$ to $SO(7,1) \times SU(1) \times U(1)$ two groups of four massless families of quarks and leptons.

ii.b. *Both groups of four families manifest* $SU(2) \times SU(2) \times U(1)$ *symmetry* as presented in Eq. (111) of Sect. 6.2.2. The three $U(1)$ singlets, $A^{Q}_{\underset{(\pm)}{78}}, A^{Y}_{\underset{(\pm)}{78}}, A^{4}_{\underset{(\pm)}{78}}$, contributing to both groups of four families, are responsible for the difference in masses of different family members, of $(u,\, d,\, \nu,\, e)$.

iii. *The vielbeins and the two kinds of the spin onection fields manifest in* $d = (3+1)$ *all the vector gauge fields and also the scalar gauge fields.*

In Ref. [15] it is shown how do the vielbeins and spin connections (if there is no condensate present any of three determines the rest two, as seen in Eq. (103)) manifest either as vector gauge fields, Sect. 6.2, Sect. 6.2.1, offering the explanation for the origin of the *standard model* vector gauge fields, or the scalar gauge fields, Sect. 6.2.2, offering the explanation for the Higgs's scalar and Yukawa couplings, and additional scalar gauge fields [34].

iii.a. In Table 11 *the triplet vector gauge field of the second* $SU(2)_{II}$ *charge* (denoted by $\vec{\tau}^2$) is presented as the *weak bosons*$_I$ and *the singlet* $U(1)_I$ *vector gauge field of the "fermion" charge* τ^4 is presented as the *hyper photon*$_I$.

The $SU(2)_{II}$ vector gauge field appears in addition to the standard model gauge fields, while the *standard model* hyper charge vector gauge field is the superposition of τ^{23} component of the $SU(2)_{II}$ vector gauge field and the $U(1)_I$ singlet vector gauge field [35].

iii.b. *The scalar gauge fields with the space index* $s = (7,8)$ *determine after the electroweak break masses of two times four families.*

The scalar gauge fields, which are the superposition of $\tilde{\omega}_{abs}, s = (7,8)$, with either $(a,b) = (0,1,2,3)$ or $(a,b) = (5,6,7,8)$, or the superposition of $\omega_{t,t',s}$, carrying the charges (Q, Y, τ^4) with respect to (a,b), Sect. 6.2.2, form two groups of two triplets (they are superposition of $\tilde{\omega}_{abs}$) and three singlets (they are superposition of $\omega_{t,t',s}$), Eq. (108).

All the triplets and the singlets with the space index $(s = (7,8))$ have the weak and the hyper charge

[33] The group $SO(13+1)$ has $\frac{d}{2} = 7$ members of the Cartan subalgebra, unifying spins and charges and relating handedness of quarks and leptons and antiquarks and antileptons to charges.

[34] The way of breaking symmetries predicts the number of massless families [129] before the electroweak break, and the symmetries of mass matrices, as well as the existence of the additional scalar fields, Eq. (101).

[35] The condensate, Table 6, of the two right handed neutrinos with the family quantum numbers of the four families which do not belong to the first four families in Table 5, couple to the triplet vector gauge field of the second $\vec{\tau}^2$ charge and to the singlet vector gauge field of the "fermion" charge τ^4, making one of the superposition of these two vector gauge fields massive, while the other remains massless as the hyper charge vector gauge field. The weak charge vector gauge field, the colour charge vector gauge field and this hyper charge vector gauge fields remain massless, since they do not couple to the condensate. The condensate has the corresponding gauge charges equal to zero.

equal to either $\tau^{13}=\frac{1}{2}$ and $Y=-\frac{1}{2}$ or $\tau^{13}=-\frac{1}{2}$ and $Y=\frac{1}{2}$, Table 8.

One group of two triplet scalar fields and the three singlet scalar fields are presented in Table 12 in the lowest part of the table. The three singlets are denoted by $A^{Q}_{\substack{78\\(\pm)}}, A^{Y}_{\substack{78\\(\pm)}}, A^{4}_{\substack{78\\(\pm)}}$ and the two triplets are denoted by $\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$ and $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$. They all origin in the simple starting action, Eq. (100), and manifest in $d=(3+1)$ as superposition of the spin connection fields of both kinds, $\omega_{tt's}$, $s=(7,8)$ and $\tilde{\omega}_{abs}$, $s=(7,8)$, Eq. (108).

iii.c. *The scalar gauge fields with the space index $s=(7,8)$ are expected to gain a constant values.*

The scalar gauge fields, the two groups of triplets and the three singlets, are expected to gain at low energies the imaginary masses (on the present stage of studying properties of the scalar gauge fields this break of symmetry is assumed as in the *standard model*, although the *spin-charge-family* does propose the origin of the scalar gauge fields and their starting couplings and consequently their properties) as presented in Eq. (112). This makes these scalar fields manifesting constant values, breaking correspondingly mass protection of quarks and leptons, since the constant values of the vacuum carry the weak and the hyper charge of $\pm\frac{1}{2},\mp\frac{1}{2}$, respectively, Table 8. This work is in progress for a few years already.

The three singlets, $A^{Q}_{\substack{78\\(\pm)}}, A^{Y}_{\substack{78\\(\pm)}}, A^{4}_{\substack{78\\(\pm)}}$ and the two triplets, $\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$ $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$, coupling to the four of the eight families of fermions presented in Table 5, determine the mass matrices of the observed three families of quarks and leptons, predicting the fourth family to the observed three.

The same three singlets, $A^{Q}_{\substack{78\\(\pm)}}, A^{Y}_{\substack{78\\(\pm)}}, A^{4}_{\substack{78\\(\pm)}}$ and the second kind of two triplets, $\vec{\tilde{A}}^{\tilde{2}}_{\substack{78\\(\pm)}}$ $\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{\substack{78\\(\pm)}}$, couple to the rest four families of Table 5, determining the mass matrices of another group of quarks and leptons. The stable family of this group form the stable baryons, which offer the explanation for the existence of the heavy *dark matter* candidates.

iii.d. *There are additional scalar fields with the space index $s=(9,10,11,12,13,14)$, which are triplets or antitriplets with respect to the space index s.*

There are additional scalar fields in the *spin-charge-family* theory, those with the space index $s=(9,10,11,12,13,14)$, Eq. (101), manifesting as a triplets or antitriplets with respect to the space index s. They cause transitions of antileptons into quarks and back and leptons into antiquarks and back, what might be responsible in the expanding universe for the matter/antimatter asymmetry and also for the proton decay, Sect. 6.2.2.

iv. *The break of the starting symmetry $SO(13+1)$ into $SO(3,1)\times SU(2)\times SU(3)\times U(1)$ is needed.*

This problem is still not solved although we have the spontaneous breaks, first from $SO(13,1)$ (and $\widetilde{SO}(13+1)$) to $SO(7,1)\times SU(3)\times U(1)$ ($\widetilde{SO}(7+1)\times\widetilde{SU}(3)\times\widetilde{U}(1)$) and then further to $SO(3,1)\times SU(2)\times SU(3)\times U(1)$ (and $\widetilde{SO}(3,1)\times\widetilde{SU}(2)\times$ $\widetilde{SU}(3)\times\widetilde{U}(1)$) all the time present.

In Refs. [132, 127, 139, 137, 136, 128, 129] we study the toy model of $d=(5+1)$, in which the spin connection fields with the space index $s=(5,6)$ force the infinite surface in the fifth and the sixth dimension to form an almost S^2 sphere, keeping rotational symmetry of the surface around one point, while fermions in $d=(3+1)$ of particular handedness keep their masslessness. These happens for all the families of fermions of particular handedness [129].

We have not yet successfully repeated the $d=(5+1)$ toy model of the spontaneous "compactification" in the case of $SO(13,1)$ (we put compactification into quotation marks, since an almost S^n sphere, as also S^2 sphere in the toy model of $d=(5+1)$ has the singular points in all infinities).

What we have to study is first to almost "compactify" the space of $s=(9,10,11,12,13,14)$ so that the space has the symmetry allowing the infinitesimal transformations of the kind around the center of the symmetry [15, 132, 127, 128, 129]

$$x'^{\mu} = x^{\mu}\,, \quad x'^{\sigma} = x^{\sigma} - i \sum_{A,i,s,t} \varepsilon^{Ai}(x^{\mu})\, c_{Ai}{}^{st} M_{st}\, x^{\sigma}\,,$$

with

$$\begin{aligned}
\tau^{Ai} &= \sum_{s,t} c^{Ai}{}_{st}\, M^{st}\,, \quad \{\tau^{Ai}, \tau^{Bj}\}_{-} = i f^{Aijk} \tau^{Ak}\, \delta^{AB}\,, \\
\vec{\tau}^{A} &= \vec{\tau}^{A\sigma}\, p_{\sigma} = \vec{\tau}^{A\sigma}{}_{\tau}\, x^{\tau}\, p_{\sigma}\,, \tau^{Ai\sigma} = \sum_{s,t} -i c^{Ai}{}_{st}\, M^{st\sigma} \\
&= \sum_{s,t} c^{Ai}{}_{st}\, (e_{s\tau}\, f^{\sigma}{}_{t} - e_{t\tau}\, f^{\sigma}{}_{s}) x^{\tau} = E^{\sigma}_{Ai}\,,
\end{aligned} \tag{118}$$

which lead to massless spinors/fermions of one particular charge and of one handedness only for each of all the $2^{\frac{7+1}{2}-1}$ families in the remaining space $d = (7+1)$. The generator of the Lorentz transformations M_{st} is equal to $M_{st} = E^{\sigma}_{st}\, p_{\sigma}$, while the Killing vectors E^{σ}_{st} fulfill the Killing equations, Ref. [15], Eq. (8).

We need to find the solutions for the equations of motion for massless fermions in $d \geq 9$ with vielbeins and the two kinds of the spin connection fields, which curve the infinite six dimensional space into an almost S^6, with the symmetry of $SU(3) \times U(1)$, in a way that they allow in $d = (7+1)$ only one massless normalizable spinor state for each $SU(3)$ coloured or colouress state of particular $U(1)$ "fermion" charges ($\tau^4 = \frac{1}{6}$ for colour triplet fermion states and $-\frac{1}{2}$ for colourless fermion states), as well as only one massless normalizable spinor state for each $SU(3)$ anticoloured or anticolourless state of particular $U(1)$ "antifermion" charges ($\tau^4 = -\frac{1}{6}$ for colour antitriplet states and $\frac{1}{2}$ for colourless states), Table 7.

These we did in Refs. [132, 127, 139, 137, 136, 128, 129] for the toy model, in which the several choices of vielbeins and spin connection fields take care of breaking the symmetry $SO(5+1)$ to $SO(3+1)$, leaving massless states of one handedness for each charge and all the families.

Then we must further break the symmetry $SO(7+1)$, and at the same time the symmetry $\widetilde{SO}(7+1)$, in equivalent way as we did the first break, to $SO(3,1)\ \times SU(2)_I \times SU(2)_{II}$, keeping correspondingly the relations among the spin, handedness and charges of quarks and leptons, and the spin, handedness and anticharges of antiquarks and antileptons, as presented in Table 7 and as assumed in Table 10.

iv.a. *The symmetry $SO(13,1)$ (and $\widetilde{SO}(13+1)$) first breaks into $SO(7,1) \times SU(3) \times U(1)$.*

The reader can notice in Table 7 that the $SO(7,1)$ content of $SO(13,1)$ is identical for quarks and leptons and identical for antiquarks and antileptons. Quarks distinguish from leptons and antiquarks from antileptons only in the $SO(6)$ content of $SO(13,1)$, that is in the colour charge and in the "fermion" charge. These two charges are described in the Clifford algebra presentation (suggested by the authors in Refs. [61, 62]) for leptons as a factor $\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$ (Eq. (87) dictates for leptons $\tau^{33} = 0$, $\tau^{38} = 0$ and $\tau^4 = -\frac{1}{2}$), and for antileptons as a factor $\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$ (Eq. (87) dictates for $\tau^{33} = 0$, $\tau^{38} = 0$ and $\tau^4 = \frac{1}{2}$). Quarks have three colour charge possibilities with $\tau^4 = \frac{1}{6}$ and antiquarks three colour anticharge possibilities with $\tau^4 = -\frac{1}{6}$. The relations of handedness to charges and handedness to anticharges follow from the Lorentz rotation of the kinds ($S^{09}, S^{39}, S^{0\,11}, S^{0\,13}$), which rotate fermions to antifermions within the same Lorentz irreducible representation. Quarks are related to antiquarks and leptons to antileptos also by the discrete symmetry operators $\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}$, presented in Subsubsect, 3.3.5, Eq. (72).

iv.b. In the next break of symmetry the $SO(7,1)$ *symmetry must break to* $SO(3,1)\times\ SU(2)\times SU(2)$.

In this break the infinite space of $d = (5,6,7,8)$ must be curved into an almost S^4 sphere with the symmetry $SU(2) \times SU(2)$. Since the left handed spinors have different $SU(2)$ weak and $SU(2)_{II}$ charge (the left handed quarks and leptons have $\tau^{13} = \pm\frac{1}{2}$, $\tau^{23} = 0$) than the right handed spinors (the right handed quarks and leptons have $\tau^{13} = 0$, $\tau^{23} = \pm\frac{1}{2}$), spinors are mass protected.

v. *The condensate makes massive the $SU(2)_{II}$ vector gauge fields and $U(1)_{\tau^4}$ vector gauge fields, as well as all the scalar gauge fields.*

The appearance of the condensate, Table 6, of two right handed neutrinos with family quantum numbers of the lower four families of eight families presented in Table 5, makes massive the $SU(2)_{II}$ vector gauge fields and $U(1)_{\tau^4}$ vector gauge fields, as well as all the scalar gauge fields, leaving massless only the $SU(3)$ colour, $U(1)_Y$ and $SU(2)$ weak vector gauge fields, Sect. 6.2, as well as the gravity in $d = (3+1)$.

Fermions — quarks and leptons and antiquarks and antileptons — remain massless and mass protected, with the spin, handedness, $SU(3)$ triplet or singlet charges, weak $SU(2)$ charge, hyper charge and family charge as presented in Tables 10, 7, "waiting for" spontaneous break of mass protection at the electroweak break.

v.a. *Spontaneous break of mass protection needs to be studied.*

Although the simple starting action of the *spin-charge-family* theory, Eqs. (100, 101), offers three singlet and twice two triplet scalar gauge fields, Sect. 6.2.2, with the space index $s = (7, 8)$ to break the mass protection of fermions causing the electroweak break (carrying with respect to the space index the weak and the hyper charge as required for Higgs's scalar in the *standard model*), yet we should show how does the electroweak break spontaneously occur when making the masses of the scalar gauge fields imaginary (in the *standard model* this is just assumed). We have not yet finish this project.

7.3 Predictions of $SO(10)$ unifying theories and *spin-charge-family* theory

There are many attempts in the literature to explain the *standard model* assumptions, using the gauge groups, discussed in Ref. [80] for ether the gauge charges or families or both.

There are many attempts in the literature to reconstruct mass matrices of quarks and leptons out of the observed masses and mixing matrices in order to learn more about properties of the fermion families [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. The most popular is the $n \times n$ mass matrix, most often with $n = 3$, close to the democratic one, predicting that $(n-1)$ families must be very light in comparison with the n^{th} one. Most of attempts treat neutrinos differently than the other family members, introducing the Majorana part and the "sea-saw" mechanism. Most often are the number of families taken to be equal to the number of the so far observed families, while symmetries of mass matrices are chosen in several different ways [77, 78, 79, 81]. Also possibilities with four families are discussed [82, 83, 84].

The existence of the *dark matter* is in the literature described mostly with the invention of new particles [86] and with the primordial black holes [87, 88, 90].

The *spin-charge-family* theory [91, 92, 93, 94, 96, 97] predicts two groups of four families of quarks and leptons, each of these two groups of four families manifesting the same $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$ symmetry of mass matrices, Eq. (108), the same for all family members, quarks and leptons. Twice two triplets of $\widetilde{SU}(2)\times \widetilde{SU}(2)$ distinguish among the upper and lower four families, the three singlets determining $U(1)$ distinguish among family members, Sect. 6.2.2. To the lower of these two groups of four families the observed three families belong. The stable of the upper four families offer the explanation for the existence of the *dark matter*.

The *spin-charge-family* theory predicts the existence of additional scalar gauge fields with the properties of the Higgs's scalar explaining the existence of the Yukawa couplings [36], the existence of the scalar fields which are with respect to the scalar index the colour triplets, offering the explanation for the matter-antimatter asymmetry of the ordinary matter, to which the first group of four families (mainly the lightest one) contributes, as well as the decay of protons.

The more work is put into the *spin-charge-family* theory, the more predictions is the theory offering.

[36] The need for the appearance of the Yukawa couplings in the *standard model* by itself predicts additional scalar fields.

7.3.1 Predictions of *spin-charge-family* theory

We have learned in the previous sections that the simple starting action, Eq. (100), in which fermions interact with gravity only — the vielbeins and the two kinds of the spin connection fields — offers the explanation for all the assumptions of the *standard model* presented in Sect. 7.1, and in a short way in Tables 10, 11, 12 and unifies all the so far known forces, with gravity included, predicting new vector gauge fields, new scalar gauge fields and new families of fermions, offering also the explanation for the second quantized postulates of Dirac.

i. The existence of the lower group of four families predicts the fourth family to the observed three, which should be seen in next experiments. The masses of quarks of these four families are determined by several scalar fields, all with the properties of the scalar higgs, which should also be observed.

The symmetry [118, 97], Eq. (111), and the values of mass matrices of the lower four families are determined with two triplet scalar fields, $\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$ and $\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$, and three singlet scalar fields, $A^{Q}_{\substack{78\\(\pm)}}$, $A^{Y}_{\substack{78\\(\pm)}}$, $A^{4}_{\substack{78\\(\pm)}}$, Eq. (108), explaining the Higgs's scalar and Yukawa couplings of the *standard model*, Refs. [97, 19, 12, 119, 14] and references therein.

Any accurate 3×3 submatrix of the 4×4 unitary matrix determines the 4×4 matrix uniquely. Since neither the quark and (in particular) nor the lepton 3×3 mixing matrix are measured accurately enough to be able to determine three complex phases of the 4×4 mixing matrix, we assume (what also simplifies the numerical procedure) [91, 92, 93, 94, 96, 97] that the mass matrices are symmetric and real and correspondingly the mixing matrices are orthogonal. We fitted the 6 free parameters of each family member mass matrix, Eq. (111), to twice three measured masses (6) of each pair of either quarks or leptons and to the 6 (from the experimental data extracted) parameters of the corresponding 4×4 mixing matrix.

We present in this paper the results for quarks only, taken from Refs. [96, 97]. The accuracy of the experimental data for leptons are not yet large enough that would allow a meaningful prediction [37]. It turns out that the experimental [116] inaccuracies are for the mixing matrices too large to tell trustworthy mass intervals for the quarks masses of the fourth family members [38]. Taking into account the calculations of Ref. [116], fitting the experimental data (and the meson decays evaluations in the literature, as well as our own evaluations) the authors of the paper [97] very roughly estimate that the fourth family quarks masses might be pretty above 1 TeV.

Since the matrix elements of the 3×3 submatrix of the 4×4 mixing matrix depend weakly on the fourth family masses, the calculated mixing matrix offers the prediction to what values will more accurate measurements move the present experimental data and also the fourth family mixing matrix elements in dependence of the fourth family masses, Eq. (119): V_{ud} will stay the same or will very slightly decrease; V_{ub} and V_{cs}, will still lower; V_{td} will lower, and V_{tb} will lower; V_{us} will slightly increase; V_{cd} will (after decreasing) slightly rise; V_{cb} will still increase and V_{ts} will (after decreasing) increase.

In Eq. (119) the matrix elements of the 4×4 mixing matrix for quarks obtained when the 4×4 mass matrices respect the symmetry of Eq. (111) while the parameters of the mass matrices are fitted to the (*exp*) experimental data [116], Ref. [97], are presented for two choices of the fourth family quark masses: $m_{u_4} = m_{d_4} = 700$ GeV (scf_1) and $m_{u_4} = m_{d_4} = 1\,200$ GeV (scf_2). In parentheses, ( ) and [ ], the changes of the matrix elements are presented, which are due to the changes of the top mass within the experimental inaccuracies: with the $m_t = (172 + 3 \times 0.76)$ GeV and $m_t = (172 - 3 \times 0.76)$,

[37] The numerical procedure, explained in the papers [96, 97], to fit free parameters of the mass matrices to the experimental data within the experimental inaccuracy of masses of the so far observed quarks and first of all within the inaccuracy of the mixing matrix elements, is very tough.

[38] We have not yet succeeded to repeat the calculations presented in Refs. [97] with the newest data from Ref. [117]. Let us say that the accuracy of the mixing matrix even for quarks remains in Ref. [117] far from needed to predict the masses of the fourth two quarks. For the chosen masses of the four family quarks the mixing matrix elements are expected to slightly change in the direction proposed by Eq. (119).

respectively (if there are one, two or more numbers in parentheses the last one or more numbers are different, if there is no parentheses no numbers are different) [arxiv:1412.5866].

$$
|V_{(ud)}| = \left(\begin{array}{l|cccc}
exp & 0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 & \\ \hline
scf_1 & 0.97423(4) & 0.22539(7) & 0.00299 & 0.00776(1) \\
scf_2 & 0.97423[5] & 0.22538[42] & 0.00299 & 0.00793[466] \\ \hline
exp & 0.225 \pm 0.008 & 0.986 \pm 0.016 & 0.0411 \pm 0.0013 & \\ \hline
scf_1 & 0.22534(3) & 0.97335 & 0.04245(6) & 0.00349(60) \\
scf_2 & 0.22531[5] & 0.97336[5] & 0.04248 & 0.00002[216] \\ \hline
exp & 0.0084 \pm 0.0006 & 0.0400 \pm 0.0027 & 1.021 \pm 0.032 & \\ \hline
scf_1 & 0.00667(6) & 0.04203(4) & 0.99909 & 0.00038 \\
scf_2 & 0.00667 & 0.04206[5] & 0.99909 & 0.00024[21] \\ \hline
scf_1 & 0.00677(60) & 0.00517(26) & 0.00020 & 0.99996 \\
scf_2 & 0.00773 & 0.00178 & 0.00022 & 0.99997[9]
\end{array}\right) . \qquad (119)
$$

Let us conclude that according to Ref. [97] the masses of the fourth family lie much above the known three. The larger are masses of the fourth family the larger are $V_{u_1 d_4}$ in comparison with $V_{u_1 d_3}$ and the more is valid that $V_{u_2 d_4} < V_{u_1 d_4}$, $V_{u_3 d_4} < V_{u_1 d_4}$. The flavour changing neutral currents are correspondingly weaker.

Let be noticed that the prediction of Ref. [114], $V_{u_1 d_4} > V_{u_1 d_3}$, $V_{u_2 d_4} < V_{u_1 d_4}$, $V_{u_3 d_4} < V_{u_1 d_4}$, agrees with the prediction of Refs. [96, 97].

In Ref. [119] the authors discuss the question why the existence of the fourth family is not (at least yet) in contradiction with the experimental data.

ii. The theory predicts the existence of besides the additional scalar fields also the additional vector gauge fields, Sect. 6.2, Subsubsects. 6.2.1, 6.2.2, Eqs. (102, 106).

Let us comment here the report of LHCb [120] from March 25^{th} 2021 on measurements of lepton decays of mesons B, showing that the decay of B-mesons to $\mu^+\mu^-$ is for $\approx 15\%$ weaker than to $e^+ + e^-$, while the *standard model* predicts that they should be very close or to be equal (with the higher corrections taken into account included). We can (so far) only say, that since the *spin-charge-family* theory predicts two kinds of Z^0_m bosons [due to the two vector gauge fields, $A^{Q'}_m = \cos\vartheta_1 A^{13}_m - \sin\vartheta_1 A^Y_m$, $A^{Q'}_m$ represents Z^0_m field, and $A^{Y'}_m = \cos\vartheta_2 A^{23}_m - \sin\vartheta_2 A^4_m$, $A^{Y'}_m$ represents the second Z'^0_m field, where $\vartheta_i, i = (1,2)$, are the two angles of breaking symmetries, Eq.(168) (where the scalar index $s \in (7,8)$ must be replaced by the vector index $m \in (0,1,2,3)$, and $A^4_m = -(\omega_{9\,10\,m} + \omega_{11\,12\,m} + \omega_{13\,14\,m})$, $A^{13}_m = (\omega_{56m} - \omega_{78m})$, $A^{23}_m = (\omega_{56s} + \omega_{78m})$, $A^Q_m = \sin\vartheta_1 A^{13}_m + \cos\vartheta_1 A^Y_m$, $A^Y_m = (g^{56}\,\omega_{56m} + A^4_m)$], and several scalar gauge field, [two triplets and three singlets, Eq. (108), all with the properties of the Higgs's scalar] there is no reason that the decay of the meson B to two leptons belonging to two different families are the same. We have not estimated yet these two decays and can not predict the ratio of the measured decays, but due to our experience it seems very unlikely that the difference would be so large.

iii. The theory predicts the existence of the upper four families of quarks and leptons and antiquarks and antileptons, with the same family members charges, Tables 7, 5, as the lower four families, interacting correspondingly with the same vector gauge fields. At low energies the upper four families are decoupled from the lower four families.

The masses of the upper four families are determined by the two triplets $(\vec{\tilde{A}}^{\tilde{2}}_{\stackrel{78}{(\pm)}}, \vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{\stackrel{78}{(\pm)}})$ and three singlets $(A^{Q}_{\stackrel{78}{(\pm)}}, A^{Q'}_{\stackrel{78}{(\pm)}}, A^{Y'}_{\stackrel{78}{(\pm)}})$, the same singlets contribute also to masses of the lower four families, Sect. 6.2.2.

The stable of the upper four families offers the explanation for the appearance of the *dark matter* in our universe.

Since the masses of the upper four families are much higher than the masses of the lower four families, the "nuclear" force among the baryons and mesons of these quarks and antiquarks differ a lot from the nuclear force of the baryons and fermions of the lower four families.

A rough estimation of properties of baryons of the stable fifth family members, of their behaviour during the evolution of the universe and when scattering on the ordinary matter, as well as a study of possible limitations on the family properties due to the cosmological and direct experimental evidences are done in Ref. [94].

In Ref. [122] the weak and "nuclear" scattering of such very heavy baryons by ordinary nucleons is studied, showing that the cross-section for such scattering is very small and therefore consistent with the observation of experiments so far, provided that the quark mass of this baryon is about 100 TeV or above.

In Ref. [94] a simple hydrogen-like model is used to evaluate properties of baryons of these heavy quarks, with one gluon exchange determining the force among the constituents of the fifth family baryons [39].

The authors of Ref. [94] study the freeze out procedure of the fifth family quarks and anti-quarks and the formation of baryons and anti-baryons up to the temperature $k_b T = 1$ GeV, when the colour phase transition starts which depletes almost all the fifth family quarks and anti-quarks, while the colourless fifth family neutrons with very small scattering cross section decouples long before (at $k_b T = 100$ GeV), Fig. 7.3.1. The cosmological evolution suggests for the mass limits the range 10 TeV $< m_{q_5} <$ a few $\cdot 10^2$

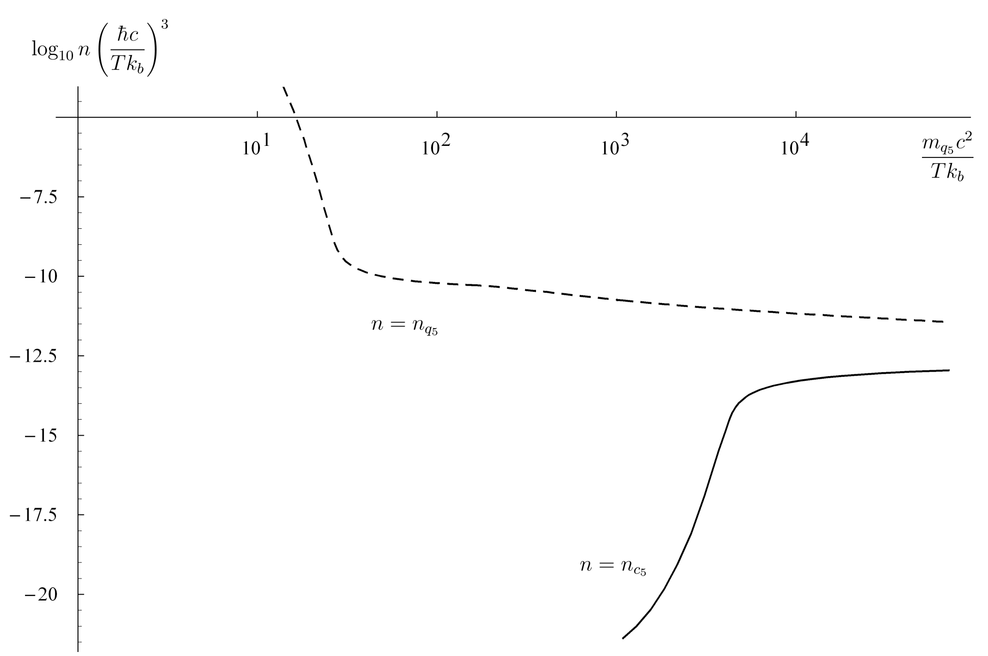

Figure 2: The dependence of the two number densities, n_{q_5} of the fifth family quarks and n_{c_5} of the fifth family clusters of quarks, as functions of $\frac{m_{q_5} c^2}{k_b T}$ is presented for the special values $m_{q_5} = 71\,\text{TeV}$. The estimated scattering cross sections, entering into Boltzmann equation, are presented in Ref. [94], Eqs. (2,3,4.5), In the treated energy (temperature $k_b T$) interval the one gluon exchange gives the main contribution to the scattering cross sections entering into the Boltzmann equations for n_{q_5} and n_{c_5}.

TeV and for the scattering cross sections $10^{-8}\,\text{fm}^2 < \sigma_{c_5} < 10^{-6}\,\text{fm}^2$. The measured density of the dark

[39] The weak force and the electromagnetic force start to be at small distances due to heavy masses of quarks of the same order of magnitude as the colour force.

matter does not put much limitation on the properties of heavy enough clusters [40].

The DAMA/LIBRA experiments [121] limit (provided that they measure the heavy fifth family clusters) the quark mass in the interval: $200\,\text{TeV} < m_{q_5} < 10^5\,\text{TeV}$, Ref. [94].

Baryons of the fifth family are heavy, forming small enough clusters with small enough scattering amplitude among themselves and with the ordinary matter to be the candidate for the dark matter.

Masses of the stable fifth family of quarks and leptons are much above the fourth family members.

Although the upper four families carry the weak (of two kinds) and the colour charge, these group of four families are completely decoupled from the lower four families up to the $< 10^{16}$ GeV, unless the breaks of symmetries recover.

iv. The spread of masses of quarks and leptons of two groups of four families from 10^{-11}GeV (ν of the first family) to 10^{15}GeV (u and d of the fourth family of the upper four families) offers the explanation for the "hierarchy problem". We have not yet in the *spin-charge-family* theory study loop corrections to the squared masses of scalar fields, of twice two triplets and three singlets which determine mass matrices of the two groups of four families, what is needed to analyze the "hierarchy" problem.

But we have various scales in our theory because we have (as the only choice) to accept different "compactification" scales for different dimensions [41], what could help to solve this "hierarchy" problem.

7.4 Massless and massive odd Clifford fermions in toy model with $d = (5+1)$

To illustrate the difference between our description of the internal space of fermions and the usual one, we demonstrate in this section on the toy model in $d = (5+1)$-dimensional space, when starting from massless fermions interacting with the gravity only and then breaking symmetries from $d = (5+1)$ to $d = (3+1)$ times an almost S^2 sphere, how to end up with the massless and charged fermions and antifermions, while a further break of symmetry, caused by scalar fields gaining constant values, leads to massive chargeless fermions manifesting as Majorana fermions, Ref. [126].

There are zweibeins and spin connection fields of particular properties in the 6^{th} and 7^{th} dimensions which make the $\mathcal{M}^{5+1}$ manifold to break into $\mathcal{M}^{3+1}$ $\times$ an almost S^2 sphere [127, 128]. The mass of spinors (fermions) originates either in the dynamics or in the vacuum expectation values of zweibein and spin connection fields in 6^{th} and 7^{th} dimensions.

We learn in Sects. 3 - 6 that creation operators and their Hermitian conjugated annihilation operators anticommute, due to the odd character of the "basis vectors" describing the internal space of fermions. Since the character of the vacuum state is even all the single fermion states have an odd character. The equations of motion, either for free or for interacting Clifford fermions, have an even character. The solutions of equations of motion are correspondingly superposition of creation operators belonging to the members of the same irreducible representation, the same family, or to different families, each of them having an odd character. The operators S^{ab} and $\tilde{S}^{ab}$, connecting members of the same family or different families, respectively, have an even Clifford character and do not change the oddness of creation operators and correspondingly of fermion states. The operators γ^a's and $\tilde{\gamma}^a$'s, both having an odd Clifford character, change the character of states from odd to even.

When describing massive states in usual cases, the ones of Dirac, γ^a's are used to transform the left handed fermion states in $d = (3+1)$ into the right handed ones to generate basis for the massive states.

[40] In the case that the weak interaction determines the cross section of the neutron n_5, the interval for the fifth family quarks would be 10 TeV $< m_{q_5}\, c^2 < 10^5$ TeV.

[41] The fact that we succeeded to fit the parameters of mass matrices to experimental data can mean, and in this way we speak about the offer of the theory to solve the "hierarchy problem", that "nature does solve this problem" and the spread of fitted masses approves our hope that we have only to find how "nature does solve the problem".

Charges are in all these cases added separately, by the choice of appropriate charge groups. Vectors representing states are just numbers which commute.

In the *spin-charge-family* theory, describing the internal space of fermions with the odd Clifford algebra creation operators, operating on the vacuum state, the multiplication of the odd creation operators, Eqs. (84, 126), by odd γ^a's leads to even Clifford algebra creation operators, which commute and do not describe fermions, as presented in App. B.1.

But if one looks at the solutions of equations of motion for chargeless massive states in the *spin-charge-family* theory case, Eq. (130), whille neglecting the charge part and correspondingly the oddness of states, what the usual way of presenting fermions does, then one easier understands the difference of the *spin-charge-family* theory way and the Dirac's way of the second quantization, presented in Sect. 3.3.1.

We present in App. B, following Ref. [24], the matrix representations of the operators γ^a's, $\tilde{\gamma}^a$'s, S^{ab} and $\tilde{S}^{ab}$, discussing their properties in our case when using odd Clifford algebra to describe the internal space of fermions appearing in families, and in the usual case when using Dirac matrices, Sect. B.2.

7.4.1 Massless and massive solutions of equations of motion in $d = (5+1)$

Let us again use for our discussions the simple toy model with odd Clifford algebra "basis vectors" describing the internal space of fermions in $d = (5+1)$-dimensional space. In Table 3 the "basis vectors" determining the creation operators and those determining the annihilation operators are presented.

In Table 13 besides the Clifford odd basis vectors also the Clifford even "basis vectors" are presented. One can notice that the application of the operators γ^a's or $\tilde{\gamma}^a$'s transforms the Clifford odd anticommuting 'basis vectors" into the Clifford even commuting "basis vectors".

How can the use of the odd Clifford algebra operators γ^a's in the usual first quantized theory be explained from the point of view of the *spin-charge-family* theory, if γ^a's transform creation operators, members of an odd irreducible representation, into members of an even irreducible representation, which neither have always the Hermitian conjugated partners which would applying on the vacuum state give zero (the product of projectors is a self adjoint operator) nor has the anticommuting properties¿

We start with the massless solutions of the Weyl equation in $d = (5+1)$ with the "basis vectors" presented in Table 3, and with the spin (or the total angular momentum) in extra dimensions, $d > (3+1)$, determining the charge in $d = (3+1)$. We then let the $\mathcal{M}^{5+1}$ manifold to break into $\mathcal{M}^{3+1}$ × an almost S^2 sphere [127, 128] due to the zweibein and spin connection fields in the 6^{th} and 7^{th} dimensions. The mass of the spinor (fermion) originates either in the dynamics in the higher dimensions or in the vacuum expectation values of the scalar fields, which are the gauge fields with the scalar index with respect to $d = (3+1)$, and therefore well defined.

With this illustration we want to make evident the fact that without knowing the action which leads to massless and massive solutions in $d = (3+1)$, the comparison of the Dirac solutions of the equation of motion which already assumes the nonzero mass and includes charges by assuming additional charge groups and the solution of the *spin-charge-family* theory does not seem meaningful.

Let us here repeat the Weyl equation of the action (Eq.(100)), the internal space of solution is described by the odd Clifford algebra objects in the case of our toy model of $d = (5+1)$ as follows

$$
\begin{aligned}
&(\gamma^m p_m + \overset{56}{(+)}\, p_{0+} + \overset{56}{(-)}\, p_{0-})\psi = 0\,,\\
&p_{0\pm} = p^5_0 \mp i\, p^6_0\,,\\
&p_{0s} = f^{\sigma}_{s}(p_\sigma - \frac{1}{2} S^{ab}\omega_{ab\sigma}) + \frac{1}{2E}\{p_\sigma, f^{\sigma}_{s} E\}_{-}\,,\\
&\overset{56}{(\pm)} = \frac{1}{2}\,(\gamma^5 \pm i\gamma^6)\,.
\end{aligned}
\tag{120}
$$

There are $2^{\frac{d}{2}-1}$ (4 in our case of $d=6$) odd "basic vectors" $\hat{b}^{m\dagger}_{f}$, $m=(1,2,3,4)$ appearing in $2^{\frac{d}{2}-1}=4$ families, $f=(I,II,III,IV)$, presented in Table 3. $S^{ab}=\frac{i}{4}(\gamma^a\gamma^b-\gamma^b\gamma^a)$ connect family members within each family, $\tilde{S}^{ab}=\frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b-\tilde{\gamma}^b\tilde{\gamma}^a)$ transform one family member of a particular family into the same family member of another family. We look for solutions of the equation of motion by superposition of the creation operators applying on the vacuum state, $\hat{b}^{s\dagger}_{f}|\psi_{oc}>|0_{\vec{p}}>=\psi^{s}_{f}(\vec{p})$, while the application of the annihilation operators on the vacuum state gives zero, $\hat{b}^{s}_{f}(\vec{p})|\psi_{oc}>|0_{\vec{p}}>=0$.

We treat here only one family, since we only want to clear up the relation between our way of describing the internal space of fermions and the usual (Dirac's) way.

Let us make a choice of the first family, $f=1$, presented in Table 3 as $\hat{b}^{m\dagger}_{f}$, with $m=(1,2,3,4)$, with ($S^{56}=\pm\frac{1}{2}, S^{12}=\pm frac12$), and let us skip the family quantum number f in this section. The Hermitian conjugated partners of the first family member are presented as $\hat{b}^{m}_{f}=(\hat{b}^{m\dagger}_{f})^{\dagger}$. In Table 3 we read that two of the "basis vectors", the first two, are right handed with the charge $\frac{1}{2}$ and the second two are left handed with the charge $-\frac{1}{2}$

$$\hat{b}^{1\dagger} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\,,\quad \hat{b}^{2\dagger}=\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\,,\quad \hat{b}^{3\dagger}=\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]}\,,\quad \hat{b}^{4\dagger}=\stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}\,. \tag{121}$$

The vacuum state $|\psi_{oc}>$ is in this case, when we do not pay attention on other families, equal to $|\psi_{oc}>=\stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}$, Eq. (47), so that the four "basis vectors" of Eq. (121) are normalized, $\hat{b}^{i}\hat{b}^{j\dagger}=\delta^{i}_{j}$, giving a nonzero contribution when applying on $|\psi_{oc}>$.

All the basic states are eigenstates of the Cartan subalgebra (of the Lorentz transformation Lie algebra), for which we take: S^{03}, S^{12}, S^{56}, with the eigenvalues, which can be read from Eq. (121) as $\frac{1}{2}$ times the numbers $\pm i$ or ± 1 in the parentheses of nilpotents $\stackrel{ab}{(k)}$ and projectors $\stackrel{ab}{[k]}$: $S^{ab}\stackrel{ab}{(k)}=\frac{k}{2}\stackrel{ab}{(k)}$, $S^{ab}\stackrel{ab}{[k]}=\frac{k}{2}\stackrel{ab}{[k]}$.

One notices that two of the states are right handed ($\psi_1=\hat{b}^{1\dagger}|\psi_{oc}>$ and $\psi_2=\hat{b}^{2\dagger}|\psi_{oc}>$) and two left handed ($\psi_3=\hat{b}^{3\dagger}|\psi_{oc}>$) and $\psi_4=\hat{b}^{4\dagger}|\psi_{oc}>$) with respect to $d=(3+1)$ (while all four carry the same, left, handedness with respect to $d=(5+1)$). The operator of handedness is defined in Eq. (172).

With the following choice of the zweibein fields we achieve that the infinite surface $d=(5,6)$ curls into an almost S^2 (with one hole [127])

$$\begin{aligned} e^{s}{}_{\sigma} &= f^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, f^{\sigma}{}_{s}=f\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, f=1+(\frac{\rho}{2\rho_0})^2\,,\\ E &= \det(e^{s}{}_{\sigma})=f^{-2}\,, e^{s}{}_{\sigma}\,f^{\sigma}{}_{t}=\delta^{s}_{t}\,,\\ x^{(5)} &= \rho\cos\phi,\quad x^{(6)}=\rho\sin\phi\,, \end{aligned}$$

while $d=(3+1)$ space remains flat ($f^{\mu}{}_{m}=\delta^{\mu}_{m}$). We choose the spin connection fields on this S^2 as

$$f^{\sigma}{}_{s'}\,\omega_{st\sigma} = iF\,f\,\varepsilon_{st}\,\frac{e_{s'\sigma}x^{\sigma}}{(\rho_0)^2}\,,\quad 0<2F\le 1\,,\quad s=5,6,\quad \sigma=(5),(6)\,, \tag{122}$$

in order to guarantee that there manifests in $d=(3+1)$ only one massless and correspondingly mass protected state [127, 126], while the rest of states are all massive. There is the whole interval for the constant F ($0<2F\le 1$), which fulfills the condition of only one massless state of the right handedness in $d=(3+1)$, which is square integrable.

When requiring that the solutions of Eq. (120) have the angular moments in $d=(5,6)$ manifesting the charges in $d=(3+1)$ ($M^{56}=x^5p^6-x^6p^5+S^{56}=-i\frac{\partial}{\partial\phi}+S^{56}$), we write the wave functions $\psi^{(6)}_{n+1/2}$ for the choice of the coordinate system $p^a=(p^0,0,0,p^3,p^5,p^6)$ as follows

$$\psi^{(6)}_{n+1/2}=(\mathcal{A}_n\,\stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}+\mathcal{B}_{n+1}\,e^{i\phi}\,\stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{[-]})\cdot e^{in\phi}e^{-i(p^0x^0-p^3x^3)}|\psi_{oc}>|0_{\vec{p}}>\,, \tag{123}$$

index ${}^{(6)}$ in $\psi^{(6)}_{n+1/2}$ is to point out that we treat the dynamics in $d = (5,6)$. Besides one massless $(\psi^{(6)}_{1/2})$ solution with $n = 0$ there is the whole series of massive solutions manifesting in $d = (3+1)$ the (Kaluza-Klein) charge $n + 1/2$: $M^{56}\,\psi^{(6)}_{n+1/2} = (n + 1/2)\,\psi^{(6)}_{n+1/2}$, and solving Eq. (120), provided that $\mathcal{A}_n$ and $\mathcal{B}_{n+1}$ are the solutions of the equations

$$
\begin{aligned}
-if\,\{\,(\frac{\partial}{\partial\rho} + \frac{n+1}{\rho}) - \frac{1}{2\,f}\,\frac{\partial f}{\partial\rho}\,(1+2F)\}\mathcal{B}_{n+1} + m\mathcal{A}_n = 0\,,\\
-if\,\{\,(\frac{\partial}{\partial\rho} - \frac{n}{\rho}) - \frac{1}{2\,f}\,\frac{\partial f}{\partial\rho}\,(1-2F)\}\mathcal{A}_n + m\mathcal{B}_{n+1} = 0\,.
\end{aligned}
\tag{124}
$$

The massless positive energy solution with spin $\frac{1}{2}$, left handedness (Eq. (172) in $d = (5+1)$, the charge in $d = (3+1)$ equal to $\frac{1}{2}$ and right handed with respect to $\Gamma^{(3+1)}$ is equal to

$$
\psi^{(6)}_{\frac{1}{2}} = \mathcal{N}_0\; f^{-F+1/2}\; \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\; e^{-i(p^0x^0-p^3x^3)}|\psi_{oc}>|0_{\vec{p}}>\;. \tag{125}
$$

For the special choice of $F = \frac{1}{2}$ (from the interval in Eq. (122) allowing only right handed square integrable massless states) the solution of Eq. (125)) simplifies to

$$
\psi^{(6)}_{\frac{1}{2}} = \mathcal{N}_0\;\; \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\; e^{-i(p^0x^0-p^3x^3)}\,. \tag{126}
$$

Massive solutions are in this special case [127, 128] expressible in terms of the associate Legendre function $P^l_n(x)$, $x = \frac{1-u^2}{1+u^2}$, $u = \frac{\rho}{2\rho_0}$, where ρ_0 is the radius of (an almost) S^2, as follows

$$
\mathcal{A}^{l(l+1)}_n \;\;=\;\; P^l_n\,,\quad \mathcal{B}^{l(l+1)}_{n+1} = \frac{-i}{\rho_0 m}\,\sqrt{1-x^2}\,\left(\frac{d}{dx} + \frac{n}{1-x^2}\right)\,\mathcal{A}^{l(l+1)}_n\,, \tag{127}
$$

with the masses [42] determined by $(\rho_0\,m)^2 = l(l+1)$ and $l = 1,2,3,\ldots,\;0 \le n < l$.

Massive chargeless solutions of the Weyl equation

Let us now assume that the scalar fields, the gauge fields of S^{56}, that is $f^{\sigma}_{s}\;\omega_{56\sigma}$, with $s = (5,6)$ and $\sigma = ((5),(6))$, gain constant values (non zero vacuum expectation values). These two scalar fields are the analogy to the complex higgs scalar of the *standard model*: The Higgs's scalar carries in the *standard model* the weak and the hyper charge, while our scalar fields carry only the "hyper" charge S^{56}. The charge, which is the spin in $d = (5,6)$, is after the scalar fields gain nonzero vacuum expectation values no longer the conserved quantity and mass protection is correspondingly removed.

In this case we replace in the Weyl equation (120) the quantities $p_{0\pm}$ with their constant values, the vacuum expectation values, $< p_{0\pm} >$, so that the equations of motion follow as

$$
< p_{0\pm} >= m_{\pm}\,,\quad (\gamma^m p_m + \overset{56}{(+)}\; m_+ + \overset{56}{(-)}\; m_-)\,\psi^{(6)} = 0\,. \tag{128}
$$

To simplify further discussions, the coordinate system in $d = (3+1)$ with $\vec{p} = 0$ is chosen. Then Eq. (128) reads

$$
\{p_0 + \gamma^0((\overset{56}{(+)}\; m_+ + \overset{56}{(-)}\; m_-)\}\psi^{(6)} = 0\,. \tag{129}
$$

[42] In the case that $d = (5,6)$ is a compact S^2 sphere these massive solutions would make infinite spectrum with quantum numbers (l,n), l defining in $d = (3+1)$ the mass and $n + \frac{1}{2}$ the Kaluza-Klein charge. In the case of an almost S^2 the spectrum start to stop when the energy approaches the strengths of the source which causes the vielbein leading to an almost S^2.

The two positive fermion massive, m, solutions with the spin in $d=(3+1)$ equal to $\pm\frac{1}{2}$, respectively, both with non conserved charges S^{56}, Eq. (129), are

$$\begin{aligned}
\psi^{(6)}_{\frac{1}{2},m} &= (\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} + \frac{m}{m_+}\, \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]})\, e^{-imx^0}\, |\psi_{oc}>|0_{\vec{p}}>\,, \\
\psi^{(6)}_{-\frac{1}{2},m} &= (\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)} + \frac{m}{m_+}\, \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]})\, e^{-imx^0}\, |\psi_{oc}>|0_{\vec{p}}>\,, \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)})\, e^{-imx^0} \\
m^2 &= m_+\, m_-\,, \quad m_+ = -m_-\,, \quad (p_0)^2 = m^2\,. \qquad (130)
\end{aligned}$$

Since the scalar gauge fields $f^{\sigma}{}_{s}\,\omega_{tt'\sigma}$ gain constant values, $\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}$ from Sect. 3.3.5 is no longer the symmetry of the equations of motion and correspondingly does not transform fermions into antifermions. It is not difficult to find the new operator which replaces $\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}$ after the break [126]. It is $(-i)\,\Gamma^{(6)}\Gamma^{(3+1)}\,\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}$.It is easy to see that in our $d=(5+1)$ case $(-i)\,\Gamma^{(6)}\Gamma^{(3+1)}\,\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}} = \gamma^5\gamma^6\gamma^0\gamma^5 I_{\vec{x}_3} I_{x^6}$, after taking into account that $m_- = -m_+$. We can check that

$$\begin{aligned}
&(-i)\,\Gamma^{(6)}\Gamma^{(3+1)}\,\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}\;(\gamma^0\,\gamma^m p_m + \gamma^0\,(\overset{56}{(+)}\; m_+ + \overset{56}{(-)}\; m_-))\;((-i)\,\Gamma^{(6)}\Gamma^{(3+1)}\,\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} \\
&= (\gamma^0\,\gamma^m p_m + \gamma^0\,((-)\,\overset{56}{(-)}\; m_+ + (-)\,\overset{56}{(+)}\; m_-)) = (\gamma^0\,\gamma^m p_m + \gamma^0\,((\overset{56}{-)}\; m_- + (\overset{56}{(+)}\; m_+))\,, \\
&\text{since}\quad m_+ = -m_-\,. \qquad (131)
\end{aligned}$$

Let us now find the antifermion states to the two positive energy states in Eq. (130)

$$\begin{aligned}
-i\,\Gamma^{(6)}\Gamma^{(3+1)}\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}\;\psi^{(6)}_{\frac{1}{2},m} &= ((-i)\;\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{[-]} + \frac{m}{m_+}\,i\;\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)})\,e^{-imx^0}\,, \\
-i\,\Gamma^{(6)}\Gamma^{(3+1)}\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}\psi^{(6)}_{-\frac{1}{2},m} &= ((-i)\;\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]} + i\,\frac{m}{m_+}\;\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)})\,e^{-imx^0}\,, \\
m^2 &= m_+ m_-\,, \quad m_+ = -m_-\,, \quad (p_0)^2 = m^2\,, \qquad (132)
\end{aligned}$$

which means that $-i\,\Gamma^{(6)}\Gamma^{(3+1)}\;\mathbb{C}_{\mathcal{N}}\cdot\,\mathcal{P}^{(d-1)}_{\mathcal{N}}\;\psi^{(6)}_{\pm\frac{1}{2},m} = i\frac{m}{m_+}\;\;\psi^{(6)}_{\pm\frac{1}{2},m}$.

The two positive solutions of the effective Weyl equations, Eq. (129), representing particles carrying no charge, are indistinguishable from the two positive energy solutions for the corresponding two antiparticles. They are indeed the Majorana particles.

In the case that there is no conserved charge due to the break of the mass protection (caused by the scalar fields ω_{56s}, which gain constant values) the Majorana fermions are described by the sum of the fermion and the corresponding antifermion state, Eq. (132). In the simple case that $\vec{p}=0$ the Majorana fermion states $\psi^{(6)}_{\pm\frac{1}{2},M}$ can be written as

$$\begin{aligned}
\psi^{(6)}_{\pm\frac{1}{2},M} &= \frac{1}{\sqrt{2}}\,(\psi^{(6)}_{\pm\frac{1}{2},m} + (\pm)\,(i)\,\Gamma^{(6)}\Gamma^{(3+1)}\mathbb{C}_{\mathcal{N}}\cdot\mathcal{P}^{(d-1)}_{\mathcal{N}}\;\psi^{(6)}_{\pm\frac{1}{2},m}(\vec{p}=0))\,, \\
(\pm)\,,&\ \text{if}\ m_+ = (\mp)\,i\,m\,, \qquad (133)
\end{aligned}$$

and have the mass equal to m [126].

Me have started in our particular case, the toy model in $d=(5+1)$, with the chain of massive fermions and the massless charged fermions of particular handedness and the corresponding antifermions, and ended with no massless states and the massive Majorana particles (in addition to the chain of massive particles).

7.5 Trials to make next step beyond *standard model* in theories

Both authors have participated in holding through 23 years a little workshop at Bled entitled "What comes beyond the standard models", at which many works were seeking to build extensions of both standard models, the electroweak and colour *standard model* and the cosmological model, which could explain the phenomena, which the electroweak and colour *standard model* can not, or even find the next step beyond both standard models.

Most attempts try to guess how to extend the *standard model* assumtions/details so that the predictions of their extensions will show up, when the accelerators reach yet higher energies, or when some particular experiments could measure their prediction.

There are attempts, like the one called "multiple point principle" of H.B.N. and his collaborators [99, 100, 101, 102, 103] [43], which try to understand the *standard model* parameters read from experiments without extending the *standard model*, just by trying to understand "why has nature made a choice" of the observed properties of quarks and leptons.

Some theories are searching for steps beyond the *standard model*, like there are $SO(10)$-unifying theories, the string theories, the Kaluza-Klein like theories, as well as the *spin-charge-family* theory, which is also a kind of Kaluza-Klein theory, with the ambition (starting from one parameter) to explain all the parameters and assumptions of the *standard model* and the cosmological models. (Although the "phase transitions" — breaks of symmetries spontaneously — are so hard job, which even if possible, requires the knowledge of boundary conditions of expanding universe at a very moment of the break of a symmetry.)

Having an appropriate theory one can do a lot with the help of observations and experiments.

In this Sect. 7.5 we intend to present only (some of) possibilities, some of the ones for which we have our own understanding of the topics.

7.5.1 To understand assumptions by string theories

Those string theories, which at all have a chance to be useful as models beyond the *standard model*, must have special fermionic modes - fields of fermions running along the string - in addition to the position and momentum degrees of freedom for the relativistic string. Otherwise there would be no fermionic states of the string, and no fermion to correspond to the fermions in the *standard model*.

But what in particular makes the (super) string theories being able to claim that there are (extremely) few assumptions once the string theory is chosen is that there are several quantum anomalies, often the symmetries, that are seemingly put in by hand by assumption, and do not come out after the quantum corrections are evaluated.

An example is that the rotational symmetry of the string theory does usually not come out from the quantum calculation, except in the case of a very special dimensionality of space time [44].

The rotational symmetry becomes a true breaking symmetry, unless the dimension of the string without the fermion mode is $d = (25 + 1)$ and with the fermion mode is $d = (9 + 1)$ [45]. The choice of the dimensionality makes the theory more predictive. It still remains to prevent that a particle with a negative squared mass, that is called a tachyon, would appear. The superstring with the fermionic modes offers this possibility — in a quite tricky way.

It then turned out that there are only 5 satisfactory string theories with no anomaly problems left and the usual physical principles such as with no tachyons and the Lorents invariance.

[43] The collaborators are D. Bennett and C.D. Froggatt after a long previous development involving also N. Brene, I. Picek, and L. Laperashvili.

[44] If one uses the infinite momentum frame formalism in which a special coordinate axis is selected and treated in a special way, one has rotational invariant theory described in a non-rotational invariant formalism.

[45] Such a trouble that a symmetry that should be there, because we put it in, gets spoiled by the quantum calculation, is called a quantum anomaly [51, 52].

The extra dimensions must be curled up to end up with $d = (3 + 1)$ observable so far. This kind of compactification appears also in the *spin-charge-family* theory.

The quantization of gravity might be the main strong point of the string theory; and then the somewhat related point that it in higher than four dimensions can have a renormalizable or meaningfull theory with quantum corrections.

But with respect to predicting phenomena at low energy physics can the string theory hardly be competitive with the *spin-charge-family* theory, which is so close to being free theory that one essentially avoids quantum corrections [46].

In getting out from string theory a detailed way of breaking the original string groups $SO(32)$ or $E_8 \times E_8$ down to the group to be observed at low energy involves typically so many choices, that it becomes hard to seriously claim that the string theory lead to the standard model group. There are also possibilities to construct models inside string theory giving the standard model group not by using the gauge groups in the string theory but brane bunches that can be made up easily to give $U(N)$ groups. But the more possibilities the more difficult to claim any clean and unique prediction.

In the *spin-charge-family* theory there is some freedom in the way how to curl up (essentially compactify) non observed dimensions one after the other. But keeping to the assumption of letting the fermions in practice sit at the origo in the extra dimensions and basically making all but a small region near this origo become unimportant, the possible choices are reduced to how big - in number of dimensions - are the successive "compactified" dimensions, and that sounds giving less possibilities than in the string theory where one does not restrict the compactification in the just mentioned way, and has a larger number of even massless particles at the outset, because of the fundamental gauge symmetry.

The Kaluza Klein way of obtaining gauge fields is in principle also available in string theory, but not so popular becuase one often has in mind truly compactified extra dimensions in which case we have a theorem by Witten making it impossible to achieve chiral fermions in the low energy theory in this way [107].

7.5.2 Flavour changing neutral currents and proton decay in theories with several scalar fields (higgss)

i. *Comments on the flavour changing neutral currents*

In theories with several scalar fields (higgses), as it is the *spin-charge-family* theory, one expects difficulties in assuring that transitions from a family member of a particular family to the same family member of another family would agree with the experiments, since such transitions, known as flavour changing neutral currents, are not observed [47]. These transitions are in the *standard model* prevented with one higgs only and with Yukawa couplings. In the *spin-charge-family* theory with the group of four families and two triplet scalar fields, coupled to family quantum numbers (the flavours), and three singlet scalar fields, coupled to family members quantum numbers, the flavour conserving neutral currents can be explained as follows:

When non-zero constants (the vacuum expectation values) of this higgses give masses to, let say, u_i quarks, i counts families, of u-quarks, the mass matrix gets proportional to the Yukawa coupling matrix. After diagonalizing the mass matrix of u_i quarks each eigenvector of the mass matrix represents one family of particular quantum number (flavour). In this new basis the Yukawa coupling matrix becomes diagonal in the family quantum numbers (flavours), and thus the coupling to the single higgs

[46] The work of one of us, H.B.N. and M. Ninomiya might help making sensible theories in higher dimensions [89].

[47] Due to A.H.G., the coauthor of N.S.M.B. in Ref. [118], any theory looking for a way beyond the *standard model* can have serious difficulties in suppressing the flavour changing neutral currents, since these processes require proper coupling constants and heavy masses of the exchange mediators. Even if in all orders of corrections the symmetry of mass matrices remains $\widetilde{SU}(2) \times \widetilde{SU}(2) \times U(1)$ and the theory reproduces the right masses and mixing matrices, this by itself does not guarantee the suppression of the flavour changing neutral currents.

can not break the family quantum number (flavour) conservation. So the higgs exchange, the one which diagonalizes the mass matrix of u-quarks, since there is only one higgs, cannot break the flavour conservation. But because the definition of the flavours were made for each member or say charge value of the quarks or leptons separately, one can easily risk that the heavy gauge bosons, which connect corresponding quarks in a quite different basis from the one defining the flavours, can violate flavour conservation. But the heavy gauge bosons of course give so called "charged" currents and they will not be conserved, only the "neutral currents" get conserved.

If there are more than one higgs acting by being exchanged in the experimentally accessible region of energies, then at least with a random coupling assumption two of the higgses will have couplings that cannot be made diagonal and thus flavour conserving with the same basis of flavour eigenstates.

The supersymmetry works so that the u_i-quarks, i denotes $(u, c, t, ..)$ quarks, get masses from different higgses than the d_i-quarks, i denotes $(d, s, b, ..)$ quarks.

The models with no supersymmetry and with two higgses [110, 111, 112, 113] typically introduce some discrete, like Z_2, symmetries to prevent couplings of some of fermions to more than one higgs.

ii. *Comments on the proton decay*

The present lower bound for the life time of the proton decay of the kind $p \to e^+ + \pi^0$ is 10^{34} years. In the *spin-charge-family*, and in the $SO(10)$-unifying theories without any supersymmetry, this life time is roughly estiated to be up to 10^{35} years.

Let be added that in the $SO(10)$-unifying theories the breaking of the $SO(10)$ group to subgroups is achieved by some Higgs's scalars, which contribute to the proton decay. In the *spin-charge-family* theory the space is (almost) compactified as discussed in Sect. 7.4, and the investigations how this compactification influences the proton decay is still in process.

One should as well carefully study in all the theories, whether baryon number conservation, which in the standard model is an accidental or non-trivially derived symmetry [4], and other symmetries [115] coming out of the assumed gauge symmetries, work.

7.5.3 Small representation requirement.

There are examples, like it is the "explanation without a model behind", or at least where the model behind is less beautiful than the assumption made to get the gauge group taken alone, but still explaining the gauge group of the *standard model*. We may mention the work of one of us (H.B.N.) and the coauthor D. Bennett [104, 105, 106, 108]. In this work the authors, Bennett and one of us set up a method of defining, what can be called the "size of a representation of a Lie group", which is essentially the volume of such a representation relative to the in the same way defined volume of the adjoint representation. The latter concept has to be extended a bit to cope with groups that are not semisimple.

Remarkably the in this way defined smallest faithful representation turns out to occur just for the gauge groups of the *standard model*. So we can claim that this looking for the smallest true group representation "explains" the standard model gauge groups. (We here emphasize the word group, because it is important that one thinks about the Lie group rather than only the Lie algebra as it is a priori the only thing that matters for a gauge theory [48].

The detailed definition of the quantity which points out the *standard model* group is up to a bit of choice for the Lie groups, which are not simple, but for simple groups it is simply the ratio of the quadratic Casimir operator expectation value for the representation in question to that of the adjoint

[48] O' Rafaitaigh [109] proposes a way to use some knowledge about the representations to select a Lie group among the usually several Lie groups corresponding to the same Lie algebra. But this O'Rafaitaigh extension really means that our prediction gives a little more information than if it would be only the Lie algebra.

representation

$$\frac{Cas_r}{Cas_{Adj}} \quad , \tag{134}$$

a quantity that is "balanced" w.r.t. it dimensional dependence in the sence that for the smallest true representations it goes to a constant for very big dimensionalities of the Lie group.

The winner of our extremization of this quantity among the non-abelian simple groups is the $SU(2)$ group, one of the ingredients in the standard model.

The other property, which our election cares for, than the requirement of having preferably gauge group components and representations with small values of $\frac{Cas_r}{Cas_{Adj}}$, is there has been divided out of the center of the say covering group an as large discrete subgroup as possible. This is equivalent to that the representations of the various representations allowed for the model having the winning gauge have favourably most strong restrictions of the type like triality for the $SU(3)$ of the representation is connected with the allowed values for the weak hyper charge. In fact one has in the standard model

$$y/2 + \frac{1}{3}\text{“}triality\text{''} + \text{“}weakisospin\text{''} \;\; = \;\; 0(\text{mod } 1). \tag{135}$$

Our winner group is strongly favoured by this type of relation being roughly so complicated as possible.

So even if one should begin to allow oneself to look for other quantities for the one we constructed, then just that the standard model is doing very well w.r.t. to low $\frac{Cas_r}{Cas_{Adj}}$ strong rules between the integerness of the weak hypercharge and the representation of the non-abelian groups, brings the choice of the winner close to the standard model.

But why should one then have such prinicple of small representations?

"The truth behind history of the work of one of us and" and it is told by me to continue: " Don Bennett leading to suggesting the principle of smallest representaions is that we worked on a rather complicated idea of a model beyond the standard model based on the dream of "Random Dynamics", which really means we assume the fundamental physics to be enourmously complicated with random couplings or parameters, but that somehow by taking a limit of say looking at only low energy expriments you can nevertheless extract some consequencies. We piled up more and more helping assumtions among which the "Multiple Point Principle" and then found that the quantity that would favour the chance for a given Lie group G to be one appearing at low energy in the "Coulomb phase"(i.e. with essentially/approximately massless gauge particles) rather than being confined or Higgesed was this ratio $\frac{Cas_r}{Cas_{Adj}}$.

That is to say, that thinking back on the way we strictly speaking arrived to the idea, we are suggested to look for a theory behind the postulate of "small representation(s)" of the type of such small representations favouring the group to appear as a gauge group in a fundamental theory put up randomly. I.e. we would say like this:

We would formally completely opposite to the model in the present article assume that the fundamental theory is extremely complicated and has lots of parameters with random values. (It sounds opposite but in a way the point should be that assuming a random model like this is about simpler than even a very simple model, and thus in a way of thinking the simplest you can propose).

Let us give the argument how a random theory might favour the "small representation" at least in some in words way:

In random theory there are only (gauge) symmetries by accident, and the main importance is that the Lagrangian or Hamiltonian - which is now a random function - varies very little when the one transforms the fields for some particle with the gauge transformation. But under such a variation the field moves around on the manifold or metric space the volume of which is what we call the size or volume of the representation. Obviously one would say: the smaller this volume the better the chance that the variation is small, and so the better the chance for an accidental gauge invariance popping up. So the gauge symmetries most likely to pop up by accident are the ones with the smallest volumes for the representations. The natural measure relative to which to normalize this volume is the corresponding volume of the adjoint representations, which is in a way the representation on the Lie algebra itself, and thus only involving the group itself.

It would be very natural to say that varying the gauge group a unit little step should be normalized by varying it a unit step in the measure of the adjoint representation. Then for the groups with "small representations" in our sense the corresponding step in the representation, of the fields, is small.

In any case it looks that the standard model group - in O'Raifeartaighs sense - has characteristic that the fields in the representations vary in our sense exceptionally little under variation of the gauge. So it would be the easiest one to get by accident."

8 Influence of second quantization in quantum physics of many body systems

Most of the recognitions and predictions of the *spin-charge-family* theory might and hopefully will influence the high energy experiments and interpretation of the high energy experiments, if the *spin-charge-family* theory offers the right next step beyond the *standard model*.

Could the recognitions and predictions presented in this paper influence the experiments and the theoretical evaluations of bound and decaying states in hadron and nuclear many body systems? Some of them might.

a. The fact that single particle states, describing quarks and leptons, anticommute, explains the necessity to use Slater determinants when describing and studying the bound states of quarks and quarks and antiquarks in hadron, nuclear and atomic physics.

b. Several scalar fields determining the masses of the observed families of quarks and leptons and the weak bosons, some of them with smaller masses than the Higss's scalar, might influence dynamics in bound and decaying states of hadrons (mesons and nucleons) and of nuclei [49].

It might still be that physicists studying properties of the bound and decaying states of the first group of (so far observed) families, bound in hadrons, will recognize that there are problems, which they could solve by taking into account the predictions of the *spin-charge-family* theory that there exist scalar fields with the properties of the Higgs's scalar, having lighter masses than it is the mass of the Higgs's scalar.

c. The existence of the second group of four families, Sect. 6.2.2, predicted to contribute to the observed *dark matter* in the universe, manifesting the "new nuclear force" among the corresponding "hadrons" of the upper four family members, might be a challenge for those who have a great experience in hadron and nuclear physics of the ordinary matter. Since the mass of nucleons (mostly of the fifth family) bound in nuclei have very high masses to this "nuclear force" the largest contribution comes from the one gluon exchange (also the weak and electromagnetic interactions start to contribute almost as much as the colour interaction [94, 122] [50].

The existence of the fourth member of the observed three families with masses larger than 1 TeV

[49]There are warnings in Sect. 7.5.2 that several higgses, in particular those of smaller masses, predict the flavour changing neutral currents (the transitions of a particular family member from one family to another) what is not in agreement with the observations. One of the authors, N.S.M.B., remains optimistic after making rough estimations.

[50]In Ref. [94, 122] the evaluation of the bound states of quarks of the fifth family into nucleons is made with the simple Bohr's model. The two triplets, the gauge fields of $\tilde{SU}(2)$ and $\tilde{SU}(2)$ together with the three singlets, the gauge fields of three $U(1)$, bring masses to the upper four families of quarks and leptons. Due to the interaction of the upper four families with the condensate, Table 6, appearing at very high energies, Sect. 6.2.2, the masses of these quarks and leptons are expected to be much higher than the masses of the members (quarks and leptons) of the lower four families.

might hardly influence the low energy regions in which properties of hadrons and nucleons are measured. Also the existence of additional scalar fields which are with respect to the space index colour triplets and antitriplets, causing proton decay and might be responsible for the matter/antimatter asymmetry in the expanding universe could not be appropriate candidates to attract physicists with experiences in hadron and nuclear physics.

9 Conclusions

The *standard model* certainly made, when postulated, a large step towards better understanding the law of nature. Now a new large step is needed to explain all the assumptions on which the *standard model* is built. *The theory is needed, which would explain all the standard model assumptions in an unique unified way.* Not only since the explanation will help to understand the law of nature but also since further experiments can hardly be correctly interpreted without making the next large enough step.

In the literature most of suggestions how to explain the assumptions of the *standard model* rely on embedding the *standard model* gauge groups into an unifying group, if possible, like there are $SO(10)$ or larger exceptional groups. When this does not seem possible or meaningful, as it is with the family group, then the multiplication of the unifying groups with the family group is often suggested. Correspondingly the vector gauge fields of the charged subgroups of the unifying group can be assumed following the *standard model* suggestions how to take into account more charges and in addition the observed number of families. And when needed the additional scalar fields are assumed.

Is the group theory approach, on which 50 years ago the *standard model* was so innovative and elegantly built, the only way or at least the best way to make the next step towards understanding the law of nature on the level of elementary fermion and boson fields?

Although the group theory approach [80] might answer several open questions which the *standard model* leaves unanswered, the authors of this review article are convinced, at least (S.N.M.B.), that the law of nature is on the level of elementary fields simple and elegant, offering in an elegant way all the answers to the questions which the *standard model* leaves open.

In this review article the authors explain step by step, illustrating steps with several examples, the building blocks of the *spin-charge-family* theory, and also its achievements so far and predictions, comparing this theory with the $SO(10)$-unifying theories:

A The simple starting action, Eq. (100), is in $d = ((d-1)+1)$-dimensional space, $d \geq (13+1)$, assumed for massless fermions, the internal space of which is described by the odd Clifford algebras, and for gravity as massless boson fields, the only bosons appearing in the theory: The vielbeins (the gauge fields of momenta), the spin connection fields (the gauge fields of the generators of the Clifford algebra operators S^{ab} and $\tilde{S}^{ab}$) [51].

A.a There are two kinds of the Clifford algebra objects determining properties of fermions in the theory. We use one kind, γ^a's, to describe the internal space of fermions, and the second kind, $\tilde{\gamma}^a$'s, to generate families, equipping each family with the "family charge".

A.a.i The odd Clifford algebra of γ^a's describes the internal space of fermions. The corresponding creation operators and their Hermitian conjugated partners annihilation operators fulfill the Dirac's postulates of the second quantized fermions, offering explanation for the Dirac's postulates, Sect. 3.3.4. Single fermion states are correspondingly anticommuting objects, manifesting that the first quantization already deals with anticommuting objects. This offers the explanation for the necessity of using Slater determinants in the first quantized many body wave functions.

A.a.ii In $d = (13+1)$ the spin and charges and families (families are unified with the spins and charges by means of our two Clifford algebras), Sect. 3, explain the appearance of quarks and leptons

[51] Let us repeat that if there are no condensate of fermions present the spin connections are expressible by vielbeins.

and antiquarks and antileptons, related to handedness as assumed by the *standard model*, appearing in families.

The Clifford operators S^{ab} transform any member of a particular family to all the other members of the same family, including fermions and antifermions, Table 7.

The Clifford operators $\tilde{S}^{ab}$ transform a family member of one family into the same family member of another family of fermions, Table 5.

In d-dimensional space there are $2^{\frac{d}{2}-1}$ families with $2^{\frac{d}{2}-1}$ family members each [52]. Spins, charges and families are correspondingly unified.

A.b Vielbeins and spin connections manifest in $d=(3+1)$, carrying the charges of subgroups of the unifying $SO(13+1)$ group, as vector and scalar gauge fields, Sect. 6.2. The ordinary gravity is the tensor gauge field of the "spinor charge" in $d=(3+1)$.

A.b.i Vector gauge fields, Sect. 6.2.1, the gauge fields of the charge groups, the gauge subgroups of $SO(13+1)$, carry in $d=(3+1)$ the space index $m=0,1,2,3$, Sect. 6.2.1.

A.b.ii Scalar gauge fields, Sect. 6.2.2, with the space index $(7,8)$ explain the appearance of the Higgs's scalar and Yukawa couplings in the *standard model*. Scalar gauge fields with the space index $s=(9,10,11,12,13,14)$, transforming antileptons into quarks and antiquarks into quarks, and back, offer the explanation for the proton decay and the *matter-antimatter asymmetry* in the expanding universe. All the scalar gauge fields have all the charges with respect to S^{ab} and $\tilde{S}^{ab}$ in the adjoint representations, as also the vector gauge fields do, while with respect to the scalar index they manifest as the weak and hyper charge doublets and colour triplets and antitriplets. There are no additional scalar fields in the simple action in $d=(13+1)$.

A.b.iii The $SO(3,1)$ "charge subgroup" of $SO(13,1)$ group determines properties of vielbeins and spin connections in $d=(3+1)$, $m=(0,1,2,3)$, representing the ordinary gravity.

A.b.iv The *spin-charge-family* theory is a free, renormalizable theory, except for gravity.

B The symmetry of the starting manifold M^{13+1} must break first to $M^{7+1}\times SU(3)\times U(1)$ to keep the starting relation among handedness and charges, what reduces the number of families to $2^{\frac{7+1}{2}-1}$ — this is done by the condensate of two right handed neutrinos with the family quantum number not belonging to the family quantum numbers of the observed families of quarks and leptons and antiquarks and antileptons, Table 6 — and then further to $M^{3+1}\times SU(2)\times SU(3)\times U(1)$. This further break is caused by the scalar fields with space index $(7,8)$, Sect. 6.2.2.

We discuss this break in the case of the toy model, in which the vielbeins and spin connections of particular properties cause the break from M^{5+1} to $M^{3+1}\times$ an almost S^2 sphere which ensures massless fermions also after the break with charges from $d=(5,6)$, Sect. 7.4. The masslessness of fermions is broken when scalar fields gain constant values. The realistic case is under consideration, and it is not yet finished.

C The *spin-charge-family* theory offers several predictions, Sect. 7.3.1:

C.a The two groups of four families are predicted, with mass matrices manifesting the $\widetilde{SU}(2)\times\widetilde{SU}(2)\times U(1)$ symmetry, what reduces the number of free parameters of mass matrices.

C.a.i To the three observed families of quarks and leptons the fourth family with masses above 1TeV is predicted.

C.a.ii The lowest of the upper four families is (almost, up to 10^{14}GeV or higher) stable, offering explanation for the existence of the *dark matter* with masses larger then a few 10 TeV. Due to high masses of quarks and leptons of the upper group of four families, "hadrons" of (mostly) the fifth family quarks experience "nuclear force", which strongly differs from the so far observed nuclear force.

C.b The *spin-charge-family* theory predicts the existence of several scalar fields, determining Higgs's

[52] When we require that quarks and leptons and antiquarks and antileptons fulfill for particular energy $p^0=|\vec{p}|$ the equations of motion, Eq. (79), then to a fermion with momentum $\vec{p}$ and particular spin the anti-fermion with momentum $-\vec{p}$ and the same spin belongs, Sect. 3.3.5, reducing the number of "physical states", Sects. 4.3.1, 4.3.2.

scalar and Yukawa couplings of the *standard model* as the superposition of two triplets and three singlets, some of them with smaller masses than the Higgs's scalar, as well as the new vector triplet gauge field — the second $SU(2)$ triplet with the mass close to 10^{14}GeV or higher, since it couples to the condensate of two right handed neutrinos appearing at this scale, Table 6.

The heavy $SU(2)$ triplet is predicted also by the $SO(10)$-unifying theories, if assuming the appropriate scalar fields, which break the symmetry of $SO(10)$.

The authors hope that the reader will recognize that the next step beyond the *standard model* might not necessarily use the group theory aproaches [80], on which already the *standard model* was built, assuming larger groups and correspondingly unifying charges in $d = (3+1)$, as well as their vector gauge fields. The *spin-charge-family* theory offers, namely, much more, while starting with the simple and correspondingly elegant action in $d \geq (13+1)$: $d = (13+1)$ is the smallest dimension, which offers the next step beyond the *standard model* in the Kaluza-Klein like theories, manifesting in $d = (3+1)$ all the properties of fermions of vector and scalar gauge fields observed so far. The *spin-charge-family* theory unifies spins, charges and families, as well as all the elementary interactions with the gravity included, explaining the second quantization postulates and treating already the first quantized fermions with anticommuting states, manifesting that there are only the second quantized fields, which in a simplified way can be presented as the first quantized fermions arranged in Slater determinants.

In Ref. [138] authors of this article prove, assuming that equations of motion are Hermitian and that the solutions form the irreducible representations of the Lorentz group, that only the metrics with the signature corresponding to q time and $d-q$ space dimensions with q odd can exist. Correspondingly in $d = (3+1)$ the only possibility is q=1.

There are open problems in the *spin-charge-family* theory, some of them shared with the Kaluza-Klein like theories [43, 44, 45, 46, 47, 48, 49, 50, 18], waiting to be solved, like:

i. The compactification of higher dimensions in the presence of families. Although the compactification problem is solved for the toy model in $d = (5+1)$ without the presence of families, Sect. 7.4, and also with the presence of families [129], the breaks of the starting symmetry in steps, leading to the observed properties of quarks and leptons and antiquarks and antileptons, to the observed properties of the scalar fields, need to be done also for $d = (13+1)$-dimensional case.

ii. It must be demonstrated how do scalar triplets and singlet fields, Sect. 6.2.2, with the properties of the Higgs's scalar, after gaining masses through interaction with the condensate, spontaneously gain imaginary masses and correspondingly the constant values, what breaks the weak and hyper charge symmetry.

iii. Although at low energies there is no difficulties with the behaviour of the vector gauge fields, the gravitational gauge field needs to be quantized.

iv. And other problems, common to all the theories, for some of which we almost see the way how to solve them and other problems which we do not yet know how to look for the solution.

The more effort and work is put into the *spin-charge-family* theory the more answers to the open questions the theory offers. Because of this at least one of us, N.S.M.B., remains optimistic. The working hypotheses of the authors of this paper (in particular of N.S.M.B.) is, since the higher dimensions used in the *spin-charge-family* theory offer in an elegant (simple) way explanations for the so many observed phenomena in elementary particle physiscs and cosmology, that higher dimensions should not be excluded by the renormalization and anomaly arguments.

At least the low energy behavior of the spin connections and vielbeins as vector and scalar gauge fields manifest as the known and more or less well defined theory.

The gravity can not be neglected just because we cannot treat it at the very high energy scales. But at low energies we can rather neglect the higher order terms and treat the *spin-charge -family* theory as renormalizable in $d = (3+1)$ dimensions only, showing the next step beyond the *standard model*.

A Action for free massless "Grassmann fermions" with integer spin [23, 20, 22]

In this appendix the action for the integer spin "Grassmann fermions" is presented, taken from Refs. [23, 20, 22], which offer the anticommuting "basis vectors", Eq. (26), if they are products of an odd number of "eigenvectors" of the Cartan subalgebra, Eq. (8).

"Basis vectors", which are products of an even number of "eigenvectors" of the Cartan subalgebra, commute. They are also presented in this appendix.

After the reduction of the Clifford space also the Grassman space is reduced, Sect. 3.2.3, and the integer spin "Grassman fermions" might have no "physical" meaning after this reduction, that is, since we do observe half integer fermions — quarks and leptons and antiquarks and atileptons of several families — the internal degrees of which is describable with one kind of the Clifford algebra objects, γ^a's, there might be after this reduction no possibilities for the integer spin "Grassmann fermions." Yet we present here a possible action for the integer spin "Grassmann fermions".

A.1 Action for free massless anticommuting "Grassmann fermions" with integer spin

In the Grassmann case the "basis vectors" of an odd Grassmann character, chosen to be the eigenvectors of the Cartan subalgebra of the Lorentz algebra in Grassmann space, Eq. (8), manifest the anticommutation relations of Eq. (26) on the algebraic level.

To compare the properties of creation and annihilation operators for "integer spin fermions", for which the internal degrees of freedom are described by the odd Grassmann algebra, with the creation and annihilation operators postulated by Dirac for the second quantized fermions depending on the spins of the internal space of fermions in $d = 2(2n + 1)$ or $4n$ (n is non negative integer) dimensional space and on the momentum space, we need to define the tensor product $*_T$ of the odd "Grassmann basis states", described by the superposition of odd products of θ^a's (with the finite degrees of freedom) and of the momentum (or coordinate) space (with the infinite degrees of freedom), taking as the new basis the tensor product of both spaces.

Statement 1: For deriving the anticommutation relations for the "Grassmann fermions", to be compared to anticommutation relations of the second quantized fermions, we define the tensor product of the Grassmann odd "basis vectors" and the momentum space

$$\mathbf{basis}_{(p^a,\theta^a)} = |p^a > \, *_T \, |\theta^a > \; . \tag{136}$$

We need even more, we need to find the Lorentz invariant action for, let say, free massless "Grassmann fermions" to define such a "basis", that would manifest the relation $p^0 = |\vec{p}|$. We follow here the suggestion of one of us (N.S.M.B.) from Ref. [20].

$$\begin{aligned} \mathcal{A}_G &= \int d^d x \; d^d\theta \; \omega \, \{\phi^\dagger \, \gamma^0_G \, \frac{1}{2} \, \theta^a p_a \phi\} + h.c. \, , \\ \omega &= \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k) \, , \end{aligned} \tag{137}$$

with $\gamma^a_G = (1 - 2\theta^a \frac{\partial}{\partial \theta_a})$, $(\gamma^a_G)^\dagger = \gamma^a_G$, for each $a = (0, 1, 2, 3, 5, \cdots, d)$. We use the integral over θ^a coordinates with the weight function ω from Eq. (165, 166). Requiring the Lorentz invariance we add after $\phi^\dagger$ the operator γ^0_G, which takes care of the Lorentz invariance. Namely

$$\begin{aligned}
\mathbf{S}^{ab\dagger}\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) &= (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{ab}\,,\\
\mathbf{S}^{\dagger}\,(1-2\theta^0\frac{\partial}{\partial\theta^0}) &= (1-2\theta^0\frac{\partial}{\partial\theta^0})\,\mathbf{S}^{-1}\,,\\
\mathbf{S} &= e^{-\frac{i}{2}\omega_{ab}(L^{ab}+\mathbf{S}^{ab})}\,,
\end{aligned} \tag{138}$$

while θ^a, $\frac{\partial}{\partial\theta_a}$ and p^a transform as Lorentz vectors.

The Lagrange density is up to the surface term equal to [53]

$$\begin{aligned}
\mathcal{L}_G &= \frac{1}{2}\phi^\dagger\,\gamma^0_G(\theta^a-\frac{\partial}{\partial\theta_a})\,(\hat{p}_a\phi)\\
&= \frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,(\theta^a-\frac{\partial}{\partial\theta_a})\,\hat{p}_a\phi-\\
&\quad (\hat{p}_a\phi^\dagger)\gamma^0_G(\theta^a-\frac{\partial}{\partial\theta_a})\phi\}\,,
\end{aligned} \tag{139}$$

leading to the equations of motion [54]

$$\frac{1}{2}\,\gamma^0_G\,(\theta^a-\frac{\partial}{\partial\theta_a})\,\hat{p}_a\,|\phi> \;=\; 0\,, \tag{140}$$

as well as the "Klein-Gordon" equation,

$$(\theta^a-\frac{\partial}{\partial\theta_a})\,\hat{p}_a\,(\theta^b-\frac{\partial}{\partial\theta_b})\,\hat{p}_b\,|\phi>=0=\hat{p}_a\hat{p}^a\,|\phi>\,.$$

The eigenstates ϕ of the equations of motion for free massless "Grassmann fermions", Eq. (140), can be found as the tensor product, Eq.(136), of the superposition of 2^{d-1} Grassmann odd "basis vectors" $\hat{b}^{\theta k\dagger}_i$ and the momentum space, represented by plane waves, applied on the vacuum state $|\,1>$. Let us remind that the "basis vectors" are the "eigenstates" of the Cartan subalgebra, Eq. (8), fulfilling (on the algebraic level) the anticommutation relations of Eq. (26). And since the oddness of the Grassmann odd "basis vectors" guarantees the oddness of the tensor products of the internal part of "Grassmann fermions" and of the plane waves, we expect the equivalent anticommutation relations also for the eigenstates of the Eq. (140), which define the single particle anticommuting states of "Grassmann fermions".

The coefficients, determining the superposition, depend on momentum p^a, $a=(0,1,2,3,5,\dots,d)$, $p^0=|\vec{p}|$, of the plane wave solution $e^{-ip_ax^a}$.

Let us therefore define the new creation operators and the corresponding single particle "Grassmann fermion" states as the tensor product of two spaces, the Grassmann odd "basis vectors" and the momentum space basis

$$\begin{aligned}
\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p}) &\stackrel{\text{def}}{=} \sum_i c^{ks}{}_i(\vec{p})\,\hat{b}^{\theta k\dagger}_i\,*_T\,\hat{b}^\dagger_{\vec{p}}\,, \qquad p^0=|\vec{p}|\,,\\
|\phi^{ks}_{tot}(\vec{p},p^0)> &= \hat{\mathbf{b}}^{\theta ks\dagger}(\vec{p},p^0)\,*_T\,|0_{\vec{p}}>|\,1>\,, \qquad p^0=|\vec{p}|\,,
\end{aligned} \tag{141}$$

[53]Taking into account the relations $\gamma^a=(\theta^a+\frac{\partial}{\partial\theta_a})$, $\tilde{\gamma}^a=i\,(\theta^a-\frac{\partial}{\partial\theta_a})$, from where one obtains $\gamma^0_G=-i\eta^{aa}\gamma^a\tilde{\gamma}^a$, the Lagrange density can be rewritten as $\mathcal{L}_G=-i\frac{1}{2}\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,(\hat{p}_a\phi)=-i\frac{1}{4}\{\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\phi\,-\hat{p}_a\phi^\dagger\,\gamma^0_G\,\tilde{\gamma}^a\,\phi\}$.

[54]Varying the action with respect to $\phi^\dagger$ and ϕ it follows: $\frac{\partial\mathcal{L}_G}{\partial\phi^\dagger}-\hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial\hat{p}_a\phi^\dagger}=0=\frac{-i}{2}\gamma^0_G\,\tilde{\gamma}^a\,\hat{p}_a\,\phi$, and $\frac{\partial\mathcal{L}_G}{\partial\phi}-\hat{p}_a\,\frac{\partial\mathcal{L}_G}{\partial(\hat{p}_a\phi)}=0=\frac{i}{2}\hat{p}_a\,\phi^\dagger\gamma^0_G\,\tilde{\gamma}^a$.

where $\hat{b}^{\dagger}_{\vec{p}}$ is defined in App. J, Eq. (211), s represents different solutions of the equations of motion and k different irreducible representations of the Lorentz group, $\vec{p}$ denotes the chosen vector $(p^0, \vec{p})$ in momentum space, and $|0_{\vec{p}}>$ is defined in Eqs. (49, 50),and also discussed in App. J.

One has further

$$|\phi^{ks}(x^0,\vec{x})> \;=\; \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, \hat{\mathbf{b}}^{\theta ks\dagger}(\vec{p})|_{p^0=|\vec{p}|} e^{-ip_a x^a} \;*_T\; |0_{\vec{p}}>|\,1>= \hat{\mathbf{b}}^{\theta ks\dagger}(\vec{x}) \;*_T\; |0_{\vec{x}}>\,|\,1>\;. \quad (142)$$

The orthogonalized states $|\phi^{ks}(\vec{p})>$ fulfill the relation

$$\begin{aligned} <\phi^{ks}(\vec{p})|\phi^{k's'}(\vec{p'})> &= \delta^{kk'}\,\delta_{ss'}\,\delta_{pp'}\,, \qquad p^0=|\vec{p}|\,,\\ <\phi^{k's'}(x^0,\vec{x'})|\phi^{ks}(x^0,\vec{x})> &= \delta^{kk'}\,\delta_{ss'}\,\delta_{\vec{x'},\vec{x}}\,. \end{aligned} \quad (143)$$

where we assumed the discretization of momenta $\vec{p}$ and coordinates $\vec{x}$.

In even dimensional spaces ($d=2(2n+1)$ and $4n$) there are 2^{d-1} Grassmann odd superposition of "basis vectors", which belong to different irreducible representations, among them twice $\frac{1}{2}\frac{d!}{\frac{d}{2}!\frac{d}{2}!}$ of the kind presented in Eqs. (23, 24) and presented in Table 2 of Sect. 3.2.1 for a particular case $d=(5+1)$, Ref. [22]. The illustration for the superposition $\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p})$ is presented, again for $d=(5+1)$, in Part I of Ref. [22].

We introduced in Eq. (141) the creation operators $\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p})$ as the tensor, $*_T$, product of the "basis vectors" of Grassmann algebra elements and the momentum basis. The Grassmann algebra elements transfer their oddness to the tensor products of these two basis. Correspondingly must $\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p})$ together with their Hermitian conjugated annihilation operators, $(\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p}))^{\dagger}=\hat{\mathbf{b}}^{\theta k\,s}(\vec{p})$, fulfill the anticommutation relations equivalent to the anticommutation relations of Eq. (26)

$$\begin{aligned} \{\hat{\mathbf{b}}^{\theta k\,s}(\vec{p}), \hat{\mathbf{b}}^{\theta k'\,s'\dagger}(\vec{p}')\}_{*_T+}\,|0_{\vec{p}}>|\,1> &= \delta^{kk'}\,\delta_{ss'}\delta(\vec{p}-\vec{p}')\,|0_{\vec{p}}>|\,1>\,,\\ \{\hat{\mathbf{b}}^{\theta k\,s}(\vec{p}), \hat{\mathbf{b}}^{\theta k'\,s'}(\vec{p}')\}_{*_T+}\,|0_{\vec{p}}>|\,1> &= 0\,\cdot\,|0_{\vec{p}}>|\,1>\,,\\ \{\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p}), \hat{\mathbf{b}}^{\theta k'\,s'\dagger}(\vec{p}')\}_{*_T+}\,|0_{\vec{p}}>|\,1> &= 0\,\cdot\,|0_{\vec{p}}>|\,1>\,,\\ \hat{\mathbf{b}}^{\theta k\,s}(\vec{p})\;*_T\;|0_{\vec{p}}>|\,1> &= 0\,\cdot\,|0_{\vec{p}}>|\,1>\,,\\ |p^0| &= |\vec{p}|\,. \end{aligned} \quad (144)$$

k labels different irreducible representations of Grassmann odd "basis vectors", s labels different — orthogonal and normalized — solutions of equations of motion and $\vec{p}$ represent different momenta fulfilling the relation $(p^0)^2=(\vec{p})^2$. Here we allow continuous momenta and take into account that

$$<\,1|\,<0_{\vec{p}}|\hat{\mathbf{b}}^{\theta k\,s}(\vec{p}) *_T \hat{\mathbf{b}}^{\theta k'\,s'\dagger}(\vec{p}')\,|0_{\vec{p}}>|\,1> \;=\; \delta^{kk'}\delta^{ss'}\delta(\vec{p}-\vec{p}')\,, \quad (145)$$

in the case of continuous values of $\vec{p}$ in even d-dimensional space.

For each momentum $\vec{p}$ there are 2^{d-1} members of the odd Grassmann character, belonging to different irreducible representations. The plane wave solutions, belonging to different $\vec{p}$, are orthogonal, defining correspondingly ∞ many degrees of freedom for each of 2^{d-1} "fermion" states, defined by $\hat{\mathbf{b}}^{\theta k\,s\dagger}(\vec{p})$ for particular $\vec{p}$, when applying on the vacuum state $|0_{\vec{p}}>|\,1>$, Eq. (22).

With the choice of the Grassmann odd "basis vectors" in the internal space of "Grassmann fermions" and by extending these "basis states" to momentum space to be able to solve the equations of motion, Eq. (140), we are able to define the creation operators $\hat{\mathbf{b}}^{\theta k\,s}(\vec{p})$ of the odd Grassmann character, which together with their Hermitian conjugated partners annihilation operators, fulfill the anticommutation relations of Eq. (144), manifesting the properties of the second quantized fermion fields. Anticommutation properties of creation and annihilation operators are due to the odd Grassmann character of the "basis vectors".

To define the Hilbert space of all possible "Slater determinants" of all possible occupied and empty fermion states, that is the tensor products $*_{T_H}$, of any number of all possible single fermion states, and to discuss the application of $\hat{\mathbf{b}}^{\theta k\, s}(\vec{p})$ and $\hat{\mathbf{b}}^{k\, s\dagger}(\vec{p})$ on the Hilbert space one can follow the procedure of Sect. 5 or can see Ref. [22].

$$\begin{aligned}
\{\hat{\mathbf{b}}^{\theta k\, s}(\vec{p})\,, \hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}')\}_{*_{T_H}+}\mathcal{H} &= \delta^{kk'}\, \delta_{ss'} \delta(\vec{p}-\vec{p}')\, \mathcal{H}\,, \\
\{\hat{\mathbf{b}}^{\theta k\, s}(\vec{p}), \hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p}')\}_{*_{T_H}+}\, \mathcal{H} &= 0\, \cdot\, \mathcal{H}\,, \\
\{\hat{\mathbf{b}}^{\theta k\, s\dagger}(\vec{p})\,, \hat{\mathbf{b}}^{\theta k'\, s'\dagger}_{(}\vec{p}')\}_{*_{T_H}+}\, \mathcal{H} &= 0\, \cdot\, \mathcal{H}\,.
\end{aligned} \tag{146}$$

Creation operators, $\hat{\mathbf{b}}^{sf\dagger}(\vec{p})$, operating on a vacuum state, as well as on the whole Hilbert space, define the second quantized integer "fermion" states.

A.2 Grassmann commuting "basis vectors" with integer spins

Grassmann even "basis vectors" manifest the commutation relations, and not the anticommutation ones as it is the case for the Grassmann odd "basis vectors". Let us use in the Grassmann even case, that is the case of superposition of an even number of θ^a's in $d = 2(2n+1)$, the notation $\hat{a}^{\theta k\dagger}_j$, again chosen to be eigenvectors of the Cartan subalgebra, Eq. (8), and let us start with one representative

$$\begin{aligned}
\hat{a}^{\theta 1\dagger}_j: &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} (\theta^0 - \theta^3)(\theta^1 + i\theta^2)(\theta^5 + i\theta^6) \\
&\cdots (\theta^{d-3} + i\theta^{d-2})\theta^{d-1}\theta^d\,.
\end{aligned} \tag{147}$$

The rest of "basis vectors", belonging to the same Lorentz irreducible representation, follow by the application of $\mathbf{S}^{ab}$. The Hermitian conjugated partner of $\hat{a}^{\theta 1\dagger}_1$ is $\hat{a}^{\theta 1}_1 = (\hat{a}^{\theta 1\dagger}_1)^\dagger$

$$\begin{aligned}
\hat{a}^{\theta 1}_1: &= (\frac{1}{\sqrt{2}})^{\frac{d}{2}-1} \frac{\partial}{\partial\theta^d}\frac{\partial}{\partial\theta^{d-1}}(\frac{\partial}{\partial\theta^{d-3}} - i\frac{\partial}{\partial\theta^{d-2}}) \\
&\cdots (\frac{\partial}{\partial\theta^0} - \frac{\partial}{\partial\theta^3})\,.
\end{aligned} \tag{148}$$

If $\hat{a}^{\theta k\dagger}_j$ represents a Grassmann even creation operator, with index k denoting different irreducible representations and index j denoting a particular member of the k^{th} irreducible representation, while $\hat{a}^{\theta k}_j$ represents its Hermitian conjugated partner, one obtains by taking into account Sect. 3.2.1, the relations

$$\begin{aligned}
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k'\dagger}_j\}_{*_A -}|\,1> &= \delta_{ij}\, \delta^{kk'}\, |\,1>\,, \\
\{\hat{a}^{\theta k}_i, \hat{a}^{\theta k`}_j\}_{*_A -}|\,1> &= 0\, \cdot\, |\,1>\,, \\
\{\hat{a}^{\theta k\dagger}_i, \hat{a}^{\theta k'\dagger}_j\}_{*_A -}\, |\,1> &= 0\, \cdot\, |\,1>\,, \\
\hat{a}^{\theta k}_i\, *_A\, |\,1> &= 0\, \cdot\, |\,1>\,, \\
\hat{a}^{\theta k\dagger}_i\, *_A\, |\,1> &= |\phi^k_{e\,i}>\,.
\end{aligned} \tag{149}$$

Equivalently to the case of Grassmann odd "basis vectors" here $\{\hat{a}^{\theta k}_i, \hat{a}^{\theta l\dagger}_j\}_{*_A -} = \hat{a}^{\theta k}_i *_A \hat{a}^{\theta l\dagger}_j - \hat{a}^{\theta l}_j *_A \hat{a}^{\theta k\dagger}_i$, with $\}_{*_A -}$ denoting commutation relations.

Also here, like in the Grassmann odd case, we can make the tensor product of the internal space of even "basis vectors" and the basis of coordinate or momentum space to form in this case the commuting creation and annihilation operators.

B Trial to compare application of γ^a matrices in usual case and in *spin-charge-family* theory

Sects. 3 - 6 show that in the *spin-charge-family* theory creation operators and their Hermitian conjugated annihilation operators anticommute, due to the odd character of the "basis vectors" describing the internal space of fermions.

The operators S^{ab} and $\tilde{S}^{ab}$, connecting in the *spin-charge-family* theory the members of the same irreducible representation (the members of a particular family) and the members of different irreducible representations (the same family member of different families), respectively, having an even Clifford character, do not change the oddness of the "basis vectors". The application of the operators γ^a's as well as $\tilde{\gamma^a}$'s, having both an odd Clifford character, do change the character of states from odd to even.

In the case of massive states the usual way, the one of Dirac, uses the γ^a's to transform the left handed fermion states in $d = (3+1)$ into the right handed ones to generate basis for the massive states. Charges are treated separately — by the choice of the charge groups.

We demonstrate in Sect. 7.4 on the toy model of $d = (5+1)$ that breaking the staring symmetry, caused by the particular scalar fields appearing in the simple starting action, Eqs. (100, 101), do lead to massless charged states in the manifold of $d = (3+1)$, while further break of symmetry, when scalar fields gain constant values, makes these massless fermions and antifermions chargeless and massive with the properties of the massive Majorana fermions, which are superposition of fermions and antifermions, Eq. (133). In this case if one forgets the spin part in the 5^{th} and 6^{th} dimensions, manifesting as charges in $d = (3+1)$, which is now not conserved any longer, there are only the spin parts manifesting in $d = (3+1)$.

Of course, the Dirac's states are not describing Majorana particles, but do also not pay attention on the oddness of the "basis states" describing the internal space of fermions as the *spin-charge-family* theory does. The vectors describing spins (S^{03} and S^{12}, or equivalently, handedness and S^{12}) in the usual text books do not have the odd character, they are columns with numbers, which commute.

But if one looks at the solutions of equations of motion for the massive states in the *spin-charge-family* theory, Eq. (130), as the states without charges, the multiplication of this massless chargeless state by γ^0, if neglecting the charge part and correspondingly the oddness of states, what the usual way of presenting fermions does, then one can easier understand the difference of the *spin-charge-family* theory way of the second quantization and the Dirac's way, presented in Sect. 3.3.1.

We present in this Sect. B, following Ref. [24], the matrix representations of the operators γ^a's, $\tilde{\gamma^a}$'s, S^{ab} and $\tilde{S}^{ab}$, manifesting how would matrices look like if families, suggested by the use of the Clifford algebra in the *spin-charge-family* theory, would be taken into account.

Since one can always embed the spin part in $d = (3+1)$ into $d \geq 5$, and if at the same time one requires the Clifford oddness of "basis vectors" in the whole space $(d-1)+1$ so that spins in higher dimensions manifest as charges in $d = (3+1)$, the corresponding creation and annihilation operators fulfill the anticommutation postulates of Dirac. The single fermion states already anticommute.

We present matrices for operators γ^a's, $\tilde{\gamma^a}$'s, S^{ab} and $\tilde{S}^{ab}$ in Sect. B.2.

Sect. B.1 demonstrates that the application of the Clifford odd operators γ^a's and $\tilde{\gamma^a}$'s, Ref. [21], on the Clifford odd anticommuting "basis vectors" transforms the Clifford odd "basis vectors" to the Clifford even "basis vector", which commute.

B.1 Clifford even creation operators

In Table 3 the Clifford odd "basis vectors" for $d = (5+1)$ case are presented as suggested by the *spin-charge-family* theory, forming four families, together with their Hermitian conjugated partners. Due to the oddness of the "basis vectors" the tensor products, $*_T$, of the "basis vectors" and the basis in the momentum/coordinate space fulfill together with their corresponding annihilation operators the anticommutation relations postulated by Dirac, explaining indeed the Dirac's anticommuting postulates.

The remaining 2^{d-1} "basis vectors" have an even Clifford character. They are presented in the lower half part of Table 13. The reader can immediately see that the multiplication of any odd "basis vector" by γ^a leads to an even "basis vector". For example, the application of γ^0 on the first family member of the first *odd I* family generates the first member of the first *even I* family.

All the Clifford even "families" with "family" members of Table 13 can be obtained also as algebraic products ($*_A$) of the Clifford odd "basis vectors" from the upper half part of the same table.

All the even families have one self adjoint "basis vector"; Being a product of projectors only this "basis vector" is its own Hermitian conjugated partner, enabling to generate and annihilate the state at the same time, when applying on the (any one) vacuum state.

Clifford even "basis vectors" commute.

The Clifford even operators S^{ab} transform family members of one family among themselves, while $\tilde{S}^{ab}$ transform one family member of any family to the same family member of another family also in the case of even "basis vectors".

We shall comment these properties in Sect. B.2.

Let us conclude: While the Grassmann algebra offers the Grassmann odd creation operators and their Hermitian conjugated partners annihilation operators, which anticommute [55], as well as Grassmann even creation operators and their Hermitian conjugated partners annihilation operators, which commute, the Clifford algebra offers only odd, anticommuting, creation and their Hermitian conjugated partners annihilation operators, Sect. A.2.

B.2 Relations between Clifford algebra and Dirac matrices in the presence of families in $d = (3+1)$ and $d = (5+1)$

We learn in Sect. 3, in particular after reduction of the Clifford algebra space, Sect. 3.2.3, and in Sects. 4, 5, that all the fermion states can be represented by the tensor products, $*_T$, of the finite number of "basis vectors" and the continuously infinite basis in the ordinary space, while the total Hilbert space consists of the tensor products, $*_{T_H}$, of all possible numbers of all possible anticommuting single particle creation operators applying on the vacuum state. We also learn in Sects. 7.4.1 and B.1 that "basis vectors" change oddness into evenness if we multiply them by Clifford odd operators γ^a's and $\tilde{\gamma}^a$'s.

The Dirac's single particle states are not Clifford odd and vectors, solving the Dirac equations, also do not manifest oddness. The fermion states gain the anticommuting character with the second quantization postulates of Dirac.

In the *spin-charge-family* theory fermions already have the anticummuting nature due to the fact that the theory uses the products of an odd number of γ^a's to describe the internal space of fermions. Correspondingly the "basis vectors" anticommute, fulfilling the anticommutation relations postulated by Dirac for the second quantized fermions, and the anticommuting character of "basis vectors" transfers also to the states, which are superposition of tensor products of the "basis vectors" and basis in the momentum or coordinate space, Eqs. (55, 57), and are solutions of equations of motion. Let us use the so far several time discussed toy model in $d = (5+1)$ to illustrate the application of the operators γ^a's, $\tilde{\gamma}^a$'s, S^{ab} and $\tilde{S}^{ab}$ on the "basis vectors".

We use in Eq. (66) the Clifford odd part of "basis vectors" in $d = (5+1)$, Table 13, describing the anticommuting odd "basis vectors" in $d = (3+1)$ with definite handedness and spin and with particular charge determined by the spin S^{56}, which in this case we neglect. These states, the solutions

[55] Like in the Clifford case also in the Grassmann case one can generate creation operators as superposition of tensor products, $*_T$, of the Grassmann odd "basis vectors" and the ordinary, momentum or coordinate, basis.

Table 13: $2^d = 64$ "eigenvectors" of the Cartan subalgebra, Eq. (8), of the Clifford odd and even algebras in $d = (5+1)$ are presented, divided into four groups, each group with four "families", each "family" with four "family" members. Two of four groups are superposition of products of an odd number of γ^a's. The "basis vectors", $\hat{b}^{m\dagger}_f$, Eq. (44), in *odd I* group, belong to four "families" ($f = 1(a), 2(b), 3(c), 4(d)$) with four members ($m = 1, 2, 3, 4$), having their Hermitian conjugated partners, $\hat{b}^m_f$, among "basis vectors" of the *odd II* part, denoted with the corresponding "family" and "family" members (a_m, b_m, c_m, d_m) quantum numbers. The "family" quantum numbers, determined by ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$), of $\hat{b}^{m\dagger}_f$ are written above each "family". The two groups with the even number of γ^a's in a product, *even I* and *even II*, have their Hermitian conjugated partners within their own group each. There are members in each of even groups, which are products of projectors only. Numbers — 03 12 56 — denote the indexes of the corresponding Cartan subalgebra members ($\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$), Eq. (8). In the columns $(7, 8, 9)$ the eigenvalues of the Cartan subalgebra members (S^{03}, S^{12}, S^{56}), Eq. (8), are presented. The last two columns tell the handedness of "basis vectors" in $d = (5+1)$, $\Gamma^{(5+1)}$, and of $d = (3+1)$, $\Gamma^{(3+1)}$, respectively, defined in Eq.(173).

odd I	m	$f = 1(a)$ $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$f = 2(b)$ $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$f = 3(c)$ $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	$f = 4(d)$ $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		03 12 56	03 12 56	03 12 56	03 12 56					
	1	$(+i)(+)(+)$	$[+i][+](+)$	$[+i](+)[+]$	$(+i)[+][+]$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	$[-i][-](+)$	$(-i)(-)(+)$	$(-i)[-][+]$	$[-i](-)[+]$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	$[-i](+)[-]$	$(-i)[+][-]$	$(-i)(+)(-)$	$[-i][+](-)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1
	4	$(+i)[-][-]$	$[+i](-)[-]$	$[+i][-](-)$	$(+i)(-)(-)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1
odd II		03 12 56 $_{fm}$	03 12 56 $_{fm}$	03 12 56 $_{fm}$	03 12 56 $_{fm}$	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
		$(-i)(+)(+)_{d_4}$	$[-i][+](+)_{d_3}$	$[-i](+)[+]_{d_2}$	$(-i)[+][+]_{d_1}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	-1
		$[+i][-](+)_{c_4}$	$(+i)(-)(+)_{c_3}$	$(+i)[-][+]_{c_2}$	$[+i](-)[+]_{c_1}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	-1
		$[+i](+)[-]_{b_4}$	$(+i)[+][-]_{b_3}$	$(+i)(+)(-)_{b_2}$	$[+i][+](-)_{b_1}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1
		$(-i)[-][-]_{a_4}$	$[-i](-)[-]_{a_3}$	$[-i][-](-)_{a_2}$	$(-i)(-)(-)_{a_1}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1
even I	m	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$[-i](+)(+)$	$(-i)[+](+)$	$[-i][+][+]$	$(-i)(+)[+]$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	-1
	2	$(+i)[-](+)$	$[+i](-)(+)$	$(+i)(-)[+]$	$[+i][-][+]$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	-1
	3	$(+i)(+)[-]$	$[+i][+][-]$	$(+i)[+](-)$	$[+i](+)(-)$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1
	4	$[-i][-][-]$	$(-i)(-)[-]$	$[-i](-)(-)$	$(-i)[-](-)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1
even II	m	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}	$\Gamma^{(5+1)}$	$\Gamma^{(3+1)}$
	1	$[+i](+)(+)$	$(+i)[+](+)$	$[+i][+][+]$	$(+i)(+)[+]$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
	2	$(-i)[-](+)$	$[-i](-)(+)$	$(-i)(-)[+]$	$[-i][-][+]$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	1
	3	$(-i)(+)[-]$	$[-i][+][-]$	$(-i)[+](-)$	$[-i](+)(-)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1
	4	$[+i][-][-]$	$(+i)(-)[-]$	$[+i](-)(-)$	$(+i)[-](-)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	-1

Table 14: In this table $2^d = 16$ vectors, describing internal space of fermions in $d = (3+1)$, are presented. Each vector carries the family member quantum number $m = (1, 2, 3, 4)$ — determined by S^{03} and S^{12}, Eqs. (8) — and the family quantum number $f = (I, II, III, IV)$ — determined by $\tilde{S}^{03}$ and $\tilde{S}^{12}$, Eq. (8). Vectors ψ^m_f are obtained by applying $\hat{b}^{m\dagger}_f$ on the vacuum state, Eq. (46, 47). Vectors, that is the family members of any family, split into even (they are sums of products of an even number of γ^a's) and odd (they are sums of products of an odd number of γ^a's). If these vectors are embedded into $d = (5+1)$ (by being multiplied by an appropriate nilpotent or projector so that they are of an odd Clifford character), they "gain" charges as presented in Table 3.

	ψ^f_m	$\gamma_0\,\psi^f_m$	$\gamma_1\,\psi^f_m$	$\gamma_2\,\psi^f_m$	$\gamma_3\,\psi^f_m$	$\tilde{\gamma}_0\,\psi^f_m$	$\tilde{\gamma}_1\,\psi^f_m$	$\tilde{\gamma}_2\,\psi^f_m$	$\tilde{\gamma}_3\,\psi^f_m$	S^{03}	S^{12}	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\Gamma^{(3+1)}$	$\tilde{\Gamma}^{(3+1)}$
ψ^a_1	$(+i)(+)$	ψ^a_3	ψ^a_4	$i\psi^a_4$	ψ^a_3	$-i\psi^b_1$	$-i\psi^c_1$	ψ^c_1	$-i\psi^b_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^a_2	$[-i][-]$	ψ^a_4	ψ^a_3	$-i\psi^a_3$	$-\psi^a_4$	$i\psi^b_2$	$i\psi^c_2$	$-\psi^c_2$	$i\psi^b_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	1	1
ψ^a_3	$[-i](+)$	ψ^a_1	$-\psi^a_2$	$-i\psi^a_2$	$-\psi^a_1$	$i\psi^b_3$	$i\psi^c_3$	$-\psi^c_3$	$i\psi^b_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^a_4	$(+i)[-]$	ψ^a_2	$-\psi^a_1$	$i\psi^a_1$	ψ^a_2	$-i\psi^b_4$	$-i\psi^c_4$	ψ^c_4	$-i\psi^b_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	-1	1
ψ^b_1	$[+i](+)$	ψ^b_3	$-\psi^b_4$	$-i\psi^b_4$	ψ^b_3	$i\psi^a_1$	$i\psi^d_1$	$-\psi^d_1$	$-i\psi^a_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^b_2	$(-i)[-]$	ψ^b_4	$-\psi^b_3$	$i\psi^b_3$	$-\psi^b_4$	$-i\psi^a_2$	$-i\psi^d_2$	ψ^d_2	$i\psi^a_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	1	-1
ψ^b_3	$(-i)(+)$	ψ^b_1	ψ^b_2	$i\psi^b_2$	$-\psi^b_1$	$-i\psi^a_3$	$-i\psi^d_3$	ψ^d_3	$i\psi^a_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^b_4	$[+i][-]$	ψ^b_2	ψ^b_1	$-i\psi^b_1$	ψ^b_2	$i\psi^a_4$	$i\psi^d_4$	$-\psi^d_4$	$-i\psi^a_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	-1	-1
ψ^c_1	$(+i)[+]$	ψ^c_3	$-\psi^c_4$	$-i\psi^c_4$	ψ^c_3	$i\psi^d_1$	$-i\psi^a_1$	$-\psi^a_1$	$i\psi^d_1$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^c_2	$[-i](-)$	ψ^c_4	$-\psi^c_3$	$i\psi^c_3$	$-\psi^c_4$	$-i\psi^d_2$	$i\psi^a_2$	ψ^a_2	$-i\psi^d_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	1	-1
ψ^c_3	$[-i][+]$	ψ^c_1	ψ^c_2	$i\psi^c_2$	$-\psi^c_1$	$-i\psi^d_3$	$i\psi^a_3$	ψ^a_3	$-i\psi^d_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1
ψ^c_4	$(+i)(-)$	ψ^c_2	ψ^c_1	$-i\psi^c_1$	ψ^c_2	$i\psi^d_4$	$-i\psi^a_4$	$-\psi^a_4$	$i\psi^d_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	-1	-1
ψ^d_1	$[+i][+]$	ψ^d_3	ψ^d_4	$i\psi^d_4$	ψ^d_3	$-i\psi^c_1$	$i\psi^b_1$	ψ^b_1	$i\psi^c_1$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^d_2	$(-i)(-)$	ψ^d_4	ψ^d_3	$-i\psi^d_3$	$-\psi^d_4$	$i\psi^c_2$	$-i\psi^b_2$	$-\psi^b_2$	$-i\psi^c_2$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	1	1
ψ^d_3	$(-i)[+]$	ψ^d_1	$-\psi^d_2$	$-i\psi^d_2$	$-\psi^d_1$	$i\psi^c_3$	$-i\psi^b_3$	$-\psi^b_3$	$-i\psi^c_3$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1
ψ^d_4	$[+i](-)$	ψ^d_2	$-\psi^d_1$	$i\psi^d_1$	ψ^d_2	$-i\psi^c_4$	$i\psi^b_4$	ψ^b_4	$i\psi^c_4$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	-1	1

of equations of motions for free massless fermions in $d = (3+1)$, are correspondingly assumed to have no charge, and are the superposition of the first two members of the third family and the first two members of the fourth families, respectively, with the charge part neglected.

The solutions of equations of motion for free massless particles in $d = (5+1)$ of Eq. (84) represent fermions and antifermions of the first family from Table 3 for $p^a = (p^0, \vec{p}, 0)$.

The solutions of equations of motion in $d = (13+1)$ for free massless right handed u-quark of the colour charge $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and the "fermion charge" $\tau^4 = \frac{1}{6}$ and its left handed antiquark, Eq. (90), manifest dynamics in $d = (3+1)$ only, if $p^a = (p^0, \vec{p}, 0, 0, 0, 0, \ldots, 0)$.

In Sect. 7.4.1 the massless and massive solutions in $d = (5+1)$ are presented.

Here we look at the $d = (3+1)$ part of the "basis vectors" belonging to $d = (5+1)$ space, appearing correspondingly in four families. In spaces with higher dimensions there would in general case be more families.

The "basis vectors" are arranged as presented in Table 14

Using the Pauli matrices from Eq. (10) as well as the unit 4×4 submatrix and the submatrix with all the matrix elements equal to zero

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0 \,, \qquad\qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \,, \tag{150}$$

and taking into account Table 14 the matrix representation for γ^a's, $\tilde{\gamma}^a$'s, S^{ab}'s and $\tilde{S}^{ab}$'s follow.

$$\gamma^0 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{smallmatrix} \end{pmatrix} , \; \gamma^1 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{smallmatrix} \end{pmatrix} , \tag{151}$$

$$\gamma^2 = \begin{pmatrix} \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} & 0 & 0 & 0 \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{smallmatrix} \end{pmatrix}, \gamma^3 = \begin{pmatrix} \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{smallmatrix} \end{pmatrix}, \tag{152}$$

manifesting the 4×4 substructure (of Dirac matrices) along the diagonal of 16×16 matrices.

The representations of $\tilde{\gamma}^a$ do not appear in the Dirac case. They manifest the off diagonal structure as follows

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}, \tilde{\gamma}^1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{153}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} \\ \begin{smallmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tilde{\gamma}^3 = \begin{pmatrix} \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{smallmatrix} & \mathbf{0} \end{pmatrix}. \tag{154}$$

Matrices S^{ab} have again along the diagonal the 4×4 substructure, repeating, up to a phase, as expected, the corresponding Dirac matrices, since the Dirac S^{ab} do not distinguish among families.

$$S^{01} = \frac{i}{2} \begin{pmatrix} \begin{smallmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{smallmatrix} \end{pmatrix}, S^{02} = \frac{i}{2} \begin{pmatrix} \begin{smallmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{smallmatrix} \end{pmatrix}, \tag{155}$$

$$S^{03} = \frac{i}{2} \begin{pmatrix} \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{smallmatrix} \end{pmatrix}, S^{12} = \frac{1}{2} \begin{pmatrix} \begin{smallmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{smallmatrix} \end{pmatrix}, \tag{156}$$

$$S^{13} = \frac{1}{2} \begin{pmatrix} \begin{smallmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\sigma^2 & 0 \\ 0 & -\sigma^2 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^2 & 0 \\ 0 & -\sigma^2 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{smallmatrix} \end{pmatrix}, S^{23} = \frac{1}{2} \begin{pmatrix} \begin{smallmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} -\sigma^1 & 0 \\ 0 & -\sigma^1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} -\sigma^1 & 0 \\ 0 & -\sigma^1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{smallmatrix} \end{pmatrix}. \tag{157}$$

$$\Gamma^{(3+1)} = -4iS^{03}S^{12} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{158}$$

The matrices of the operators $\tilde{S}^{ab}$ have again off diagonal 4×4 substructure, except $\tilde{S}^{03}$ and $\tilde{S}^{12}$, which are diagonal.

$$\tilde{S}^{01} = -\frac{i}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tilde{S}^{02} = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tilde{S}^{03} = \frac{i}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{pmatrix}, \tag{159}$$

$$\tilde{S}^{12} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{pmatrix}, \tilde{S}^{13} = \frac{i}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tilde{S}^{23} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{160}$$

$$\tilde{\Gamma}^{(3+1)} = -4i\tilde{S}^{03}\tilde{S}^{12} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}. \tag{161}$$

Let us now conclude this appendix: We start with the operators γ^a's and $\tilde{\gamma}^a$'s and express the "basis vectors" as superposition of and an odd number of γ^a's. Here we present the matrix representation of the operators γ^a's, $\tilde{\gamma}^a$'s, S^{ab}'s and $\tilde{S}^{ab}$'s, operating on the "basis vectors", which are eigenvectors of the Cartan subalgebra of the Lorentz algebra, S^{ab}'s and $\tilde{S}^{ab}$'s. This matrices can be compared with the Pauli matrices and γ^m's and S^{mn}'s matrices of Dirac, after we neglect the charge part in the 6^{th} and the 7^{th} dimension of the internal space of fermions.

Since there are several families the representations of the operators γ^a's, $\tilde{\gamma}^a$'s, S^{ab}'s and $\tilde{S}^{ab}$'s manifest the existence of families.

The states on which the Pauli matrices and the Dirac's γ^a matrices apply are just columns of numbers, with no anticommuting character, which "basis vectors" do have.

The Clifford algebra enables to create creation operators which define, applying on the vacuum state, the single particle states with the odd Clifford character. Since the oddness of (the finite number of) "basis vectors" transfers in the tensor products, $*_T$, with the continuously infinite basis in ordinary space to the creation and annihilation operators, making them Clifford odd, the Dirac postulates are not needed. The use of the Clifford algebra for the description of the internal space explains the Dirac's second quantized postulates.

In Sect. 3.3.4 we compare the way of the second quantization of fermions in the ıspin-charge-family theory with the Dirac's one.

C Understanding triangle anomalies cancellation in standard model

We clarify here the difference in explaining the "miraculous" anomalies cancellation in the *standard model* in the $SO(10)$ unifying theories and the *spin-charge-family* theory. Since the $SO(10)$ group is subgroup of $SO(13,1)$ group, the difference occurs from the fact that in the *spin-charge-family* theory the spins and charges are unified, while in the $SO(10)$ theories they are not. One see in Table 7 that handedness and charges of quarks and leptons and of antiquarks and antileptons are uniquely determined — fermions and anti-fermions belong to the same irreducible representation of the Lorentz group, while in the unifying theories handedness and charges must be correlated "by hand". We follow here mainly Ref. [18].

In $d = (2n)$-dimensional space-time massless fermions contribute through the one-loop (n + 1)-angle diagram in general an anomalous (infinite) function, which causes the current non-conservation and contributes to the gauge non-invariance of the action [53, 54].

To the triangle anomalies the right-handed spinors (fermions) and the right handed anti-spinors (anti-fermions) contribute with the opposite sign than the left handed spinors and the left handed anti-spinors. Their common contribution to anomalies is proportional to [53]

$$\Big(\sum_{(A,i,B,j,C,k)_{L\,\bar{L}}} Tr[\tau^{Ai}\,\tau^{Bj}\,\tau^{Ck}] - \sum_{(A,i,B,j,C,k)_{R\,\bar{R}}} Tr[\tau^{Ai}\,\tau^{Bj}\,\tau^{Ck}] \Big), \tag{162}$$

where τ^{Ai} are in the *standard model* the generators of the infinitesimal transformation of the groups $SU(3), SU(2)$ and $U(1)$, while in the *spin-charge-family* theory τ^{Ai} are the infinitesimal generators of the irreducible subgroups of the starting orthogonal group $SO(2(2n+1)-1,1)$, $n=3$ (which include all the *standard model* groups, offering correspondingly the explanation for their origin), Eqs. (85, 86, 87). The traces run over the representations of one massless family of the left handed fermions and anti-fermions, denoted by $_{L\bar{L}}$, and the right handed fermions and anti-fermions, denoted by $_{R\bar{R}}$.

Embedding the *standard model* groups into the orthogonal group $SO(13,1)$ explains elegantly the "miraculous" cancellation of the triangle anomalies in the *standard model.* Embedding the *standard model* groups into the orthogonal group $SO(10)$ group explains the "miraculous" cancellation of the triangle anomalies if one correlates charges and handedness.

Table 7 presents spinor handedness ($\Gamma^{(3,1)}$), their spin (S^{12}), weak charge (τ^{13}), the second $SU(2)_{II}$ (τ^{23}) charge (arising together with $SU(2)_I$ from $SO(4)$), their colour charge (τ^{33}, τ^{38}) (arising together with $U(1)_{II}$ from $SO(6)$), and the "fermion charge" (τ^4, the generator of $U(1)_{II}$). The hyper charge is $Y = (\tau^{23} + \tau^4)$, the electromagnetic charge is $Q = (\tau^{13} + Y)$.

The triangle anomaly of the *standard model* occurs if the traces in Eq.(162) are not zero for either the left handed quarks and leptons and anti-quarks and anti-leptons or the right handed quarks and leptons and anti-quarks and anti-leptons for the Feynman triangle diagrams in which the gauge vector fields of the type

$$\begin{aligned}
&U(1) \times U(1) \times U(1)\,, \\
&SU(2) \times SU(2) \times U(1)\,, \\
&SU(3) \times SU(3) \times SU(3)\,, \\
&SU(3) \times SU(3) \times U(1)\,, \\
&U(1) \times \text{gravitational}
\end{aligned} \tag{163}$$

contribute to the triangle anomaly.

Table 15 presents in the first seven columns (up to ||) by the *standard model* assumed properties of the members of any massless family, running in the triangle. The last two columns, taken from Table 7, describe additional properties which quarks and leptons and anti-quarks and anti-leptons would have, if the *standard model* groups $SO(3,1)$, $SU(2), SU(3)$ and $U(1)$ are embedded into the $SO(13,1)$ group. As already pointed out, the $SO(10)$ unifying theories do from the point of the charges the same, but do correlate handedness and charges in the same way as in the ŝtandard model. We comment these last two columns in particular in Sect. 7.

In the *spin-charge-family* theory the family quantum numbers are determined by the second kind of the Clifford algebra objects $\tilde{S}^{ab}$, which commute with S^{ab} describing spins and charges. Correspondingly the spins and charges are the same for all the families.

To calculate the traces required in Eq. (162) for the triangle anomalies of Eq. (163) the quantum numbers of the left handed spinors and anti-spinors, as well as of the right handed spinors and anti-spinors, presented in Table 15, are needed.

Let us calculate the traces, Eq. (163), for possible anomalous triangle diagrams in $d = (3+1)$. One must evaluate the trace of the product of three generators and sum the trace over all the states of either the left handed members — 16 states presented in the first part of Table 15 — or the right handed members — 16 states presented in the second part of Table 15. Let us recognize again that in the case of embedding the *standard model* groups into $SO(13,1)$ we have $Y = (\tau^4 + \tau^{23})$.

For the triangle Feynman diagram, to which three hyper $U(1)$ boson fields contribute, we must evaluate $\sum_i Tr(Y_i)^3$, in which the sum runs over all the members ($i = (1,..,16)$) of the left handed spinors and anti-spinors, and of the right handed spinors and anti-spinors separately. In the case of

Table 15: Properties of the left handed quarks and leptons and of the left handed anti-quarks and anti-leptons (the first 16 lines) and of the right handed quarks and leptons and the right handed anti-quarks and anti-leptons (the second 16 lines), as assumed by the *standard model*, are presented in the first eight columns. In the last two columns the two quantum numbers are added, which fermions and anti-fermions would have if the *standard model* groups $SO(3,1)$, $SU(2)$, $SU(3)$ and $U(1)$ are embedded into the $SO(13,1)$ group. All the infinitesimal generators of the subgroups of the orthogonal group $SO(13,1)$, representing charges in $d=(3+1)$, are in the *spin-charge-family* theory the superposition of the generators S^{st}, Eqs. (85, 86, 87). The handedness is defined in Eq.(173). The whole quark part appears, due to the colour charges, three times. These quantum numbers are the same for all the families.

i_L	name	hand-edness $\Gamma^{(3,1)}$	weak charge τ^{13}	hyper charge Y	colour τ^{33}	charge τ^{38}	elm charge Q	$SU(2)_{II}$ charge τ^{23}	$U(1)_{II}$ charge τ^4
1_L	u_L	-1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$	0	$\frac{1}{6}$
2_L	d_L	-1	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{3}$	0	$\frac{1}{6}$
3_L	u_L	-1	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$	0	$\frac{1}{6}$
4_L	d_L	-1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{3}$	0	$\frac{1}{6}$
5_L	u_L	-1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$	0	$\frac{1}{6}$
6_L	d_L	-1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{3}$	0	$\frac{1}{6}$
7_L	ν_L	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	$-\frac{1}{2}$
8_L	e^L	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	-1	0	$-\frac{1}{2}$
9_L	$\bar{u}L$	-1	0	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$
10_L	$\bar{d}L$	-1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$
11_L	$\bar{u}L$	-1	0	$-\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$
12_L	$\bar{d}L$	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$
13_L	$\bar{u}L$	-1	0	$-\frac{2}{3}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$
14_L	$\bar{d}L$	-1	0	$\frac{1}{3}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{6}$
15_L	$\bar{\nu}_L$	-1	0	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$
16_L	$\bar{e}_L$	-1	0	1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$
1_R	u_R	1	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
2_R	d_R	1	0	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$
3_R	u_R	1	0	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
4_R	d_R	1	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$
5_R	u_R	1	0	$\frac{2}{3}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
6_R	d_R	1	0	$-\frac{1}{3}$	0	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$
7_R	ν_R	1	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
8_R	e_R	1	0	-1	0	0	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
9_R	$\bar{u}_R$	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$	0	$-\frac{1}{6}$
10_R	$\bar{d}_R$	1	$\frac{1}{2}$	$-\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	0	$-\frac{1}{6}$
11_R	$\bar{u}_R$	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$	0	$-\frac{1}{6}$
12_R	$\bar{d}_R$	1	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{3}$	0	$-\frac{1}{6}$
13_R	$\bar{u}_R$	1	$-\frac{1}{2}$	$-\frac{1}{6}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$	0	$-\frac{1}{6}$
14_R	$\bar{d}_R$	1	$\frac{1}{2}$	$-\frac{1}{6}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{3}$	0	$-\frac{1}{6}$
15_R	$\bar{\nu}_R$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$
16_R	$\bar{e}_R$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0	$\frac{1}{2}$

embedding the *standard model* groups into $SO(13,1)$ we have

$$
\begin{aligned}
\sum_{i_{L,R}} (Y_{i_{L,R}})^3 &= \sum_{i_{L,R}} (\tau^4_{i_{L,R}} + \tau^{23}_{i_{L,R}})^3 \\
&= \sum_{i_{L,R}} (\tau^4_{i_{L,R}})^3 + \sum_{i_{L,R}} (\tau^{23}_{i_{L,R}})^3 \\
&+ \sum_{i_{L,R}} 3 \cdot (\tau^4_{i_{L,R}})^2 \cdot \tau^{23}_{i_{L,R}} + \sum_{i_{L,R}} 3 \cdot \tau^4_{i_{L,R}} \cdot (\tau^{23}_{i_{L,R}})^2 \,,
\end{aligned}
\tag{164}
$$

for either the left, i_L, or the right, i_R, handed members. Table 15 demonstrates clearly that $(Y_{i_{L,R}})^3 = 0$ without really making any algebraic evaluation. Namely, the last column of Table 15 manifests that $\sum_{i_L} (\tau^4_{i_L})^3 = 0$ [in details: $\sum_{i_L} (\tau^4_{i_L})^3 = 2\cdot 3 \cdot (\frac{1}{6})^3 + 2 \cdot 3 \cdot (-\frac{1}{6})^3 + 2 \cdot (-\frac{1}{2})^3 + 2 \cdot (\frac{1}{2})^3 = 0$]. Table 15 also demonstrates (the last but one column) that $\sum_{i_L} (\tau^{23}_{i_L})^3 = 0$ [$= (3+1)\cdot ((-\frac{1}{2})^3 + (\frac{1}{2})^3))$], and that also $\sum_{i_R} (\tau^{23}_{i_R})^3 = 0$ [$= (3+1) \cdot ((\frac{1}{2})^3 + (-\frac{1}{2})^3)$].

From Table 15 one sees also (without calculating) that $\sum_{i_L} 3 \cdot (\tau^4_{i_L})^2 \cdot \tau^{23}_{i_L} = 0$, in particular $\sum_{i_L} 3 \cdot (\tau^4_{i_L})^2 \cdot \tau^{23}_{i_L} = 3.\{((\frac{1}{2})^2 \cdot (-\frac{1}{2} + \frac{1}{2}) + 3 \cdot (-\frac{1}{6})^2 \cdot (-\frac{1}{2} + \frac{1}{2})\}$, as well as that $\sum_{i_R} 3 \cdot (\tau^4_{i_R})^2 \cdot \tau^{23}_{i_R} = 0$ [$= 3 \cdot \{((-\frac{1}{2})^2 \cdot (\frac{1}{2} + (-\frac{1}{2})) + 3 \cdot (\frac{1}{6})^2 \cdot (\frac{1}{2} + (-\frac{1}{2}))\}$].

That the last term in Eq. (164) is zero for either the left or the right handed spinors can also easily be seen just by looking at Table 15 [or in details: $\sum_{i_L} 3 \cdot \tau^4_{i_L} \cdot (\tau^{23}_{i_L})^2 = 0 = 3 \cdot \{ (\frac{1}{2}((\frac{1}{2})^2 + (-\frac{1}{2})^2) + 3 \cdot (-\frac{1}{6})((\frac{1}{2})^2 + (-\frac{1}{2})^2))\}$, as well as that $\sum_{i_R} 3 \cdot \tau^4_{i_L} \cdot (\tau^{23}_{i_R})^2 = 0 = 3 \cdot \{(-\frac{1}{2}((\frac{1}{2})^2 + (-\frac{1}{2})^2) + 3 \cdot (\frac{1}{6})((\frac{1}{2})^2 + (-\frac{1}{2})^2))\}$].

Since all the members belong to one spinor representation, it is straightforwardly that all the triangle traces are zero, if the *standard model* groups are the subgroups of the orthogonal group $SO(13,1)$.

There is no need for a detailed calculations, since a look in Table 15 gives immediately the answer.

From only the *standard model* assumptions point of view the cancellation of the triangle anomalies does look miraculous. For our $\sum_{i_{L,R}} (Y_{i_{L,R}})^3$ one obtains for the left handed members: [$3 \cdot 2 \cdot (\frac{1}{6})^3 + 2 \cdot (-\frac{1}{2})^3 + 3 \cdot ((-\frac{2}{3})^3 + (\frac{1}{3})^3) + 1^3)$], and for the right handed members: [$3 \cdot ((\frac{2}{3})^3 + (-\frac{1}{3})^3) + (-1)^3) + 3 \cdot 2 \cdot (-\frac{1}{6})^3 + 2 \cdot (\frac{1}{2})^3$].

D Norms in Grassmann space and Clifford space

Let us define the integral over the Grassmann space [8] of two functions of the Grassmann coordinates $< \mathbf{B}|\theta >< \mathbf{C}|\theta >$, $< \mathbf{B}|\theta > = < \theta|\mathbf{B} >^\dagger$,

$$
< \mathbf{B}|\theta > = \sum_{k=0}^{d} b_{a_1 \ldots a_k} \theta^{a_1} \cdots \theta^{a_k} \,,
$$

by requiring

$$
\begin{aligned}
&\{d\theta^a, \theta^b\}_+ = 0 \,, \quad \int d\theta^a = 0 \,, \quad \int d\theta^a \theta^a = 1 \,, \\
&\int d^d\theta \; \theta^0 \theta^1 \cdots \theta^d = 1 \,, \\
&d^d\theta = d\theta^d \ldots d\theta^0 \,, \quad \omega = \prod_{k=0}^{d} (\frac{\partial}{\partial \theta_k} + \theta^k) \,,
\end{aligned}
\tag{165}
$$

with $\frac{\partial}{\partial \theta_a}\theta^c = \eta^{ac}$. We shall use the weight function [8] $\omega = \prod_{k=0}^{d}(\frac{\partial}{\partial \theta_k} + \theta^k)$ to define the scalar product in Grassmann space $<\mathbf{B}|\mathbf{C}>$

$$
\begin{aligned}
<\mathbf{B}|\mathbf{C}> &= \int d^d\theta^a \;\omega <\mathbf{B}|\theta><\theta|\mathbf{C}> \\
&= \sum_{k=0}^{d} \int b^*_{b_1\dots b_k} c_{b_1\dots b_k}\,. \qquad (166)
\end{aligned}
$$

To define norms in Clifford space Eq. (165) can be used as well.

E Expressions for scalar fields in term of $\omega_{s'ts}$ and $\tilde{\omega}_{abs}$

As presented in Sect. 6.2.2 the *spin-charge-family* theory offers the explanation for the appearance of the scalar higss, with the weak charge $\pm\frac{1}{2}$ and the hyper charge $\mp\frac{1}{2}$, and Yukawa couplings, as well as for the matter/antimatter asymmetry in the universe. There are namely spin connections of two kinds, ω_{abs} and $\tilde{\omega}_{abs}$, with the space index $s \geq 5$, which manifest in $d = (3+1)$ as scalars with the weak charge $\pm\frac{1}{2}$ and the hyper charge $\mp\frac{1}{2}$ and with the space index $s = (7,8)$, and there are also scalars which manifest with respect to the space index $t = (9,10,11,12,13,14)$ as colour triplets or antitriplets.

E.1 Scalar fields with the space index $s = (7,8)$

The scalar fields, responsible for masses of the family members and of the heavy bosons [16, 17] after gaining nonzero vacuum expectation values and triggering the electroweak break, are presented in the second line of Eq. (101). These scalar fields are included in the covariant derivatives of momenta as $-\frac{1}{2}$ $S^{s's"}\omega_{s's"s}$ $-\frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{abs}$, $s \in (7,8)$, $(a,b), \in (0,\dots,3),(5,\dots,8)$.

One can express the scalar fields carrying the quantum numbers of the subgroups of the family groups, expressed in terms of $\tilde{\omega}_{abs}$ (they contribute to mass matrices of quarks and leptons and to masses of the heavy bosons), if taking into account Eqs. (85, 86, 88),

$$
\begin{aligned}
\sum_{a,b} -\frac{1}{2}\tilde{S}^{ab}\,\tilde{\omega}_{abs} &= -(\vec{\tilde{\tau}}^{\tilde{1}}\,\vec{\tilde{A}}^{\tilde{1}}_s + \vec{\tilde{N}}_{\tilde{L}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s + \vec{\tilde{\tau}}^{\tilde{2}}\,\vec{\tilde{A}}^{\tilde{2}}_s + \vec{\tilde{N}}_{\tilde{R}}\,\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s)\,,\\
\vec{\tilde{A}}^{\tilde{1}}_s &= (\tilde{\omega}_{58s} - \tilde{\omega}_{67s},\,\tilde{\omega}_{57s} + \tilde{\omega}_{68s},\,\tilde{\omega}_{56s} - \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_s &= (\tilde{\omega}_{23s} + i\,\tilde{\omega}_{01s},\,\tilde{\omega}_{31s} + i\,\tilde{\omega}_{02s},\,\tilde{\omega}_{12s} + i\,\tilde{\omega}_{03s})\,,\\
\vec{\tilde{A}}^{\tilde{2}}_s &= (\tilde{\omega}_{58s} + \tilde{\omega}_{67s},\,\tilde{\omega}_{57s} - \tilde{\omega}_{68s},\,\tilde{\omega}_{56s} + \tilde{\omega}_{78s})\,,\\
\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_s &= (\tilde{\omega}_{23s} - i\,\tilde{\omega}_{01s},\,\tilde{\omega}_{31s} - i\,\tilde{\omega}_{02s},\,\tilde{\omega}_{12s} - i\,\tilde{\omega}_{03s})\,,\\
& \quad (s \in (7,8))\,. \qquad (167)
\end{aligned}
$$

Scalars, expressed in terms of ω_{abc} (contributing as well to the mass matrices of quarks and leptons and to masses of the heavy bosons) follow, if using Eqs. (86, 87, 88)

$$
\begin{aligned}
\sum_{s',s''} -\frac{1}{2} S^{s's''}\, \omega_{s's''s} &= -(g^{23}\, \tau^{23}\, A^{23}_{s} + g^{13}\, \tau^{13}\, A^{13}_{s} + g^{4}\, \tau^{4}\, A^{4}_{s})\,,\\
g^{13}\, \tau^{13}\, A^{13}_{s} + g^{23}\, \tau^{23}\, A^{23}_{s} + g^{4}\, \tau^{4}\, A^{4}_{s} &= g^{Q}\, Q A^{Q}_{s} + g^{Q'}\, Q' A^{Q'}_{s} + g^{Y'}\, Y'\, A^{Y'}_{s}\,,\\
A^{4}_{s} &= -(\omega_{9\,10\,s} + \omega_{11\,12\,s} + \omega_{13\,14\,s})\,,\\
A^{13}_{s} &= (\omega_{56s} - \omega_{78s})\,, \quad A^{23}_{s} = (\omega_{56s} + \omega_{78s})\,,\\
A^{Q}_{s} &= \sin\vartheta_1\, A^{13}_{s} + \cos\vartheta_1\, A^{Y}_{s}\,, \quad A^{Q'}_{s} = \cos\vartheta_1\, A^{13}_{s} - \sin\vartheta_1\, A^{Y}_{s}\,,\\
A^{Y'}_{s} &= \cos\vartheta_2\, A^{23}_{s} - \sin\vartheta_2\, A^{4}_{s}\,,\\
&\quad (s \in (7,8))\,. \qquad (168)
\end{aligned}
$$

Scalar fields from Eq. (167) interact with quarks and leptons and antiquarks and antileptons through the family quantum numbers, while those from Eq. (168) interact through the family members quantum numbers. In Eq. (168) the coupling constants are explicitly written in order to see the analogy with the gauge fields of the *standard model*.

Let be shown that the scalar fields $A^{Ai}_{\substack{78\\(\pm)}}$ are *triplets* as the gauge fields of the family quantum numbers ($\vec{\tilde{N}}_R$, $\vec{\tilde{N}}_L$, $\vec{\tilde{\tau}}^2$, $\vec{\tilde{\tau}}^1$; Eqs. (85, 86, 107)) or singlets as the gauge fields of $Q = \tau^{13} + Y$, $Q' = -\tan^2\vartheta_1 Y + \tau^{13}$ and $Y' = -\tan^2\vartheta_2 \tau^4 + \tau^{23}$, for $\tilde{A}^{N_L i}_{\substack{78\\(\pm)}}$ and for $A^{Q}_{\substack{78\\(\pm)}}$, taking into account Eq. (85) (where we replace S^{ab} by $\mathcal{S}^{ab}$) and Eq. (107), while recognizing that $\tilde{A}^{N_L \boxplus}_{\substack{78\\(\pm)}} = \tilde{A}^{N_L 1}_{\substack{78\\(\pm)}} \boxplus i\, \tilde{A}^{N_L 2}_{\substack{78\\(\pm)}}$, and that

$$
\begin{aligned}
\tilde{A}^{\tilde{N}_L \boxplus}_{\substack{78\\(\pm)}} &= \{(\tilde{\omega}_{23\substack{78\\(\pm)}} + i\,\tilde{\omega}_{01\substack{78\\(\pm)}}) \boxplus i\,(\tilde{\omega}_{31\substack{78\\(\pm)}} + i\,\tilde{\omega}_{02\substack{78\\(\pm)}})\}\,,\\
\tilde{A}^{\tilde{N}_L 3}_{\substack{78\\(\pm)}} &= (\tilde{\omega}_{12\substack{78\\(\pm)}} + i\,\tilde{\omega}_{03\substack{78\\(\pm)}})\,,\\
A^{Q}_{\substack{78\\(\pm)}} &= \omega_{56\substack{78\\(\pm)}} - (\omega_{9\,10\substack{78\\(\pm)}} + \omega_{11\,12\substack{78\\(\pm)}} + \omega_{13\,14\substack{78\\(\pm)}})\,,
\end{aligned}
$$

the scalar fields are chosen (with respect to the space index $s = (7,8)$) to be eigenstates of the weak charge, Eq. (109).

One finds

$$
\begin{aligned}
\tilde{N}^3_L\, \tilde{A}^{\tilde{N}_L \boxplus}_{\substack{78\\(\pm)}} &= \boxplus\, \tilde{A}^{\tilde{N}_L \boxplus}_{\substack{78\\(\pm)}}\,, \quad \tilde{N}^3_L\, \tilde{A}^{\tilde{N}_L 3}_{\substack{78\\(\pm)}} = 0\,,\\
Q\, A^{Q}_{\substack{78\\(\pm)}} &= 0\,, \qquad (169)
\end{aligned}
$$

taking into account $Q = \mathcal{S}^{56} + \tau^4 = \mathcal{S}^{56} - \frac{1}{3}(\mathcal{S}^{9\,10} + \mathcal{S}^{11\,12} + \mathcal{S}^{13\,14})$, and with τ^4 defined in Eq. (87), if replacing S^{ab} by $\mathcal{S}^{ab}$ from Eq. (107). Similarly one finds properties with respect to the Ai quantum numbers for all the scalar fields $A^{Ai}_{\substack{78\\(\pm)}}$.

E.2 Scalar fields with the space index $s = (9, \ldots, 14)$

Expressions for the vector gauge fields in terms of the spin connection fields and the vielbeins, which correspond to the colour charge ($\vec{A}^3_m$), the $SU(2)_{II}$ charge ($\vec{A}^2_m$), the weak $SU(2)_I$ charge ($\vec{A}^1_m$) and the $U(1)$ charge originating in $SO(6)$ ($\vec{A}^4_m$), can be found by taking into account Eqs. (86, 87). Equivalently one finds the vector gauge fields in the "tilde" sector, or one just uses the expressions from Eqs. (168, 167), if replacing the scalar index s with the vector index m.

The expression for $\sum_{tab} \gamma^t \; \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abt}$, used in Eq. (114) ($\tilde{S}^{ab}$ are the infinitesimal generators of either $\widetilde{SO}(3,1)$ or $\widetilde{SO}(4)$, while $\tilde{\omega}_{abt}$ belong to the corresponding gauge fields with $t = (9, \ldots, 14)$), and obtained by using Eqs. (85 - 89), are

$$\begin{aligned}
&\sum_{abt} \gamma^t \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{abt} = \sum_{+-tt'ab} \overset{tt'}{(\oplus)} \; \frac{1}{2} \tilde{S}^{ab} \tilde{\omega}_{ab \overset{tt'}{(\oplus)}} = \\
&\sum_{+-tt'} \overset{tt'}{(\oplus)} \; \{ \tilde{\tau}^{2+} \tilde{A}^{2+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{2-} \tilde{A}^{2-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{23} \tilde{A}^{23}_{\overset{tt'}{(\oplus)}} + \\
&\tilde{\tau}^{1+} \tilde{A}^{1+}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{1-} \tilde{A}^{1-}_{\overset{tt'}{(\oplus)}} + \tilde{\tau}^{13} \tilde{A}^{13}_{\overset{tt'}{(\oplus)}} + \\
&\tilde{N}^{+}_{R} \tilde{A}^{N_R+}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{R} \tilde{A}^{N_R-}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{R} \tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} + \\
&\tilde{N}^{+}_{L} \tilde{A}^{N_L+}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{-}_{L} \tilde{A}^{N_L-}_{\overset{tt'}{(\oplus)}} + \tilde{N}^{3}_{L} \tilde{A}^{N_L 3}_{\overset{tt'}{(\oplus)}} \} \,,
\end{aligned}$$

$$\begin{aligned}
\tilde{A}^{N_R \boxplus}_{\overset{tt'}{(\oplus)}} &= (\tilde{\omega}_{23 \overset{tt'}{(\oplus)}} - i \, \tilde{\omega}_{01 \overset{tt'}{(\oplus)}}) \boxminus i (\tilde{\omega}_{31 \overset{tt'}{(\oplus)}} - i \, \tilde{\omega}_{02 \overset{tt'}{(\oplus)}}) \,, \quad \tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{12 \overset{tt'}{(\oplus)}} - i \, \tilde{\omega}_{03 \overset{tt'}{(\oplus)}}) \,, \\
\tilde{A}^{N_L \boxplus}_{\overset{tt'}{(\oplus)}} &= (\tilde{\omega}_{23 \overset{tt'}{(\oplus)}} + i \, \tilde{\omega}_{01 \overset{tt'}{(\oplus)}}) \boxminus i (\tilde{\omega}_{31 \overset{tt'}{(\oplus)}} + i \, \tilde{\omega}_{02 \overset{tt'}{(\oplus)}}) \,, \quad \tilde{A}^{N_R 3}_{\overset{tt'}{(\oplus)}} = (\tilde{\omega}_{12 \overset{tt'}{(\oplus)}} + i \, \tilde{\omega}_{03 \overset{tt'}{(\oplus)}}) \,.
\end{aligned} \tag{170}$$

The term $\sum_{tt't''} \gamma^t \; \frac{1}{2} S^{t't"} \, \omega_{t't"t}$ in Eq. (101) can be rewritten with respect to the generators $S^{t't"}$ and the corresponding gauge fields $\omega_{s's"t}$ as one colour octet scalar field and one $U(1)_{II}$ singlet scalar field (Eq. 87)

$$\begin{aligned}
\sum_{tt''t'''} \gamma^t \, \frac{1}{2} S^{t"t'"} \, \omega_{t"t'"t} &= \sum_{+,-} \sum_{(t\,t')} \overset{tt'}{(\oplus)} \; \{ \vec{\tau}^3 \cdot \vec{A}^3_{\overset{tt'}{(\oplus)}} + \tau^4 \cdot A^4_{\overset{tt'}{(\oplus)}} \} \,, \\
(t\,t') &\in ((9\,10), 11\,12), 13\,14)) \,.
\end{aligned} \tag{171}$$

F Handedness in Grassmann and Clifford space

The handedness $\Gamma^{(d)}$ is one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group S^{ab}, defined as

$$\Gamma^{(d)} \;=\; \alpha \, \varepsilon_{a_1 a_2 \ldots a_{d-1} a_d} \, S^{a_1 a_2} \cdot S^{a_3 a_4} \cdots S^{a_{d-1} a_d} \,, \tag{172}$$

with α, which is chosen so that $\Gamma^{(d)} = \pm 1$.

In the Grassmann case S^{ab} is defined in Eq. (19), while in the Clifford case Eq. (172) simplifies, if we take into account that $S^{ab}|_{a \neq b} = \frac{i}{2} \gamma^a \gamma^b$ and $\tilde{S}^{ab}|_{a \neq b} = \frac{i}{2} \tilde{\gamma}^a \tilde{\gamma}^b$, as follows

$$\Gamma^{(d)} : \;=\; (i)^{d/2} \qquad \prod_a \quad (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if} \quad d = 2n \,. \tag{173}$$

G Discrete symmetries of vector and scalar gauge fields in $d = (3+1)$ from $d = ((d-1)+1)$ [125]

We follow here Ref. [125]. In Sect. 3.3.5 our definition [125] of the discrete symmetry operators $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathbb{T}_{\mathcal{N}}$, Eq. (72), is presented and their application on the second quantized states of fermions and antifermions, living in $d = ((d-1)+1)$ spaces and manifesting in $d = (3+1)$ as the observed fermions and antifermions, is discussed.

In this App. G we apply the discrete symmetry operators $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathbb{T}_{\mathcal{N}}$, Eq. (72), on vielbeins and spin connection fields, leaving in $d = ((d-1)+1)$ spaces and manifesting $d = (3+1)$ as the vector and scalar gauge fields (as it is in the Kaluza-Klein theories) [15]. We again follow Ref. [125].

Let us treat the simple starting Lagrange density for a spinor in $d = ((d-1)+1)$ dimensional space, which carries, like in the Kaluza-Klein theories, the spins and no charges, Eqs. (100, 101), Sect. 6

$$\begin{aligned}
\mathcal{L}_{\{} &= \frac{1}{2} E\, \Psi^{\dagger}\, \gamma^0\, \gamma^a\, p_{0a}\, \Psi + \, h.c.\,, \\
p_{0a} &= f^{\alpha}_{a} p_{\alpha} + \frac{1}{2E} \{p_{\alpha}, f^{\alpha}_{a} E\}_{-} - \frac{1}{2} S^{cd}\, f^{\alpha}_{a}\, \omega_{cd\alpha}\,.
\end{aligned} \tag{174}$$

f^{α}_{a} are vielbein and $\omega_{cd\alpha}$ spin connection fields, the gauge fields of p^a and S^{ab}, respectively. The families quantum numbers, determined by $\tilde{S}^{ab}$, commute, Eq. (29), with in Sect. 3.3.5 defined discrete symmetries operators [56].

Let the vielbeins and spin connections be responsible for the break of symmetry of $M^{(d-1)+1}$ into $M^{3+1}\times\, M^{d-4}$ so that the manifold M^{d-4} is (almost) compactified [127, 128, 129] and let the spinors (fermions) manifest in $d = (3+1)$ the ordinary spin and the charges and the families [57]. Looking for the subgroups (denoted by A) of the $SO((d-1)+1)$ group and assuming no gravity in $d = (3+1)$, the Lagrange density of Eq. (174) can be rewritten in a more familiar shape

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} E\, \psi^{\dagger}\, \gamma^0\, (\gamma^m\, p_{0m} + \gamma^s\, p_{0s})\, \psi\, + h.c.\,, \\
p_{0m} &= p_m - \sum_{A} \vec{\tau}^{A}\, \vec{A}^{A}_{m}\,, \\
p_{0s} &= f^{\sigma}_{s}\, p_{\sigma} + \frac{1}{2E} \{p_{\sigma}, f^{\sigma}_{s} E\}_{-} - \sum_{B} \vec{\tau}^{B}\, \vec{A}^{B}_{s}\,,
\end{aligned} \tag{175}$$

with $m = (0,1,2,3)$, $s = (5,6,\dots,d)$ and $\tau^{Ai} = \sum_{st} c^{Ai}{}_{st}\, S^{st}$, $\sum_A \vec{\tau}^A\, \vec{A}^A_m = \frac{1}{2} \sum_{st} S^{st}\, \omega_{stm}$, $\sum_A \vec{\tau}^A\, \vec{A}^A_s = \frac{1}{2} \sum_{st} S^{st}\, f^{\sigma}_{s} \omega_{st\sigma}$.

Taking into account Eqs. (86, 87) (and recognizing that all $\tau^{Ai} = \sum_{A,i} c^{Ai}{}_{st}\, S^{st}$ include only $S^{st} = \frac{i}{2}\gamma^s\gamma^t$ with either γ^s or γ^t imaginary and the remaining one is real, Eq. (18)) one easily finds that

$$\begin{aligned}
\mathbb{C}_{\mathcal{N}}\; \tau^{Ai}\; \mathbb{C}^{-1}_{\mathcal{N}} &= -\tau^{Ai}\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; \tau^{Ai}\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} &= -\tau^{Ai}\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\; \tau^{Bi}\, (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= (-\tau^{Bi})\,.
\end{aligned} \tag{176}$$

Taking into account how do the vector gauge fields transform according to prescription in Sect. 3.3.5, Eq. (72), one further finds

$$\begin{aligned}
\mathbb{C}_{\mathcal{N}}\; A^{Ai}_{m}(x^0, \vec{x}_3)\; \mathbb{C}^{-1}_{\mathcal{N}} &= -A^{Ai}_{m}(x^0, \vec{x}_3)\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\; A^{Ai}_{m}(x^0, \vec{x}_3)\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}})^{-1} &= -A^{Ai\,m}(x^0, -\vec{x}_3)\,, \\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\; \tau^{Ai}\, A^{Ai}_{m}(x)\; (\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= (-\tau^{Ai})\, (-A^{Ai*}_{m}(-x))\,,
\end{aligned} \tag{177}$$

for τ^{Ai} from the Cartan subalgebra for each A. It is always true that $\tau^{Ai}\, A^{Ai}_{m}$ transforms either to $(-\tau^{Ai})\, (-A^{Ai}_{m})$ or to $\tau^{Ai}\, A^{Ai}_{m}$, for each Ai, all in agreement with the standard knowledge for the gauge vector fields and charges in $d = (3+1)$ [134].

[56] The discrete symmetry operators for the "Grassmann fermions" are presented in Refs. [21, 22, 25].

[57] In Refs. [127, 128, 129] it is demonstrated on the toy model how such an almost compactification could occur.

One can check that

$$
\begin{aligned}
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\,\gamma^{a}(\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= \gamma^{a}\,,\\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}S^{ab}\,(\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= -S^{ab}\,,\\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\,f^{\alpha}{}_{a}(x)\,p_{\alpha}\,(\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= f^{\alpha\,*}{}_{a}(-x)\,p_{\alpha}\,,\\
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\omega_{abc}(x)\,(\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= -\omega^{*}_{abc}(-x)\,,\\
\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\,\tilde{\omega}_{abc}(x)\,(\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} &= -\tilde{\omega}^{*}_{abc}(-x)\,.
\end{aligned}
\tag{178}
$$

From Eqs. (176, 177, 178) it follows that

$$
\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}\,\tau^{Ai}\,A^{Ai}_{s}(x)(\mathbb{C}_{\mathcal{N}}\mathcal{P}^{(d-1)}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}})^{-1} = (-\tau^{Ai})\,(-A^{Ai*}_{s})(-x))\,,
$$

concerning in $d=(3+1)$ the gauge scalar fields, which determine in the *spin-charge-family* theory massless and massive solutions for fermions (quarks and leptons and antiquarks and antileptons) and, and when some of them gaining constant values — the one with the space index $s=(7,8)$, contribute not only to masses of spinors but also to those gauge fields, to which they couple.

The Hermiticity requirement for the Lagrange density (Eq. (175)), $\mathcal{L}^{\dagger}=\mathcal{L}$, leads to

$$
\omega^{*}_{abc}(x) \;=\; (\mp)\,\omega_{abc}(x)\,;\,(-)\,\mathrm{if}\,a=c\,\mathrm{or}\,b=c\,,\,(+)\,\mathrm{otherwise}\,,
\tag{179}
$$

which is to be taken into account together with the $\mathbb{C}_{\mathcal{N}}\mathcal{P}_{\mathcal{N}}\mathcal{T}_{\mathcal{N}}$ invariance.

For additional arguments about the generality of our proposal for the discrete symmetry operators $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}^{(d-1)}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{N}}$ the reader is invited to follow Ref. [125].

H Symmetries of mass matrices on tree level and beyond manifesting $SU(2)\times SU(2)\times U(1)$ symmetry in *spin-charge-family* theory [96, 97]

Let states of massless quarks and leptons for two times four families — presented in Table 5 with creation operators, creating the internal part of the right handed u-quark and right handed ν-leptons if applying on the vacuum state $|\psi_{oc}>$ — be denoted by ψ^{α}_{f}, $\alpha=(u,d,\nu,e)_{L,R}$ and f is the family quantum number. Table 7 represents all the members of the first family of Table 5. All the members of any of two times four families follows by the application of S^{ab}.

The scalar gauge fields ($\vec{\tilde{\tau}}^{1}\cdot\vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$, $\vec{\tilde{N}}_{L}\cdot\vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$) with the space index $s=(7,8)$ transform the right handed members into the left handed ones, and causing as well transformations among families of the first group.

The scalar gauge fields ($\vec{\tilde{\tau}}^{2}\cdot\vec{\tilde{A}}^{\tilde{2}}_{\substack{78\\(\pm)}}$, $\vec{\tilde{N}}_{R}\cdot\vec{\tilde{A}}^{\tilde{N}_{\tilde{R}}}_{\substack{78\\(\pm)}}$) behave equivalently on the second group of the four families.

The scalar gauge fields $(Q\cdot A^{Q}_{\substack{78\\(\pm)}}, Y\cdot A^{Q]Y}_{\substack{78\\(\pm)}}, \tau^{4}\cdot A^{4}_{\substack{78\\(\pm)}})$ "see" only the family members quantum numbers, independently of the family quantum number. They are diagonal and apply on both groups of four families equivalently.

Although the two groups of scalar fields might have different properties, different different masses and different coupling constants, they bring to the two groups of four families the equivalent $SU(2)\times SU(2)\times U(1)$ symmetry [58]

[58]The infinitesimal generators $\tilde{N}^{i}_{L}\,,\,i=(1,2,3)$ and $\tilde{N}^{i}_{R}\,,\,i=(1,2,3)$ determine the algebra of the two invariant subgroups of the $\widetilde{SO}(1,3)$ group, while $\tilde{\tau}^{1i}\,,\,i=(1,2,3)$ and $\tilde{\tau}^{2i}\,,\,i=(1,2,3)$ determine the two invariant subgroups of the $\widetilde{SO}(4)$ group. The first four families of Table 5, carrying the quantum numbers of $\tilde{N}^{i}_{L}$ and $\tilde{\tau}^{1i}$, are influenced by the corresponding scalar gauge fields, the second four families are influenced by the scalar fields of $\tilde{N}^{i}_{R}$ and $\tilde{\tau}^{2i}$.

Let us therefore treat only one group of four families, the first one, interacting with ($\vec{\tilde{\tau}}^{1} \cdot \vec{\tilde{A}}^{\tilde{1}}_{\substack{78\\(\pm)}}$, $\vec{\tilde{N}}_{L} \cdot \vec{\tilde{A}}^{\tilde{N}_{\tilde{L}}}_{\substack{78\\(\pm)}}$). The operators

$$\begin{aligned} \tilde{N}^{i}_{L}\,,\; i=(1,2,3)\,, &\qquad \tau^{1i}\,,\; i=(1,2,3)\,,\\ \{\tilde{N}^{i}_{L},\tilde{N}^{j}_{L}\}_{-} = i\,\varepsilon^{ijk}\tilde{N}^{k}_{L}\,,\;\; \{\tilde{\tau}^{1i},\tilde{\tau}^{1j}\}_{-} &= i\,\varepsilon^{ijk}\tilde{\tau}^{1k}\,,\{\tilde{N}^{i}_{L},\tilde{\tau}^{1j}\}_{-}=0\,, \end{aligned} \tag{180}$$

where ε^{ijk} is the totally antisymmetric tensor, transform the basic vectors ψ^{α}_{i}, into one another as follows

$$\begin{aligned} \tilde{N}^{3}_{L}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= \frac{1}{2}\,(-\psi^{\alpha}_{1},\psi^{\alpha}_{2},-\psi^{\alpha}_{3},\psi^{\alpha}_{4})\,,\\ \tilde{N}^{+}_{L}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= (\psi^{\alpha}_{2},\;\;0,\psi^{\alpha}_{4},\;\;0)\,,\\ \tilde{N}^{-}_{L}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= (0\;\;,\psi^{\alpha}_{1},\;\;0,\psi^{\alpha}_{3})\,,\\ \tilde{\tau}^{13}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= \frac{1}{2}\,(-\psi^{\alpha}_{1},-\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4})\,,\\ \tilde{\tau}^{1+}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= (\psi^{\alpha}_{3},\psi^{\alpha}_{4},\;\;0,\;\;0)\,,\\ \tilde{\tau}^{1-}\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) &= (\;\;0,\;\;0,\psi^{\alpha}_{1},\psi^{\alpha}_{2})\,. \end{aligned} \tag{181}$$

The three $U(1)$ operators (Q, Y and τ^{4}) (or any three superposition of them, like Q, Q', Y') commute with the family operators $\vec{\tilde{N}}_{L}$ and $\vec{\tilde{\tau}}^{1}$, distinguishing only among family members α.

$$\begin{aligned} &\{\tilde{N}^{i}_{L}\,,(Q,Q',Y')\}_{-} = (0,0,0)\,,\\ &\{\tilde{\tau}^{1i}\,,(Q,Q',Y')\}_{-} = (0,0,0)\,,\\ &(Q\,,Y\,,\tau^{4})\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4}) = (Q^{\alpha}\,,Y^{\alpha}\,,\tau^{4\alpha})\,(\psi^{\alpha}_{1},\psi^{\alpha}_{2},\psi^{\alpha}_{3},\psi^{\alpha}_{4})\,, \end{aligned} \tag{182}$$

giving the same eigenvalues for all the families.

The nonzero vacuum expectation values of the scalar gauge fields of $\tilde{N}^{i}_{L}$ ($\tilde{A}^{\tilde{N}_{L}i}_{s} = \tilde{c}^{\tilde{N}_{L}i}{}_{\tilde{m}\tilde{n}}\,\tilde{\omega}^{\tilde{m}\tilde{n}}{}_{s}$, $(\tilde{m},\tilde{n})=(0,1,2,3)$), of $\tilde{\tau}^{1i}$ ($\tilde{A}^{\tilde{1}i}_{s} = \tilde{c}^{\tilde{1}i}{}_{\tilde{t}\tilde{t}'}\,\tilde{\omega}^{\tilde{t}\tilde{t}'}{}_{s}$, $(\tilde{t},\tilde{t}')=(5,6,7,8)$) and of the three singlet scalar gauge fields of (Q, Y and τ^{4}), which all are superposition of $\omega_{t,t',s}$ ($A^{Q}_{s} = c^{Q}{}_{tt'}\,\omega^{tt'}{}_{s}$, $A^{Y}_{s} = c^{Y}{}_{tt'}\,\omega^{tt'}{}_{s}$ and $A^{4}_{s} = c^{Y'}{}_{tt'}\,\omega^{tt'}{}_{s}$, $s=(7,8)$, $(t,t')=(5,6,7,8)$), determine on the tree level, together with the corresponding coupling constants, the $SU(2)\times SU(2)\times U(1)$ symmetry and the strength of the mass matrix of each family member α, Eq. (111). In loop corrections the scalar fields $\tilde{A}^{\tilde{N}_{L}i}_{s}$, $\tilde{A}^{\tilde{1}i}_{s}$, $A^{Q}_{s}, A^{Y}_{s}, A^{4}_{s}$ contribute to all the matrix elements, keeping the symmetry unchanged, Eq. (111). The twice two zeros on the tree level obtain in loop corrections the value b.

One easily checks that a change of the phases of the left and the right handed members, there are $(2n-1)$ possibilities, causes changes in phases of matrix elements in Eq. (111).

All the scalars are doublets with respect to the weak charge, contributing to the weak and the hyper charge of the fermions so that they transform the right handed members into the left handed onces, Sect. 6.2.2, what is in the *standard model* just required.

H.1 Properties of non Hermitian mass matrices

This pedagogic presentation of well known properties of non Hermitian matrices can be found in many textbooks. We repeat this topic here only to make our discussions transparent.

Let us take a non Hermitian mass matrix M^{α} as it follows from the *spin-charge-family* theory, α denotes a family member.

We always can diagonalize a non Hermitian M^α with two unitary matrices, S^α ($S^{\alpha\,\dagger} S^\alpha = I$) and T^α ($T^{\alpha\,\dagger} T^\alpha = I$)

$$S^{\alpha\,\dagger} M^\alpha T^\alpha \;=\; \mathbf{M}^\alpha_d = (m^\alpha_1 \ldots m^\alpha_i \ldots m^\alpha_n). \tag{183}$$

The proof is added below.

Changing phases of the basic states, those of the left handed one and those of the right handed one, the new unitary matrices $S'^\alpha = S^\alpha F_{\alpha S}$ and $T'^\alpha = T^\alpha F_{\alpha T}$ change the phase of the elements of diagonalized mass matrices $\mathbf{M}^\alpha_d$

$$\begin{aligned} S'^{\alpha\,\dagger} M^\alpha T'^\alpha &= F^\dagger_{\alpha S} \mathbf{M}^\alpha_d F_{\alpha T} = \\ &\quad diag(m^\alpha_1 e^{i(\phi^{\alpha S}_1 - \phi^{\alpha T}_1)} \ldots m^\alpha_i e^{i(\phi^{\alpha S}_i - \phi^{\alpha T}_i)}, \ldots m^\alpha_n e^{i(\phi^{\alpha S}_n - \phi^{\alpha T}_n)}), \\ F_{\alpha S} &= diag(e^{-i\phi^{\alpha S}_1}, \ldots, e^{-i\phi^{\alpha S}_i}, \ldots, e^{-i\phi^{\alpha S}_n}), \\ F_{\alpha T} &= diag(e^{-i\phi^{\alpha T}_1}, \ldots, e^{-i\phi^{\alpha T}_i}, \ldots, e^{-i\phi^{\alpha T}_n}). \end{aligned} \tag{184}$$

In the case that the mass matrix is Hermitian T^α can be replaced by S^α, but only up to phases originating in the phases of the two basis, the left handed one and the right handed one, since they remain independent.

The non Hermitian mass matrices can be diagonalized in two ways, that is either one diagonalizes $M^\alpha M^{\alpha\,\dagger}$ or $M^{\alpha\dagger} M^\alpha$

$$\begin{aligned} (S^{\alpha\dagger} M^\alpha T^\alpha)(S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger &= S^{\alpha\dagger} M^\alpha M^{\alpha\,\dagger} S^\alpha = \mathbf{M}^{\alpha 2}_{dS}, \\ (S^{\alpha\dagger} M^\alpha T^\alpha)^\dagger (S^{\alpha\dagger} M^\alpha T^\alpha) &= T^{\alpha\dagger} M^{\alpha\,\dagger} M^\alpha T^\alpha = \mathbf{M}^{\alpha 2}_{dT}, \\ \mathbf{M}^{\alpha\,\dagger}_{dS} &= \mathbf{M}^\alpha_{dS}, \quad \mathbf{M}^{\alpha\,\dagger}_{dT} = \mathbf{M}^\alpha_{dT}. \end{aligned} \tag{185}$$

The proof that $\mathbf{M}^\alpha_{dS} = \mathbf{M}^\alpha_{dT}$ proceeds as follows. Let us define two Hermitian (H^α_S, H^α_T) and two unitary matrices (U^α_S, H^α_T)

$$\begin{aligned} H^\alpha_S &= S^\alpha \mathbf{M}^\alpha_{dS} S^{\alpha\,\dagger}, \qquad H^\alpha_T = T^\alpha \mathbf{M}^{\alpha\dagger}_{dT} T^{\alpha\,\dagger}, \\ U^\alpha_S &= H^{\alpha -1}_S M^\alpha, \qquad U^\alpha_T = H^{\alpha -1}_T M^{\alpha\,\dagger}, \end{aligned} \tag{186}$$

It is easy to show that $H^{\alpha\,\dagger}_S = H^\alpha_S$, $H^{\alpha\,\dagger}_T = H^\alpha_T$, $U^\alpha_S U^{\alpha\,\dagger}_S = I$ and $U^\alpha_T U^{\alpha\,\dagger}_T = I$. Then it follows

$$\begin{aligned} S^{\alpha\dagger} H^\alpha_S S^\alpha &= \mathbf{M}^\alpha_{dS} = \mathbf{M}^{\alpha\,\dagger}_{dS} = S^{\alpha\dagger} M^\alpha U^{\alpha\,-1}_S S^\alpha = S^{\alpha\dagger} M^\alpha T^\alpha, \\ T^{\alpha\dagger} H^\alpha_T T^\alpha &= \mathbf{M}^\alpha_{dT} = \mathbf{M}^{\alpha\,\dagger}_{dT} = T^{\alpha\dagger} M^{\alpha\,\dagger} U^{\alpha\,-1}_T T^\alpha = T^{\alpha\dagger} M^{\alpha\dagger} S^\alpha, \end{aligned} \tag{187}$$

where we recognized $U^{\alpha\,-1}_S S^\alpha = T^\alpha$ and $U^{\alpha\,-1}_T T^\alpha = S^\alpha$. Taking into account Eq. (184) the starting basis can be chosen so, that all diagonal masses are real and positive.

I Statements and proofs of statements [23]

There are two kinds of the Clifford algebra objects, γ^a's and $\tilde{\gamma}^a$'s, both expressible with the Grassmann algebra objects, θ^a's and their derivatives $\frac{\partial}{\partial\theta_a}$'s. In Grassmann d-dimensional space there are d anticommuting operators θ^a, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, .., d)$, and d anticommuting derivatives with respect to θ^a, $\frac{\partial}{\partial\theta_a}$, $\{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of θ^a and another half corresponding superposition of $\frac{\partial}{\partial\theta_a}$.

$$\begin{aligned} \{\theta^a, \theta^b\}_+ = 0\,, &\qquad \{\frac{\partial}{\partial\theta_a}, \frac{\partial}{\partial\theta_b}\}_+ = 0\,, \\ \{\theta_a, \frac{\partial}{\partial\theta_b}\}_+ &= \delta_{ab}\,, (a,b) = (0,1,2,3,5,\cdots,d)\,. \end{aligned} \tag{188}$$

Defining [20]

$$(\theta^a)^\dagger = \eta^{aa}\frac{\partial}{\partial\theta_a}\,, \quad \text{it follows}\,, \quad (\frac{\partial}{\partial\theta_a})^\dagger = \eta^{aa}\theta^a\,. \tag{189}$$

The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$ is assumed.

One can define new operators, expressed with θ^a's and $\frac{\partial}{\partial\theta_a}$'s

$$\begin{aligned}
\gamma^a &= (\theta^a + \frac{\partial}{\partial\theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})\,,\\
\theta^a &= \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial\theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,.
\end{aligned} \tag{190}$$

One can make 2^d products of superpositions of γ^a's and 2^d products of superposition of $\tilde{\gamma}^a$'s, all together $2\cdot 2^d$ objects.

Statement 1. γ^a's and $\tilde{\gamma}^a$'s define two independent Clfford algebras.

To prove this statement one only needs to take into account Eqs. (189, 16).

$$\begin{aligned}
\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\
\{\gamma^a, \tilde{\gamma}^b\}_+ &= 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\
(\gamma^a)^\dagger &= \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,.
\end{aligned} \tag{191}$$

Eq. (191) demonstrates that these two Clifford algebra objects obviously define two independent "basis vectors". Either γ^a's or $\tilde{\gamma}^a$'s are, up to a sign, self adjoint operators.

The generators of the Lorentz transformations in the Grassmann algebra space are defined as follows

$$\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})\,, \quad (\mathbf{S}^{ab})^\dagger = \eta^{aa}\eta^{bb}\mathbf{S}^{ab}\,. \tag{192}$$

Statement 2: The sum of the generators of the Lorentz transformations in each of the two Clifford algebra spaces, $S^{ab} = i\frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and $\tilde{S}^{ab} = i\frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, respectively, are equal to the generators of the Lorentz transformation in the Grassmann algebra space $\mathbf{S}^{\dashv\lfloor} = \mathcal{S}^{\dashv\lfloor} + \tilde{\mathcal{S}}^{\dashv\lfloor}$.

To prove this statement one only has to express in the sum $S^{ab} + \tilde{S}^{ab}$ first S^{ab} with γ^a and γ^b and $\tilde{S}^{ab}$ with $\tilde{\gamma}^a$ and $\tilde{\gamma}^b$, and then γ^a and $\tilde{\gamma}^a$ with θ^a and $\frac{\partial}{\partial\theta_a}$, using Eq. (16).

One obtains that $S^{ab} = i\frac{1}{2}(\frac{\partial}{\partial\theta_a}\theta^b + \theta^a\frac{\partial}{\partial\theta_b})$ and $\tilde{S}^{ab} = i\frac{1}{2}(\theta^a\frac{\partial}{\partial\theta_b} + \frac{\partial}{\partial\theta_a}\theta^b)$, leading to $S^{ab} + \tilde{S}^{ab} = \mathbf{S}^{ab}$.

We conclude

$$\begin{aligned}
S^{ab} &= \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)\,, \quad \tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)\,,\\
\mathbf{S}^{ab} &= S^{ab} + \tilde{S}^{ab}\,, \quad \{S^{ab}, \tilde{S}^{ab}\}_- = 0\,,\\
\{S^{ab}, \gamma^c\}_- &= i(\eta^{bc}\gamma^a - \eta^{ac}\gamma^b)\,,\\
\{\tilde{S}^{ab}, \tilde{\gamma}^c\}_- &= i(\eta^{bc}\tilde{\gamma}^a - \eta^{ac}\tilde{\gamma}^b)\,,\\
\{S^{ab}, \tilde{\gamma}^c\}_- &= 0\,, \quad \{\tilde{S}^{ab}, \gamma^c\}_- = 0\,.
\end{aligned} \tag{193}$$

Statement 2a: The eigenvectors of the operators $\mathbf{S}^{ab} = i\,(\theta^a\frac{\partial}{\partial\theta_b} - \theta^b\frac{\partial}{\partial\theta_a})$ can be written as follows

$$\begin{aligned}
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b) &= k\,\frac{1}{\sqrt{2}}(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)\,,\\
\mathbf{S}^{ab}\,\frac{1}{\sqrt{2}}\,(1 + \frac{i}{k}\theta^a\theta^b) &= 0\,,
\end{aligned} \tag{194}$$

while the corresponding eigenvectors of S^{ab} and $\tilde{S}^{ab}$ in each of the two spaces are

$$\begin{aligned}
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) &= \frac{k}{2}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,,\\
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) &= \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b) &= \frac{k}{2}\frac{1}{2}(\tilde{\gamma}^a + \frac{\eta^{aa}}{ik}\tilde{\gamma}^b)\,,\\
\tilde{S}^{ab}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b) &= \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\tilde{\gamma}^a\tilde{\gamma}^b)\,.
\end{aligned} \tag{195}$$

with $k^2 = \eta^{aa}\eta^{bb}$.

The Eq. (20) can be proven by applying $\mathbf{S}^{ab}$ on $\frac{1}{\sqrt{2}}\,(\theta^a + \frac{\eta^{aa}}{ik}\theta^b)$, leading to
$i\frac{1}{\sqrt{2}}(-\theta^b\eta^{aa} + \frac{\eta^{aa}\eta^{bb}}{ik}\theta^a) = \frac{ik^2}{ik\sqrt{2}}(\theta^a - \frac{\eta^{aa}ik}{k\cdot 2})$.

The application of $\mathbf{S}^{ab}$ on either a constant or on a constant$\cdot\theta^a\theta^b$ gives 0. In all these cases it is assumed that $a \neq b$.

The proof of Eq.(195) goes similarly, again $a \neq b$ is assumed:
$\frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\frac{1}{2}(-\eta^{aa}\gamma^b + \frac{\eta^{aa}\eta^{bb}}{ik}\gamma^a) = \frac{k}{2}\frac{1}{2}(\gamma^a - \eta^{aa}\frac{i}{k}\gamma^b)$.
$\frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{i}{2}\frac{1}{2}(\gamma^a\gamma^b - \frac{i}{k}\eta^{aa}\eta^{bb}) = \frac{k}{2}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)$.

Replacing S^{ab} with $\tilde{S}^{ab}$, and γ^a's with $\tilde{\gamma}^a$'s goes through the same steps.

Statement 2b: The members of any irreducible representation of S^{ab} follow from the starting one by the application of S^{cd}, which do not belong to the Cartan subalgebra of the Lorentz algebra.

The proof follows if we apply γ^a on nilpotents and projectors, since $S^{ab} = \frac{i}{2}\gamma^a\gamma^b$.

$$\gamma^a \stackrel{ab}{(k)}= \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)}= -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]}=\stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]}= -ik\eta^{aa} \stackrel{ab}{(-k)}\;. \tag{196}$$

Correspondingly, any transformation on "basis vectors" of the kind, which do not change sign as is required in Eq. (196), lead to another irreducible representations.

Statement 3: Postulating that $\tilde{\gamma}^a$'s operate on γ^a's as follows [62, 8, 17, 19, 20]

$$\tilde{\gamma}^a B \;=\; (-)^B\, i\, B\gamma^a\,,$$

with $(-)^B = -1$, if B is (a function of) an odd product of γ^a's, otherwise $(-)^B = 1$ [62], the reduction of the Clifford space and correspondingly also the reduction of the Grassmann space follows.

Eq. (38) requires

$$\begin{aligned}
&[\tilde{\gamma}^a(a_0 + a_{bc}\gamma^b\gamma^c + a_{bcde}\gamma^b\gamma^c\gamma^d\gamma^e + \cdots) = i(a_0 + a_{bc}\gamma^b\gamma^c + a_{bcde}\gamma^b\gamma^c\gamma^d\gamma^e + \cdots)\gamma^a]|\psi_{oc}>\,,\\
&[\tilde{\gamma}^a(a_b\gamma^b + a_{bcd}\gamma^b\gamma^c\gamma^d + \cdots) = -i(a_b\gamma^b + a_{bcd}\gamma^b\gamma^c\gamma^d + \cdots)\gamma^a]|\psi_{oc}>\,,
\end{aligned}$$

To prove *Statement 3.* let us evaluate what does Eq. (38) require when we use Eq.(16)on $|\psi_{oc}>$: $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ and $\tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial\theta_a})$, with $|\psi_{oc}>$ expressed as well with θ^a's and $\frac{\partial}{\partial\theta_a}$.

Let us point out that $|\psi_{oc}>$, expressed in terms of θ^a's and $\frac{\partial}{\partial\theta_a}$'s is an even function of θ^a's, $|\psi_{oc}>= (1 + \theta^a\theta^b + \cdots)$, while $\frac{\partial}{\partial\theta_a}$, applying on identity, gives zero.

The proof is needed for any even and any odd summand of B, appearing in Eq. (38) and for an arbitrary $|\psi_{oc}>$.

Let us start with $[\tilde{\gamma}^a a_0 = i a_0\gamma^a]|\psi_{oc}>$, with $|\psi_{oc}>= (1 + \theta^a\theta^b + \cdots)$ and a_0 an arbitrary constant. This relation requires that $[i(\theta^a - \frac{\partial}{\partial\theta_a}) = i(\theta^a + \frac{\partial}{\partial\theta_a})]|\psi_{oc}>$, leading to $-2i\frac{\partial}{\partial\theta_a}|\psi_{oc}>\Rightarrow 0,\ \forall\frac{\partial}{\partial\theta_a}$. This last relation can only be true if $\frac{\partial}{\partial\theta_a}|\psi_{oc}>\Rightarrow 0,\ \forall\frac{\partial}{\partial\theta_a}$.

Evaluating $[\tilde{\gamma}^a a_{bc}\gamma^b\gamma^c = i a_{bc}\gamma^b\gamma^c\gamma^a]|\psi_{oc}>$ we end up again with the requirement $\frac{\partial}{\partial\theta_a}|\psi_{oc}>\Rightarrow 0,\ \forall\frac{\partial}{\partial\theta_a}$. Applying $\tilde{\gamma}^a$ on any even products of γ^a's we end up with the same requirement $\frac{\partial}{\partial\theta_a}|\psi_{oc}>\Rightarrow 0,\ \forall\frac{\partial}{\partial\theta_a}$.

Application of $\tilde{\gamma}^a$ on any odd products of γ^a's, while $|\psi_{oc}>= (1+\theta^a\theta^b+\cdots)$,

$$[\tilde{\gamma}^a(a_b\gamma^b + a_{bcd}\gamma^b\gamma^c\gamma^d+\cdots) = -i(a_b\gamma^b + a_{bcd}\gamma^b\gamma^c\gamma^d+\cdots)\gamma^a]|\psi_{oc}>\,, \tag{197}$$

it follows again, after expressing $\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial\theta_a})$ and $\gamma^a = (\theta^a + \frac{\partial}{\partial\theta_a})$ into Eq. 197, that $\frac{\partial}{\partial\theta_a}|\psi_{oc}>\Rightarrow 0$ is the only solution, leading to .

$$\theta^a \Rightarrow \gamma^a\,,$$

which does not mean that θ^a is equal to γ^a but rather that the whole Grassmann algebra reduces to only one of the two Clifford algebras, the one, in which the "basis vectors" are superposition of products of (odd when describing fermions) number of γ^a's. It also does not mean that θ^a's are equal to $i\tilde{\gamma}^a$'s, since the application of $\tilde{\gamma}^a$'s depend on properties of $B(\gamma^a)$, on which $\tilde{\gamma}^a$'s apply.

Statement 3a: The relations of Eq. (29) remain valid also after the reduction of the Clifford space.

Let us check Eq. (29): $\{\tilde{\gamma}^a,\tilde{\gamma}^b\}_+ = 2\eta^{ab} = \tilde{\gamma}^a\tilde{\gamma}^b + \tilde{\gamma}^b\tilde{\gamma}^a = \tilde{\gamma}^a i\gamma^b + \tilde{\gamma}^b i\gamma^a = i\gamma^b(-i)\gamma^a + i\gamma^a(-i)\gamma^b = 2\eta^{ab}$. $\{\tilde{\gamma}^a,\gamma^b\}_+ = 0 = \tilde{\gamma}^a\gamma^b + \gamma^b\tilde{\gamma}^a = \gamma^b(-i)\gamma^a + \gamma^b i\gamma^a = 0$. For a particular case one has $\{\tilde{\gamma}^a,\gamma^a\}_+ = 0 = \tilde{\gamma}^a\gamma^a + \gamma^a\tilde{\gamma}^a = \gamma^a(-i)\gamma^a + \gamma^a i\gamma^a = 0$.

The application of $\tilde{\gamma}^a$ obviously depends on the space on which it applies, Eq. (38) namely requires: $[\tilde{\gamma}^a(a_0 + a_b\gamma^b + a_{bc}\gamma^b\gamma^c+\cdots) = (ia_0\gamma^a + (-i)a_b\gamma^b\gamma^a + ia_{bc}\gamma^b\gamma^c\gamma^a+\cdots)]|\psi_{oc>}$. Statement 3b is proved.

Statement 3b: Taking into account in Eq. (38) required application of $\tilde{\gamma}^a$'s on the Clifford space of γ^a's (causing the reduction of the Clifford space and at the same time as well the reduction of the Grassmann space), it follows that the eigenvalues of $\tilde{S}^{ab}$ on the eigenvectors of S^{ab} agree with the eigenvalues of S^{ab} on nilpotents, while the eigenvalues of $\tilde{S}^{ab}$ and S^{ab} on projectors, which are eigenvectors of S^{ab}, have opposite sign.

Let us check Eq.(39).

$\tilde{S}^{ab}\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\tilde{\gamma}^a\tilde{\gamma}^b\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b) = \frac{i}{2}\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\gamma^b\gamma^a = \frac{i}{2}\frac{1}{2}(-\eta^{aa}\gamma^b+\frac{\eta^{aa}\eta^{bb}}{ik}\gamma^a) = \frac{k}{2}\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)$,

$\tilde{S}^{ab}\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b) = \frac{i}{2}\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\gamma^b\gamma^a = \frac{i}{2}\frac{1}{2}(-\gamma^a\gamma^b+\frac{i}{k}\eta^{aa}\eta^{bb}) = -\frac{k}{2}\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)$, where it is taken into account that $k^2 = \eta^{aa}\eta^{bb}$.

This proves *Statement 3b*.

Statement 4: The algebraic product of $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$ is the same for all m of a particular irreducible representation f.

To prove this we take into account that $\hat{b}^{m\dagger}_f$ follows from $\hat{b}^{m-1\dagger}_f$ by the application of a particular $2iS^{eg}$. Then $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f = \hat{b}^{m-1}_f(2\,S^{eg})^\dagger *_A (2\,S^{eg})\hat{b}^{m-1\dagger}_f = \hat{b}^{m-1}_f *_A \hat{b}^{m-1\dagger}_f$, due to the relation $(-2iS^{ab})^\dagger(-2iS^{ab}) = 1)$. Repeating this procedure for each m proves the statement.

Statement 5: Each irreducible representation has its own algebraic product $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$.

We pay attention to the Clifford odd representations, but the proof is as well valid for the Clifford even representations.

To prove this statement let us start with $d = 2(2n+1)$ with the irreducible representation f and the member m equal to $\hat{b}^{m\dagger}_f = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{(+)}$. To obtain the $2^{\frac{d}{2}-1}-1$ rest irreducible representations we must transform each of possible pairs $\stackrel{ab}{(k)}\stackrel{cd}{(k)}$ into $\stackrel{ab}{[k]}\stackrel{cd}{[k]}$. Let us start with the first two. One obtains $\hat{b}^{m\dagger}_{f'} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\cdots\stackrel{d-1\,d}{(+)}$. The two algebraic products, $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}$ and $\hat{b}^m_{f'} *_A \hat{b}^{m\dagger}_{f'} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[-]}\cdots\stackrel{d-1\,d}{[-]}$, distinguish in the first two projectors. When replacing a pair by a pair in $\hat{b}^{m\dagger}_f$, we end up with $2^{\frac{d}{2}-1}$ different $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$, differing in all possible pairs $\stackrel{ab}{[+]}\stackrel{a'b'}{[+]}$ replacing $\stackrel{ab}{[-]}\stackrel{a'b'}{[-]}$.

For $d = 4n$ and the Clifford odd representations we must start with $\hat{b}^{m\dagger}_f = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{[+]}$, and then repeat all steps. We shall again obtain $2^{\frac{d}{2}-1}$ different $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$.

Statement 5a: There are $2^{\frac{d}{2}-1}$ different algebraic products $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$.

Since due to *Statement 4.* all the members of a particular irreducible representation have the same algebraic product $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$, we can conclude that there are $2^{\frac{d}{2}-1}$ different algebraic products $\hat{b}^m_f *_A \hat{b}^{m\dagger}_f$.

Statement 5b: Each creation operator $\hat{b}^{m\dagger}_f$ gives nonzero contribution when applied on $|\psi_{oc}>$.

There is one summand in $|\psi_{oc}>$, namely, $\hat{b}^m_f \hat{b}^{m\dagger}_f = \hat{b}^{m'}_f \hat{b}^{m'\dagger}_f$, $\forall m'$, on which $\hat{b}^{m\dagger}_f$ gives a nonzero contribution: $\sum_{f'=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m\dagger}_f \;\, \hat{b}^m_{f'} \hat{b}^{m\dagger}_{f'} = \sum_{f'=1}^{2^{\frac{d}{2}-1}} \delta^{ff'} \hat{b}^{m\dagger}_{f'}$.

Statement 5c: In odd representations the algebraic product of any two annihilation operators $\hat{b}^m_f {}_{*_A} \hat{b}^{m'}_{f'}$ gives zero, as also the algebraic product of any two creation operators $\hat{b}^{m\dagger}_f {}_{*_A} \hat{b}^{m'\dagger}_{f'}$ gives zero. Correspondingly the application of the annihilation operators on $|\psi_{oc}>$ gives zero contribution.

All annihilation operators are "orthogonal", as also all the creation operators are: $\hat{b}^m_f {}_{*_A} \hat{b}^{m'}_{f'} = 0$, $\hat{b}^{m\dagger}_f {}_{*_A} \hat{b}^{m'\dagger}_{f'} = 0$. Within the same irreducible representation at least one nilpotent of the two creation operators or of the two annihilation operators are the same. Among different irreducible representations of each kind separately, either one nilpotent is the same in both operators appearing in the product, or $\stackrel{ab}{(k)}$ multiplies $\stackrel{ab}{[k]}$ or $\stackrel{ab}{[-k]}$ multiplies $\stackrel{ab}{(k)}$ or $\stackrel{ab}{[k]}$ multiplies $\stackrel{ab}{[-k]}$, since one irreducible representation differs from the other in a pair $\stackrel{ab}{(k)}\stackrel{ef}{(k')}$ going to $\stackrel{ab}{[k]}\stackrel{ef}{[k']}$ or $\stackrel{ab}{(k)}\stackrel{ef}{[k']}$ going to $\stackrel{ab}{[k]}\stackrel{ef}{(k')}$ or $\stackrel{ab}{[k]}\stackrel{ef}{(k')}$ going to $\stackrel{ab}{(k)}\stackrel{ef}{[k']}$.

Statement 6: The operator $\tilde{S}^{cd}$, which does not belong to the Cartan subalgebra of Eq, (8), generates after the reduction of the Clifford space a new irreducible representation, carrying different family quantum number.

The proof of *Statement 5.* contains also the proof for *Statement 6.*. All the members of one irreducible representation are reachable by the application of S^{ab}'s. Let us start in $d = 2(2n+1)$ with the Clifford odd representation containing the member $\hat{b}^{m\dagger}_f = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{bd}{(+)}\cdots\stackrel{b'd'}{(+)}\cdots\stackrel{d-1\,d}{(+)}$ (with the family quantum numbers $(\frac{i}{2},\frac{1}{2},\cdots,\frac{1}{2},\cdots,\frac{1}{2},\cdots,\frac{1}{2})$, determined by $\tilde{S}^{ab}$'s from Eq. (8)). Operator $\tilde{S}^{bb'}$ transforms $\hat{b}^{m\dagger}_f$ into $\stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{bd}{[+]}\cdots\stackrel{b'd'}{[+]}\cdots\stackrel{d-1\,d}{(+)}$. This new creation operator belongs to new irreducible representation (with the family quantum numbers $(\frac{i}{2},\frac{1}{2},\cdots,-\frac{1}{2},\cdots,-\frac{1}{2},\cdots,\frac{1}{2})$), since it is not reachable by S^{ab} (which generate all the rest members of the same irreducible representation). Transforming all pairs into the new ones, one obtains $2^{\frac{d}{2}-1}$ families.

For $d = 4n$ we start for odd representations by $\hat{b}^{m\dagger}_f = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{bd}{(+)}\cdots\stackrel{b'd'}{(+)}\cdots\stackrel{d-1\,d}{[+]}$ and repeat the above procedure (by taking into account that $\tilde{S}^{bb'}$ transforms $\stackrel{bd}{(+)}\cdots\stackrel{b'd'}{[+]}$ into $\stackrel{bd}{[+]}\cdots\stackrel{b'd'}{(+)}$).

Similarly we can find all the families of the Clifford even representations, if taking into account Eq. (38)

Statement 7: Creation operators $\hat{b}^{m\dagger}_f$ and their Hermitian conjugated partners annihilation operators, appearing in Eq. (36), have the properties

$$
\begin{aligned}
\hat{b}^m_f {}_{*_A} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\,,\\
\hat{b}^{m\dagger}_f {}_{*_A} |\psi_{oc}> &= |\psi^m_f>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'}_{f'}\}_{*_A +} |\psi_{oc}> &= 0\, |\psi_{oc}>\,,\\
\{\hat{b}^m_f, \hat{b}^{m'\dagger}_f\}_{*_A +} |\psi_{oc}> &= \delta^{mm'} |\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_f, \hat{b}^{m'\dagger}_{f'}\}_{*_A +} |\psi_{oc}> &= 0 \cdot |\psi_{oc}>\;.
\end{aligned}
\tag{198}
$$

Let us prove this statement step by step:

a. The last line of Eq. (198) requires that $[\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'} = 0]|\psi_{oc}>$.

a.i. The product of two equal creation operators is zero, since the product of two nilpotents of the same kind, $(\stackrel{ab}{(k)})^2 = 0$, gives zero.

a.ii. All the creation operators can be obtained from the starting one by the application S^{cd}. Since at this application at least one nilpotent remains the same, it follows that all creation operators within the same irreducible representation are orthogonal ($\hat{b}^{m\dagger}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f} = 0$), by themselves and then also on $|\psi_{oc}>$.

a.iii. The creation operators of two different irreducible representations $f \neq f'$ can be obtained from the starting one by replacing two nilpotents $\stackrel{ab}{(k)} {}_{*_A} \stackrel{ef}{(k')}$ by $\stackrel{ab}{[k]} {}_{*_A} \stackrel{ef}{[k']}$, a nilpotent $\stackrel{ab}{(k)}$ and a projector $\stackrel{ab}{[k]}$ by the projector $\stackrel{ab}{[k]}$ and the nilpotent $\stackrel{ab}{(k)}$, or two projectors $\stackrel{ab}{[k]}\,\stackrel{ab}{[k']}$ by two nilpotents. Since at least one of the nilpotents remains the same, it follows that all the same members of different irreducible representations are orthogonal, ($\hat{b}^{m\dagger}_{f}{}_{*_A}\hat{b}^{m\dagger}_{f'} = 0$), by themselves and then also on $|\psi_{oc}>$.

a.iv. The creation operators, belonging to two different irreducible representations (f, f') and to two different members (m, m') have the property ($\hat{b}^{m\dagger}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f'} = 0$) (are orthogonal), due to the way how they are created ($\hat{b}^{m\dagger}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f'} = S^{mm'}\hat{b}^{m'\dagger}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f'} = 0$, as it is proven under iii.); Either the two have the same nilpotent or there appear a product of two projectors of the same type with opposite k ($\stackrel{ab}{[k]}$, $\stackrel{ab}{[-k]}$), or it appears ($\stackrel{ab}{[k]} {}_{*_A} \stackrel{ef}{(-k)}= 0$) or ($\stackrel{ab}{(k)} {}_{*_A} \stackrel{ef}{[k]}= 0$), Eq. (30).

b. The first line requires $\hat{b}^{m}_{f}{}_{*_A}|\psi_{oc}>= 0$, which is equivalent to requiring $\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m'}_{f'}{}_{*_A}\hat{b}^{m'\dagger}_{f'} = 0$, $\forall\, m'$ and f', since the vacuum state is equal to $|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^{m}_{f}{*_A}\hat{b}^{m\dagger}_{f}$, independent of the choice of m. The proofs from a.i.-a.iv. guarantee that $\hat{b}^{m}_{f}{}_{*_A}|\psi_{oc}>= 0$, since $\hat{b}^{m}_{f} = (\hat{b}^{m\dagger}_{f})^{\dagger}$.

c. The second line requires that $\hat{b}^{m'\dagger}_{f'}{}_{*_A}|\psi_{oc}>= |\psi^{m'}_{f'}>$. Namely, $\hat{b}^{m'\dagger}_{f'}{}_{*_A}\hat{b}^{m}_{f'}{}_{*_A}\hat{b}^{m\dagger}_{f'} = \hat{b}^{m'\dagger}_{f'}\ \forall m$, what follows from the way how $|\psi_{oc}>$ is created, while $\hat{b}^{m'\dagger}_{f'}{}_{*_A}\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m\dagger}_{f} = 0$, $\forall\, f \neq f'$, since the application of $\hat{b}^{m'\dagger}_{f'}{}_{*_A}$ on $\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m\dagger}_{f'}$ gives zero due to the orthogonality of the members of different irreducible representations.

d. The third line requires of Eq. (198) requires that $[\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'} = 0]|\psi_{oc}>$. Since $\hat{b}^{m'}_{f'} = (\hat{b}^{m'\dagger}_{f'})^{\dagger}$, the proof is the same as in the case a..

e. The fourth line requires that $\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f}\}_{*_A+}|\psi_{oc}>= \delta^{mm'}|\psi^{m'}_{f}>$, which means that $(\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f} + \hat{b}^{m'\dagger}_{f}{}_{*_A}\hat{b}^{m}_{f})_{*_A}|\psi_{oc}>= \delta^{mm'}|\psi^{m'}_{f}>$. This proofs follows from a. and b..

Allowing however, that also $f \neq f'$, one finds that the term $\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m'\dagger}_{f'}{}_{*_A}|\psi_{oc}>$ gives a nonzero contribution — what is an even Clifford object and not zero. But after reducing the Clifford algebra space, Eq. (38), the two irreducible representations (f, f') carry two different family quantum numbers. The algebraic product has no meaning any longer: Two different families, reachable from each other by $\tilde{S}^{ab}$, are orthogonal in the sense of the tensor product.

(As an example let us demonstrate this on the case $d = (5+1)$, presented on Table 3: $\hat{b}^{m=1}_{f=II} = (-)\,\stackrel{56}{(-)}\ |\ \stackrel{12}{[+]}\stackrel{03}{[+i]}$, $\hat{b}^{m=1\dagger}_{f=I} = \stackrel{03}{(+i)}\,\stackrel{12}{(+)}\ |\ \stackrel{56}{(+)}$. One obtains $\hat{b}^{m=1}_{f=II*_A}\hat{b}^{m=1\dagger}_{f=I}{}_{*_A}|\psi_{oc}> = \stackrel{03}{(+i)}\,\stackrel{12}{(+)}\stackrel{56}{[-]}$, what represents a even Clifford object and not a nonzero contribution.)

(c.i. Let us show as an example, that $\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m'}_{f} = 0$. Each annihilation operator of the same irreducible representation f and different member m' follows from $\hat{b}^{m}_{f}$, $\hat{b}^{m'}_{f} = (S^{m'm}\hat{b}^{m\dagger}_{f})^{\dagger} = \hat{b}^{m}_{f}(S^{mm'})^{\dagger}$. Correspondingly we have $\hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m'}_{f'} = \hat{b}^{m}_{f}{}_{*_A}\hat{b}^{m}_{f}(S^{mm'})^{\dagger} = 0$.)

Statement 8: $< \psi^{\mathbf{s}'}{}_{\mathbf{f}'}(\tilde{\mathbf{p}'})|\psi^{\mathbf{s}}{}_{\mathbf{f}}(\tilde{\mathbf{p}}) >= = \delta^{ss'}\delta_{ff'}\,\delta(\vec{p}' - \vec{p})$.

Let us prove this relation.

Since the "basis vectors" in internal space of fermions are orthogonal according to Eq. (36)

$$\{\hat{b}^{m}_{f}{}_{*A}\,,\,\hat{b}^{m'\dagger}_{f'}{}_{*A}\}_{+}|\psi_{oc}> \;=\; \hat{b}^{m}_{f}{}_{*A}\,\hat{b}^{m'\dagger}_{f'}{}_{*A}|\psi_{oc}>=\delta^{mm'}\delta_{ff'}\,|\psi_{oc}>\,,$$

it follows by taking into account Eq. (213) of App. J [59], here we leave out the tensor product between states of ordinary space and internal space of fermions.

$$\begin{aligned}
&< 0_{\vec{p}}|<\psi_{oc}|\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}')\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})|\psi_{oc}>|0_{\vec{p}}>\\
&=< 0_{\vec{p}}|<\psi_{oc}|\sum_{m,m'} c^{*s'm'}{}_{f'}(\vec{p}')\,\hat{b}_{\vec{p}'}\,\hat{b}^{m'}_{f'}\,c^{sm}{}_{f}(\vec{p})\,\hat{b}^{m\dagger}_{f}\,\hat{b}^{\dagger}_{\vec{p}}|\psi_{oc}>|0_{\vec{p}}>\\
&=\sum_{m,m'}<\psi_{oc}|\hat{b}^{m'}_{f'}{}_{*A}\,\hat{b}^{m\dagger}_{f}\,|\psi_{oc}>< 0_{\vec{p}}|c^{*s'm'}{}_{f'}(\vec{p}')\,\hat{b}_{\vec{p}'}\,\hat{b}^{\dagger}_{\vec{p}}c^{sm}{}_{f}(\vec{p})|0_{\vec{p}}>=\\
&\sum_{m} c^{ms*}{}_{f}(\vec{p},|p^0|=|\vec{p}|)\;\; c^{ms'}{}_{f'}(\vec{p},|p^0|=|\vec{p}|)\,\delta_{ff'}\,\delta(\vec{p}'-\vec{p})\\
&=\delta^{ss'}\delta_{ff'}\,\delta(\vec{p}'-\vec{p})\,,
\end{aligned}\tag{199}$$

since $<\psi_{oc}|\hat{b}^{m'}_{f'}{}_{*A}\,\hat{b}^{m\dagger}_{f}\,|\psi_{oc}>=\delta^{mm'}\delta_{ff'}$ and $< 0_{\vec{p}}|\hat{b}_{\vec{p}'}\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>=\delta(\vec{p}'-\vec{p})$, App. J, while

$$\sum_{m} c^{ms*}{}_{f}(\vec{p},|p^0|=|\vec{p}|)\;\; c^{ms'}{}_{f'}(\vec{p},|p^0|=|\vec{p}|)=\delta^{ss'}\delta_{ff'}\;. \tag{200}$$

J Continuous spectra in momentum and coordinate space

Creation and annihilation operators for a single fermion state must include besides the internal space of fermions, described by the superposition of odd products of Clifford algebra operators γ^a's, denoted by $\hat{b}^{m\dagger}_{f}$, also the momentum or coordinate part. We make the tensor product, $*_T$, of both parts, of the anticommuting internal one and the commuting momentum or coordinate one, as suggested in Eq. (48),

$$\{\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p}) \;=\; \hat{b}^{m\dagger}_{f}\;\; *_T\;\hat{b}^{\dagger}_{\vec{p}}\}\,|\psi_{oc}>*_T|0_{\vec{p}}>\;. \tag{201}$$

The quantum numbers (m,f) represent the internal part of anticommuting "basis states" $\hat{b}^{m\dagger}_{f}$ of the family member m of the family f, the vacuum state written here as $|\psi_{oc}>*_T|0_{\vec{p}}>$ is the vacuum for the starting single particle state, from which one obtains other single particle states with the same internal part, by the operators, $\hat{b}^{\dagger}_{\vec{p}}$, which pushes the momentum by an amount $\vec{p}$.

We study free massless fermions in $(d=(d-1)+1)$-dimensional space, for any d. We treat the continuous spectrum of $\vec{p}$ assuming that solutions of the Euler-Lagrange equations relate momentum $\vec{p}$ and energy p^0, $(p^0)^2=(\vec{p})^2$.

We follow up to Eq. (211) more or less Ref. [135], while neither Eq. (211) nor the rest of equations are from Ref. [135] or we have found in the literature.

The commutation relations of the Hermitian operators $\hat{\vec{x}}$ and $\hat{\vec{p}}$, $\hat{\vec{x}}^{\dagger}=\hat{\vec{x}}$ and $\hat{\vec{p}}^{\dagger}=\hat{\vec{p}}$, in units in which $\hbar=1=c$, are as follows

$$\{\hat{x}^i\,,\hat{x}^j\}_{-} \;=\; 0\,,\quad \{\hat{x}^i\,,\hat{p}^j\}_{-}=\delta^{ij}\,,\quad \{\hat{p}^i\,,\hat{p}^j\}_{-}=0\,. \tag{202}$$

Let us write down some useful relations, following from Eq. (202),

$$\{\hat{\vec{x}}\,,f(\hat{\vec{x}},\hat{\vec{p}})\}_{-} \;=\; i\frac{\partial f(\hat{\vec{x}},\hat{\vec{p}})}{\partial\hat{\vec{p}}}\,,\quad \{\hat{\vec{p}}\,,f(\hat{\vec{x}},\hat{\vec{p}})\}_{-}=-i\frac{\partial f(\hat{\vec{x}},\hat{\vec{p}})}{\partial\hat{\vec{x}}}\,, \tag{203}$$

[59] In Eq. (213) one finds the relation $<\vec{p}|f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p}'})|\vec{p}'>=f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p}'})\delta(\vec{p}-\vec{p}')$.

Let us solve the eigenvalue equations for a state $|\vec{p}>$ and $|\vec{x}>$, both changing continuously with $\vec{p}$ and $\vec{x}$, respectively,

$$\begin{aligned}
\hat{\vec{p}}\,|\vec{p}> &= \vec{p}\,|\vec{p}>\,, \qquad \hat{\vec{x}}\,|\vec{x}>=\vec{x}\,|\vec{x}>\,,\\
|\vec{p}> &= e^{i\hat{\vec{x}}\cdot\vec{p}}\,|0_{\vec{p}}>\,, \qquad |\vec{x}>=e^{-i\hat{\vec{p}}\cdot\vec{x}}\,|0_{\vec{x}}>\,,\\
<\vec{p}| &= <0_{\vec{p}}|\,e^{-i\hat{\vec{x}}\cdot\vec{p}}\,, \qquad <\vec{x}|=<0_{\vec{x}}|\,e^{i\hat{\vec{p}}\cdot\vec{x}}\,,
\end{aligned} \tag{204}$$

the operator $e^{i\hat{\vec{x}}\cdot\vec{p}}$ translates $|0_{\vec{p}}>$ to $|\vec{p}>$ and operator $e^{-i\hat{\vec{p}}\cdot\vec{x}}$ translates $|0_{\vec{x}}>$ to $|\vec{x}>$, where $|0_{\vec{p}}>$ and $|0_{\vec{x}}>$ are states with eigenvalues $\vec{0}$ in both cases.

The continuous spectra are normalized as follows

$$<\vec{p}\,|\,\vec{p'}> \;=\; \delta(\vec{p}-\vec{p'})=\prod_{i=1,d-1}\delta(p^i-p'^i)\,,\quad <\vec{x}\,|\,\vec{x'}>=\delta(\vec{x}-\vec{x'})=\prod_{i=1,d-1}\delta(x^i-x'^i)\,, \tag{205}$$

and fulfill the completeness relations

$$\int|\vec{p}>d^{d-1}p<\vec{p}| \;=\; I\,,\quad \int|\vec{x}>d^{d-1}x<\vec{x}|=I\,. \tag{206}$$

$$\tag{207}$$

Due to the fact that $\hat{\vec{p}}$ and $\hat{\vec{x}}$ do not commute, Eq. (202), we have to evaluate $<\vec{p}|\vec{x}>$, using Eq. (204) and the relations $f(\hat{\vec{p}})\,|\vec{p}>=f(\vec{p})\,|\vec{p}>$ and $f(\hat{\vec{x}})\,|\vec{x}>=f(\vec{x})\,|\vec{x}>$. It follows

$$\begin{aligned}
<\vec{p}\,|\,\vec{x}> &= <\vec{p}|e^{-i\hat{\vec{p}}\cdot\vec{x}}|0_{\vec{x}}>=e^{-i\vec{p}\cdot\vec{x}}<\vec{p}\,|\,0_{\vec{x}}>\\
&= e^{-i\vec{p}\cdot\vec{x}}<0_{\vec{p}}\,|\,e^{-i\hat{\vec{x}}\cdot\vec{p}}|0_{\vec{x}}>=e^{-i\vec{p}\cdot\vec{x}}<0_{\vec{p}}\,|\,0_{\vec{x}}>\;.
\end{aligned} \tag{208}$$

Taking into account $<\vec{p}\,|\,\vec{x}>^{*}=<\vec{x}\,|\,\vec{p}>$, and the orthogonality and completeness relations, we evaluate the relation $|<0_{\vec{p}}\,|\,0_{\vec{x}}>|=\frac{1}{(\sqrt{2\pi})^{d-1}}$,

$$\begin{aligned}
\int<\vec{p}|\vec{x}>d\vec{x}<\vec{x}\,|\vec{p'}> &= \delta(\vec{p}-\vec{p'})=|<0_{\vec{p}}|0_{\vec{x}}>|^2\int e^{-i(\vec{p}-\vec{p'})\cdot\vec{x}}d\vec{x}\\
&= |<0_{\vec{p}}|0_{\vec{x}}>|^2(2\pi)^{d-1}\,,
\end{aligned} \tag{209}$$

from where we find

$$<\vec{p}|\vec{x}> \;=\; e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}=<\vec{x}|\vec{p}>^{*} \tag{210}$$

We started this section with the intention to define the creation operators for fermions, which would include both spaces, the internal one and the momentum (or coordinate) one, as presented in Eq. (201).

Therefore we rewrite Eqs. (204, 205) as follows

$$\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>=\hat{b}_{-\vec{p}}\,|0_{\vec{p}}>\,, \qquad |\vec{x}>=\hat{b}^{\dagger}_{\vec{x}}\,|0_{\vec{x}}>=\hat{b}_{-\vec{x}}\,|0_{\vec{x}}>\,,\\
<\vec{p}| &= <0_{\vec{p}}|\,\hat{b}_{\vec{p}}=<0_{\vec{p}}|\,\hat{b}^{\dagger}_{-\vec{p}}\,, \qquad <\vec{x}|=<0_{\vec{x}}|\,\hat{b}_{\vec{x}}=<0_{\vec{x}}|\,\hat{b}^{\dagger}_{-\vec{x}}\,,\\
<\vec{p}|\vec{p'}> &= <0_{\vec{p}}|\,\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p'}}\,|0_{\vec{p}}>=\delta(\vec{p}-\vec{p'})\,,\;<\vec{x}|\vec{x'}>=<0_{\vec{x}}|\,\hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{x'}}\,|0_{\vec{x}}>=\delta(\vec{x}-\vec{x'})\,,\\
\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p'}} &= \delta(\vec{p}-\vec{p'})\,, \qquad \hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{x'}}=\delta(\vec{x}-\vec{x'})\,,
\end{aligned} \tag{211}$$

with $<0_{\vec{p}}|0_{\vec{p}}>=1$ and $<0_{\vec{x}}|0_{\vec{x'}}>=1$ normalize to unity, so that operating in the single particle space by $\hat{b}^{\dagger}_{\vec{p}}$ shifts the momentum by the amount $\vec{p}$, while the operator $\hat{b}_{\vec{p}}$ shifts the momentum by the amount

$-\vec{p}$. It then follows

$$
\begin{aligned}
<\vec{p}|\vec{x}> &= <0_{\vec{p}}|\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}>= (<0_{\vec{x}}|\hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>)^{\dagger}\\
\{\hat{b}^{\dagger}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p'}}\}_{-} &= 0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p'}}\}_{-}=0\,, \qquad \{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{p'}}\}_{-}=0\,,\\
\{\hat{b}^{\dagger}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x'}}\}_{-} &= 0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x'}}\}_{-}=0\,, \qquad \{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x'}}\}_{-}=0\,,\\
&\text{while}\\
\{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{x}}\}_{-} &= e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \qquad ,\{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{p}}\}_{-}=e^{-i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,,
\end{aligned}
\tag{212}
$$

since $<\vec{p}|\vec{x}>=<0_{\vec{p}}|\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}>=e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}$ and $<0_{\vec{p}}|\hat{b}^{\dagger}_{\vec{x}}\,\hat{b}_{\vec{p}}|0_{\vec{x}}>=0$.

Since the two spaces, the internal one and the ordinary one, are independent, it is proven that our assumption in Eq. (201) is justified.

Let us present here some useful relations

$$
\begin{aligned}
f(\hat{\vec{p}})|\vec{p}> &= f(\hat{\vec{p}})\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>=f(\vec{p})\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}}>\,, \quad f(\hat{\vec{x}})|\vec{x}>=f(\hat{\vec{x}})\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}>=f(\vec{x})\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}>\,,\\
<\vec{p}|f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p'}})|\vec{p'}> &= f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p'}})\delta(\vec{p}-\vec{p'})\,, \quad <\vec{x}|f^{*}(\hat{\vec{x}})\,f(\hat{\vec{x'}})|\vec{x'}>=f^{*}(\hat{\vec{x}})\,f(\hat{\vec{x'}})\delta(\vec{x}-\vec{x'})\,,\\
<\vec{p}|f(\hat{\vec{x}})|\vec{p'}> &= f(-i\frac{\partial}{\partial\vec{p}})\,\delta(\vec{p}-\vec{p'})\,, \quad <\vec{x}|f(\hat{\vec{p}})|\vec{x'}>=f(i\frac{\partial}{\partial\vec{x}})\,\delta(\vec{x}-\vec{x'})\,,\\
<\vec{p}|\psi_{\vec{p}}> &= \frac{1}{(\sqrt{2\pi})^{d-1}}\int d^{d-1}x\,e^{-i\vec{p}\cdot\vec{x}}<\vec{x}|\psi_{\vec{x}}>\,,\\
<\vec{x}|\psi_{\vec{x}}> &= \frac{1}{(\sqrt{2\pi})^{d-1}}\int d^{d-1}p\,e^{i\vec{p}\cdot\vec{x}}<\vec{p}|\psi_{\vec{p}}>\,,
\end{aligned}
\tag{213}
$$

For

$$
\begin{aligned}
<\vec{p}|\psi_{\vec{p}}> &= f(\hat{\vec{p}})|\vec{p}>\,, \quad <\vec{x}|\psi_{\vec{x}}>=g(\hat{\vec{x}})|\vec{x}>\,,\\
&\text{it follows}\\
<\vec{x}|\psi_{\vec{x}}> &= \frac{1}{(\sqrt{2\pi})^{d-1}}\int d^{d-1}p\,e^{i\vec{p}\cdot\vec{x}}\,f(\hat{\vec{p}})|\vec{p}>=g(\hat{\vec{x}})|\vec{x}>\,,\\
<\vec{p}|\psi_{\vec{p}}> &= \frac{1}{(\sqrt{2\pi})^{d-1}}\int d^{d-1}p\,e^{-i\vec{p}\cdot\vec{x}}\,g(\hat{\vec{x}})|\vec{x}>=f(\hat{\vec{p}})|\vec{p}>\,,
\end{aligned}
\tag{214}
$$

Since the "basis vectors" in internal space of fermions are orthogonal according to Eq. (36)

$$
\{\hat{b}^{m}_{f}\,{}_{*A}\,,\,\hat{b}^{m'\dagger}_{f'}\,{}_{*A}\}_{+}|\psi_{oc}> \;=\; \hat{b}^{m}_{f}\,{}_{*A}\,\hat{b}^{m'\dagger}_{f'}\,{}_{*A}|\psi_{oc}>=\delta^{mm'}\delta_{ff'}\,|\psi_{oc}>\,,
\tag{215}
$$

it follows by taking into account Eq. (213) of App. J [60]

$$
\begin{aligned}
&<0_{\vec{p}}|*_{T}<\psi_{oc}|\hat{\mathbf{b}}^{s'}_{f'}(\vec{p'})\,*_{T}\,\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\;*_{T}\;|\psi_{oc}>\;*_{T}\,|0_{\vec{p}}>\\
&=<0_{\vec{p}}|*_{T}<\psi_{oc}|\sum_{m,m'}c^{*s'm'}{}_{f'}(\vec{p'})\,\hat{b}_{\vec{p'}}\,*_{T}\,\hat{b}^{m'}_{f'}\,c^{sm}{}_{f}(\vec{p})\,\hat{b}^{m\dagger}_{f}\,*_{T}\,\hat{b}_{\vec{p}}\,*_{T}\;|\psi_{oc}>\;*_{T}\,|0_{\vec{p}}>\\
&=\sum_{m,m'}<\psi_{oc}|\hat{b}^{m'}_{f'}\,*_{A}\,\hat{b}^{m\dagger}_{f}\,|\psi_{oc}><0_{\vec{p}}|c^{*s'm'}{}_{f'}(\vec{p'})\,\hat{b}_{\vec{p'}}\,\hat{b}^{\dagger}_{\vec{p}}\,c^{sm}{}_{f}(\vec{p})|0_{\vec{p}}>=\\
&\sum_{m}c^{ms*}{}_{f}(\vec{p},|p^{0}|=|\vec{p}|)\;\;c^{ms'}{}_{f'}(\vec{p},|p^{0}|=|\vec{p}|)\,\delta(\vec{p'}-\vec{p})\\
&=\delta^{ss'}\delta_{ff'}\,\delta(\vec{p'}-\vec{p})\,,
\end{aligned}
\tag{216}
$$

[60] In Eq. (213) one finds the useful relation $<\vec{p}|f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p'}})|\vec{p'}>=f^{*}(\hat{\vec{p}})\,f(\hat{\vec{p'}})\delta(\vec{p}-\vec{p'})$.

due to $< 0_{\vec{p}}|\hat{b}_{\vec{p}'}\hat{b}^{\dagger}_{\vec{p}}|0_{\vec{p}} >= \delta(\vec{p}' - \vec{p})$, App. J, while

$$\sum_{m} c^{ms*}{}_{f}(\vec{p}, |p^0| = |\vec{p}|) \;\; c^{ms'}{}_{f'}(\vec{p}, |p^0| = |\vec{p}|) = \delta^{ss'}\delta_{ff'} \; . \tag{217}$$

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