Mass as a Dynamical Variable

XXIV Bled Workshop
What Comes Beyond the Standard Models?

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Fixed Particle Masses — Axiom or Convenience?

J. J. Thomson cathode ray experiments (1897)

Cathode rays = beam of discrete particles with fixed e/m

R. Millikan and H. Fletcher oil-drop experiment (1909)

Minimum electron charge $\longrightarrow$ fixed electron mass

Particle Data Group (2020)

Variation in measured mass: $\Delta m_e \simeq 10^{-8}$

Convention: treat one particle mass m as fixed by *a priori* constraint

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu \qquad (i\partial\!\!\!/ - eA\!\!\!/ - m)\,\psi = 0 \qquad d^4p\,\delta(\underbrace{p^\mu p_\mu + m^2}_{\text{mass shell}}) = \frac{d^3\mathbf{p}}{2\sqrt{\mathbf{p}^2 + m^2}}$$

$$\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$$

Fixed Particle Masses — Axiom or Convenience?

Complications from the Standard Model

Higgs mechanism

Elementary particles $\sim$ massless asymptotic states $\quad \bar{\psi}\,(i\partial\!\!\!/ - f\,G\!\!\!/auge)\,\psi$

Particle masses $\longleftarrow$ interactions with Higgs field $\quad m = f\,\langle \text{Higgs} \rangle_0$

Fixed masses $\in$ effective theories

Some one-particle masses sharper than others (PDG 2020)

$\Delta m \sim 10^{-10}$ for composite p, n

$\Delta m \sim 25\%$ for constituent u, d quarks

Holding masses fixed $\longleftrightarrow$ issues and quirks

Constrained mechanics

Problem of time

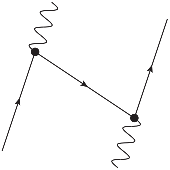
Flavor oscillations

Missing mass / energy

Stueckelberg-Feynman Antiparticle

Particle propagates backward in time

Feynman diagram in QED (1948)



Interaction vertex

Virtual $e^{-}(E>0) \longrightarrow e^{-}(E<0)+\gamma$

Appears in laboratory as $e^{-} \longrightarrow e^{+}+\gamma$

Future timelike trajectory $\longrightarrow$ past timelike trajectory

Fock particle trajectory $x^{\mu}(\tau)$ (1937)

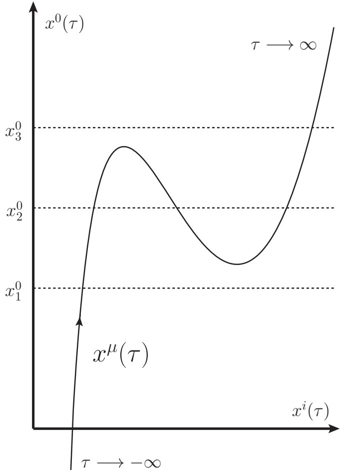
$$-\dot{x}^2 = \left(c\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right)^2 = -\left(c\frac{dt}{d\tau}\right)^2 \left(1, \frac{1}{c}\frac{d\mathbf{x}}{dt}\right)^2 = c^2\dot{t}^2\left(1-\frac{\mathbf{v}^2}{c^2}\right)$$

Mass shell constraint

Fix timelike trajectory $-M\dot{x}^2 = Mc^2 \longrightarrow \dfrac{dt}{d\tau} = \dot{t} = \pm\dfrac{1}{\sqrt{1-\dfrac{\mathbf{v}^2}{c^2}}}$

Stueckelberg-Feynman Antiparticle

Particle evolves backward in time



Stueckelberg trajectory in relativistic classical mechanics (1941)

- $x^\mu(\tau),\ \dot{x}^\mu(\tau)$ all independent
- Pair processes
  - Continuous evolution
    - $\dot{t} > 0 \longrightarrow \dot{t} < 0$
    - Somewhere $\dot{t} = 0$
  - $\dot{x}^\mu$ crosses spacelike lightcone
  - $\dot{x}^2(\tau) = \dot{x}^\mu \dot{x}_\mu$ **dynamical**
- External $\tau \neq$ proper time
  - $\dot{x}^2$ changes sign
  - $ds = \sqrt{-\dot{x}^2} d\tau$ not meaningful

Stueckelberg's proposed classical Lorentz force

$$D_\tau \left(M\dot{x}^\mu \right) = M \left(\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho \right) = eF^{\mu\nu} g_{\nu\rho} \dot{x}^\rho + G^\mu$$

Usual metric $g_{\mu\nu}(x) \underset{\text{flat}}{\longrightarrow} \text{diag}(-1,1,1,1)$ and connection $\Gamma^\mu_{\nu\rho}$ for $\mu,\nu,\rho = 0,\cdots,3$

Classical off-shell propagation

$$\frac{d}{d\tau} \left(\frac{1}{2} M\dot{x}^2 \right) = M\dot{x}_\mu D_\tau \dot{x}^\mu = e\ \dot{x}_\mu F^{\mu\nu} \dot{x}_\nu + \dot{x}_\mu G^\mu = \dot{x}_\mu G^\mu$$

$$G^\mu = 0 \ \longrightarrow \ \frac{1}{2} M\dot{x}^2 = \text{constant}$$

Mass shell constraint $\longrightarrow$ conservation law (for standard electrodynamics)

What could be source for G^μ?

Upgrade nonrelativistic classical and quantum mechanics

$$\left.\begin{array}{c}\text{Newtonian time } t \\ + \\ \text{Unconstrained } \left\{x^i, \dfrac{dx^j}{dt}\right\} \\ + \\ \text{Galilean symmetry} \\ + \\ \text{Scalar Hamiltonian } H\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{External time } \tau \\ + \\ \text{Unconstrained } \left\{x^\mu, \dfrac{dx^\nu}{d\tau}\right\} \\ + \\ \text{Poincaré symmetry} \\ + \\ \text{Scalar Hamiltonian } K\end{array}\right.$$

Inherit nonrelativistic methods and insights

$$\frac{\partial H}{\partial t} = 0 \Rightarrow \text{energy conserved} \quad \longrightarrow \quad \frac{\partial K}{\partial \tau} = 0 \Rightarrow \text{mass conserved}$$

Free particle

$$K = \frac{1}{2M} p^\mu p_\mu \quad \longrightarrow \quad \dot{x}^\mu = \frac{p^\mu}{M} \; , \; \dot{p}^\mu = 0 \quad \longrightarrow \quad \dot{x}^2 = \text{constant}$$

Physical spacetime event $x^\mu(\tau)$

Irreversible occurrence **at** time τ

$$\tau_2 > \tau_1 \implies \begin{cases} x^\mu(\tau_2) \text{ occurs } \textbf{after } x^\mu(\tau_1) \\ x^\mu(\tau_2) \textbf{ cannot change } x^\mu(\tau_1) \\ \text{No grandfather paradox} \end{cases}$$

Evolution

4D block universe $\mathcal{M}(\tau)$ **occurs** at τ

Infinitesimally close 4D block universe $\mathcal{M}(\tau + d\tau)$ occurs at $\tau + d\tau$

$$\mathcal{M}(\tau) \xrightarrow[\text{Hamiltonian } K \text{ generates evolution in } \tau]{} \mathcal{M}(\tau + d\tau)$$

$$\left.\begin{array}{r} \text{scalar } K \\ \text{external } \tau \end{array}\right\} \implies \text{No conflict with general diffeomorphism invariance}$$

Overview of Talk

Classical electrodynamics

Five τ-dependent potentials $A_\mu(x) \longrightarrow a_\alpha(x,\tau)$ $\qquad \alpha = 0,1,2,3,5$

Lorentz force permits mass exchange between particle and field

Total mass and momentum of particle and field conserved

Self-interaction restores on-shell mass

Maxwell electrodynamics $\sim$ τ-independent equilibrium

Quantum electrodynamics

First order unconstrained quantization

Retarded causality in $\tau \longrightarrow$ no matter loops

Super-renormalizable with suppression of mass exchange

General relativity

4D metric $g_{\mu\nu}(x)$ on $\mathcal{M} \longrightarrow g_{\alpha\beta}(x,\tau)$ on $\mathcal{M}(\tau)$

4+1 formalism generalizes Arnowitt-Deser-Misner (ADM)

Evolving metric $\longrightarrow$ mass exchange across spacetime

Horwitz-Piron Covariant Mechanics

Covariant Lagrangian and Hamiltonian mechanics (1973)

Classical Lagrangian on 8D unconstrained phase space

$$L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + e \dot{x}^\mu A_\mu(x) - V(x) \qquad \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = 0$$

Generalized Lorentz force

$$M \left(\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho \right) = e F^{\mu\nu} \dot{x}_\nu - \partial^\mu V \quad \longrightarrow \quad G^\mu = -\partial^\mu V$$

where

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M \dot{x}_\mu + e A_\mu(x)$$

Manifestly covariant Hamiltonian mechanics

$$K = \dot{x}^\mu p_\mu - L = \frac{1}{2M} (p^\mu - e A^\mu)(p_\mu - e A_\mu) + V$$

Classical: $\dot{x}^\mu = \dfrac{\partial K}{\partial p_\mu} \quad \dot{p}^\mu = -\dfrac{\partial K}{\partial x_\mu}$ Quantum: $i \partial_\tau \psi(x,\tau) = K \psi(x,\tau)$

Hamiltonian

$$K = \frac{p_{1\mu}p_1^\mu}{2M_1} + \frac{p_{2\mu}p_2^\mu}{2M_2} + V(x_1, x_2) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho) = \frac{P^\mu P_\mu}{2M} + K_{rel}$$

Center of mass and relative motion

$$P^\mu = p_1^\mu + p_2^\mu \qquad p^\mu = \frac{M_2 p_1^\mu - M_1 p_2^\mu}{M} \qquad M = M_1 + M_2 \qquad m = \frac{M_1 M_2}{M}$$

Generalized central force

$$V(x_1, x_2) = V(\rho) \qquad \text{where} \qquad \rho = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 - (t_1 - t_2)^2}$$

Relativistic models with mass as dynamical variable

Bound states and scattering — Horwitz and Arshansky (1989)

Selection rules, radiative transitions, perturbation theory, Zeeman and

Stark effects, bound state decay — Horwitz and Land (1993, 1995, 2001)

Entanglement and interference in time — Horwitz (2018)

Classical Off-Shell Electrodynamics: Framework

Where does potential $V(x)$ come from?

Extend classical electrodynamics

Invariance under τ-dependent gauge transformations

Requires τ-dependent gauge potentials $A_\mu(x) \longrightarrow a_\mu(x,\tau)$

Requires *fifth* τ-dependent gauge potential $ea_5(x,\tau) = -V$

τ-dependent field equations

Lorentz force: 4 independent components

E and $\mathbf{p}$ unconstrained

Particles and fields can exchange mass

Total mass of particles + fields conserved

Concatenation of events along worldline

Integration over $\tau \longrightarrow$ Maxwell equations

Extracts equilibrium theory from microscopic dynamics

Classical Off-Shell Electrodynamics: Framework

5D gauge theory — Horwitz, Saad, and Arshansky (1989), Land (2020)

Classical gauge freedom

$$S = \int d\tau\, L \;\longrightarrow\; \int d\tau \left[L + \frac{d}{d\tau}\Lambda\left(x,\tau\right) \right] = \int d\tau \left[L + \delta\big(\underbrace{\dot{x}^\mu \partial_\mu \Lambda + \partial_\tau \Lambda}_{\dot{x}^\alpha \partial_\alpha \Lambda} \big) \right]$$

Generalized interaction: 5 gauge potentials

$$\frac{e}{c}\dot{x}^\mu A_\mu\left(x\right) - V(x) \;\longrightarrow\; \frac{e}{c}\dot{x}^\mu a_\mu\left(x,\tau\right) + \frac{e}{c}\dot{x}^5 a_5\left(x,\tau\right) = \frac{e}{c}\dot{x}^\alpha a_\alpha\left(x,\tau\right)$$

$$\boxed{\lambda,\mu,\nu = 0,1,2,3 \qquad \text{and} \qquad \alpha,\beta,\gamma = 0,1,2,3,5}$$

$$\boxed{x^5 = c_5\tau \quad \text{for} \quad \dot{x}^5 = \text{constant} = c_5 \ll c}$$

Conserved 5-current and Maxwell current

$$j^\alpha\left(x,\tau\right) = c\dot{x}^\alpha \delta^4\left(x - X\left(\tau\right)\right) \qquad\qquad \partial_\alpha j^\alpha = \partial_\mu j^\mu + \partial_5 j^5 = 0$$

$$J^\mu(x) = \int d\tau\, j^\mu\left(x,\tau\right) \quad\longrightarrow\quad \partial_\mu J^\mu = 0$$

Classical Off-Shell Electrodynamics: Framework

Lorentz force (Land 1991)

Lagrangian

$$L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e}{c} \dot{x}^\alpha a_\alpha \left(x, \tau \right)$$

break 5D symmetry: $\dot{x}^5$ not dynamical

Equations of motion

$$M \ddot{x}_\mu = \frac{e}{c} \dot{x}^\beta f_{\mu\beta} = \frac{e}{c} \left(\dot{x}^\nu f_{\mu\nu} - c_5 f_{5\mu} \right)$$

$$\frac{d}{d\tau} \left(-\frac{1}{2} M \dot{x}^\mu \dot{x}_\mu \right) = c_5 \frac{e}{c} \dot{x}^\mu f_{5\mu}$$

5D field strength

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \qquad \alpha, \beta = 0, 1, 2, 3, 5$$

$f_{\mu\nu} \left(x, \tau \right) \longrightarrow F_{\mu\nu} \left(x \right)$ at equilibrium (τ-independence)

$\varepsilon^\mu \left(x, \tau \right) = f^{5\mu} \left(x, \tau \right) = \partial^5 a^\mu - \partial^\mu a^5$ induces mass exchange

In co-moving frame of one particle

For $\tau < 0$

$$x(\tau) = (c\tau, \mathbf{0}) \quad \longrightarrow \quad \dot{x}(\tau) = (c, \mathbf{0}) \quad \longrightarrow \quad -M\dot{x}^2 = Mc^2$$

$$a_\alpha(x,\tau) = 0 \quad \longrightarrow \quad f_{\alpha\beta} = 0 \quad \longrightarrow \quad M\ddot{x} = 0$$

For $\tau > 0$

$$a^\mu(x,\tau) = 0 \quad \longrightarrow \quad f^{\mu\nu} = 0$$

$$a_5 = -\varepsilon\, x_0 \quad \longrightarrow \quad f_{5k} = 0 \qquad f_{50} = -\partial^0 a_5 = \varepsilon \qquad (\varepsilon = \text{constant})$$

Acceleration in x^0 coordinate $\Rightarrow$ mass acceleration

$$M\ddot{\mathbf{x}}_\mu = 0 \qquad M\ddot{x}^0 = e\frac{c_5}{c}\,\varepsilon \qquad \frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^\mu\dot{x}_\mu\right) = e\frac{c_5}{c}\,\dot{x}^0\,\varepsilon$$

Classical Off-Shell Electrodynamics: Framework

Electromagnetic Action (Horwitz, Saad, Arshansky 1989, Land 2001)

Expand interaction term

$$\dot{X}^\alpha a_\alpha \longrightarrow \int d^4x \; \dot{X}^\alpha(\tau)\delta^4\left(x - X(\tau)\right) a_\alpha(x,\tau) = \frac{1}{c}\int d^4x \; j^\alpha(x,\tau) a_\alpha(x,\tau)$$

Action

$$S_{\text{em}} = \int d^4x d\tau \left\{ \frac{e}{c^2} j^\alpha(x,\tau) a_\alpha(x,\tau) - \int \frac{ds}{\lambda}\frac{1}{4c}\left[f^{\alpha\beta}(x,\tau)\Phi(\tau - s) f_{\alpha\beta}(x,s)\right]\right\}$$

Interaction kernel (non-local in time τ)

$$\Phi(\tau) = \delta(\tau) - (\xi\lambda)^2\delta''(\tau)$$ removes singularity in Coulomb law

$\lambda =$ constant with dimensions of time — a correlation time

$$\xi = \frac{1}{2}\left[1 + \left(\frac{c_5}{c}\right)^2\right]$$

Classical Off-Shell Electrodynamics: Framework

5D pseudo-metric

Terms $j^\alpha a_\alpha$ and $f^{\alpha\beta} f_{\alpha\beta}$ suggest 5D symmetry

Must break to O(3,1) in presence of matter $\longrightarrow$ O(4,1) or O(3,2)

Raising and lowering 5th index

$$g_{\alpha\beta} \underset{\text{flat}}{\longrightarrow} \eta_{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma) \qquad \eta_{55} = \sigma = \pm 1$$

$\delta''(\tau)$ term in action breaks higher symmetry

$$\int d\tau ds\ f^{\alpha\beta}(x,\tau)\delta''(\tau - s) f_{\alpha\beta}(x,s) = -\int d\tau \left[\partial_\tau f^{\alpha\beta}(x,\tau)\right] \partial_\tau f_{\alpha\beta}(x,\tau)$$

Expanding

$$f^{\alpha\beta} f_{\alpha\beta} = f^{\mu\nu} f_{\mu\nu} + 2\eta^{55} f_5^{\ \ \mu} f_{5\mu}$$

Interpret $\sigma = \eta^{55}$ as relative sign of vector-vector kinetic term

Classical Off-Shell Electrodynamics: Framework

pre-Maxwell field equations

Inverse interaction kernel

$$\varphi(\tau) = \lambda \Phi^{-1}(\tau) = \lambda \int \frac{d\kappa}{2\pi} \frac{e^{-i\kappa\tau}}{1+(\xi\lambda\kappa)^2} = \frac{1}{2\xi} e^{-|\tau|/\xi\lambda}$$

$$\int \frac{ds}{\lambda}\, \varphi(\tau - s)\, \Phi(s) = \delta(\tau) \qquad\qquad \int \frac{d\tau}{\lambda}\, \varphi(\tau) = 1$$

Vary action with respect to $a_\alpha(x,\tau)$

$$\partial_\beta f^{\alpha\beta}(x,\tau) = \frac{e}{c} \int ds\; \varphi(\tau - s)\, j^\alpha(x,s) = \frac{e}{c}\, j^\alpha_\varphi(x,\tau)$$

$$\partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} = 0 \qquad\qquad \text{(identically)}$$

$\varphi = \frac{1}{2\xi} e^{-|\tau|/\xi\lambda}$ smooths sharp current $j^\alpha(x,\tau) = c\dot{X}^\alpha \delta^4(x - X(\tau))$

Classical Off-Shell Electrodynamics: Framework

Comparing pre-Maxwell and Maxwell equations

Maxwell equations in 3+1 (space + time) components

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} = \frac{e}{c}\ \mathbf{J} \qquad\qquad \nabla \cdot \mathbf{E} = \frac{e}{c}\ J^0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0$$

pre-Maxwell in 4+1 (spacetime + τ) components

$$\partial_\nu f^{\mu\nu} - \frac{1}{c_5}\frac{\partial}{\partial \tau} f^{5\mu} = \frac{e}{c}\ j^\mu_\varphi \qquad\qquad \partial_\mu f^{5\mu} = \frac{e}{c}\ j^5_\varphi$$

$$\partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \qquad\qquad \partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5}\frac{\partial}{\partial \tau} f_{\mu\nu} = 0$$

Field decomposition

$f_{5\mu} \sim$ electric field sourced by j^5 in Gauss law

$f^{\mu\nu} \sim$ magnetic field induced by "curl" and τ variation of $f_{5\mu}$

Electromagnetic action

$$S_{\mathrm{em}} = \int d^4x d\tau \left\{ \frac{e}{c^2} j^\alpha(x,\tau) a_\alpha(x,\tau) - \frac{1}{4c} \left[f_\Phi^{\alpha\beta}(x,\tau) f_{\alpha\beta}(x,\tau) \right] \right\}$$

$$f_\Phi^{\alpha\beta}(x,\tau) = \int \frac{ds}{\lambda} \Phi(\tau - s) f^{\alpha\beta}(x,s)$$

Mass-energy-momentum tensor

$$T_\Phi^{\alpha\beta} = \frac{1}{c} \left(f_\Phi^{\alpha\gamma} f^\beta{}_\gamma - \frac{1}{4} g^{\alpha\beta} f_\Phi^{\delta\gamma} f_{\delta\gamma} \right)$$

Translation invariance $\longrightarrow$ Noether symmetry

$$\partial_\alpha T_\Phi^{\alpha\beta} = -\frac{e}{c^2} f^{\beta\alpha} j_\alpha$$

For particle current

$$\partial_\alpha T^{\alpha\beta}_\Phi = -\frac{e}{c^2} f^{\beta\alpha} j_\alpha = -\frac{e}{c} f^{\beta\alpha}\, \dot{X}_\alpha \delta^4\left(x - X(\tau)\right)$$

Spacetime integral

LHS $$\int d^4x\ \partial_\alpha T^{\alpha\beta} = \int d^4x\ \partial_\mu T^{\mu\beta} + \int d^4x\ \partial_5 T^{5\beta} = \frac{1}{c_5}\frac{d}{d\tau}\int d^4x\ T^{5\beta}$$

RHS $$-\frac{e}{c}\int d^4x\ f^{\beta\alpha}\, \dot{X}_\alpha \delta^4\left(x - X(\tau)\right) = -\frac{e}{c} f^{\beta\alpha}\left(X, \tau\right)\, \dot{X}_\alpha$$

Lorentz force

$$\frac{e}{c} f^{\mu\alpha} \dot{X}_\alpha = \frac{d}{d\tau}\left(M \dot{X}^\mu\right) \qquad \frac{e}{c} f^{5\mu} \dot{X}_\mu = \frac{1}{c_5}\frac{d}{d\tau}\left(-\sigma \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu\right)$$

Total mass-energy-momentum of particle + field conserved

$$\frac{d}{d\tau}\left(\int d^4x\ T^{5\mu} + M\dot{x}^\mu\right) = 0 \qquad \frac{d}{d\tau}\left(\int d^4x\ T^{55} - \sigma \frac{1}{2} M \dot{x}^2\right) = 0$$

Concatenation — sum contributions to $G(x)$ from events $g_\alpha(x,\tau)$

Integrate function $g_\alpha(x,\tau)$ along worldline: $G_\alpha(x) = \int_{-\infty}^{\infty} \frac{d\tau}{\lambda}\, g_\alpha(x,\tau)$

Boundary condition $g_5(x,\pm\infty) = 0$

Divergenceless Maxwell current

$$\partial_\alpha j_\varphi^\alpha = \partial_\mu j_\varphi^\mu + \partial_5 j_\varphi^5 = 0 \quad \longrightarrow \quad \partial_\mu J^\mu(x) = \partial_\mu \int_{-\infty}^{\infty} \frac{d\tau}{\lambda}\, j_\varphi^\mu(x,\tau) = 0$$

Field equations

$$\left.\begin{array}{c} \partial_\beta f^{\alpha\beta}(x,\tau) = \frac{e}{c} j_\varphi^\alpha(x,\tau) \\ \\ \partial_{[\alpha} f_{\beta\gamma]} = 0 \end{array}\right\} \xrightarrow[\int \frac{d\tau}{\lambda}]{} \left\{\begin{array}{c} \partial_\nu F^{\mu\nu}(x) = \frac{e}{c} J^\mu(x) \\ \\ \partial_{[\mu} F_{\nu\rho]} = 0 \end{array}\right.$$

Concatenation extracts on-shell Maxwell theory as equilibrium limit

pre-Maxwell equations lead to 5D wave equation

Green's function contains two parts

$G_{Maxwell}$ support on 4D lightcone at instantaneous τ separation

$G_{Correlation}$ support on timelike/spacelike separations ($\sigma = \pm 1$)

$G_{Correlation}$ falls off much faster than $G_{Maxwell}$

Coulomb problem

'Static' source moving uniformly on t-axis $\longrightarrow$ Yukawa potential

Mass spectrum of photon $\sim$ range of possible mass exchange

Provides limit on λ

Liénard-Wiechert potential$= \varphi\left(\tau - \tau_R\right) \times$Maxwell result

Lorentz force $\longrightarrow$ Maxwell result as $c_5 \to 0$

From pre-Maxwell equations in Lorenz gauge

$$\partial_\beta \partial^\beta a^\alpha = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) a^\alpha = (\partial_\mu \partial^\mu + \frac{g_{55}}{c_5^2}\, \partial_\tau^2) a^\alpha = -\frac{e}{c}\, j_\varphi^\alpha\, (x, \tau)$$

Green's function

$$G_P(x,\tau) = -\frac{1}{2\pi}\delta(x^2)\delta(\tau) - \frac{c_5}{2\pi^2}\frac{\partial}{\partial x^2}\theta(-g_{55}g_{\alpha\beta}x^\alpha x^\beta)\frac{1}{\sqrt{-g_{55}g_{\alpha\beta}x^\alpha x^\beta}}$$

$$= \; G_{Maxwell} + G_{Correlation}$$

$G_{Correlation}$

Smaller than $G_{Maxwell}$ by c_5/c and drops off as $1/\left|\mathbf{x}\right|^2$

May be neglected at at low energy

Spacelike support for $\sigma = -1$

Timelike support for $\sigma = +1$

$$\boxed{\int d\tau \, G_P = -\frac{1}{2\pi}\delta(x^2)}$$

"Static" source event

$X(\tau) = (c\tau, 0, 0, 0)$ evolves along x^0-axis

Induces current

$$j^0_\varphi(x,\tau) = c^2 \varphi(t-\tau)\,\delta^3(\mathbf{x}) \qquad \mathbf{j}_\varphi(x,\tau) = 0 \qquad j^5_\varphi(x,\tau) = \frac{c_5}{c} j^0_\varphi(x,\tau)$$

Potential using $G_{Maxwell} = -\frac{1}{2\pi}\delta(x^2)\delta(\tau)$

$$a^0(x,\tau) = \frac{e}{4\pi|\mathbf{x}|}\varphi\left(\tau - \left(t - \frac{|\mathbf{x}|}{c}\right)\right) \qquad \mathbf{a} = 0 \qquad a^5(x,\tau) = \frac{c_5}{c} a^0(x,\tau)$$

Test event

Observer on parallel trajectory $x(\tau) = (c\tau, \mathbf{x})$

$\varphi(\tau) \longrightarrow$ Yukawa-type potential $\quad a^0(x,\tau) = \dfrac{e}{4\pi|\mathbf{x}|}\dfrac{1}{2\xi}e^{-|\mathbf{x}|/\xi\lambda c}$

Classical Off-Shell Electrodynamics: Field Solutions

Liénard-Wiechert potential (Land 2016)

Arbitrary source event $X^\mu(\tau) \longrightarrow$ current

$$j^\alpha_\varphi(x,\tau) = -\frac{e}{c}\int ds\ \varphi(\tau - s)\,\dot{X}^\alpha(s)\,\delta^4(x - X(s))$$

Potential using $G_{Maxwell} = -\frac{1}{2\pi}\delta(x^2)\delta(\tau)$

$$\begin{aligned} a^\alpha(x,\tau) &= \frac{e}{2\pi}\int ds\ \varphi(\tau - s)\,\dot{X}^\alpha(s)\,\delta\left((x - X(s))^2\right)\theta^{ret} \\ &= \frac{e}{4\pi}\varphi(\tau - \tau_R)\,\frac{u^\alpha}{|u\cdot z(\tau_R)|} \end{aligned}$$

where $\quad u^\mu = \dot{X}^\mu(\tau) \qquad z^\mu = x^\mu - X^\mu(\tau) \qquad z^2(\tau_R) = 0$

τ-dependence in $\varphi(\tau - \tau_R)$

$$a^\mu(x,\tau) = \varphi(\tau - \tau_R)\,A^\mu_{\text{Liénard-Wiechert}}(x)$$

From Liénard-Wiechert potential

$$M\ddot{x}^\mu = \frac{e}{c}\left[f^\mu{}_\nu(x,\tau)\dot{x}^\nu + f^{5\mu}(x,\tau)\dot{x}^5\right]$$
$$= \frac{e}{c}\frac{e}{4\pi}\varphi\left(\tau-\tau_R\right)\left[\mathcal{F}^\mu{}_\nu(x,\tau)\dot{x}^\nu + c_5^2\ \mathcal{F}^{5\mu}(x,\tau)\right]$$

where $\mathcal{F}^{\mu\nu}$ and $\mathcal{F}^{5\mu}$ do not contain c_5

$$\mathcal{F}^{\mu\nu} = \frac{\left(z^\mu u^\nu - z^\nu u^\mu\right)u^2}{\left(u\cdot z\right)^3} + \left[\frac{\left(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu\right)\left(u\cdot z\right) - \left(z^\mu u^\nu - z^\nu u^\mu\right)\left(\dot{u}\cdot z\right)}{\left(u\cdot z\right)^3} + \frac{\epsilon\left(\tau-\tau_R\right)}{\lambda}\frac{z^\mu u^\nu - z^\nu u^\mu}{\left(u\cdot z\right)^2}\right]$$
$$\mathcal{F}^{5\mu} = \frac{z^\mu u^2 - u^\mu\left(u\cdot z\right)}{\left(u\cdot z\right)^3} - \frac{\left(\dot{u}\cdot z\right)z^\mu}{\left(u\cdot z\right)^3} + \frac{\epsilon\left(\tau-\tau_R\right)}{\lambda}\frac{z^\mu - u^\mu\left(u\cdot z\right)}{\left(u\cdot z\right)^2}$$

Using $\varphi(\tau) = \frac{1}{2\xi}e^{-|\tau|/\xi\lambda}$ and $\xi = \frac{1}{2}\left[1+\left(\frac{c_5}{c}\right)^2\right]$

$$M\ddot{x}^\mu = \frac{e^2}{4\pi c}\ e^{-|\tau-\tau_R|/\xi\lambda}\ \frac{\mathcal{F}^\mu{}_\nu\dot{x}^\nu + c_5^2\ \mathcal{F}^{5\mu}}{1+\left(c_5/c\right)^2}$$

Lorentz force from Liénard-Wiechert potential

$$M\ddot{x}^\mu = \frac{e^2}{4\pi c} \, e^{-|\tau-\tau_R|/\xi\lambda} \, \frac{\mathcal{F}^\mu{}_\nu \dot{x}^\nu + c_5^2 \, \mathcal{F}^{5\mu}}{1+(c_5/c)^2} \xrightarrow[c_5\to 0]{} \frac{e^2}{4\pi c} \, e^{-2|\tau-\tau_R|/\lambda} \, \mathcal{F}^\mu{}_\nu \dot{x}^\nu$$

Homogeneous pre-Maxwell equations

$$f^{\alpha\beta} = \partial^\alpha a^\beta - \partial^\beta a^\alpha \ \Rightarrow \ \partial_\mu f_{\nu\rho} + \partial_\nu f_{\rho\mu} + \partial_\rho f_{\mu\nu} = 0 \ \text{ satisfied identically}$$

$$\partial_\nu f_{5\mu} - \partial_\mu f_{5\nu} + \frac{1}{c_5}\frac{\partial}{\partial\tau} f_{\mu\nu} = 0 \ \Rightarrow \ \partial_\tau f_{\mu\nu} = 0 \ \Rightarrow \ \partial_\tau \varphi \ \longrightarrow \ 0 \ \Rightarrow \ \lambda \to \infty$$

Inhomogeneous pre-Maxwell equations decouple

$$\lambda \to \infty \ \Rightarrow \ \varphi = 1 \ \Rightarrow \ \frac{\partial}{\partial\tau} f^{5\mu} = 0 \ , \ j^\mu_\varphi(x,\tau) = J^\mu(x) \ \Rightarrow \ \partial_\nu f^{\mu\nu} = \frac{e}{c} \, J^\mu$$

$$\mathcal{F}^{\mu\nu} = \frac{(z^\mu u^\nu - z^\nu u^\mu)\, u^2}{(u\cdot z)^3} + \left[\frac{(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu)\,(u\cdot z) - (z^\mu u^\nu - z^\nu u^\mu)\,(\dot{u}\cdot z)}{(u\cdot z)^3} \right]$$

Photon mass in Coulomb potential

Yukawa-type potential $a^0(x,\tau) = \dfrac{e}{4\pi|\mathbf{x}|}\dfrac{1}{2\xi}e^{-|\mathbf{x}|/\xi\lambda c}$

Photon mass spectrum $m_\gamma c^2 \sim \hbar/\xi\lambda$

Experimental error for photon mass $\sim 10^{-18}eV \;\longrightarrow\; \lambda > 10^4$ seconds

Field strengths from Yukawa potential

$$f^{k0}(x,\tau) = a^0 \qquad f^{k5}(x,\tau) = \frac{c_5}{c}a^0 \qquad f^{ij}(x,\tau) = f^{50}(x,\tau) = 0$$

Lorentz force for $e^- + e^+ \longrightarrow e^- + e^+$ and $e^- + e^- \longrightarrow e^- + e^-$

$$M\ddot{\mathbf{x}} = \mp e^2\,\frac{1 \pm \eta_{55}\left(c_5/c\right)^2}{1+\left(c_5/c\right)^2}\nabla\left(\frac{e^{-|\mathbf{x}|/\xi\lambda c}}{4\pi\,|\mathbf{x}|}\right)$$

$$\frac{\sigma\left(e^- + e^+ \longrightarrow e^- + e^+\right)}{\sigma\left(e^- + e^- \longrightarrow e^- + e^-\right)} = 1 \pm \text{experimental error} \simeq \left[\frac{1 \mp \eta_{55}\left(\frac{c_5}{c}\right)^2}{1+\left(\frac{c_5}{c}\right)^2}\right]^2$$

Mass Shifts and Mass Restoration

Toy model for mass change from interaction

Particle experiences stochastic perturbative interaction

Small periodic amplitude at high frequency added to position

Large velocity perturbations

Possible macroscopic mass perturbation

Self-interaction $\longrightarrow$ mass restoration

Particle x^0 varies in co-moving frame $\longrightarrow$ mass acceleration

For $\sigma = +1$, $G_{Correlation}$ has timelike support

Particle interacts in future with its own field

Interaction damps mass acceleration to zero

Self-interaction vanishes when mass remains on-shell

Mass Shift by Stochastic Perturbation

On-shell event enters dense region of charged particles

Uniformly propagating event

$$x(\tau) = u\tau = \left(u^0, \mathbf{u}\right) \qquad\qquad u^2 = -c^2$$

Dense region of charged particles

Small stochastic perturbation $X(\tau) \longrightarrow x(\tau) = u\tau + X(\tau)$

Typical distance d between force centers $\longrightarrow$ roughly periodic perturbation

Characteristic period $= \dfrac{d}{|\mathbf{u}|} = \dfrac{\text{very short distance}}{\text{moderate velocity}} =$ very short time

fundamental frequency $= \omega_0 = 2\pi \dfrac{|\mathbf{u}|}{d} =$ very high frequency

amplitude $= |X^\mu(\tau)| \sim \alpha d$

macroscopic factor $= \alpha < 1$

Mass Shift by Stochastic Perturbation

Perturbed motion

Expand perturbation in Fourier series

$$X(\tau) = \text{Re} \sum_n a_n \ e^{in\omega_0 \tau}$$

Write four-vector coefficients as

$$a_n = \alpha d s_n = \alpha d \left(s_n^0, \mathbf{s}_n \right) = \alpha d \left(c s_n^t, \mathbf{s}_n \right)$$

where s_n represent normalized Fourier series $(s_0^\mu \sim 1)$

Perturbed motion on microscopic scale d

$$X(\tau) = \alpha d \ \text{Re} \sum_n s_n^\mu \ e^{in\omega_0 \tau}$$

Perturbed velocity on macroscopic scale $\alpha |\mathbf{u}|$

$$\dot{x}^\mu(\tau) = u^\mu + \alpha |\mathbf{u}| \ \text{Re} \sum_n 2\pi n \ s_n^\mu \ i e^{in\omega_0 \tau}$$

Mass Shift by Stochastic Perturbation

Perturbed mass

Unperturbed on-shell mass

$$m = -\frac{M\dot{x}^2(\tau)}{c^2} = M$$

Perturbed mass, neglecting α^2

$$m = -\frac{M\dot{x}^2(\tau)}{c^2} = -\frac{M}{c^2}\left(u + \alpha\,|\mathbf{u}|\ \text{Re}\sum_n 2\pi n\ s_n\ ie^{in\omega_0\tau}\right)^2$$

$$\simeq M\left(1 + 4\pi\alpha\,|\mathbf{u}|\,\text{Re}\sum_n n\ s_n^t\ ie^{in\omega_0\tau}\right)$$

$$m \longrightarrow m\left(1 + \frac{\Delta m}{m}\right) \qquad \frac{\Delta m}{m} = 4\pi\alpha\,|\mathbf{u}|\,\text{Re}\sum_n n\ s_n^t\ ie^{in\omega_0\tau}$$

Larger mass shifts if $\alpha > 1 \ \Rightarrow\ \alpha^2$ becomes significant

Arbitrarily moving event $X^\mu(\tau)$

In co-moving frame $\quad X(\tau) = (ct(\tau), \mathbf{0}) \qquad \dot{X}(\tau) = (c\dot{t}(\tau), \mathbf{0})$

Produces current $j^\alpha_\varphi(x,\tau)$ and field $f^{\alpha\beta}(x,\tau)$

At time $\tau^* > \tau$

$G_{Maxwell} = 0$ on timelike separation $X(\tau^*) - X(\tau) = c(t(\tau^*) - t(\tau), \mathbf{0})$

$G_{Correlation}$ has timelike support for $\sigma = +1$

Particle interacts with its own induced potential

$$a^\alpha(X(\tau^*), \tau^*) = \frac{ec_5}{2\pi^2 c^3} \int ds \; \dot{X}^\alpha(s) \left(\frac{1}{2} \frac{\theta(g(s))}{(g(s))^{3/2}} - \frac{\delta(g(s))}{(g(s))^{1/2}} \right) \theta(\tau^* - s)$$

$$c^2 g(s) = -\left((X(\tau) - X(s))^2 + c_5^2 (\tau - s)^2 \right) = c^2 \left((t(\tau^*) - t(s))^2 - \frac{c_5^2}{c^2} (\tau^* - s)^2 \right)$$

Self-Interaction for Mass Stability

On-shell trajectory

Particle evolves uniformly in co-moving frame to $t\left(\tau^*\right) = \tau^*$

$$g(s) = \left(1 - c_5^2/c^2\right)\left(\tau^* - s\right)^2$$

Potential

$$a\left(X\left(\tau^*\right), \tau^*\right) = \frac{ec_5}{2\pi^2 c^3}\left(c, \mathbf{0}, c_5\right) \int_{-\infty}^{\tau^*} \left(\frac{\theta\left(\left(1 - \frac{c_5^2}{c^2}\right)\left(\tau^* - s\right)^2\right)}{2\left(\left(1 - \frac{c_5^2}{c^2}\right)\left(\tau^* - s\right)^2\right)^{3/2}} - \frac{\delta\left(\left(1 - \frac{c_5^2}{c^2}\right)\left(\tau^* - s\right)^2\right)}{\left(\left(1 - \frac{c_5^2}{c^2}\right)\left(\tau^* - s\right)^2\right)^{1/2}} \right)$$

$$= \frac{ec_5}{2\pi^2 c^3}\left(c, \mathbf{0}, c_5\right) \lim_{s \to \tau^*} \left(\frac{1}{2\left(\tau^* - s\right)^2} - \frac{\frac{1}{2}}{\left(\tau^* - s\right)^2} \right)$$

$$= 0$$

No self-interaction for $\dot{x}^2 =$ constant

Self-Interaction for Mass Stability

Arbitrary trajectory in co-moving frame

Trajectory

$$\dot{X}(\tau) = \left(c\dot{t}(\tau), \mathbf{0}\right) \longrightarrow a^i = \partial_i a^0 = \partial_i a^5 = f^{\mu\nu} = f^{5i} = 0$$

Field strength

$$f^{50} = \frac{1}{c_5}\partial_{\tau^*} a^0 + \frac{1}{c}\partial_t a^5 = f_\theta^{50} + f_\delta^{50} + f_{\delta'}^{50}$$

$$f_\theta^{50} = \frac{3e}{4\pi^2}\frac{c_5^2}{c^4}\int ds\ \frac{\theta\left(g(s)\right)}{\left(g(s)\right)^{5/2}}\ \theta^{ret}\ \Delta\left(\tau^*, s\right)$$

$$f_\delta^{50} = -\frac{e}{\pi^2}\frac{c_5^2}{c^4}\int ds\ \frac{\delta\left(g(s)\right)}{\left(g(s)\right)^{3/2}}\ \theta^{ret}\ \Delta\left(\tau^*, s\right)$$

$$f_{\delta'}^{50} = -\frac{e}{\pi^2}\frac{c_5^2}{c^4}\int ds\ \frac{\delta'\left(g(s)\right)}{\left(g(s)\right)^{1/2}}\ \theta^{ret}\ \Delta\left(\tau^*, s\right)$$

where
$$\Delta\left(\tau^*, s\right) = \dot{t}(s)(\tau^* - s) - \left(t\left(\tau^*\right) - t\left(s\right)\right)$$
$$g(s) = \left(t\left(\tau^*\right) - t\left(s\right)\right)^2 - \left(c_5^2/c^2\right)\left(\tau^* - s\right)^2$$

At constant velocity

$$x^0(\tau) = u^0\tau \qquad \Rightarrow \qquad \Delta(\tau^*, s) = \frac{u^0}{c}(\tau^* - s) - \left(\frac{u^0}{c}\tau^* - \frac{u^0}{c}s\right) = 0$$

Expand $t(\tau)$ to order $\ddot{t}$

$$\begin{aligned} t(\tau^*) - t(s) &= t(s) + \dot{t}(s)(\tau^* - s) + \frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right) - t(s) \\ &= \dot{t}(s)(\tau^* - s) + \frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right) \end{aligned}$$

From which

$$\Delta(\tau^*, s) = -\frac{1}{2}\ddot{t}(s)(\tau^* - s)^2 + o\left((\tau^* - s)^3\right)$$

$\Delta(\tau^*, s) \neq 0 \ \Rightarrow \ x^0(\tau)$ accelerates in rest frame $\Rightarrow$ mass shift

Small, sudden jump in mass at $\tau = 0$

$$t(\tau) = \begin{cases} \tau & , \quad \tau < 0 \\ (1+\beta)\,\tau & , \quad \tau > 0 \end{cases} \qquad \Rightarrow \qquad \dot{t}(\tau) = \begin{cases} 1 & , \quad \tau < 0 \\ 1+\beta & , \quad \tau > 0 \end{cases}$$

For $\tau^* < 0$

$$\theta^{ret} \;\Rightarrow\; s < 0 \quad \longrightarrow \quad \dot{t}(\tau^*) = \dot{t}(s) = 1 \quad \longrightarrow \quad \Delta(\tau^*, s) = 0$$

For $\tau^* > 0$

$$s > 0 \quad \longrightarrow \quad \dot{t}(\tau^*) = \dot{t}(s) = 1 + \beta \quad \longrightarrow \quad \Delta(\tau^*, s) = 0$$

$$s < 0 \quad \longrightarrow \quad \Delta(\tau^*, s) = \dot{t}(s)\,(\tau^* - s) - ((1+\beta)\,(\tau^*) - s) = -\beta\tau^*$$

Solve $g(s^*) = 0 \;\longrightarrow\; s^* = \left(1 + \frac{\beta}{1 - \frac{c_5}{c}}\right)\tau^* > \tau^*$

$$g(s) > 0 \;\text{ on }\; s < 0 < \tau^* \quad \Rightarrow \quad f_{\delta}^{50} = f_{\delta'}^{50} = 0$$

Support of self-interaction

$$\theta(g(s)) = 1 \ \text{ for } \ s < \tau^* \text{ and } \ \Delta(\tau^*, s) = \begin{cases} -\beta\tau^* & , \quad \text{for } s < 0 \\ 0 & , \quad \text{for } s > 0 \end{cases}$$

Field strength

$$\begin{aligned} f^{50} = f^{50}_\theta &= (-\beta\tau^*) \frac{3e}{4\pi^2} \frac{c_5^2}{c^4} \int_{-\infty}^{0} ds \ \frac{1}{[g(s)]^{5/2}} \\ &= (-\beta\tau^*) \frac{3e}{4\pi^2} \frac{c_5^2}{c^4} \int_{-\infty}^{0} ds \ \frac{1}{\left[\left((1+\beta)\,\tau^* - s \right)^2 - \frac{c_5^2}{c^2} (\tau^* - s)^2 \right]^{5/2}} \\ &= \frac{e}{4\pi^2} \frac{1}{c_5^2 \left(\beta\tau^*\right)^3} \ Q\left(\beta, \frac{c_5^2}{c^2}\right) \end{aligned}$$

$Q\left(\beta, \frac{c_5^2}{c^2}\right)$ positive, dimensionless, finite for $c_5 < c$, $\ Q\left(\beta, \frac{c_5^2}{c^2}\right) \xrightarrow[c_5 \to 0]{} 0$

Self-Interaction for Mass Stability

Factor Q

$$Q\left(\beta, \frac{c_5^2}{c^2}\right) = \left[2\left(1-\frac{c_5^2}{c^2}\right)^{3/2} \left(1 - \frac{\left(1-\frac{c_5^2}{c^2}\right)^{1/2}\left(1+\frac{\beta}{\left(1-\frac{c_5^2}{c^2}\right)}\right)}{\left[1+\frac{2\beta}{1-\frac{c_5^2}{c^2}}+\frac{\beta^2}{1-\frac{c_5^2}{c^2}}\right]^{1/2}} \right) \right.$$

$$\left. + \frac{\beta^2 \; \frac{c_5^2}{c^2}\left(1+\frac{c_5^2}{c^2}\frac{\beta}{1-\frac{c_5^2}{c^2}}\right)}{\left(1-\frac{c_5^2}{c^2}\right)^{1/2}\left[1+\frac{2\beta}{1-\frac{c_5^2}{c^2}}+\frac{\beta^2}{1-\frac{c_5^2}{c^2}}\right]^{3/2}} \right]$$

Lorentz force

$$f^{\mu\nu} = 0 \longrightarrow \; M\ddot{x}^{\mu} = -ec_5 f^{5\mu}$$

Self-interaction

$$M\ddot{x}^0 = -c_5 e f^{50} = \begin{cases} 0 & , \quad \tau^* < 0 \\ -\dfrac{\lambda e^2}{4\pi^2} \dfrac{1}{c_5 \left(\beta\tau^*\right)^3} \, Q\left(\beta, \dfrac{c_5^2}{c^2}\right) & , \quad \tau^* > 0 \end{cases}$$

$$M\ddot{x}^i = -c_5 e f^{5i} \dot{x}_i = 0$$

$$\frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^2\right) = e f^{5\mu}\dot{x}_\mu = -ec f^{50}\dot{t} = -\frac{\lambda e^2}{4\pi^2} \frac{c}{c_5^2 \left(\beta\tau^*\right)^3} \, Q\left(\beta, \frac{c_5^2}{c^2}\right) \dot{t}$$

Emergent picture

Self-interaction $\longrightarrow$ force opposing mass exchange

Mass damps back to on-shell value

Force vanishes when $\dot{t} = 1$

Off-Shell Quantum Electrodynamics

First order Lagrangian in particle and field

- $\psi^* i\partial_\tau \psi$ kinetic term for particle
- No $\dot{a}_5$ term $\longrightarrow$ Gauss law and eliminates longitudinal modes
- Unconstrained Lagrangian
- 4-Momentum + mass states $d^k d\kappa$

Natural cut-off

- Mass shift undetermined (analogous to scattering angle)
- Interaction kernel Φ puts $\dfrac{1}{1+\lambda^2\kappa^2}$ into photon propagator
- λ restricts mass exchange as in classical Coulomb
- Only one photon loop to renormalize
- Particles propagate with τ-retarded causality — no matter loops

Recovers standard Klein-Gordon if $\Delta m = 0$ or $\sqrt{s} \gg M$

Stueckelberg-Schrodinger equation

$$\left(i\hbar\partial_\tau + e\frac{c_5}{c}a_5\right)\ \psi\left(x,\tau\right) = \frac{1}{2M}\left(p^\mu - \frac{e}{c}a^\mu\right)\left(p_\mu - \frac{e}{c}a_\mu\right)\ \psi\left(x,\tau\right)$$

Local 5D gauge invariance

$$\text{Wavefunction} \qquad \psi(x,\tau) \quad \to \quad \exp\left[\frac{ie}{\hbar c}\Lambda(x,\tau)\right]\ \psi(x,\tau)$$

$$\text{Potential} \qquad a_\alpha(x,\tau) \quad \to \quad a_\alpha(x,\tau) + \partial_\alpha\Lambda(x,\tau)$$

Global gauge invariance $\longrightarrow$ conserved current $\quad \partial_\alpha j^\alpha = 0$

$$j^\mu = -\frac{i\hbar}{2M}\left\{\psi^*\left(\partial^\mu - \frac{ie}{c}a^\mu\right)\psi - \psi\left(\partial^\mu + \frac{ie}{c}a^\mu\right)\psi^*\right\} \qquad j^5 = c_5\left|\psi(x,\tau)\right|^2$$

Lagrangian

$$\mathcal{L} = \psi^*(i\partial_\tau + ea_5)\psi - \frac{1}{2M}\psi^*(-i\partial_\mu - ea_\mu)(-i\partial^\mu - ea^\mu)\psi - \frac{\lambda}{4} f^{\alpha\beta} f^{\Phi}_{\alpha\beta}$$

$$f^{\Phi}_{\alpha\beta}(x,\tau) = \int ds\ \Phi(\tau - s)\ f_{\alpha\beta}(x,s)$$

Jackiw first order constrained quantization — introduce $\epsilon^\mu = f^{5\mu}$

$$\mathcal{L} = i\psi^*\dot{\psi} - \lambda \dot{a}^\mu \epsilon^\Phi_\mu - \frac{1}{2M}\psi^*(-i\partial_\mu - ea_\mu)(-i\partial^\mu - ea^\mu)\psi - \frac{\lambda}{4} f^{\mu\nu} f^{\Phi}_{\mu\nu}$$

$$- \frac{\lambda}{2}\epsilon^\mu \epsilon^\Phi_\mu + a_5\left(e\psi^*\psi - \lambda\partial^\mu\epsilon^\Phi_\mu\right)$$

Path integral

$$\mathcal{Z} = \frac{1}{\mathcal{N}}\int \mathcal{D}\psi^*\ \mathcal{D}\psi\ \mathcal{D}a_\mu\ \mathcal{D}a_5\ \mathcal{D}\epsilon_\mu e^{iS}$$

No $\dot{a}^5$ term $\Rightarrow \int \mathcal{D}a_5 \longrightarrow$ Gauss law constraint $\delta(\partial^\mu\epsilon^\Phi_\mu - e\psi^*\psi)$

Solve constraint + gauge transformation

$$\mathcal{L} = i\psi^* \dot{\psi} - \frac{1}{2M}\psi^*(-i\partial_\mu - ea_{\perp\,\mu})(-i\partial^\mu - ea_\perp^\mu)\psi + \frac{1}{2}a_{\perp\,\mu}\left(\Box + \sigma\partial_\tau^2\right)a_\perp^{\Phi\,\mu}$$

Feynman rules

Matter field propagator $\qquad \dfrac{1}{(2\pi)^5}\dfrac{-i}{\frac{1}{2M}p^2 - P - i\epsilon}$

Photon propagator $\qquad \left[g^{\mu\nu} - \dfrac{k^\mu k^\nu}{k^2}\right]\dfrac{-i}{k^2+\kappa^2-i\epsilon}\;\;\dfrac{1}{1+\lambda^2\kappa^2}$

Three-particle interaction

$$\frac{e}{2M}i(p+p')^\nu\;(2\pi)^5\delta^4(p-p'-k)\delta(P-P'-\kappa)$$

Four-particle interaction

$$\frac{-ie^2}{M}(2\pi)^5 g_{\mu\nu}\delta^4(k-k'-p'+p)\delta(-\kappa+\kappa'+P'-P)$$

Off-Shell Quantum Electrodynamics
Super-renormalizable

Matter propagator

$$G(x,\tau) = \int \frac{d^4k\,d\kappa}{(2\pi)^5} \frac{e^{i(k\cdot x-\kappa\tau)}}{\frac{1}{2M}k^2-\kappa-i\epsilon} = i\theta(\tau)\int \frac{d^4k}{(2\pi)^4} e^{i(k\cdot x-\frac{1}{2M}k^2+i\epsilon)}$$

Retarded causality $\longrightarrow$ no matter loops

Feynman: extract stationary eigenstate of mass operator $-i\hbar\partial_\tau$

$$\int_{-\infty}^{\infty} d\tau e^{-i(m^2/2M)\tau} G(x,\tau) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{\frac{1}{2M}(k^2+m^2)-i\epsilon} = 2M\ \Delta_{\mathrm{F}}(x)$$

Photon propagator

Interaction kernel $\Phi(\tau) \longrightarrow$ cut-off $\left(1+\lambda^2K^2\right)^{-1}$

One divergent photon loop renormalized by shifting mass term $i\psi^*\partial_\tau\psi$

$$G_0^{(2)}(p)\ \left((2\pi)^5\frac{ie_0^2}{M}\right)^2 \frac{1}{\lambda}\int d^4q\,dQ\,\frac{-i}{q^2+Q^2-i\epsilon}\ \ \frac{1}{1+\lambda^2Q^2}\ G_0^{(2)}(p)$$

Identical particles

$$m_1^{\text{in}} = m_2^{\text{in}} = M$$

Mass exchange

$$\Delta m = m_1^{\text{out}} - m_2^{\text{out}}$$

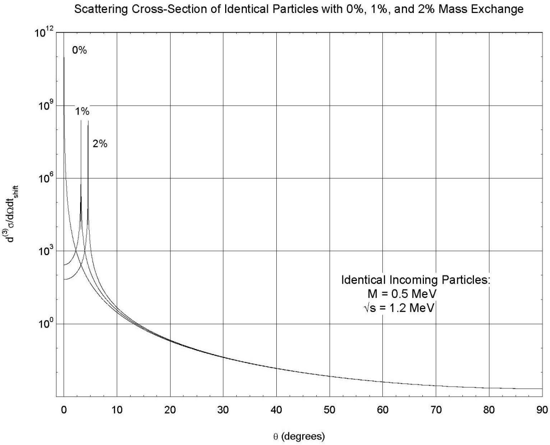
Shifts pole from 0°

Not fixed kinematically

Restricted by cutoff

$$\Delta m < \text{photon mass}$$

$$\lesssim \frac{\hbar}{c^2}\frac{1}{\lambda}$$



Recover Klein-Gordon cross-section

$$\Delta m = 0 \text{ or } \sqrt{s} \gg M$$

General Relativity with τ-Evolution

How to find $g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x,\tau)$ — Land 2018 - 2021

Evolving spacetime

Block universe $\mathcal{M}(\tau)$ evolves to block universe $\mathcal{M}(\tau + d\tau)$

Spacetime metric $g_{\mu\nu}(x,\tau)$ should evolve to $g_{\mu\nu}(x,\tau + d\tau)$

Hint from electrodynamics: could evolve as $g_{\alpha\beta}(x,\tau)$

Must break 5D symmetry — no geodesic equation for $\dot{x}^5$

Leads to $\frac{dK}{d\tau} \neq 0 \ \Rightarrow$ particle mass not conserved

Linearized system $\longrightarrow$ post-Newtonian model with mass acceleration

4+1 formalism (generalizes ADM formalism)

Find $\mathcal{M}$ by foliation of $\mathcal{M}_5 = \mathcal{M} \times \tau$-line

Parameterize $g_{\alpha\beta}(x,\tau)$ as $\gamma_{\mu\nu}(x,\tau)$, lapse, shift

Einstein equations $\longrightarrow$ evolution equations for $\gamma_{\mu\nu}(x,\tau)$ and constraints

Components $g_{5\beta}(x,\tau)$ must be small

Preserve 5D geometry of Ricci tensor — break matter symmetry to 4+1

General Relativity with τ-Evolution

Geometry, evolution, and trajectory

Geometry

Neighboring events in spacetime $\mathcal{M}$ (instantaneous displacement)

Interval $\delta x^2 = \gamma_{\mu\nu} \delta x^\mu \delta x^\nu = (x_2 - x_1)^2$

Invariance of interval: geometrical statement about $\mathcal{M}$

Evolution

$$\mathcal{M}(\tau) \xrightarrow[\text{Hamiltonian } K \text{ generates } \tau\text{-evolution}]{} \mathcal{M}(\tau + d\tau)$$

Symmetries: dynamical statements about K

Trajectory in δx and $\delta\tau$

Neighboring events $X_1 = (x_1, c_5\tau_1) \qquad X_2 = (x_2, c_5(\tau_1 + \delta\tau))$

Distance $X_2 - X_1 = \left(\delta x + \frac{dx(\tau)}{d\tau} \delta\tau, c_5 \delta\tau \right)$

Interval $dX^2 = \left(\delta x + \frac{dx(\tau)}{d\tau} \delta\tau \right)^2 + \sigma c_5^2 \delta\tau^2 = g_{\alpha\beta}(x, \tau)\, \delta x^\alpha \delta x^\beta$

General Relativity with τ-Evolution

Break 5D symmetry $\longrightarrow$ 4D+1

Constrain non-dynamical scalar $x^5 \equiv c_5 \tau$

$$L = \frac{1}{2} M g_{\alpha\beta}(x,\tau) \dot{x}^\alpha \dot{x}^\beta = \frac{1}{2} M g_{\mu\nu} \, \dot{x}^\mu \dot{x}^\nu + M c_5 \, g_{\mu 5} \dot{x}^\mu + \frac{1}{2} M c_5^2 \, g_{55}$$

Euler-Lagrange $\longrightarrow$ geodesic equations

$$0 = \frac{D\dot{x}^\alpha}{D\tau} = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma \longrightarrow \begin{cases} \ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma} \dot{x}^\lambda \dot{x}^\sigma + 2 c_5 \Gamma^\mu_{5\sigma} \dot{x}^\sigma + c_5^2 \Gamma^\mu_{55} = 0 \\ \ddot{x}^5 = \dot{c}_5 \equiv 0 \end{cases}$$

Hamiltonian

$$K = p_\mu \dot{x}^\mu - L = \frac{1}{2} M g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} M c_5^2 \, g_{55} = L - M c_5^2 \, g_{55}$$

$$\frac{dK}{d\tau} = -\frac{1}{2} M \dot{x}^\mu \dot{x}^\nu \frac{\partial g_{\mu\nu}}{\partial \tau} - \frac{1}{2} M c_5^2 \frac{\partial g_{55}}{\partial \tau}$$

particle mass not generally conserved

Non-thermodynamic dust

Number of events per spacetime volume $= n(x,\tau)$

Particle mass density $= \rho(x,\tau) = Mn(x,\tau)$

5-component event current $= j^\alpha(x,\tau) = \rho(x,\tau)\dot{x}^\alpha(\tau) = Mn(x,\tau)\dot{x}^\alpha(\tau)$

Matter current is vector $j^\mu(x,\tau)$ and scalar $j^5(x,\tau)$

Continuity equation

$$\nabla_\alpha j^\alpha = \nabla_\mu j^\mu + \frac{1}{c^5}\partial_\tau \rho c^5 = \nabla_\mu j^\mu + \partial_\tau \rho$$

Mass-energy-momentum tensor

$$\nabla_\beta T^{\alpha\beta} = 0 \qquad\qquad T^{\alpha\beta} = \rho\dot{x}^\alpha\dot{x}^\beta \longrightarrow \begin{cases} T^{\mu\nu} = \rho\dot{x}^\mu\dot{x}^\nu \\ T^{5\beta} = c_5 j^\beta \end{cases}$$

Einstein equations in 5D (unmodified)

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Small perturbation to flat metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \longrightarrow \partial_\gamma g_{\alpha\beta} = \partial_\gamma h_{\alpha\beta} \qquad \left(h_{\alpha\beta}\right)^2 \approx 0 \qquad h \simeq \eta^{\alpha\beta} h_{\alpha\beta}$$

Define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$ and impose gauge condition $\partial_\lambda \bar{h}^{\alpha\lambda} = 0$

Einstein equations: $\dfrac{16\pi G}{c^4} T_{\alpha\beta} = -\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} = -\left(\partial^\mu \partial_\mu + \dfrac{\eta_{55}}{c_5^2}\partial_\tau^2\right)\bar{h}_{\alpha\beta}$

Solve using leading term in Green's function for 5D wave equation

$$\bar{h}_{\alpha\beta}(x,\tau) = \frac{4G}{c^4} \int d^3x' \, \frac{T_{\alpha\beta}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau\right)}{|\mathbf{x}-\mathbf{x}'|}$$

Evolving spacetime event

$X^{\alpha}(\tau) = \left(X^{\mu}\left(\tau\right), c_5\tau\right)$ with notation $\xi^{\alpha}(\tau) = \frac{1}{c}u^{\alpha}\left(\tau\right) = \frac{1}{c}\dot{X}^{\alpha}$

5D interval conserved

$$\frac{d}{d\tau}u^2 = 2u_{\alpha}\frac{Du^{\alpha}}{D\tau} = 0$$

Choose value in rest frame: $u = (c, 0, 0, 0, c_5)$

$$u^2 = c^2\xi^2 = -c^2 + \sigma c_5^2 \longrightarrow \xi^2 = -1 + \sigma\xi_5^2 \approx -1$$

Spacetime particle density

$$\rho\left(x, \tau\right) = \rho\left(x - X\left(\tau\right)\right)$$

Mass-energy-momentum tensor

$$T^{\alpha\beta} = m\rho\left(x, \tau\right)\dot{X}^{\alpha}\dot{X}^{\beta} = m\rho\left(x, \tau\right)u^{\alpha}u^{\beta} = mc^2\rho\left(x, \tau\right)\xi^{\alpha}\xi^{\beta}$$

Metric from leading term in Green's function

$$\bar{h}_{\alpha\beta}(x,\tau) = \mathcal{G}\left[T_{\alpha\beta}\right] = \frac{4Gm}{c^2}\xi_\alpha\xi_\beta \int d^3x' \frac{\rho\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}', \tau\right)}{|\mathbf{x}-\mathbf{x}'|}$$

Full metric from trace

$$\eta^{\alpha\beta}\bar{h}_{\alpha\beta} = \eta^{\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h\right) = \frac{2-D}{2}h \longrightarrow h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{D-2}\eta_{\alpha\beta}\bar{h}$$

In $D = 5$ for static source $\xi = (1, \mathbf{0}, c_5/c)$

$$h_{00} = \frac{2}{3}\mathcal{G}\left[T_{00}\right] \qquad h_{05} = \frac{2}{3}\sigma\xi_5\mathcal{G}\left[T_{00}\right]$$

$$h_{ij} = \frac{1}{3}\delta_{ij}\mathcal{G}\left[T_{00}\right] \qquad h_{55} = \frac{1}{3}\sigma\mathcal{G}\left[T_{00}\right]$$

Expect: $h_{00} \sim h_{ii}$ and $h_{55} \ll h_{00}$ for consistency with standard GR

Modified $\eta_{\alpha\beta}$ explicitly breaks 5D symmetry in presence of matter

$$R_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta} \qquad \bar{\eta}_{\mu\nu} = \eta_{\mu\nu} \qquad \bar{\eta}_{5\alpha} = 0$$

Trace reversed linearized Einstein equation

$$R_{\mu\nu} = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}\bar{\eta}_{\mu\nu}\bar{T}\right)$$

$$R_{5\alpha} = -\frac{1}{2}\partial^\gamma\partial_\gamma h_{5\alpha} = \frac{8\pi G}{c^4}T_{5\alpha}$$

Modified solution

$$h_{00} = \frac{1}{2}\mathcal{G}\left[T_{00}\right] \qquad h_{05} = \sigma\xi_5\mathcal{G}\left[T_{00}\right]$$

$$h_{ij} = \frac{1}{2}\delta_{ij}\mathcal{G}\left[T_{00}\right] \qquad h_{55} = \sigma\xi_5^2\mathcal{G}\left[T_{00}\right]$$

For a source $X(\tau) = (c\tau, \mathbf{0})$ and taking $\rho(\mathbf{x}) = \delta^3(\mathbf{x})$

$$g_{\mu\nu} = \left(-1 + \frac{2Gm}{c^2 r}, \left(1 + \frac{2Gm}{c^2 r}\right)\delta_{ij}\right)$$
$$\approx \left(-\left(1 - \frac{2Gm}{c^2 r}\right), \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\delta_{ij}\right)$$

Consistent with spherically symmetric Schwarzschild metric

$$g_{55} = \sigma\left(1 + \sigma\xi_5^2\left(\frac{2Gm}{c^2 r}\right)\right) = \sigma + o\left(\frac{c_5^2}{c^2}\right)$$

Approach distinguishes evolution from geometry

Preserves 5D symmetry of Ricci tensor $R_{\alpha\beta}$ (geometry)

Breaks 5D symmetry in relationship between $R_{\alpha\beta}$ and $T_{\alpha\beta}$ (physics)

Point source in co-moving frame: $\dot{T} \neq 1 \;\Rightarrow\;$ mass acceleration

$$X = (cT(\tau), \mathbf{0}) \qquad \dot{T} = 1 + \alpha\,(\tau)\,/2 \qquad \alpha^2 \approx 0$$

Mass distribution

$$M(x,\tau) = m\,\delta^3\,(\mathbf{x})\,\rho\,(t - T\,(\tau))$$

$$T^{00} = M(x,\tau)c^2\,\dot{T}^2 \qquad T^{\alpha i} = 0 \qquad T^{55} = \frac{c_5^2}{c^2}T^{00} \approx 0$$

Metric and connection perturbations from Green's function

$$\bar{h}^{00}\,(x,\tau) = \frac{4GM}{c^2R}\dot{T}^2 \qquad \bar{h}^{\alpha i}\,(x,\tau) = 0 \qquad \bar{h}^{55}\,(x,\tau) = \frac{c_5^2}{c^2}\bar{h}^{00} \approx 0$$

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} \qquad \Gamma^{\mu}_{50} = \frac{1}{2c_5}\eta^{\mu 0}\partial_\tau h_{00}$$

Acceleration in time coordinate (neglecting velocity of test particle)

$$\ddot{t} = \left(\partial_\tau h_{00}\right)\dot{t} + \dot{\mathbf{x}}\cdot\left(\nabla h_{00}\right)\dot{t}^2 \approx \frac{2GM}{c^2 R}\left(1+\frac{\alpha\left(\tau\right)}{2}\right)\dot{\alpha}\left(\tau\right)\dot{t}$$

Angular and radial equations

Put $\theta = \pi/2$

$$\ddot{\mathbf{x}} = \frac{c^2}{2}\left(\nabla h_{00}\right)\dot{t}^2 \longrightarrow \begin{cases} 2\dot{R}\dot{\phi} + R\ddot{\phi} = 0 \;\longrightarrow\; \dot{\phi} = \dfrac{L}{MR^2} \\ \ddot{R} - \dfrac{L^2}{M^2R^3} = -\dfrac{GM}{R^2}\dot{t}^2\dot{T}^2 \end{cases}$$

Angular momentum conserved

Post-Newtonian term on RHS of radial equation

$\alpha = 0 \;\Rightarrow\; \dot{T} = 1 \;\Rightarrow\; \dot{t}^2\dot{T}^2 = 1$ recovers Newtonian gravitation

Neglecting $\dot{R}/c \ll 1$ and $\partial_\tau \rho \approx 0$

$$\dot{t} = \exp\left[\frac{2GM}{c^2 R}\left(\alpha + \frac{1}{4}\alpha^2\right)\right] \longrightarrow \dot{t}^2 \dot{T}^2 \simeq 1 + \frac{1}{2}\left(1 + \frac{2GM}{c^2 R}\right)\alpha$$

Using solution to t equation in radial equation

$$\frac{d}{d\tau}\left\{\frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{L^2}{M^2 R^2} - \frac{GM}{R}\left(1 + \frac{1}{2}\alpha(\tau)\right)\right\} = -\frac{GM}{2R}\frac{d}{d\tau}\alpha(\tau)$$

LHS is $\dfrac{d}{d\tau}$ (particle Hamiltonian) $= \dfrac{d}{d\tau}$ (particle mass)

$\dot{T} \neq 1 \Rightarrow$ energy change without in source rest frame $\Rightarrow$ mass change

Mass transfer across spacetime

- Source transfers mass to perturbed metric field $h_{\alpha\beta}$
- Test particle absorbs mass from $h_{\alpha\beta} \longrightarrow$ particle mass not conserved

$\alpha = 0$ recovers mass conservation

Modified field equation

$$R_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}R = \frac{8\pi G}{c^4}T_{\alpha\beta} \quad \longleftrightarrow \quad R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}T\right)$$

Trace using $\bar{\eta}^{\alpha\beta} \quad (\bar{\eta}_{\mu\nu} = \eta_{\mu\nu}, \;\; \bar{\eta}_{5\alpha} = 0)$

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4}\eta^{\mu\nu}T_{\mu\nu} \;\longrightarrow\; -R = \frac{8\pi G}{c^4}\left(T - \sigma T_{55}\right)$$

Write trace reversed Einstein equation

$$R_{\alpha\beta} + \bar{\eta}_{\alpha\beta}\Lambda = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}\bar{\eta}_{\alpha\beta}T\right)$$

Identifying the mass density Λ as a cosmological term

$$\Lambda = -\frac{8\pi G}{c^4}\sigma T_{55} \qquad\qquad T_{55} \sim \frac{c_5^2}{c_2}T_{00} \ll T_{00}$$

General Relativity with τ-Evolution

4+1 formalism for metric evolution

Extension of 3+1 / ADM formalism

Decompose $\mathcal{M}$ into spacelike hypersurface + normal time direction

Einstein equations $\longrightarrow$ 6 t-evolution equations for γ_{ij} + 4 constraints

Construct $\mathcal{M}_5 = \mathcal{M} \times R$ with coordinates $X = (x, c_5\tau)$

Admixture of 4D spacetime geometry and τ-evolution

In flat pseudo-spacetime $g_{\alpha\beta} \to \eta_{\alpha\beta} = \text{diag}\,(-1, 1, 1, 1, \sigma)$ where $\sigma = \pm 1$

External τ $\longrightarrow$ natural foliation

Decompose $\mathcal{M}_5$ into spacetime hypersurface $\sum_\tau$ + normal τ direction

5D metric $g_{\alpha\beta} \longrightarrow \{\gamma_{\mu\nu}(x,\tau)$, lapse N, and (tangent) shift $N^\mu\}$

Einstein equations $\longrightarrow$ 10 τ-evolution equations for $\gamma_{\mu\nu}$ + 5 constraints

4D hypersurface

$$\sum_{\tau_0} = \left\{ X \in \mathcal{M}_5 \;\middle|\; S(X) = 0 \right\} \text{ where } S(X) = X^5 / c_5 - \tau_0$$

Rank 4 Jacobian

$$E^\alpha_\mu = \left(\frac{\partial X^\alpha}{\partial x^\mu} \right)_{\tau_0} \longrightarrow E_\mu = \partial_\mu = \partial / \partial x^\mu \quad \text{as basis for tangent space of } \sum_{\tau_0}$$

Unit normal to $\sum_{\tau_0}$

$$n_\alpha = \sigma \left| g^{55} \right|^{-1/2} \partial_\alpha S\left(X\right) \longrightarrow \begin{cases} n \cdot E_\mu = n_\alpha E^\alpha_\mu = 0 \\ n^2 = g^{\alpha\beta} n_\alpha n_\beta = \sigma \end{cases}$$

Induced metric on $\sum_{\tau_0}$

$$ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta = g_{\alpha\beta} \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu = \gamma_{\mu\nu} dx^\mu dx^\nu$$

Parameterize time evolution

$$\tau \longrightarrow \tau + \delta\tau \ \Rightarrow X^\alpha \longrightarrow X^\alpha + \left(\frac{\partial X^\alpha}{\partial \tau}\right)_{x_0} \delta\tau = X^\alpha + \left(N n^\alpha + N^\mu E^\alpha_\mu\right) \delta\tau$$

Under spacetime displacement

N: lapse
N^μ: shift

$$X^\alpha \longrightarrow X^\alpha + \left(\frac{\partial X^\alpha}{\partial x^\mu}\right)_{\tau_0} \delta x^\mu = X^\alpha + E^\alpha_\mu \delta x^\mu$$

5D displacement and interval

$$dX^\alpha = N n^\alpha c_5 d\tau + E^\alpha_\mu \left(N^\mu c_5 d\tau + dx^\mu\right) \qquad ds^2 = g_{\alpha\beta} dX^\alpha dX^\beta$$

Decomposition of metric using $n^2 = \sigma \qquad n_\alpha E^\alpha_\mu = 0 \qquad \gamma_{\mu\nu} = g_{\alpha\beta} E^\alpha_\mu E^\beta_\nu$

$$g_{\alpha\beta} = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} \qquad g^{\alpha\beta} = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix}$$

Projector onto tangent 4D hypersurface $\sum_{\tau_0}$

$$P_{\alpha\beta} = g_{\alpha\beta} - \sigma n_\alpha n_\beta \qquad n_\alpha = \text{unit normal}$$

Projected covariant derivative

$$\text{For } V^\beta \in \mathcal{M}_5 \qquad D_\alpha V_\beta = P_\alpha^\gamma P_\beta^\delta \nabla_\gamma V_\delta$$

Projected curvature

$$\left[D_\alpha, D_\beta\right] V_\perp^\gamma = \bar{R}^\gamma_{\delta\alpha\beta} V_\perp^\delta$$

Extrinsic curvature: evolution of the unit normal

$$K_{\alpha\beta} = -P_\alpha^\gamma P_\beta^\delta \, \nabla_\delta n_\gamma$$

Express 4D Ricci tensor in terms of $\bar{R}_{\mu\nu}$ and $K_{\mu\nu}$

Lie derivatives of $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ $\longrightarrow$ integrable first order PDEs

$\mathcal{L}_\tau$: Lie derivative in τ direction
$K_{\mu\nu}$: Extrinsic curvature
$S_{\mu\nu}$: Spacetime projection of $T_{\alpha\beta}$
κ: Mass density T_{55}
$R_{\mu\nu}$: Projection of Ricci tensor

Evolution equation for spacetime metric

$$\frac{1}{c_5}\mathcal{L}_\tau \gamma_{\mu\nu} - \mathcal{L}_{\mathbf{N}} \gamma_{\mu\nu} = -2NK_{\mu\nu}$$

Evolution equation for extrinsic curvature

$$\left(\frac{1}{c_5}\mathcal{L}_\tau - \mathcal{L}_{\mathbf{N}}\right) K_{\mu\nu} = -D_\mu D_\nu N + N\left\{-\sigma\bar{R}_{\mu\nu} + KK_{\mu\nu} - 2K^\lambda_\mu K_{\nu\lambda} + \sigma\frac{8\pi G}{c^4}\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S + \sigma\kappa\right)\right]\right\}$$

Hamiltonian Constraint

$$\bar{R} - \sigma\left(K^2 - K^{\mu\nu}K_{\mu\nu}\right) = -\sigma\frac{16\pi G}{c^4}\kappa$$

Momentum Constraint

$$D_\mu K^\mu_\nu - D_\nu K = \frac{8\pi G}{c^4}p_\nu$$

Linearized metric

$$\|g_{\alpha\beta}\| = \begin{bmatrix} \gamma_{\mu\nu} & N_\mu \\ N_\mu & \sigma N^2 + \gamma_{\mu\nu} N^\mu N^\nu \end{bmatrix} = \begin{bmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{\mu 5} & \eta_{55} + h_{55} \end{bmatrix}$$

$$\left\|g^{\alpha\beta}\right\| = \begin{bmatrix} \gamma^{\mu\nu} + \sigma \frac{1}{N^2} N^\mu N^\nu & -\sigma \frac{1}{N^2} N^\mu \\ -\sigma \frac{1}{N^2} N^\mu & \sigma \frac{1}{N^2} \end{bmatrix} \approx \begin{bmatrix} \eta^{\lambda\nu} - h^{\mu\nu} & -\sigma h_5^\mu \\ -\sigma h_5^\mu & \sigma \left(1 - \sigma h_{55}\right) \end{bmatrix}$$

Leads to

$$N = \frac{1}{\sqrt{1 - \sigma h_{55}}} \approx 1 + \frac{1}{2} \sigma h_{55} \qquad N_\mu = h_{5\mu}$$

$$n_\alpha = \sigma \left(1 + \frac{1}{2} \sigma h_{55}\right) \delta_\alpha^5 \qquad n^\alpha = -h_5^\mu \delta_\mu^\alpha + \left(1 - \frac{1}{2} \sigma h_{55}\right) \delta_5^\alpha$$

By direct calculation

$$K_{\alpha\beta} = -P_\alpha^\gamma P_\beta^\delta \ \nabla_\delta \left[\sigma \left(1 + \frac{1}{2}\sigma h_{55} \right) \delta_5^\delta \right] = \sigma \Gamma_{\mu\nu}^5$$

Evolution equation for metric in 4+1 decomposition

$$-\frac{1}{2} \left(\frac{1}{c_5} \partial_\tau - \mathcal{L}_\mathbf{N} \right) \gamma_{\mu\nu} = N K_{\mu\nu}$$

Discarding terms $\left(h_{\alpha\beta}\right)^2 \approx 0$

$$\mathcal{L}_\mathbf{N} \gamma_{\mu\nu} = D_\mu N_\nu + D_\nu N_\mu \approx \partial_\mu N_\nu + \partial_\nu N_\mu = \partial_\mu h_{5\nu} + \partial_\nu h_{5\mu}$$

Evolution equation becomes

$$-\frac{1}{2} \left(\partial_5 \gamma_{\mu\nu} - \partial_\mu h_{5\nu} - \partial_\nu h_{5\mu} \right) = N K_{\mu\nu} \approx K_{\mu\nu}$$

LHS $= \sigma \Gamma_{\mu\nu}^5 \ \longrightarrow$ automatically satisfied

Bianchi identity for linearized theory

$$\nabla_\alpha G^{\alpha\beta} = \nabla_\alpha \left(R^{\alpha\beta} - \frac{1}{2}\bar{\eta}^{\alpha\beta} R \right) = \partial_\alpha \left(R^{\alpha\beta} - \frac{1}{2}\bar{\eta}^{\alpha\beta} R \right) + o\left(h^2_{\alpha\beta} \right) = 0$$

Rearranged as $\frac{1}{c_5}\partial_\tau G^{5\beta} = -\partial_\mu G^{\mu\beta} + o\left(h^2_{\alpha\beta} \right)$

RHS must contain terms in $g_{\alpha\beta}$, $\partial_\tau g_{\alpha\beta}$, and $\partial^2_\tau g_{\alpha\beta}$

$G^{5\beta}$ contains no second order τ-derivatives of $g_{\alpha\beta}$

Constraints

Initial conditions for second order PDE are $g_{\alpha\beta}$, $\partial_\tau g_{\alpha\beta}$, $T_{\alpha\beta}$

$G^{5\beta}$ field equation is relationship among initial conditions

Five constraint equations: propagate without evolving

$$G_{5\beta} = R_{5\beta} - \frac{1}{2}\bar{\eta}_{5\beta} R = \frac{8\pi G}{c^4} T_{5\beta}$$

Separate components of 5D Ricci tensor

$$R^{(5)}_{\alpha\beta} = \frac{1}{2}\Big(\partial_\alpha\partial^\lambda h_{\lambda\beta} + \partial_\beta\partial^\sigma h_{\alpha\sigma} - \partial^\lambda\partial_\lambda h_{\alpha\beta} - \partial_\alpha\partial_\beta\bar{\eta}^{\lambda\sigma}h_{\lambda\sigma}$$
$$+\partial_\alpha\partial^5 h_{5\beta} + \partial_\beta\partial^5 h_{\alpha 5} - \partial^5\partial_5 h_{\alpha\beta} - \partial_\alpha\partial_\beta\bar{\eta}^{55}h_{55}\Big)$$

Spacetime components

$$R^{(5)}_{\mu\nu} = R^{(4)}_{\mu\nu} + \sigma\partial_5 \underbrace{\frac{1}{2}\left(\partial_\mu h_{5\nu} + \partial_\nu h_{\mu 5} - \partial_5 h_{\mu\nu}\right)}_{K_{\mu\nu}} - \frac{1}{2}\partial_\mu\partial_\nu\bar{\eta}^{55}h_{55}$$

4D Ricci tensor $R^{(4)}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial^\lambda h_{\lambda\nu} + \partial_\nu\partial^\sigma h_{\mu\sigma} - \partial^\lambda\partial_\lambda h_{\mu\nu} - \partial_\mu\partial_\nu\eta^{\lambda\sigma}h_{\lambda\sigma}\right)$

5 components

$$R_{5\beta} = \frac{1}{2}\Big(\partial_5\partial^\lambda h_{\lambda\beta} + \partial_\beta\partial^\sigma h_{5\sigma} - \partial^\lambda\partial_\lambda h_{5\beta} - \partial_5\partial_\beta\bar{\eta}^{\lambda\sigma}h_{\lambda\sigma}$$
$$+\partial_\beta\partial^5 h_{55} - \partial_\alpha\partial_\beta\bar{\eta}^{55}h_{55}\Big)$$

Spacetime components of 5D Ricci tensor

$$R^{(5)}_{\mu\nu} = R^{(4)}_{\mu\nu} + \sigma \partial_5 K_{\mu\nu} - \frac{1}{2} \partial_\mu \partial_\nu \bar{\eta}^{55} h_{55}$$

Spacetime part of modified field equation

$$R^{(5)}_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} \bar{\eta}_{\mu\nu} \bar{T} \right)$$

Rearranging terms

$$\partial_5 K_{\mu\nu} = \frac{1}{2} \partial_\mu \partial_\nu \bar{\eta}^{55} h_{55} - \sigma R^{(4)}_{\mu\nu} + \sigma \frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{T} \right)$$

Evolution equation for $K_{\mu\nu}$ in 4+1 decomposition

Under $\left(h_{\alpha\beta}\right)^2 \approx 0 \qquad \eta^{55} \longrightarrow \bar{\eta}^{55} = 0 \qquad \bar{T} + \eta_{55}\kappa \longrightarrow \bar{T}$

Applying Lorenz condition

$$\partial^\lambda h_{\alpha\lambda} = \frac{1}{2}\partial_\alpha \bar{\eta}^{\lambda\sigma} h_{\lambda\sigma} + \frac{1}{2}\partial_\alpha \eta^{55} h_{55} - \partial^5 h_{\alpha 5}$$

and field equation

$$R_{5\alpha} = -\frac{1}{2}\partial^\gamma \partial_\gamma h_{5\alpha} = \frac{8\pi G}{c^4} T_{5\alpha}$$

confirms

$$R_{5\beta} = \frac{1}{2}\left(-\partial_5 \partial^5 h_{\beta 5} - \partial^\lambda \partial_\lambda h_{5\beta}\right) = -\frac{1}{2}\partial^\gamma \partial_\gamma h_{5\beta} = \frac{8\pi G}{c^4} T_{5\beta}$$

providing expressions for the non-dynamical shift and lapse N^μ and N

Constraint: Gradient of extrinsic curvature

$$\partial^\mu K_{\mu\nu} = \frac{1}{2}\left(\partial^\mu \partial_\mu h_{5\nu} + \partial^\mu \partial_\nu h_{5\mu} - \partial_5 \partial^\mu h_{\mu\nu}\right) = \frac{1}{2}\partial^\gamma \partial_\gamma h_{5\nu} = \frac{8\pi G}{c^4} T_{5\nu}$$

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Thank You

For Your

Patience