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Ultraviolet divergences in supersymmetric theories regularized by higher derivatives

Althoght supersymmetry has not yet been discovered experimentally, at present there are some indirect evidences that it is really present in high energy physics. Namely,

- In supersymmetric theories running coupling constants are unified in agreement with the prediction of Grand Unification Theories.
- Supersymmetry explains absence of experimental data on the proton decay, which is inevitable in Grand Unification Theories.
- Supersymmetry forbids quadratically divergent quantum correction to a mass of the Higgs boson and does not require its fine tuning at the Grand Unification scale.
- Supersymmetry predicts existence of a light Higgs boson with a mass close to m_Z.

Investigation of quantum corrections in supersymmetric theories is very important both for theory and for phenomenology.
In supersymmetric theories possible ultraviolet divergences are restricted by some non-renormalization theorems. The most known of them are the following:

- $\mathcal{N}=1$ superpotential does not receive divergent quantum corrections.
- $\mathcal{N}=2$ supersymmetric gauge theories are finite starting from the two-loop approximation.
- $\mathcal{N}=4$ supersymmetric Yang–Mills theory is finite in all loops.

Other non-renormalization theorems will be discussed below.

The non-renormalization theorems allow constructing finite theories with $\mathcal{N}<4$ supersymmetry. For $\mathcal{N}=2$ supersymmetric theories it is made by a special choice of a gauge group and representations for the matter superfields. For $\mathcal{N}=1$ supersymmetric thories and theories with softly broken supersymmetry is also necessary to make a special tuning of a renormalization scheme.

The exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function can also be considered as a non-renormalization theorem.

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

It relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

Three- and four-loop calculations in $\mathcal{N}=1$ supersymmetric theories made with dimensional reduction supplemented by modified minimal subtraction (i.e. in the so-called $\overline{\text{DR}}$-scheme)

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

revealed that the NSVZ relation in the $\overline{\text{DR}}$-scheme holds only in the one- and two-loop approximations, where the β-function is scheme independent. (The NSVZ relation relates the two-loop β-function to the one-loop anomalous dimension, which is also scheme independent.)

However, in the three- and four-loop approximations it is possible to restore the NSVZ relation with the help of a specially tuned finite renormalization of the gauge coupling constant. Note that a possibility of making this finite renormalization is highly nontrivial.

This implies that the NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\text{DR}}$-scheme is not NSVZ.

The higher covariant derivative regularization

Here we would like to derive the exact NSVZ β-function in all loops by direct summation of the perturbative series and to formulate an all-loop renormalization prescription which gives an NSVZ scheme.

The main ingredient which allows doing this is the higher covariant derivative regularization proposed by A.A.Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301;
Theor.Math.Phys. **13** (1972) 1064.

By construction, it includes insertion of the Pauli–Villars determinants for removing residual one-loop divergencies

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

Unlike dimensional reduction, this regularization is self-consistent. It can be formulated in a manifestly supersymmetric way in terms of $\mathcal{N}=1$ superfields

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

The simplest example of an $\mathcal{N}=1$ supersymmetric gauge theory is $\mathcal{N}=1$ SQED with N_f flavors, which in the massless limit is described by the action

$$S=\frac{1}{4e_0^2}\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_\alpha^*e^{2V}\phi_\alpha+\widetilde\phi_\alpha^*e^{-2V}\widetilde\phi_\alpha\Big),$$

where V is a real gauge superfield, ϕ_α and $\widetilde\phi_\alpha$ with $\alpha=1,\ldots,N_f$ are chiral matter superfields. The supersymmetric field strength in the Abelian case is defined as $W_a=\bar D^2D_aV/4$. For this theory $C_2=0,\quad C(R)=I,\quad T(R)=2N_f,\quad r=1$, where I is the $2N_f\times 2N_f$ identity matrix.

In this case the NSVZ β-function takes the form

$$\beta(\alpha)=\frac{\alpha^2N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. **B 166** (1986) 334.

This equation relates the L-loop β-function to the $(L-1)$-loop anomalous dimension of the matter superfields $\gamma(\alpha)$.

The higher derivative regularization for $\mathcal{N}=1$ supersymmetric electrodynamics

To regularize $\mathcal{N}=1$ SQED by higher derivatives, we first add to its action a term containing higher derivatives. Then the regularized action takes the form

$$S_{\text{reg}} = \frac{1}{4e_0^2}\text{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a + \sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_\alpha^* e^{2V}\phi_\alpha + \widetilde{\phi}_\alpha^* e^{-2V}\widetilde{\phi}_\alpha\Big),$$

where $R(\partial^2/\Lambda^2)$ is a regulator function, e.g., $R = 1 + \partial^{2n}/\Lambda^{2n}$.

After the adding of the higher derivative term divergences survive only in the one-loop approximation. To regularize the residual one-loop (sub)divergences, we insert the Pauli–Villars determinants into the generating functional,

$$Z[J,j,\widetilde{j}] = \int D\mu\,\Big(\det PV(V,M)\Big)^{N_f} \exp\Big\{iS_{\text{reg}} + iS_{\text{gf}} + S_{\text{sources}}\Big\}.$$

Masses of the Pauli–Villars superfields should satisfy the important condition $M = a\Lambda$ with $a \neq a(e_0)$.

Different definition of renormalization group functions

It is important to distinguish renormalization group functions (RGFs) defined in terms of the bare coupling constant α_0,

$$\beta(\alpha_0) \equiv \frac{d\alpha_0(\alpha, \Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}}; \qquad \gamma(\alpha_0) \equiv -\frac{d\ln Z(\alpha, \Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha=\text{const}},$$

and RGFs standardly defined in terms of the renormalized coupling constant α,

$$\widetilde{\beta}(\alpha) \equiv \frac{d\alpha(\alpha_0, \Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}; \qquad \widetilde{\gamma}(\alpha) \equiv \frac{d\ln Z(\alpha_0, \Lambda/\mu)}{d\ln\mu}\Big|_{\alpha_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare coupling constant do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defrined in terms of the renormalized coupling constant depend both on regularization and on a renormalization prescription.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha)\Big|_{\mbox{HD+MSL}} = \beta(\alpha_0\to\alpha);\qquad \widetilde{\gamma}(\alpha)\Big|_{\mbox{HD+MSL}} = \gamma(\alpha_0\to\alpha).$$

A key observation needed for derivation of the NSVZ relation is that in the case of using the higher derivative regularization the integrals giving the β-function defined in terms of the bare coupling constant are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories.

A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The three-loop β-function of $\mathcal{N}=1$ SQED as an integral of double total derivatives

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = N_f \frac{d}{d\ln\Lambda}\Big\{2\pi\int \frac{d^4Q}{(2\pi)^4}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\frac{\ln(Q^2+M^2)}{Q^2} + 4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{e^2}{K^2R_K^2}$$

$$\times\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\left(\frac{1}{Q^2(K+Q)^2} - \frac{1}{(Q^2+M^2)((K+Q)^2+M^2)}\right)\Big[R_K\Big(1+\frac{e^2N_f}{4\pi^2}\ln\frac{\Lambda}{\mu}\Big)$$

$$-2e^2N_f\Big(\int\frac{d^4L}{(2\pi)^4}\frac{1}{L^2(K+L)^2} - \int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2+M^2)((K+L)^2+M^2)}\Big)\Big]$$

$$+4\pi\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e^4}{K^2R_KL^2R_L}\frac{\partial}{\partial Q^\mu}\frac{\partial}{\partial Q_\mu}\Big\{\Big(-\frac{2K^2}{Q^2(Q+K)^2(Q+K+L)^2}$$

$$\times\frac{1}{(Q+L)^2} + \frac{2}{Q^2(Q+K)^2(Q+L)^2}\Big) - \Big(-\frac{2(K^2+M^2)}{((Q+K)^2+M^2)((Q+L)^2+M^2)}$$

$$\times\frac{1}{(Q^2+M^2)((Q+K+L)^2+M^2)} + \frac{2}{(Q^2+M^2)((Q+K)^2+M^2)((Q+L)^2+M^2)}$$

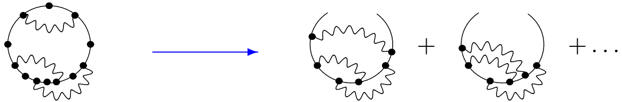
$$-\frac{4M^2}{(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)}\Big) + O(e^6)\Big\}$$

Integrals of double total derivatives and a graphical interpretation of the NSVZ relation for $\mathcal{N}=1$ SQED

The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decrease at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4} \frac{\partial}{\partial Q^\mu} \frac{\partial}{\partial Q_\mu} \Big(\frac{f(Q^2)}{Q^2}\Big) = \int\limits_{S^3_\varepsilon} \frac{dS^\mu}{(2\pi)^4} \Big(- \frac{2Q_\mu}{Q^4} f(Q^2) + \frac{2Q_\mu}{Q^2} f'(Q^2)\Big)$$
$$= \frac{1}{4\pi^2} f(0) \ne 0.$$

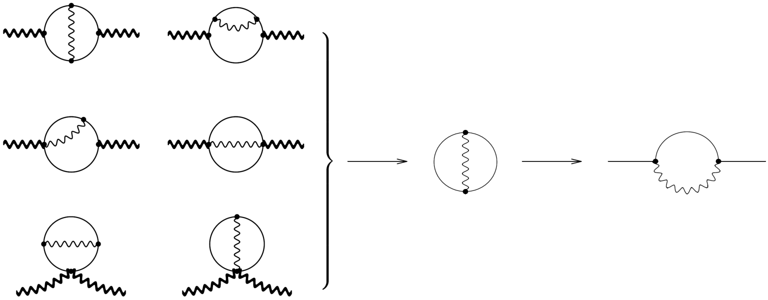
Due to similar equations the double total derivatives effectively cut a loop of the matter superfields. As a result we obtain diagrams contributing to the anomalous dimension of the matter superfields, in which a number of loops is less by 1.



This allows to give a simple graphical interpretation of the NSVZ relation for the considered Abelian case.

For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the gauge superfield to the corresponding contribution to the anomalous dimension of matter superfields obtained by cuts of the matter line:

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.



RGFS defined in terms of the bare coupling constant obtained after calculating the integrals of double total derivatives and the integrals which determine the two-loop anomalous dimension are given by the expressions

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} + \frac{\alpha_0 N_f}{\pi^2} - \frac{\alpha_0^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha_0^3);$$

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{\pi^2}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha_0^3).$$

where

$$A \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{R(x)}; \qquad a = \frac{M}{\Lambda}.$$

They do not depend on finite constants b_i and g_i, which specify the renormalization scheme an satisfy the NSVZ relation.

RGFs defined in terms of the renormalized coupling constant are written as

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} + N_f(b_2 - b_1)\Big) + O(\alpha^3)$$

$$\widetilde{\gamma}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} - N_f b_1 + N_f g_1\Big) + O(\alpha^3).$$

We see that RGFs defined in terms of the renormalized coupling constant depend on a renormalization scheme due to the dependence of the finite constants b_i and g_i. The constants b_i in the considered three-loop approximation are defined by the equation

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_1\Big) - \frac{\alpha N_f}{\pi^2}\Big(\ln\frac{\Lambda}{\mu} + b_2\Big) - \frac{\alpha^2 N_f}{\pi^3}\Big(\frac{N_f}{2}\ln^2\frac{\Lambda}{\mu}$$
$$-\ln\frac{\Lambda}{\mu}\Big(N_f\ln a + N_f + \frac{N_f A}{2} + \frac{1}{2} - N_f b_1\Big) + b_3\Big) + O(\alpha^3).$$

Similarly, the finite constants g_i appear in the two-loop expression for the renormalization constant of the matter superfields Z, which is not also uniquely defined,

$$Z = 1 + \frac{\alpha}{\pi}\Big(\ln\frac{\Lambda}{\mu} + g_1\Big) + \frac{\alpha^2(N_f+1)}{2\pi^2}\ln^2\frac{\Lambda}{\mu}$$
$$-\frac{\alpha^2}{\pi^2}\ln\frac{\Lambda}{\mu}\Big(N_f\ln a - N_f b_1 + N_f + \frac{N_f A}{2} + \frac{1}{2} - g_1\Big) + \frac{\alpha^2 g_2}{\pi^2} + O(\alpha^3).$$

The choice of the constants b_i and g_i fixes a renormalization scheme in the considered approximation.

In the HD+MSL scheme all these finite constants vanish,

$$g_2 = b_1 = b_2 = b_3 = 0,$$

and both definition of RGFs give the same functions up to the formal replacing of arguments. In particular in the considered approximation

$$\frac{\widetilde{\beta}(\alpha)}{\alpha^2} = \frac{N_f}{\pi} + \frac{\alpha N_f}{\pi^2} - \frac{\alpha^2 N_f}{\pi^3}\Big(N_f \ln a + N_f + \frac{N_f A}{2} + \frac{1}{2}\Big) + O(\alpha^3) = \frac{\beta(\alpha)}{\alpha^2};$$

$$\widetilde{\gamma}(\alpha) = \frac{d\ln Z}{d\ln\mu} = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(N_f + \frac{N_f A}{2} + \frac{1}{2} + N_f \ln a\Big) + O(\alpha^3) = \gamma(\alpha).$$

That is why in this scheme the NSVZ equation is valid. It turns out that it is so in all orders of the perturbation theory.

Below we will compare explicit expressions for RGFs for some special renormalization schemes. For the HD+MSL and MOM schemes they were obtained in

A.L.Kataev, K.S., Phys.Lett. **B730** (2014) 184; Theor.Math.Phys. **181** (2014) 1531; A.E.Kazantsev, K.S., JHEP **06** (2020) 108.

The HD+MSL-scheme

$$\widetilde{\gamma}_{\text{HD+MSL}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{HD+MSL}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3)\Big).$$

The MOM-scheme (The result is the same of dimensional reduction and the higher derivative regularization.)

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1 + N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1 - \zeta(3))\Big) + O(\alpha^3)\Big).$$

The $\overline{\text{DR}}$-scheme

I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. **B386** (1996) 138.

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2 + 2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2 + 3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

The exact expression for the β-function of $\mathcal{N}=1$ supersymmetric electrodynamics as an integral of total derivatives

An all-loop expression for the β-function (defined in terms of the bare coupling constant in the case of using the higher derivative regularization) was obtained in

K.S., Nucl.Phys. **B 852** (2011) 71.

It is convenient to write in in the coordinate representation,

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{\pi i}{\mathcal{V}_4}\frac{d}{d\ln\Lambda}\Big\langle 4(N_f)^2\Big(\mbox{Tr}\Big[x^\mu, \bar\theta^{\dot b}(\gamma^\mu)_{\dot b}{}^a\theta_a v\Big(\ln(*)-\ln(\widetilde{*})\Big)-(PV)\Big]\Big)^2$$
$$+N_f\,\mbox{Tr}\,\theta^4 v^2\Big[x_\mu,\Big[x^\mu,\,\ln(*)+\ln(\widetilde{*})-(PV)\Big]\Big]\Big\rangle_{\mbox{1PI}} - \mbox{singularities.}$$

In this form the factorization of double total derivatives with respect to loop momenta is evident, because (after the Wick rotation)

$$\mbox{Tr}[x_\mu,[x^\mu,B]] - \mbox{singularity} \to i\int \frac{d^4Q}{(2\pi)^4}\,\frac{\partial^2 B}{\partial Q^\mu \partial Q_\mu}.$$

On the previous slide we used the notations:

$$* \equiv \frac{1}{1-(e^{2V}-1)\bar{D}^2 D^2/16\partial^2}; \qquad \widetilde{*} \equiv \frac{1}{1-(e^{-2V}-1)\bar{D}^2 D^2/16\partial^2}$$

are the operators encoding sequences of matter propagators and vertices;

$$\langle X[V]\rangle \equiv \frac{1}{Z}\int DV\, X[V]\Big(\mbox{Det}(PV,M)\mbox{Det}(*)\mbox{Det}(\widetilde{*})\Big)^{N_f}$$

$$\exp\Big\{\frac{i}{4e_0^2}\int d^4x\, d^4\theta\, V\Big[\partial^2 + \Big(1-\frac{1}{\xi_0}\Big)\Big(\frac{\bar{D}^2 D^2}{16}+\frac{D^2\bar{D}^2}{16}\Big)\Big]R(\partial^2/\Lambda^2)V\Big\}$$

is a functional integration, which converts products of V-s into products of gauge propagators;

v is almost constant function descreasing at a very large scale $R\to\infty$, for example, $v = v_0\exp\big(-(X^\mu)^2/2R^2\big)$;

$$\mathcal{V}_4 \equiv \int d^4x\, v^2 \sim R^4.$$

The NSVZ β-function for $\mathcal{N}=1$ supersymmetric electrodynamics as a sum of singularities

Taking into account that $\mathrm{Tr}[x^\mu, Y] = 0$ we see that the expression for the β-function is determined by singular contributions, which are proportional to δ-functions in the momentum space. Their sum has been calculated in

K.S., Nucl.Phys. **B 852** (2011) 71.

exactly in all orders of the perturbation theory starting from the above all-loop expression for the β-function. The result is the NSVZ relation

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big).$$

Therefore, RGFs defined in terms of the bare coupling constant satisfy the NSVZ relation in all orders for an arbitrary ξ-gauge and for an arbitrary renormalization prescription which supplements the higher derivative regularization.

Consequently, for RGFs defined in terms of the renormalized coupling constant the HD+MSL prescription gives one of the NSVZ scheme, so that

$$\frac{\widetilde\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi}\Big(1-\widetilde\gamma(\alpha)\Big).$$

Renormalizable non-Abelian $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S = \frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mathrm{c.c.}\Big\}.$$

We assume that the gauge group is simple, and the chiral matter superfields ϕ_i lie in its representation R. The gauge and Yukawa coupling constants are denoted by e_0 and λ_0^{ijk}, respectively. The strength of the gauge superfield V is defined by the equation

$$W_a \equiv \frac{1}{8}\bar{D}^2\Big(e^{-2V}D_a e^{2V}\Big).$$

The theory under consideration is gauge invariant if the (bare) masses and Yukawa couplings satisfy the conditions

$$m_0^{im}(T^A)_m{}^j + m_0^{mj}(T^A)_m{}^i = 0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k + \lambda_0^{imk}(T^A)_m{}^j + \lambda_0^{mjk}(T^A)_m{}^i = 0.$$

For quantizing the theory it is convenient to use the background field method. Moreover, it is necessary to take into account nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

This can be done with the help of the replacement $e^{2V} \to e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}$, where $\boldsymbol{V}$ and V are the background and quantum gauge superfields, respectively, and the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization. In the lowest order

J.W.Juer and D.Storey, Phys.Lett. **119B** (1982) 125; Nucl. Phys. **B216** (1983) 185.

$$\mathcal{F}(V)^A = V^A + e_0^2\, y_0\, G^{ABCD}\, V^B V^C V^D + \ldots,$$

where y_0 is one of the constant entering this set, and G^{ABCD} is a certain function of the structure constants.
The background gauge invariance

$$\phi_i \to (e^A)_i{}^j \phi_j; \qquad V \to e^{-A^+} V e^{A^+}; \qquad e^{2\boldsymbol{V}} \to e^{-A^+} e^{2\boldsymbol{V}} e^{-A}.$$

parameterized by a chiral superfield A remains a manifest symmetry of the effective action.

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\, d^2\theta\, W^a \left(e^{-2\boldsymbol{V}} e^{-2\mathcal{F}(V)}\right)_{Adj} R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\left(e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\right)_{Adj} W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big) e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\Big]_i{}^j \phi_j$$
$$+\Big[\int d^4x\, d^2\theta\, \Big(\frac{1}{4} m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a; \qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\, \bar D_{\dot a} e^{-2\boldsymbol{V}} e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\, d^4\theta\, \boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj} \bar{\boldsymbol{\nabla}}^2 V.$$

Also it is necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

The Pauli–Villars determinants in the non-Abelian case

For regularizing residual one-loop sivergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \mathsf{Det}(PV, M_\varphi)^{-1} \mathsf{Det}(PV, M)^c \times \exp\Big\{ i\Big(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}} \Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\mathsf{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\mathsf{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

Hedre we use chiral commuting Pauli–Villars superfields.
The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lies in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\text{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

The all-loop derivation of the NSVZ equation: the main steps

1. First, one proves the ultraviolet finiteness of triple vertices with two external lines of the Faddeev–Popov ghosts and one external line of the quantum gauge superfield.
2. Next, it is necessary to rewrite the NSVZ relation in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

K.S., Nucl.Phys. **B909** (2016) 316.

3. After this we prove that the β-function is determined by integrals of double total derivatives with respect to loop momenta and present a method for constructing this integrals.

K.S., JHEP **10** (2019) 011.

4. Then the NSVZ equation is obtained by summing singular contributions.
5. Finally, an NSVZ scheme is constructed.

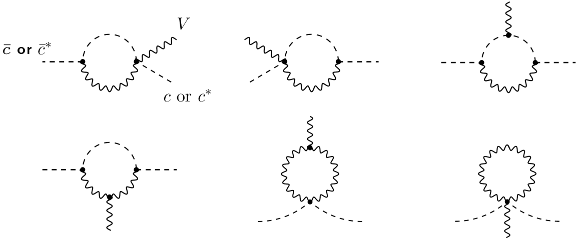
K.S., Eur.Phys.J. **C80** (2020) 10, 911.

Non-renormalization of the three-point gauge-ghost vertices

An important statement needed for proving the NSVZ equation in the non-Abelian case is the all-order finiteness of triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield.

K.S., Nucl.Phys. **B909** (2016) 316.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The ultraviolet finiteness of their sum has been verified by an explicit calculation



Example: a part of the one-loop expression for one of the triple gauge-ghost vertices

A part of the effective action corresponding to the $\bar{c}^+ Vc$ vertex is written as

$$\frac{ie_0}{4} f^{ABC} \int d^4\theta \, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \Big(f(p,q) \partial^2 \Pi_{1/2} V^B(\theta, -p)$$

$$+ F_\mu(p,q) (\gamma^\mu)_{\dot{a}}{}^b D_b \bar{D}^{\dot{a}} V^B(\theta, -p) + F(p,q) V^B(\theta, -p) \Big) c^C(\theta, -q).$$

After the Wick rotation the sum of the tree and one-loop contributions to the function F is given by

$$F(P,Q) = 1 + \frac{e_0^2 C_2}{4} \int \frac{d^4K}{(2\pi)^4} \Big\{ - \frac{(Q+P)^2}{R_K K^2 (K+P)^2 (K-Q)^2} - \frac{\xi_0 \, P^2}{K_K K^2 (K+Q)^2}$$
$$\times \frac{1}{(K+P+Q)^2} + \frac{\xi_0 \, Q^2}{K_K K^2 (K+P)^2 (K+Q+P)^2} + \Big(\frac{\xi_0}{K_K} - \frac{1}{R_K} \Big)$$
$$\times \Big(- \frac{2(Q+P)^2}{K^4 (K+Q+P)^2} + \frac{2}{K^2 (K+Q+P)^2} - \frac{1}{K^2 (K+Q)^2} - \frac{1}{K^2 (K+P)^2} \Big) \Big\}$$
$$+ O(\alpha_0^2, \alpha_0 \lambda_0^2).$$

We see that this expression is finite in the ultraviolet region.

The all-loop proof is based on the superfield Feynman rules and the Slavnov–taylor identities. It is valid in the case of using the superfield quantization for an arbitrary ξ-gauge, (unlike similar statements proved earlier

D. Dudal, H. Verschelde, S.P. Sorella, Phys. Lett. **B555** (2003) 126; M.A.L. Capri, D.R. Granado, M.S. Guimaraes, I.F. Justo, L. Mihaila, S.P. Sorella, D. Vercauteren, Eur.Phys.J. **C74** (2014), 2844.

in the Landau gauge $\xi \to 0$ for some theories formulated in terms of usual fields).

There are 4 vertices of the considered structure, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$. All of them have renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0, \qquad \text{where}$$

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1} \bar{c}_R c_R.$$

Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

The non-Abelian NSVZ equation can be equivalently rewritten as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The β-function in the right hand side can be expressed in terms of the charge renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Using the finiteness of the gauge-ghost vertices we obtain

$$\beta(\alpha_0,\lambda_0) = -2\alpha_0\frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big),$$

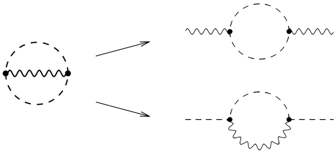
where γ_c and γ_V are the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare couplings), respectively.

Substituting this expression into the the right hand side we obtain the equivalent form of the NSVZ equation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, becauase the right ahmd side does not contain a denominator depending on couplings.

The new form of the NSVZ equation has a graphical interpretation similar to the Abelian case:



A very important step needed for the perturbative derivation of the NSVZ equation is a proof of the fact that (with the higher covariant derivative regularization) the function $\beta(\alpha_0,\lambda_0)/\alpha_0^2$ is determined by integrals of double total derivatives. Also it is important to formulate a method for constructing these integrals.

For making the proof we first introduce some auxiliary parameters to the action. Next, the function $\beta(\alpha_0,\lambda_0)/\alpha_0^2$ should be extracted from a part of the effective action corresponding to the two-point Green function of the background gauge superfield,

$$\Gamma^{(2)}_{\boldsymbol{V}} = -\frac{1}{8\pi}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,\boldsymbol{V}(-p,\theta)\partial^2\Pi_{1/2}\boldsymbol{V}(p,\theta)\,d^{-1}(\alpha_0,\lambda_0,\Lambda/p).$$

This can be done with the help of the formal substitution $\boldsymbol{V}^A \to v^A\theta^4$,

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = \frac{d}{d\ln\Lambda}\Big(d^{-1}-\alpha_0^{-1}\Big)\Big|_{\alpha,\lambda=\mathbf{const};\,p\to 0} = -\frac{2\pi}{\mathcal{V}_4}\frac{d\Delta\Gamma^{(2)}_{\boldsymbol{V}}}{d\ln\Lambda}\Big|_{\alpha,\lambda=\mathbf{const};\,\boldsymbol{V}=\theta^4 v},$$

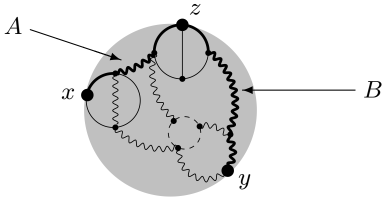
where $\Delta\Gamma \equiv \Gamma - S_{\text{reg}}$ and we also use the notation

$$\mathcal{V}_4 = \int d^4x\,(v^A)^2 \sim R^4 \to \infty.$$

Next, it is necessary to make some algebraic transformations involving the important identity

$$\theta^2 AB\theta^2 + 2(-1)^{P_A+P_B}\theta^a A\theta^2 B\theta_a - \theta^2 A\theta^2 B - A\theta^2 B\theta^2 = O(\theta).$$

where A and B are sequences of propagators and vertices connecting the points x, z, and y in a certain supergraph.



As a result of these transformation it is possible to write the function $\beta(\alpha_0,\lambda_0)/\alpha_0^2$ in the form of an expression which formally vanishes due to the Slavnov–Taylor identity for the background gauge invariance with the parameter

$$A = ia_\mu^B t^B y^\mu; \qquad A^+ = -ia_\mu^B t^B (y^\mu)^*.$$

The β-function of $\mathcal{N}=1$ supersymmetric gauge theories as an integral of double total derivatives

In the above equation a_μ^B are some constants, and $y^\mu \equiv x^\mu + i\bar\theta^{\dot a}(\gamma^\mu)_{\dot a}{}^b\theta_b$ are chiral coordinates. Therefore, this parameter of the gauge transformations rapidly increase at infinity. This leads to the violation of the Slavnov–Taylor identity which follows from the background gauge invariance,

$$\int d^8x\, \big(\bar\theta^{\dot a}(\gamma^\mu)_{\dot a}{}^b\theta_b\big)_x \frac{\delta\Gamma}{\delta \boldsymbol{V}_x}\bigg|_{\text{quant.fields}=0} = -i\int d^8x\, \big(y^\mu - (y^\mu)^*\big)_x \frac{\delta\Gamma}{\delta \boldsymbol{V}_x}\bigg|_{\text{quant.fields}=0} \ne 0.$$

This occurs due to singular contributions of the form

$$\frac{\partial^2}{\partial Q_\mu \partial Q^\mu}\Big(\frac{1}{Q^2}\Big) = -4\pi^2\delta^4(Q).$$

In the momentum space the formally vanishing expression for the function $(\beta - \beta_{\text{1-loop}})/\alpha_0^2$ is given by integrals of double total derivatives. The analysis of the Feynman rules made in

K.S., JHEP **10** (2019) 011.

revealed that these integrals can be constructed according to a certain formal prescription presented in the next slide.

The method for constructing integrals of double total derivatives

1. We consider a vacuum supergraph. A contribution coming from all superdiagrams obtained from it by adding two external lines of the background gauge superfield to the function

$$\frac{1}{\alpha_0^2}\Big(\beta(\alpha_0,\lambda_0)-\beta_{\textbf{1-loop}}(\alpha_0)\Big)$$

can be obtained with the help of the following formal operations:
2. We insert a factor $\theta^4(v^B)^2$ to an arbitrary point of the supergraph.
3. The resulting expression is calculated. Terms in which derivatives act on v^B should be omitted.
4. We mark L propagators with the momenta Q_i^μ which are considered as independent. Their product is proportional to $\prod_{i=1}^L \delta_{a_i}^{b_i}$.
5. In the integrand we make the formal substitution

$$\prod_{i=1}^L \delta_{a_i}^{b_i} \ \to\ \sum_{k,l=1}^L \prod_{i\ne k,l} \delta_{a_i}^{b_i}\,(T^A)_{a_k}{}^{b_k}(T^A)_{a_l}{}^{b_l}\,\frac{\partial^2}{\partial Q_k^\mu\,\partial Q_l^\mu}.$$

6. To the resulting expression we apply the operator

$$-\frac{2\pi}{r\mathcal{V}_4}\frac{d}{d\ln\Lambda}.$$

By construction, the result has the form of an integral of double total derivatives.

A modification of the method needed for summing singular contributions

For summing singular contributions it is necessary to slightly modify the step 5. It can be equivalently formulated as follows:

5. a) We consider a set of vacuum supergraphs which differ from the original one only by marking one of different momenta.

b) Next, it is necessary to calculate D-algebra and present the result as a sum of scalar integrals. All terms proportional to $1/q^{2n}$ with $n\ge 2$, should be omitted.

c) For one of the propagators with the marked momentum we make the formal substitution

$$\frac{1}{q^2} \quad\to\quad C_2\,\frac{\partial^2}{\partial q_\mu \partial q^\mu}\Big(\frac{1}{q^2}\Big) \qquad \mbox{for gauge and ghost propagators;}$$

$$\delta_i^j\,\frac{1}{q^2} \quad\to\quad C(R)_i{}^j\,\frac{\partial^2}{\partial q_\mu \partial q^\mu}\Big(\frac{1}{q^2}\Big) \qquad \mbox{for propagators of the matter superfields.}$$

In this case the manifest factorization into integrals of douible derivatives is lost, but the result remains the same.

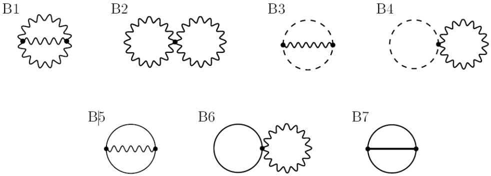
The correctness of both version of the method for calculating the β-function has been confirmed by a certain number of explicit calculations. Some of them will be described below.

An example: the two-loop contribution to the β-function of $\mathcal{N}=1$ supersymmetric gauge theories

The method described above simplifies explicit calculations of the β-function in a great extent. For instance, the total two-loop contribution to the β-function of $\mathcal{N}=1$ supersymmetric Yang–Mills theory with matter superfields in an arbitrary ξ-gauge has been calculated in

K.S., Proceedings of the Steklov Institute of Mathematics **309** (2020) 284.

It is generated by the supergraphs



To obtain usual superdiagrams which determine the β-function, we need attach two external lines of the background gauge superfield in all possible ways. However, the new method allows to calculate only (specially modified) vacuum supergraphs.

The result (for the β-function defined in terms of the are couplings) is given by the expression

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{(2\pi)^2}\Big[-3C_2^2 + \frac{1}{r}\,C_2\,\mbox{tr}\,C(R) + \frac{2}{r}\,\mbox{tr}\,\Big(C(R)^2\Big)\Big] - \frac{1}{8\pi^3 r}C(R)_i{}^j\lambda^*_{0jmn}\lambda_0^{imn} + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4).$$

The gauge dependence dissappears, and the result agrees with the one found aelier with the help of the dimensional technique. Moreover, it turns out that the NSVZ equations

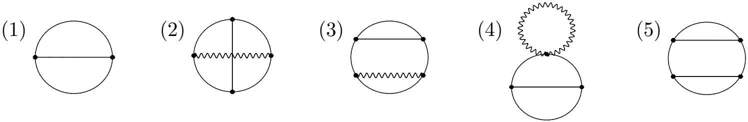
$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) - 2C_2\gamma_V(\alpha_0,\lambda_0) + \frac{1}{r}C(R)_i{}^j\left(\gamma_\phi\right)_j{}^i(\alpha_0,\lambda_0)\Big) + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4);$$

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j\left(\gamma_\phi\right)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi(1-C_2\alpha_0/2\pi)} + O(\alpha_0^2,\alpha_0\lambda_0^2,\lambda_0^4).$$

are valid even for the loop integrals. However, in this approximation the scheme dependence does not manifest itself.

An example: a three-loop contribution to the β-function containing the Yukawa couplings

The explicit three-loop calculation is very complicated and was done only for contributions containing the Yukawa couplings. The relevant two- and three-loop supergraphs are presented below:



The standard calculation of the corresponding superdiagrams with two external lines of the background gauge superfield was done in

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

Subsequently, a similar calculation was done with the help of a new method,

K.S., JHEP 10 (2019) 011.

It allows to verify if the new method correctly reproduces the results of the above calculation of the β-function.

As an example we present the expression for term quartic in the Yukawa couplings calculated with the help of the new method:

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int \frac{d^4K}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4}\, \lambda_0^{imn} \lambda^*_{0jmn} \frac{\partial}{\partial Q_\mu} \frac{\partial}{\partial Q^\mu} \Big(\frac{1}{K^2}$$

$$\times \frac{1}{F_K\, Q^2 F_Q\, (Q+K)^2 F_{Q+K}}\Big) + \frac{4\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \Big[\lambda_0^{iab}$$

$$\times \lambda^*_{0kab} \lambda_0^{kcd} \lambda^*_{0jcd} \Big(\frac{\partial}{\partial K_\mu} \frac{\partial}{\partial K^\mu} - \frac{\partial}{\partial Q_\mu} \frac{\partial}{\partial Q^\mu}\Big) + 2\lambda_0^{iab} \lambda^*_{0jac} \lambda_0^{cde} \lambda^*_{0bde} \frac{\partial}{\partial Q_\mu} \frac{\partial}{\partial Q^\mu}\Big]$$

$$\times \frac{1}{K^2 F_K^2\, Q^2 F_Q\, (Q+K)^2 F_{Q+K}\, L^2 F_L\, (L+K)^2 F_{L+K}} = -\frac{1}{2\pi r} C(R)_i{}^j (\Delta\gamma_\phi)_j{}^i.$$

As the other similar expressions it coincides with the result of the standard calculation and at the level of loop integrals satisfies the NSVZ relation. This calculation and other similar calculations confirm the correctness of the general argumentation discussed above in the approximation where the dependence on a regularization and/or a renormalization prescription is essential.

Deriving the NSVZ β-function by summing of singularities: introducing of an auxiliary parameter

The method for constructing integrals of double total derivatives described above can be used for deriving the exact NSVZ β-function in all orders of the perturbation theory

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

Let us modify the action by introducing an auxiliary real parameter g,

$$S_g \equiv S_{\text{total}} + \Delta S_g,$$

where $S_{\text{total}} \equiv S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}}$ and

$$\Delta S_g \equiv \frac{1}{4}\Big(\frac{1}{g} - 1\Big) \int d^8x \Big[- V^A R(\partial^2/\Lambda^2)\, \partial^2\Pi_{1/2} V^A - \frac{1}{8\xi_0} D^2 V^A \\ \times K(\partial^2/\Lambda^2) \bar D^2 V^A + \bar c^{+A} c^A - \bar c^A c^{+A} + \phi^{*i} F(\partial^2/\Lambda^2)\phi_i \Big].$$

After this all vertices coincide with the vertices of the original theory, and propagators of the quantum superfields (except for the Nielsen–Kallosh ghosts) are multiplied by the factor g. For $g=1$ the action S_g coincides with the original action S.

Summation of singularities: the derivative of the effective action with espect to the auxiliary parameter

The derivative of the effective action Γ_g (which is constructed from S_g) with respect to the parameter g at $g=1$ can be expressed in terms of the inverse two-point Green functions,

$$\frac{\partial\Gamma_g}{\partial g}\bigg|_{g=1;\,\text{fields}=0} = \frac{i}{4}\int d^8x\,\Big\{\Big[\partial^2\Pi_{1/2}R(\partial^2/\Lambda^2) + \frac{1}{16\xi_0}\Big(\bar{D}^2D^2 + D^2\bar{D}^2\Big)$$
$$\times K(\partial^2/\Lambda^2)\Big]_x\Big(\frac{\delta^2\Gamma}{\delta V_x^A\delta V_y^A}\Big)^{-1} - F(\partial^2/\Lambda^2)_x\Big(\frac{\delta^2\Gamma}{\delta\phi^{*i}_{,x}\delta\phi_{i,y}}\Big)^{-1} - \Big(\frac{\delta^2\Gamma}{\delta c_x^A\,\delta\bar{c}_y^{+A}}\Big)^{-1}$$
$$-\Big(\frac{\delta^2\Gamma}{\delta\bar{c}_x^A\,\delta c_y^{+A}}\Big)^{-1}\Big\}\bigg|_{y=x}.$$

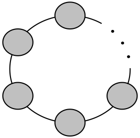
This expression is a sum of all vacuum supergraphs with one marked propagator. (Therefore, each supergraph is multiplied by a number of propagators.)

If a usual two-point Green function contains a function $G=1+\Delta G$ of the external momentum, then the inverse Green function will contain

$$G^{-1} = 1 + \sum_{n=1}^{\infty}(-1)^n(\Delta G)^n.$$

Comparing the structure of contributions to the β-function and to the derivative of the effective action with respect to the parameter g

The term containing $(\Delta G)^n$ corresponds to the sum of supergraphs which have n propagators with the momenta coinciding to the momentum of the marked propagator (including it).



Therefore, if we consider a corresponding sum of vacuum supergraphs in which only one of the propagators with coinciding momenta can be marked, then it will be equal to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\Delta G)^n = -\ln G.$$

This implies that for calculating the β-function it is necessary to make the substitution $G^{-1} \to -\ln G$. For example, for supergraphs with a marked matter propgator

$$\Big(\frac{\delta^2\Gamma}{\delta\phi^{*i}_{,x}\,\delta\phi_{i,y}}\Big)^{-1}\Big|_{\text{fields}=0} \quad\to\quad \frac{D_x^2 \bar D_x^2}{4\partial^2 F}\Big(\ln\frac{G_\phi}{F}\Big)_i{}^i \delta^8_{xy}.$$

Deriving the new form of the exact NSVZ β-function by summing singular contributions

Then the total contribution of the considered supergraphs to the function β/α^2 constructed according to the method described above takes the form

$$\Delta_{\text{matter}}\Big(\frac{\beta}{\alpha_0^2}\Big) = -\frac{i\pi}{8r\mathcal{V}_4} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int d^8x\, d^8y\, (\theta^4)_x (v^B)_x^2 \int \frac{d^4q}{(2\pi)^4}\, \delta^8_{xy}(q)$$
$$\times \Big(\ln\frac{G_\phi}{F}\Big)_j{}^i \frac{\partial^2}{\partial q_\mu \partial q^\mu}\Big(\frac{1}{q^2}\Big) D_x^2 \bar D_x^2 \delta^8_{xy} = -\frac{1}{2\pi r} C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha_0,\lambda_0),$$

where $\delta^8_{xy}(q) \equiv \delta^4(\theta_x-\theta_y) e^{iq_\mu(x^\mu-y^\mu)}$.

As a result of similar calculations we obtained

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} - \frac{\beta_{\text{1-loop}}(\alpha_0)}{\alpha_0^2}$$
$$= \underbrace{\frac{1}{\pi} C_2 \gamma_V(\alpha_0,\lambda_0)}_{\text{gauge propagators}} + \underbrace{\frac{1}{\pi} C_2 \gamma_c(\alpha_0,\lambda_0)}_{\text{Faddeev–Popov ghost propagators}} - \underbrace{\frac{1}{2\pi r} C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)}_{\text{matter propagators}}.$$

Thus, we obtain the main result:

The NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1 - C_2\alpha_0/2\pi)}.$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

The two-loop anomalous dimension defined in terms of the bare coupling constant for $\mathcal{N}=1$ supersymmetric theories regularized by higher derivatives has been calculated in

A.E.Kazantsev, K.S., JHEP **2006** (2020) 108.

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) = -\frac{\alpha_0}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{2\pi^2}\left[C(R)^2\right]_i{}^j - \frac{1}{16\pi^4}$$
$$\times\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde} - \frac{3\alpha_0^2}{2\pi^2}C_2C(R)_i{}^j\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) + \frac{\alpha_0^2}{2\pi^2}T(R)C(R)_i{}^j$$
$$\times\Big(\ln a + 1 + \frac{A}{2}\Big) - \frac{\alpha_0}{8\pi^3}\lambda^*_{0lmn}\lambda_0^{jmn}C(R)_i{}^l(1-B+A) + \frac{\alpha_0}{4\pi^3}\lambda^*_{0imn}\lambda_0^{jml}$$
$$\times C(R)_l{}^n(1-A+B) + O\Big(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6\Big),$$

where

$$A = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad B = \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{F^2(x)}\qquad a = \frac{M}{\Lambda};\qquad a_\varphi = \frac{M_\varphi}{\Lambda}.$$

If the anomalous dimension of the matter superfields defined in terms of the bare couplings has been calculated in L-loops with the higher derivative regularization, then it is possible to construct the $(L+1)$-loop β-function from the NSVZ equation without loop calculations. For example, in the three-loop approximation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R)\Big) + \frac{\alpha_0}{4\pi^2}\Big\{-3C_2^2 + \frac{1}{r}C_2\,\mathrm{tr}\,C(R) + \frac{2}{r}\,\mathrm{tr}\left[C(R)^2\right]\Big\}$$
$$-\frac{1}{8\pi^3 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0^2}{8\pi^3}\Big\{-3C_2^3 + \frac{1}{r}C_2^2\,\mathrm{tr}\,C(R) - \frac{2}{r}\,\mathrm{tr}\left[C(R)^3\right] + \frac{2}{r}$$
$$\times C_2\,\mathrm{tr}\left[C(R)^2\right]\Big(3\ln a_\varphi + 4 + \frac{3A}{2}\Big) - \frac{2}{r^2}\,\mathrm{tr}\,C(R)\,\mathrm{tr}\left[C(R)^2\right]\Big(\ln a + 1 + \frac{A}{2}\Big)\Big\}$$
$$-\frac{\alpha_0 C_2}{16\pi^4 r}C(R)_j{}^i\lambda^*_{0imn}\lambda_0^{jmn} + \frac{\alpha_0}{16\pi^4 r}\left[C(R)^2\right]_j{}^i\lambda^*_{0imn}\lambda_0^{jmn}\Big(1 + A - B\Big) - \frac{\alpha_0}{8\pi^4 r}$$
$$\times C(R)_j{}^i C(R)_l{}^n\lambda^*_{0imn}\lambda_0^{jml}\Big(1 - A + B\Big) + \frac{1}{32\pi^5 r}C(R)_j{}^i\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde}$$
$$+O\Big(\alpha_0^3,\,\alpha_0^2\lambda_0^2,\,\alpha_0\lambda_0^4,\,\lambda_0^6\Big).$$

Certainly, RGFs defined in terms of the renormalized couplings can also be calculated for an arbitrary renormalization prescription.

Obtaining RGFs defined in terms of the renormalized couplings

To calculate RGFs defined in terms of the renormalized couplings, first, we integrate the equations

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}};\qquad (\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d(\ln Z_\phi)_i{}^j}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and obtain the expressions for the renormalized gauge coupling constant and $(\ln Z_\phi)_i{}^j$. They depend on a set of finite constants which determine a subtraction scheme in the considered approximation. Next, we substitute the expressions obtained in this way into the equations

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}};\qquad (\widetilde\gamma_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d(\ln Z_\phi)_i{}^j}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

These RGFs will nontrivially depend on the finite constants due to the scheme dependence.

Here (at the next slide) we only present the result for one particular case, namely, for one-loop finite $\mathcal{N}=1$ supersymmetric theories, see

P.West, Phys.Lett. **B 137** (1984) 371;
A.Parkes, P.West, Phys.Lett. **B 138** (1984) 99.

An important particular case is theories finite in the one-loop approximation which satisfy the conditions

$$T(R) = 3C_2; \qquad \lambda^*_{imn}\lambda^{jmn} = 4\pi\alpha C(R)_i{}^j.$$

In this case the two-loop anomalous dimension and the three-loop β-function defined in terms of the renormalized couplings have the form

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) = -\frac{3\alpha^2}{2\pi^2}C_2 C(R)_i{}^j\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) - \frac{\alpha}{4\pi^2}\Big(\frac{1}{\pi}\lambda^*_{imn}\lambda^{jml}C(R)_l{}^n$$
$$+2\alpha\left[C(R)^2\right]_i{}^j\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big);$$
$$\frac{\widetilde{\beta}(\alpha,\lambda)}{\alpha^2} = \frac{3\alpha^2}{4\pi^3 r}C_2\,\mathrm{tr}\left[C(R)^2\right]\Big(\ln\frac{a_\varphi}{a} - b_{11} + b_{12}\Big) + \frac{\alpha}{8\pi^3 r}\Big(\frac{1}{\pi}C(R)_j{}^i C(R)_l{}^n$$
$$\times\lambda^*_{imn}\lambda^{jml} + 2\alpha\,\mathrm{tr}\left[C(R)^3\right]\Big)\Big(A - B - 2g_{12} + 2g_{11}\Big) + O\Big(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6\Big).$$

We see that in this case the NSVZ equation is satisfied in the lowest nontrivial approximation for an arbitrary renormalization presription,

$$\frac{\beta(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda) + O(\alpha^3,\alpha^2\lambda^2,\alpha\lambda^4,\lambda^6).$$

The NSVZ equation for theories finite in the lowest loops

For $\mathcal{N}=1$ supersymmetric theories finite in the one-loop approximation it is possible to tune a subtraction scheme so that the theory will be all loop finite

D.I.Kazakov, Phys. Lett. B **179** (1986) 352; A.V.Ermushev, D.I.Kazakov, O.V.Tarasov, Nucl.Phys. B **281** (1987) 72; C.Lucchesi, O.Piguet, K.Sibold, Helv.Phys.Acta **61** (1988) 321; Phys.Lett. B **201** (1988) 241.

If a subtraction scheme is tuned in such a way that the β-function vanishes in the first L loops and the anomalous dimension for the matter superfields vanishes in the first $(L-1)$ loops, then

K.S., Eur.Phys.J. C **81** (2021) 571.

for an arbitrary renormalization prescription the $(L+1)$-loop gauge β-function satisfies the equation

$$\frac{\beta_{L+1}(\alpha,\lambda)}{\alpha^2} = -\frac{1}{2\pi r}C(R)_i{}^j(\gamma_{\phi,L})_j{}^i(\alpha,\lambda),$$

Therefore, if a theory is finite in a certain approximation, its β-function vanishes in the next order. This exactly agrees with the earlier known result of

A.J.Parkes, P.West, Nucl.Phys. B **256** (1985) 340;
M.T.Grisaru, B.Milewski and D.Zanon, Phys.Lett. **155B** (1985) 357.

$\mathcal{N}=2$ supersymmetric theories can be considered as a particular case of $\mathcal{N}=1$ supersymmetric theories. Therefore, they can be formulated in terms of $\mathcal{N}=1$ superfields,

$$S=\frac{1}{2e_0^2}\mbox{tr}\Big(\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\int d^4x\,d^4\theta\,\Phi^+e^{2V}\Phi\,e^{-2V}\Big)+\frac{1}{4}\int d^4x\,d^4\theta$$
$$\times\Big(\phi^+e^{2V}\phi+\widetilde\phi^+e^{-2V^T}\widetilde\phi\Big)+\Big[\int d^4x\,d^2\theta\,\Big(\frac{i}{\sqrt{2}}\widetilde\phi^t\Phi\phi+\frac{1}{2}m_0\widetilde\phi^T\phi\Big)+\mbox{c.c.}\Big]$$

Here the chiral superfield Φ in the adjoint representation is an $\mathcal{N}=2$ superpartner of the gauge superfield V. The chiral superfields ϕ and $\widetilde\phi$ in the representations R_0 and $\bar R_0$ form an $\mathcal{N}=2$ hypermultiplet.

Therefore, we obtain an $\mathcal{N}=1$ supersymmetric theory with chiral matter superfields in the reducible representation

$$R=Adj+R_0+\bar R_0,$$

containing nontrivial Yukawa interaction.

In this formulation only $\mathcal{N}=1$ supersymmetry is manifest, while the second supersymmetry is hidden.

A higher derivative term S_Λ invariant under both supersymmetries has been constructed in

I.L.Buchbinder and K.S., Nucl.Phys. **B883** (2014) 20.

However, with the help of the $\mathcal{N}=1$ superfield technique it is impossible to quantize a theory in the $\mathcal{N}=2$ supersymmetric way. Therefore, in this case quantum corrections can break the hidden supersymmetry.

Due to the manifest background gauge invariance the background gauge superfield is not renormalized, $\boldsymbol{V}=\boldsymbol{V}_R$. The chiral matter superfields are renormalized as

$$\Phi=\sqrt{Z_\Phi}\,\Phi_R;\qquad \phi=\sqrt{Z_\phi}\,\phi_R;\qquad \widetilde{\phi}=\sqrt{Z_\phi}\,\widetilde{\phi}_R.$$

Due to the non-renormalization of the superpotential these renormalization constants are related by the equation $Z_\Phi^{1/2}=Z_\phi^{-1}$. Consequently,

$$(\gamma_\phi)_i{}^j(\alpha_0)\equiv-\frac{d\ln Z_\phi}{d\ln\Lambda}\cdot\delta_i^j=\frac{1}{2}\,\frac{d\ln Z_\Phi}{d\ln\Lambda}\cdot\delta_i^j=-\frac{1}{2}\,\gamma_\Phi(\alpha_0)\cdot\delta_i^j\equiv\gamma_\phi(\alpha_0)\cdot\delta_i^j.$$

The anomalous dimensions and group factors for $\mathcal{N}=2$ supersymmetric theories

Components of the chiral superfield $\Phi = e_0\Phi^A T^A$ are renormalized as $\Phi^A = Z_A{}^B(\Phi_R)^B$, where $Z_A{}^B \equiv (Z_\Phi Z_3)^{1/2}\delta_A{}^B$. Consequently,

$$\gamma(\alpha_0)_A{}^B = -2\cdot\frac{d\ln Z_A{}^B}{d\ln\Lambda} = -\frac{d\ln(Z_3 Z_\Phi)}{d\ln\Lambda}\delta_A{}^B = \Big(\frac{\beta(\alpha_0)}{\alpha_0} + \gamma_\Phi(\alpha_0)\Big)\delta_A{}^B.$$

Therefore, the matrix of anomalous dimensions takes the form

$$\gamma_i{}^j(\alpha_0) = \begin{pmatrix} (\beta(\alpha_0)/\alpha_0 + \gamma_\Phi(\alpha_0))\cdot\delta_A^B & 0 & 0 \\ 0 & -\gamma_\Phi(\alpha_0)/2\cdot\delta_i^j & 0 \\ 0 & 0 & -\gamma_\Phi(\alpha_0)/2\cdot\delta_i^j \end{pmatrix}$$

Moreover, to investigate the NSVZ β-function for $\mathcal{N}=2$ supersymmetric gauge theories, we note that for the representation $R = Adj + R_0 + \bar{R}_0$

$$T(R) = C_2 + 2T(R_0);\qquad C(R)_i{}^j = \begin{pmatrix} C_2\cdot\delta_A^B & 0 & 0 \\ 0 & C(R_0) & 0 \\ 0 & 0 & C(R_0) \end{pmatrix}.$$

The NSVZ relation for $\mathcal{N}=2$ supersymmetric theories

Substituting the expressions for $T(R)$, $C(R)_i{}^j$, and $\gamma_i{}^j(\alpha_0)$ into the NSVZ relation we obtain the equation

$$\beta(\alpha_0) = -\frac{\alpha_0^2\Big[2C_2 - 2T(R_0) + C_2\Big(\beta(\alpha_0)/\alpha_0 + \gamma_\Phi(\alpha_0)\Big) - T(R_0)\gamma_\Phi(\alpha_0)\Big]}{2\pi(1-C_2\alpha_0/2\pi)}.$$

Solving it for $\beta(\alpha_0)$ we conclude that the β-function of the considered theory is given by the expression

$$\beta(\alpha_0) = -\frac{\alpha_0^2}{\pi}\Big(C_2 - T(R_0)\Big)\Big(1+\frac{1}{2}\,\gamma_\Phi(\alpha_0)\Big) = -\frac{\alpha_0^2}{\pi}\Big(C_2 - T(R_0)\Big)\Big(1-\gamma_\phi(\alpha_0)\Big).$$

This implies that, in general, higher loop $(L>1)$ contributions to the β-function do not vanish and are determined by the function $\gamma_\Phi(\alpha_0)$.

If $\mathcal{N}=2$ supersymmetry were not broken, then the superfield Φ would renormalize exactly as the background gauge superfield and $\gamma_\Phi(\alpha_0)=0$. Then the β-function would be exausted at the one-loop,

$$\beta(\alpha_0) = -\frac{\alpha_0^2}{\pi}\Big(C_2 - T(R_0)\Big).$$

Explicitly $\mathcal{N} = 2$ supersymmetric formulation in the harmonic superspace

$\mathcal{N} = 2$ supersymmetry is a manifest symmetry in the case of using $\mathcal{N} = 2$ harmonic superspace

A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky and E.Sokatchev, Class.Quant.Grav. **1** (1984) 469.

with the coordinates $(x^\mu, \theta^i_a, \bar\theta_{i\dot a}, u_i^\pm)$, where $u_i^- = (u^{+i})^*$ and $u^{+i}u_i^- = 1$. With the help of the harmonic superspace one can quantize the theory in a manifestly $\mathcal{N} = 2$ supersymmetric way. That is why the harmonic superspace technique together with the background superfield method allow having manifest $\mathcal{N} = 2$ supersymmetry and gauge invariance at all steps of calculating quantum corrections.

A.S.Galperin, E.A.Ivanov, V.I.Ogievetsky and E.S.Sokatchev, Harmonic superspace. Cambridge University Press (2001) 306p.

The higher covariant derivative regularization can also be formulated in the harmonic superspace

I.L.Buchbinder, N.G.Pletnev and K.S., Phys.Lett. **B751** (2015) 434.

The higher covariant derivative regularization allows to prove simply the $\mathcal{N}=2$ non-renormalization theorem starting from the NSVZ β-function.

The degree of divergence (for non-regularized theory) in the harmonic superspace is written as

I.L.Buchbinder, S.M.Kuzenko and B.A.Ovrut, Phys.Lett. **B433** (1998) 335.

$$\omega = -N_\phi - N_c - \frac{1}{2}N_D,$$

where N_ϕ is a number of external hypermultiplet lines, N_c is a number of external ghost lines, and N_D is a number of spinor derivatives acting on external lines. Therefore, all superdiagrams containing hypermultiplet external lines are finite, so that $\gamma_\phi(\alpha_0)=0$. Consequently,

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = -\frac{1}{\pi}\Big(C_2 - T(R)\Big)\Big(1-\gamma_\phi(\alpha_0)\Big) = -\frac{1}{\pi}\Big(C_2 - T(R)\Big).$$

This implies that the β-function is non-trivial only in the one-loop approximation.

Conclusion

- In the case of using the regularization by higher covariant derivatives RGFs defined in terms of the bare couplings satisfy the NSVZ equation in all orders for any renormalization prescription.
- RGFs defined in terms of the renormalized couplings satisfy the NSVZ equation in the HD+MSL scheme, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms.
- The β-function of $\mathcal{N}=1$ supersymmetric gauge theories is determined by integrals of double total derivatives in the momentum space.
- The triple gauge-ghost vertices are UV finite in all orders. This allows to rewrite the NSVZ relation in an equivalent form, which relates the β-function to the anomalous dimensions of the quantum superfields.
- Non-renormalization theorems for theories with extended supersymmetry follows from the NSVZ relation in the case of using the higher covariant derivative regularization and $\mathcal{N}=2$ supersymmetric quantization.

Thank you for the attention!