Modules over Clifford algebras as a basis for the theory of second quantization of spinors

Vadim Monakhov
Associate Professor
Saint-Petersburg State University
v.v.monahov@spbu.ru

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Presentation plan

- Introduction: Clifford algebras, spinor modules and algebraic spinors.
- Part 1. Transformation of spinor field operators. Spinor module of CAR algebra.
- Part 2. Reflections and P, T, C inversions.

Mathematical theories of spinors

- **Spinors** were discovered by Élie Cartan. They are two-valued elements of the representation space of the Lie algebra of infinitesimal rotations. Matrix representation: columns and rows.
- **Algebraic spinors** - theory of Clifford algebras and Clifford modules generated by idempotents. Matrix representation: d=2m dimensional complex space in the form of square matrices $2^m \cdot 2^m$.
- **Superalgebraic spinors** – extension of the theory of algebraic spinors. Grassmann variables and derivatives with respect to them. CAR algebra of second quantization of fermions.

Real Clifford algebra and its representations

- Physicists often say that Dirac gamma matrices generate a Clifford algebra. This is not true. They generate a matrix representation of the Clifford algebra – a linear map.
- **Real Clifford algebra:**

Vector $X=e_\nu\, x^\nu$; e_ν – basis vectors.

Complex conjugation $X^*=e_\nu\, x^\nu$; $e_\nu^*= e_\nu$, $(x^\nu)^*= x^\nu$.

- **Matrix representation of the Clifford algebra:**

Vector $X=\gamma_\nu\, x^\nu$; γ_ν – gamma matrices.

Complex conjugation $X^*= \gamma_\nu^*\, x^\nu$; $\gamma_\nu^*\neq \gamma_\nu$, $(x^\nu)^*= x^\nu$.

- Each linear representation of a Clifford algebra is one-to-one associated with a module over this algebra.

Modules over Clifford algebra (Clifford modules)

- A module is an Abelian group over an operator ring. This is a generalization of the concept of a vector space over a field of real or complex numbers, when the field is replaced by a ring of operators.
- A module over a Clifford algebra is a homomorphism of this algebra into a linear vector space. A homomorphism is a mapping that preserves basic operations and relations.
- A module is a linear representation of the algebra. An irreducible module is a module that does not contain submodules.
- Every Clifford algebra is a bilateral (left and right) module over itself.
- Quantum mechanics state vectors are elements of modules. Bra vectors $<\Psi|$ are elements of the right module. Ket vectors $|\Psi>$ are elements of the left module.

Irreducible Clifford module and spinor space as minimal left ideal

- Clifford module can be decomposed into a sum of irreducible Clifford modules.
- Minimal left (right) ideal of the Clifford algebra is a spinor space. It is generated by a primitive idempotent and is an irreducible Clifford module.
- Clifford algebra is a reducible Clifford module. The identity of the Clifford algebra is decomposed into the sum of minimal left (right) ideals of the algebra generated by primitive idempotents.
- These idempotents are constructed using elements of the Cartan subalgebra whose squares are equal to one.

Decomposition of the identity of the Clifford algebra into a sum of minimal ideals

Let $(t,s)=(1,d-1)$, d – dimension of the space

Cartan subalgebra generators $\gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^5\gamma^6, \ldots, \gamma^{d-1}\gamma^d$.

$(\gamma^0\gamma^3)^2 = 1, (\gamma^1\gamma^2)^2 = (\gamma^5\gamma^6)^2 = \ldots = (\gamma^{d-1}\gamma^d)^2 = -1$

=> 2^d orthogonal idempotents $(I_{\ldots})^2 = I_{\ldots}$:

$$I_{\pm 03,\pm 12,\pm 56,\ldots,} = \frac{1\pm\gamma^0\gamma^3}{2}\frac{1\pm \mathrm{i}\gamma^1\gamma^2}{2}\frac{1\pm \mathrm{i}\gamma^5\gamma^6}{2}\cdots\frac{1\pm \mathrm{i}\gamma^{d-1}\gamma^d}{2},$$

$$I_{03,12,56,\ldots,} + I_{-03,12,56,\ldots,} + I_{03,-12,56,\ldots,} + I_{-03,-12,56,\ldots,} + \ldots = 1.$$

Minimal left ideals: $J_{03,12,56,\ldots,} = Cl(t,s)\cdot I_{03,12,56,\ldots,};$

$$J_{-03,12,56,\ldots,} = Cl(t,s)\cdot I_{-03,12,56,\ldots,};$$

$\ldots$

Primitive idempotents and spinor spaces, d=4

Cartan subalgebra generators $\gamma^0\gamma^3$, $\mathrm{i}\gamma^1\gamma^2$ or γ^5, $\mathrm{i}\gamma^1\gamma^2$ or $\gamma^0, \mathrm{i}\gamma^1\gamma^2$.

They are equivalent because $\gamma^0\gamma^3\gamma^1\gamma^2 = -\mathrm{i}\gamma^5$; 4 idempotents:

$$I_{\pm 0,\pm 12} = \frac{1 \pm \gamma^0}{2}\frac{1 \pm \mathrm{i}\gamma^1\gamma^2}{2}; \qquad I_{0,12} + I_{-0,12} + I_{0,-12} + I_{-0,-12} = 1.$$

$$I_{0,12} \cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; I_{0,-12} \cong \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; I_{-0,12} \cong \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{-0,-12} \cong \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Spinor space } \Psi I_{0,12} \cong \begin{pmatrix} \psi^1 & 0 & 0 & 0 \\ \psi^2 & 0 & 0 & 0 \\ \psi^3 & 0 & 0 & 0 \\ \psi^4 & 0 & 0 & 0 \end{pmatrix}$$

Problems in the theory of spinors

- Why Clifford algebra of the spacetime is real? More general Clifford algebra of field operators is complex due to the existence of momentum and factors $\exp(-i\, p_\nu x^\nu)$.
- Element $i\gamma^1\gamma^2$ also exists only in complex Clifford algebra.
- There are 4 spinor spaces with incomprehensible physical meaning of these 4 spinors. Are they related to the presence of 4 generations of spinors, 4 colors, or something else?
- There are 4 spinor spaces, but there is no spinor vacuum. Idempotents are spinor components with nonzero spin. The identity is a superposition (sum) of 4 spinor fields.
- Conjugated spinor is a sum of 4 non-conjugated spinors.

Problems in the theories of spinors, continued

- Spinors are representations of the Clifford algebra of spacetime – but not of the CAR algebra. (CAR is Canonical Anticommutation Relations of the creation and annihilations operators of fermions).
- What is algebraic construction of the vacuum state vector?
- The existence of a spin follows from the Clifford algebra. What is the origin of momentum and electric and other charges of spinors?
- Why are P, T, C symmetries of spinors broken?

Theory of superalgebraic spinors solves most of these problems.

Part 1. Transformation of spinor field operators. Spinor module of CAR algebra

1. Small and large Clifford algebra. Superalgebraic spinors.
2. Lie group and Lorentz transformations.
3. Infinitesimal transformation of basis field operators.
4. Gamma operators – analogs of gamma matrices.
   Two additional matrices compared to the Dirac theory.
5. Operator of generalized Dirac conjugation. Spinor vacuum.
6. Main decomposition: general form of transformation of the field operators. Dirac equation.

Theory of superalgebraic spinors: publications

1. M. Pavšič. A theory of quantized fields based on orthogonal and symplectic Clifford algebras. *Advances in Applied Clifford Algebras*, 2012, v.22, p.449-481.
2. V. Monakhov. Superalgebraic representation of Dirac matrices. *Theoretical and Mathematical Physics*. 2016. v. 186. p.70–82.
3. V. Monakhov. Dirac matrices as elements of superalgebraic matrix algebra. *Bulletin of the Russian Academy of Sciences: Physics*, 2016, v.80, p. 985–988.
4. V. Monakhov. Superalgebraic structure of Lorentz transformations. *J. of Physics: Conf. Series*, 2018, v.1051, 012023.
5. V.Monakhov. Generalization of Dirac conjugation in the superalgebraic theor y of spinors *Theoretical and Mathematical Physics*, 2019, v.200, p.1026-1042.
6. V. Monakhov. Vacuum and spacetime signature in the theory of superalgebraic spinors. *Universe*, 2019, v.5(7), 162.

Theory of superalgebraic spinors: publications

7. V. Monakhov. Spacetime and inner space of spinors in the theory of superalgebraic spinors. *Journal of Physics: Conference Series,* 2020, v.1557(1), 12031.
8. V. Monakhov. Generation of Electroweak Interaction by Analogs of Dirac Gamma Matrices Constructed from Operators of the Creation and Annihilation of Spinors. *Bulletin of the Russian Academy of Sciences: Physics*, 2020, v. 84(10), pp. 1216–1220.
9. V. Monakhov. The Dirac Sea, T and C Symmetry Breaking, and the Spinor Vacuum of the Universe, *Universe,* 2021, v. 7(5), 124.
10. V. Monakhov. A.Kozhedub. Algebra of Superalgebraic Spinors as Algebra of Second Quantization of Fermions. *Geom. Integrability & Quantization*, 2021, vol. 22, p.165-187.

Small and large Clifford algebra

- There are two Clifford algebras in the theory of fermions:

1. Finite-dimensional Clifford algebra of gamma operators (their representations are gamma matrices). We called it small Clifford algebra.
2. Infinite-dimensional Clifford algebra of creation $a(p)^+$ and annihilation $a(p)$ operators with basis Clifford vectors $(a+a^+)/\sqrt{2}$ and $(a-a^+)/\sqrt{2}$. It is CAR algebra. We called it large Clifford algebra.

- Previously, these two algebras were thought to be independent. We found that gamma operators can be constructed from elements of the large Clifford algebra.

Creation and annihilation operators. Discretization of momentum space

$$a_j(p) = \frac{\partial}{\partial\theta^j(p)}, \quad p \approx p' \approx 0;$$

$$a_j(p)^+ = \theta^j(p);$$

$$\{a_i(p), a_j(p')^+\} = \delta_i^j \delta(p - p').$$

$$\text{Quasi - continuous spectrum } p = p_j, \quad \delta(p_i - p_j) \cong \frac{1}{\Delta^3 p_i}\delta_j^i,$$

$$\{\frac{\partial}{\partial\theta^k(p_i)}, \theta^l(p_j)\} = \delta_k^l \frac{1}{\Delta^3 p_i}\delta_j^i,$$

$$\{\frac{\partial}{\partial\theta^k(p_i)}, \frac{\partial}{\partial\theta^l(p_j)}\} = \{\theta^k(p_i), \theta^l(p_j)\} = 0$$

Lie group and Lorentz transformations

$$\Psi_1\Psi_2\ldots\Psi_n \to \mathrm{e}^{dG}\Psi_1\mathrm{e}^{-dG}\,\mathrm{e}^{dG}\Psi_2\mathrm{e}^{-dG}\ldots\mathrm{e}^{dG}\Psi_n\mathrm{e}^{-dG} =$$

$$= (\mathrm{e}^{d\hat{G}}\Psi_1)(\mathrm{e}^{d\hat{G}}\Psi_2)\ldots(\mathrm{e}^{d\hat{G}}\Psi_n) - \text{Lie group inner automorphism}$$

$$d\hat{G} = [dG,\bullet], \quad d\hat{G}\Psi = [dG,\Psi],$$

$$(\mathrm{e}^{d\hat{G}}\Psi) = \mathrm{e}^{dG}\Psi\mathrm{e}^{-dG} = (1+[dG,\bullet])\Psi = \Psi + (d\hat{G}\Psi).$$

$$dG = \frac{1}{4}\gamma^{\mu}\gamma^{\nu}d\omega_{\mu\nu} = \frac{1}{4}\gamma^{\mu\nu}d\omega_{\mu\nu} \text{ generates}$$

$$\text{Lorentz transformation of the field operator: } \mathrm{e}^{\hat{\gamma}^{\mu\nu}d\omega_{\mu\nu}/4}\Psi.$$

$$dG^{+} = \frac{1}{4}(\gamma^{\nu})^{+}(\gamma^{\mu})^{+}d\omega_{\mu\nu} = -\frac{1}{4}(\gamma^{\mu})^{+}(\gamma^{\nu})^{+}d\omega_{\mu\nu}.$$

$$\mu,\nu = 1,2,3 \Rightarrow -dG^{+} = dG;\ \mu = 0, \nu = 1,2,3 \Rightarrow -dG^{+} = -dG.$$

Lorentz transformation of the creation and annihilation operators

Spacetime signature (t, s). Consider Lie group transformations :

$$b_i(p_j) \equiv \mathrm{e}^{dG} a_i(0) \mathrm{e}^{-dG} \,|_{p=0\to p=p_j} = \mathrm{e}^{d\hat{G}} a_i(0) \,|_{p=0\to p=p_j} = \mathrm{e}^{\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu}/4} a_i(0) \,|_{p=0\to p=p_j}$$

$$b_i(p)^+ = (\mathrm{e}^{dG} a_i(0) \mathrm{e}^{-dG})^+ \,|_{p=0\to p=p_j} = (\mathrm{e}^{-dG^+} a_i(0)^+ \mathrm{e}^{dG^+}) \,|_{p=0\to p=p_j}$$

$$\Rightarrow b_i(p)^+ \neq (\mathrm{e}^{dG} a_i(0)^+ \mathrm{e}^{-dG}) \,|_{p=0\to p=p_j} \text{ if } \; t \neq 0 \text{ or } s \neq 0.$$

$\Rightarrow b_i(p)^+$ does not transform according to the Lie group if $t \neq 0$ or $s \neq 0$.

$\overline{b}_i(p) = (\mathrm{e}^{dG} a_i(0)^+ \mathrm{e}^{-dG}) \,|_{0\to p} = (\mathrm{e}^{\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu}/4} a_i(0)^+) \,|_{0\to p}$ is element of the Lie group.

$$\{b_k(p_i), \overline{b}_l(p_j)\} = \mathrm{e}^{dG} \{a_k(0), a_l(0)^+\} \mathrm{e}^{-dG} \,|_{p=0\to p=p_i} \; \delta_k^l \delta_j^i = \delta_k^l \frac{1}{\Delta^3 p_i} \delta_j^i \;\Rightarrow$$

$$\{b_k(p), \overline{b}_l(p')\} = \delta_k^l \delta(p - p') \;\Rightarrow \text{ CAR algebra}$$

Second quantized Dirac theory, p=0

$$p=0,\, p_0=m,\quad \Psi(0)=\begin{pmatrix} a_1(0)\\ 0\\ 0\\ 0\end{pmatrix} e^{-imx^0},\;\; a_1(0)\,\Psi_{\mathrm{V}}=0;\; a_k(0)\,\Psi_{\mathrm{V}}=0$$

$$=>\Psi_{\mathrm{V}}=a_1(0)a_2(0)a_3(0)\;a_4(0)\ldots=\frac{\partial}{\partial\theta^1(0)}\frac{\partial}{\partial\theta^2(0)}\frac{\partial}{\partial\theta^3(0)}\frac{\partial}{\partial\theta^4(0)}\ldots;$$

$$\mathrm{degree}(\theta^k(0))=+1,\,\mathrm{degree}(\frac{\partial}{\partial\theta^k(0)})=-1,\,\mathrm{degree}(\frac{\partial}{\partial\theta^k(0)}\theta^k(0))=0.$$

Ψ_{V} is scalar => degree of Ψ_{V} must be $=0$ for all k.

$$=>\Psi_{\mathrm{V}}=a_1(0)a_1(0)^{+}a_2(0)a_2(0)^{+}\;a_3(0)\;a_3(0)^{+}a_4(0)a_4(0)^{+}\ldots$$

Second quantized Dirac theory, p≠0, discrete momentums

Example for (t, s) = (1,3) : consider Lorentz transformation $e^{\gamma^{01}\varphi}$:

$$p = 0 \to p = p_j,\ a_k(0) \to a_k(p_j),\ \ \Psi_{\mathrm{V}} \to \Psi_{\mathrm{V}},\ a_k(p)\Psi_{\mathrm{V}} = 0$$

$$\Psi(0) \to \Psi(p) = \begin{pmatrix} a_1(p)\cosh\varphi \\ 0 \\ 0 \\ a_1(p)\sinh\varphi \end{pmatrix} e^{-ip_\mu x^\mu} = b_1(p) e^{-ip_\mu x^\mu}.$$

=> annihilation operator is transformed into annihilation one.

Signature $t = 0$ or $s = 0$ => $a_k(0)^+ \to a_k(p_j)^+$

$$=> \ \Psi_{\mathrm{V}} \sim \prod_{p_j} \prod_{k=1,2,3,4} a_k(p_j) a_k(p_j)^+$$

4-component spinor field operators

Spinor field operator $\Psi = \int d^3 p \left(\psi^{\alpha}(p) \frac{\partial}{\partial \theta^{\alpha}(p)} + \psi^{\tau}(p) \theta^{\tau}(p) \right)$

$$\frac{\partial}{\partial \theta^1(p)} \cong \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \frac{\partial}{\partial \theta^2(p)} \cong \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \theta^3(p) \cong \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \theta^4(p) \cong \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\theta^1(p) \cong (1 \;\; 0 \;\; 0 \;\; 0), \qquad \theta^2(p) \cong (0 \;\; 1 \;\; 0 \;\; 0),$$

$$\frac{\partial}{\partial \theta^3(p)} \cong (0 \;\; 0 \;\; 1 \;\; 0), \quad \frac{\partial}{\partial \theta^4(p)} \cong (0 \;\; 0 \;\; 0 \;\; 1).$$

Superalgebraic spinor field operators (Grassmann variables and their derivatives) are more general than columns and rows.

Infinitesimal transformation of basis field operators for signature (t,s)=(4,0); $c_{\mathrm{kl}} \rightarrow 0$

$$b_1 = \frac{\partial'}{\partial\theta^1} = \frac{\partial}{\partial\theta^1} + c_{11}\frac{\partial}{\partial\theta^1} + c_{12}\frac{\partial}{\partial\theta^2} + c_{13}\theta^3 + c_{14}\theta^4 + c_{15}\theta^1 + c_{16}\theta^2 + c_{17}\frac{\partial}{\partial\theta^3} + c_{18}\frac{\partial}{\partial\theta^4}$$

$$b_2 = \frac{\partial'}{\partial\theta^2} = \frac{\partial}{\partial\theta^2} + c_{21}\frac{\partial}{\partial\theta^1} + c_{22}\frac{\partial}{\partial\theta^2} + c_{23}\theta^3 + c_{24}\theta^4 + c_{25}\theta^1 + c_{26}\theta^2 + c_{27}\frac{\partial}{\partial\theta^3} + c_{28}\frac{\partial}{\partial\theta^4}$$

$$b_3^+ = \theta^3{}' = \theta^3 + c_{31}\frac{\partial}{\partial\theta^1} + c_{32}\frac{\partial}{\partial\theta^2} + c_{33}\theta^3 + c_{34}\theta^4 + c_{35}\theta^1 + c_{36}\theta^2 + c_{37}\frac{\partial}{\partial\theta^3} + c_{38}\frac{\partial}{\partial\theta^4}$$

$$b_4^+ = \theta^4{}' = \theta^4 + c_{41}\frac{\partial}{\partial\theta^1} + c_{42}\frac{\partial}{\partial\theta^2} + c_{43}\theta^3 + c_{44}\theta^4 + c_{45}\theta^1 + c_{46}\theta^2 + c_{47}\frac{\partial}{\partial\theta^3} + c_{48}\frac{\partial}{\partial\theta^4}$$

Hermitian conjugated:

$$b_1^+ = \theta^1{}' = \theta^1 + c_{11}^*\theta^1 + c_{12}^*\theta^2 + c_{13}^*\frac{\partial}{\partial\theta^3} + c_{14}^*\theta^4\frac{\partial}{\partial\theta^4} + c_{15}^*\frac{\partial}{\partial\theta^1} + c_{16}^*\frac{\partial}{\partial\theta^2} + c_{17}^*\theta^3 + c_{18}^*\theta^4$$

$$b_2^+ = \theta^2{}' = \theta^2 + c_{21}^*\theta^1 + c_{22}^*\theta^2 + c_{23}^*\frac{\partial}{\partial\theta^3} + c_{24}^*\theta^4\frac{\partial}{\partial\theta^4} + c_{25}^*\frac{\partial}{\partial\theta^1} + c_{26}^*\frac{\partial}{\partial\theta^2} + c_{27}^*\theta^3 + c_{28}^*\theta^4$$

$$b_3 = \frac{\partial'}{\partial\theta^3} = \frac{\partial}{\partial\theta^3} + c_{31}^*\theta^1 + c_{32}^*\theta^2 + c_{33}^*\frac{\partial}{\partial\theta^3} + c_{34}^*\theta^4\frac{\partial}{\partial\theta^4} + c_{35}^*\frac{\partial}{\partial\theta^1} + c_{36}^*\frac{\partial}{\partial\theta^2} + c_{37}^*\theta^3 + c_{38}^*\theta^4$$

$$b_4 = \frac{\partial'}{\partial\theta^4} = \frac{\partial}{\partial\theta^4} + c_{41}^*\theta^1 + c_{42}^*\theta^2 + c_{43}^*\frac{\partial}{\partial\theta^3} + c_{44}^*\theta^4\frac{\partial}{\partial\theta^4} + c_{45}^*\frac{\partial}{\partial\theta^1} + c_{46}^*\frac{\partial}{\partial\theta^2} + c_{47}^*\theta^3 + c_{48}^*\theta^4$$

Equations for coefficients c_{kl}

$$\{b_k(p), b_l(p')\} = 0, \quad \{b_k(p), \overline{b_l}(p')\} = \delta_k^l \delta(p - p').$$

$$\{b_1, b_1\} = 0 \;=>\; c_{15} = c_{26} = c_{37} = c_{48} = 0;$$

$$\{b_1, b_2\} = 0 \;=>\; c_{25} + c_{16} = 0;$$

$$\{b_1, b_3^+\} = 0 \;=>\; c_{35} + c_{17} = 0;$$

$$\{b_1, b_4^+\} = 0 \;=>\; c_{45} + c_{18} = 0;$$

$$\{b_1, b_1^+\} = \{\frac{\partial}{\partial \theta^1}, \theta^1\} \;=>\; c_{11}^* + c_{11} = 0;$$

$$\{b_1, b_2^+\} = 0 \;=>\; c_{21}^* + c_{12} = 0;$$

$$\ldots$$

Transformation operators following from CAR

$$\mathrm{CAR:}\quad \{\frac{\partial}{\partial\theta^k(p)},\theta^l(p')\}=\delta_k^l\delta(p-p'),$$

$$\{\frac{\partial}{\partial\theta^k(p)},\frac{\partial}{\partial\theta^l(p')}\}=\{\theta^k(p),\theta^l(p')\}=0.$$

$$\frac{\partial}{\partial\theta^k(p)}\to\frac{\partial}{\partial\theta^l(p)}\qquad => \hat{G}=\int d^3p'[\frac{\partial}{\partial\theta^l(p')}\theta^k(p'),\bullet],$$

$$\frac{\partial}{\partial\theta^k(p)}\to\theta^l(p),\ k\neq l\ => \hat{G}=\int d^3p'[\theta^l(p')\theta^k(p'),\bullet],$$

$$\theta^k(p)\to\theta^l(p)\qquad => \hat{G}=\int d^3p'[\theta^l(p')\frac{\partial}{\partial\theta^k(p')},\bullet],$$

$$\theta^k(p)\to\frac{\partial}{\partial\theta^l(p)},\ k\neq l\ => \hat{G}=\int d^3p'[\frac{\partial}{\partial\theta^l(p')}\frac{\partial}{\partial\theta^k(p')},\bullet].$$

Main decomposition: general form of transformation of the field operators

$$\Psi' = (1 + i\hat{\gamma}^{a} d\omega_{a} + \frac{1}{4}\hat{\gamma}^{ab} d\omega_{ab})\Psi,$$

$$a,b = 0,1,2,3,4,6,7.$$

$$\Psi' = (1 + i\hat{\gamma}^{\mu} d\omega_{\mu} + \frac{1}{4}\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu} + \frac{1}{2}\hat{\gamma}^{67} d\omega_{67} + i\hat{\gamma}^{g} d\omega_{g} + \frac{1}{4}\hat{\gamma}^{\mu g} d\omega_{\mu g} +$$

$$+ \frac{1}{4}\hat{\gamma}^{46} d\omega_{46} + \frac{1}{4}\hat{\gamma}^{47} d\omega_{47})\Psi, \quad \mu,\nu = 0,1,2,3; g = 4,6,7; \; \hat{\gamma}^{4} = i\hat{\gamma}^{5}.$$

Term $i\hat{\gamma}^{\mu} d\omega_{\mu}$ – decomposition by momentums.

Term $\frac{1}{4}\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu}$ – Lorentz transformations.

$\hat{\gamma}^{67} = i\hat{Q}$; $\hat{Q}$ – operator of electric charge. – *See later*.

Gamma operators (analogs of matrices): 7 basis Clifford vectors, (t,s)=(7,0)

$$\hat{\gamma}^0 = -\int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\theta^2(p) + \frac{\partial}{\partial\theta^3(p)}\theta^3(p) + \frac{\partial}{\partial\theta^4(p)}\theta^4(p), \bullet],$$

$$\hat{\gamma}^1_+ = -i\int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^2(p), \bullet],$$

$$\hat{\gamma}^2_+ = \int \mathrm{d}^3 p\ [-\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^2(p), \bullet],$$

$$\hat{\gamma}^3_+ = -i\int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^1(p) - \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p)\ , \bullet],$$

$$\hat{\gamma}^5 = \int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p)\ , \bullet],$$

$$\hat{\gamma}^6_+ = \int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} + \theta^2(p)\theta^1(p) - \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p)\ , \bullet],$$

$$\hat{\gamma}^7_+ = -i\int \mathrm{d}^3 p\ [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} - \theta^2(p)\theta^1(p) + \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p)\ , \bullet].$$

Gamma operators for (t,s)=(1,6) : two additional ones compared to Dirac's theory

$$\hat{\gamma}^0 = \int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\theta^2(p) + \frac{\partial}{\partial\theta^3(p)}\theta^3(p) + \frac{\partial}{\partial\theta^4(p)}\theta^4(p), \bullet],$$

$$\hat{\gamma}^1 = \int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^2(p), \bullet],$$

$$\hat{\gamma}^2 = i\int \mathrm{d}^3 p \; [-\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^2(p), \bullet],$$

$$\hat{\gamma}^3 = \int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} - \theta^3(p)\theta^1(p) - \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p) \;, \bullet],$$

$$\hat{\gamma}^4 = i\hat{\gamma}^5 = i\int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^3(p)} + \theta^3(p)\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\frac{\partial}{\partial\theta^4(p)} + \theta^4(p)\theta^2(p) \;, \bullet],$$

$$\hat{\gamma}^6 = i\int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} + \theta^2(p)\theta^1(p) - \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p) \;, \bullet],$$

$$\hat{\gamma}^7 = \int \mathrm{d}^3 p \; [\frac{\partial}{\partial\theta^1(p)}\frac{\partial}{\partial\theta^2(p)} - \theta^2(p)\theta^1(p) + \frac{\partial}{\partial\theta^3(p)}\frac{\partial}{\partial\theta^4(p)} - \theta^4(p)\theta^3(p) \;, \bullet].$$

Operator of generalized Dirac conjugation. Spinor vacuum

$$b_\alpha(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\frac{\partial}{\partial\theta^\alpha(0)}|_{p=0\to p=p_i}, \quad \bar{b}_\alpha(p_i) = \exp(\hat{\gamma}^{0k}\varphi_k)\theta^\alpha(0)|_{p=0\to p=p_i}$$

$$\bar{\Psi} = (M\Psi)^+$$

Signature $(+------)$, $(t,s)=(1,6) \Leftrightarrow M = \hat{\gamma}^0$.

$$\bar{\Psi} = (\hat{\gamma}^0\Psi)^+.$$

$$\Psi_{\mathrm{V}}(0) = (\Delta^3 p|_{p=0})^4 \frac{\partial}{\partial\theta^1(0)}\theta^1(0)\frac{\partial}{\partial\theta^2(0)}\theta^2(0)\frac{\partial}{\partial\theta^3(0)}\theta^3(0)\frac{\partial}{\partial\theta^4(0)}\theta^4(0)$$

$$\Psi_{\mathrm{V}}(p_i) = (\Delta^3 p_i)^4\, b_1(p_i)\bar{b}_1(p_i)b_2(p_i)\bar{b}_2(p_i)b_3(p_i)\bar{b}_3(p_i)b_4(p_i)\bar{b}_4(p_i)$$

$\Psi_{\mathrm{V}} = \prod_i \Psi_{\mathrm{V}}(p_i)$ – it is primitive idempotent.

Fermionic vacuum as primitive idempotent of the large Clifford algebra

- Ψ_V is primitive idempotent of the large Clifford algebra.
- The action of the annihilation operator on Ψ_V yields zero.
- Matrix representations of the gamma operators γ^ν, ν=0,1,2,3,5 are Dirac gamma matrices γ^ν. Gamma operators transform the spinor field operators of the large Clifford algebra in the same way that the Dirac gamma matrices transform the spinor states of the small Clifford algebra.
- Ψ_V is invariant under Lorentz transformations of the Lie group. It has zero spin, 4-momentum and electric charge.
- Ψ_V is not invariant under reflections. Instead, inversions are needed - see below.

Decomposition in momentums. Dirac equation

$$\Psi' = (1 + i\hat{\gamma}^{\mu} d\omega_{\mu} + \frac{1}{4}\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu} + \frac{1}{2} i\hat{Q}\, d\omega_{67})\Psi, \quad \mu,\nu = 0,1,2,3.$$

Let $d\omega_{\mu} = -m\, dx_{\mu}$, and $d\omega_{67} = qA^{\mu} dx_{\mu}$.

If 4 - vector $\hat{\gamma}^{\mu} dx_{\mu}$ is timelike we do Lorentz rotation to $\hat{\gamma}^{0} dx_{0}$.

Define parameter $p = 0$ for this state. Let $A^{\mu} = 0$.

$$\Psi(0) = \psi^{1} \frac{\partial}{\partial \theta^{1}(0)} + \psi^{2} \frac{\partial}{\partial \theta^{2}(0)} + \psi_{3}\theta^{3}(0) + \psi_{4}\theta^{4}(0) = \Psi_{+} + \Psi_{-}.$$

$$\Psi(0)' = (1 - i\hat{\gamma}^{0} m\, dx_{\mu}')\Psi(0) = \exp(-ip^{0}' dx_{0})\Psi_{+} + \exp(ip^{0}' dx_{0})\Psi_{-}.$$

Lorentz rotation backward to $\hat{\gamma}^{\mu} dx_{\mu}$

$$=> \Psi(p)' = \exp(-ip^{\mu} dx_{\mu})\Psi_{+}(p) + \exp(p^{\mu}\, dx_{\mu})\Psi_{-}(p).$$

$$\hat{\gamma}^{\mu} i\partial_{\mu}\Psi' = m\Psi'. \quad \text{If } A^{\mu} \neq 0 => \hat{\gamma}^{\mu}(i\partial_{\mu} - \frac{q}{2}\hat{Q}A_{\mu})\Psi' = m\Psi'.$$

Main decomposition: additional terms, no-go theorems

$$\Psi' = (1 + i\hat{\gamma}^{\mu} d\omega_{\mu} + \frac{1}{4}\hat{\gamma}^{\mu\nu} d\omega_{\mu\nu} + \frac{1}{2} i\hat{Q}\, d\omega_{67} + i\hat{\gamma}^{g} d\omega_{g} + \frac{1}{4}\hat{\gamma}^{\mu g} d\omega_{\mu g} + \frac{1}{4}\hat{\gamma}^{46} d\omega_{46} +$$

$$+ \frac{1}{4}\hat{\gamma}^{47} d\omega_{47})\Psi = (1 + d\hat{G})\Psi, \quad \mu,\nu = 0,1,2,3; g = 4,6,7, \ \hat{\gamma}^{4} = i\hat{\gamma}^{5}.$$

$d\omega_{\mu\nu} = A_{\mu\nu\lambda} dx^{\lambda} \Rightarrow$ gravitational interaction (~ Lorentz transformations).

$d\omega_{g} = p_{g} dx^{g} \neq 0 \Rightarrow$ new dimensions of the spacetime.

No - go theorems : McGlinn - O'Raifeartaigh - Jost and Coleman - Mandula theorems.

Generators of internal symmetries must commute with generators of the Lorentz group.

$d\omega_{\mu g} = A_{\mu\nu g} dx^{\nu} \neq 0, [\frac{1}{4}\hat{\gamma}^{\mu g} d\omega_{\mu g}, \hat{\gamma}^{\mu\nu}] \neq 0 \Rightarrow$ violation of Lorentz invariance or

rotation in new dimensions.

$d\omega_{46} = A_{46\nu} dx^{\nu} \neq 0$ or $d\omega_{47} = A_{47\nu} dx^{\nu} \neq 0 \Rightarrow [d\hat{G}, \hat{Q}] \neq 0 \Rightarrow$

violation of conservation of electric charge.

New vector fields $A_{46\nu}, A_{47\nu}$ and tenzor fields $A_{\mu\nu 4}, A_{\mu\nu 6}, A_{\mu\nu 7}$ or new dimensions.

Conclusions on part 1

- The proposed theory unambiguously follows from the transformation laws of the creation and annihilation operators by the Lie group (Lorentz group).
- In this theory, spinors are representations of both the spacetime Clifford algebra and the CAR algebra.
- Seven gamma operators can be constructed from the creation and annihilation operators of spinor and antispinor.
- Five of them are analogs of the Dirac gamma matrices, and two correspond to the internal degrees of freedom of the spinor and generate an electric charge.
- The generalized Dirac conjugation operator is one-to-one given by the signature of the spacetime and the internal space of spinors.

Conclusions on part 1, continued

- The spinor vacuum state vector Ψ_V has zero spin, 4-momentum and electric charge.
- The requirement to preserve the CAR algebra under transformations of the Lie group leads to the decomposition of spinor field operator in terms of momentums, Lorentz rotations, electromagnetic interaction, and as yet incomprehensible terms. It defines a Lie group that is wider than the Lorentz group.
- New possible mechanisms of violation of the Lorentz invariance and the law of conservation of electric charge are found.
- Due to Ψ_V, the properties of discrete symmetry operations differ significantly from the usual theory of spinors.

Part 2. Reflections and *P*, *T*, *C* inversions

- R-operators (reflections, rotations, reverse).
- Operators of charge Q, spatial reflection P_1 and spatial inversion P.
- Alternative spinor vacuum.
- Operators of time reflection T_1 and time inversion T.
- Operator of charge conjugation C.

R-operators (reflections, rotations, reverse)

Lie group generator $d\hat{G} = [dG, \bullet]$

$$(1 + d\hat{G})\Psi_1\Psi_2 \ldots \Psi_k = 1 + [dG, \Psi_1]\Psi_2 \ldots \Psi_k + \Psi_1[dG, \Psi_2] \ldots \Psi_k + \ldots =$$

$$= (e^{d\hat{G}}\Psi_1)(e^{d\hat{G}}\Psi_2) \ldots (e^{d\hat{G}}\Psi_k)$$

$$e^{\hat{G}}\Psi_1\Psi_2 \ldots \Psi_k = (e^{\hat{G}}\Psi_1)(e^{\hat{G}}\Psi_2) \ldots (e^{\hat{G}}\Psi_k)$$

$R_{\hat{G}} = e^{\hat{G}}$ – operator acts on all factors simultaneously.

Examples of R - operators : $e^{i\hat{\gamma}^0 \pi/2} = R_{i\hat{\gamma}^0},\ e^{i\hat{\gamma}^\mu \varphi_\mu}, e^{\hat{\gamma}^{\mu\nu} \varphi_{\mu\nu}},$

$R_{-x}:\ x^k \to -x^k, x^0 \to x^0$; complex conjugation $(\bullet)^*$;

transposition $(\bullet)^T$; Hermitian conjugation $(\bullet)^+ = (\bullet)^T (\bullet)^*$.

Clifford algebra: operators of reflection

Reflection operator A transforms Clifford vector X as

$X' = AXA^{-1} = (\lambda A)X(\lambda A)^{-1}$, $\lambda \neq 0$ – any complex number.

Operator A transforms spinor Ψ as

$$\Psi' = A\Psi.$$

Reflection should keep norm => $(\Psi', \Psi') = (\Psi, \Psi)$

$$=> \quad \lambda^* \lambda = 1 => \lambda = e^{i\varphi}.$$

$A = i\hat{\gamma}^0$ => reflects all $\hat{\gamma}^k, k \neq 0$:

$$\hat{\gamma}^0{}' = \hat{\gamma}^0; \hat{\gamma}^k{}' = -\hat{\gamma}^k, k = 1,2,3,4,6,7. \quad \hat{\gamma}^4 = i\hat{\gamma}^5$$

$A = \hat{\gamma}^a \hat{\gamma}^b = \hat{\gamma}^{ab}$ => reflects $\hat{\gamma}^a$ and $\hat{\gamma}^b$.

Spinors: no phase arbitrariness

Reflection operator A transforms spinor Ψ as

$$\Psi' = A\Psi.$$

and transforms conjugated spinor Ψ as

$$\overline{\Psi}' = A\overline{\Psi},$$

since $\overline{\Psi}$ is transformed by the same Lie group operators as Ψ.

$$\lambda = e^{i\varphi} \Rightarrow \overline{\Psi}' = e^{i\varphi} A\overline{\Psi} = e^{-i\varphi} A\overline{\Psi}$$

$\Rightarrow \lambda = \pm 1.$

CAR algebra, $\theta^k(p)' = \lambda\theta^k(p)$, $\dfrac{\partial'}{\partial\theta^k(p)} = \lambda\dfrac{\partial}{\partial\theta^k(p)} \Rightarrow$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

Operators of charge *Q*, spatial reflection *P₁* and spatial inversion *P*

$$\hat{Q} = i\hat{\gamma}^6\hat{\gamma}^7 =$$

$$\int d^3p\left[\frac{\partial}{\partial\theta^1(p)}\theta^1(p) + \frac{\partial}{\partial\theta^2(p)}\theta^2(p) - \frac{\partial}{\partial\theta^3(p)}\theta^3(p) - \frac{\partial}{\partial\theta^4(p)}\theta^4(p), \bullet\right]$$

$-$ generator of rotations in the plane $\hat{\gamma}^6, \hat{\gamma}^7$.

Operator of charge in the theory of second quantization.

$$\hat{Q}\Psi = \Psi, \qquad \hat{Q}\overline{\Psi} = -\overline{\Psi},$$

$$e^{i\hat{Q}\varphi}\Psi = e^{i\varphi}\Psi, \quad e^{i\hat{Q}\varphi}\overline{\Psi} = e^{-i\varphi}\overline{\Psi}.$$

$P_1 = R_{\hat{\gamma}^0\hat{Q}}$ $-$ spatial reflection. Keeps $\hat{\gamma}^0, \hat{\gamma}^6, \hat{\gamma}^7$ invariant.

Breaks Dirac equation. $R_{-x}: \ x^k \to -x^k, x^0 \to x^0$.

$P = R_{-x}R_{\hat{\gamma}^0\hat{Q}}$ $-$ spatial inversion. Keeps Dirac equation.

Alternative spinor vacuum

$\hat{\gamma}^1, \hat{\gamma}^2, \hat{\gamma}^3, \hat{\gamma}^5, \hat{\gamma}^6, \hat{\gamma}^7$ – change Ψ_{V} to Ψ_{altV}

$\hat{\gamma}^0$ – keeps Ψ_{V}

$$\Psi_{\mathrm{altV}}(0) = (\Delta^3 p\,|_{p=0})^4\, \theta^1(0)\frac{\partial}{\partial\theta^1(0)}\theta^2(0)\frac{\partial}{\partial\theta^2(0)}\theta^3(0)\frac{\partial}{\partial\theta^3(0)}\theta^4(0)\frac{\partial}{\partial\theta^4(0)}$$

$$\Psi_{\mathrm{altV}}(p_i) = (\Delta^3 p_i)^4\, \bar{b}_1(p_i) b_1(p_i) \bar{b}_2(p_i) b_2(p_i) \bar{b}_3(p_i) b_3(p_i) \bar{b}_4(p_i) b_4(p_i)$$

$$\Psi_{\mathrm{altV}} = \prod_i \Psi_{\mathrm{altV}}(p_i)$$

$\bar{b}_k(p_i)$ – annihilation operator

$b_k(p_i)$ – creation operator

Operators of time reflection T_1 and time inversion T

$$T_1 = R_{-x^0} R_{\hat{\gamma}^1 \hat{\gamma}^3} (\bullet)^*, \qquad \Psi_V \to \Psi_V$$

"Rewinding the film", annihilation operator must become creation one, and vice versa

$$R = R_{\hat{\gamma}^{05}} R_{\hat{\gamma}^{26}} \left(\bullet\right)^T - \text{reverse } R\,\Psi_1 \Psi_2 \ldots \Psi_k = \Psi_k \ldots \Psi_2 \Psi_1$$

$$R\,\Psi_{\mathrm{V}} = \Psi_{\mathrm{altV}}, \quad R\,\Psi = \Psi, \quad R\overline{\Psi} = \overline{\Psi}$$

$$T = RT_1 = R_{-x^0} R_{\hat{\gamma}^7} (\bullet)^+, \quad \Psi_V \to \Psi_{altV}$$

=> T cannot be a symmetry of Nature.

Interaction of spinor with Ψ_{altV} and with Ψ_V differs.

Operator of charge conjugation C

It is rigorously proven (Jost) that the PTC operator is antiunitary. P - unitary, T - antiunitary => C is unitary.

Formulas with the antiunitary operator C are correct when using the concept of the Dirac Sea.

$C_1 = R_{i\hat{\gamma}^{56}}$, $\Psi_V \to \Psi_V$, errors for multyparticle states.

Proper results for multyparticle field operators :

$$C = RC_1 = R_{-i\hat{\gamma}^{02}}\left(\bullet\right)^T, \quad \Psi_V \to \Psi_{altV}$$

=> C cannot be a symmetry of Nature.

Interaction of spinor with Ψ_{altV} and with Ψ_V differs.

Conclusions on part 2

$$P = R_{-x^k} R_{\hat{\gamma}^0 \hat{Q}}, \qquad \Psi_V \to \Psi_V,$$

$$T = R_{-x^0} R_{\hat{\gamma}^7} (\bullet)^+, \qquad \Psi_V \to \Psi_{altV}, \text{breaks symmetry}$$

$$C = R_{-i\hat{\gamma}^{02}} \left(\bullet\right)^T, \quad \Psi_V \to \Psi_{altV}, \text{breaks symmetry}$$

$$CPT = R_{-x^\mu} J_+, \quad \Psi_V \to \Psi_V$$

$$J_+ = R_{\hat{\gamma}^{26}} \left(\bullet\right)^* - \text{operator of real structure}$$

(charge conjugation) in Krein spaces.

Conclusions on part 2, continued

- Operators T and C are not consistent with vacuum of the Universe.
- They can only be approximate symmetry operators.
- The symmetry breaking is small when spinor is independent particle.
- Vacuum is similar to the Dirac Sea.
- P, TC, CPT can be exact symmetry operators of spinors.