In odd dimensional space Clifford algebra manifest properties of the Fadeev-Popov ghosts

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Some publications:

- *Phys. Lett.* **B 292**, 25-29 (**1992**), *J. Math. Phys.* **34**, 3731-3745 (**1993**), *Mod. Phys. Lett.* **A 10**, 587-595 (**1995**),
- Journal of Physics: Conference Series, Volume 2482/2023, Proceedings https://iopscience.iop.org/issue/1742-6596/2482/1/012012
- Rev. Artile in **Progress in Particle and Nuclear Physics**, **http://doi.org/10.1016.j.ppnp.2021.103890** ,
- **Symmetry 2023,15,818-12-V2 94818, https:doi.org/10.3390/sym15040818**, [arxiv.org/abs/2301.04466]

More than **50 years ago** the **electroweak (and colour) standard model** offered an **new step** in **understanding the origin of fermions** and **bosons**.

We are still with a lot of open questions, among them:

- **Do we understand internal spaces of fermions and bosons in an unique way?**
- **Do we understand why fermions appear in families while bosons do not?**
- **Do we understand the postulates of the second quantized fields**; **why fermion fields anti-commute while boson fields commute?**
- **Do we understand the appearance of vector and scalar gauge fields?**

- A brief introduction into the description of the internal degrees of freedom of fermions and bosons with the Clifford odd and even algebra, respectively.

- In Grassmann d-dimensional space there are d anti-commuting (operators) θ^a, and d anti-commuting operators which are derivatives with respect to θ^a, $\frac{\partial}{\partial \theta_a}$,

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0\,,$$

$$\{\theta_a, \frac{\partial}{\partial \theta_b}\}_+ = \delta_{ab}\,,\ (a,b) = (0,1,2,3,5,\cdots,d)\,.$$

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- **There are two kinds of the Clifford algebra objects, γ^a and $\tilde{\gamma}^a$, in any d .**

$$\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})\,,$$

$$\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,$$

offering together $2 \cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d of $\tilde{\gamma}^a$.

- The two kinds of the Clifford algebra objects anticommute

$$
\begin{aligned}
\{\gamma^{a}, \gamma^{b}\}_{+} &= 2\eta^{ab} = \{\tilde{\gamma}^{a}, \tilde{\gamma}^{b}\}_{+}, \\
\{\gamma^{a}, \tilde{\gamma}^{b}\}_{+} &= 0,
\end{aligned}
$$

- The Grassmann algebra can describe commuting and anti-commuting integer spin fields.

  There are no integer spin spin fermions observed so far.

  There is only one kind of fermions with half integer spin observed so far.

- **The postulate**

$$
\begin{aligned}
(\tilde{\gamma}^{\mathbf{a}}\mathbf{B} &= \mathbf{i}(-)^{\mathbf{n_B}}\mathbf{B}\gamma^{\mathbf{a}})\,|\psi_0>, \\
(\mathbf{B} &= a_0 + a_a\gamma^a + a_{ab}\gamma^a\gamma^b + \cdots + a_{a_1\cdots a_d}\gamma^{a_1}\ldots\gamma^{a_d})|\psi_o>
\end{aligned}
$$

with $(-)^{n_B} = +1, -1$, if B has a Clifford even or odd character, respectively, $|\psi_o>$ is a vacuum state on which the operators γ^a **apply**,
reduces the Clifford space for fermions for the factor of two,
while the operators $\tilde{\gamma}^a\tilde{\gamma}^b = -2i\tilde{S}^{ab}$ **equip each irreducible representation of internal Lorentz group with the family quantum numbers**.

- It is convenient to write all the **"basis vectors" describing the internal space of either fermion fields or boson fields as products of nilpotents and projectors, which are eigenvectors of the chosen Cartan subalgebra**

$$
\begin{aligned}
&S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\; d}\,,\\
&\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\; d}\,,\\
&\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,.
\end{aligned}
$$

nilpotents

$$
S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \stackrel{ab}{(\mathbf{k})} := \frac{\mathbf{1}}{\mathbf{2}}(\gamma^{\mathbf{a}} + \frac{\eta^{\mathbf{aa}}}{\mathbf{ik}}\gamma^{\mathbf{b}})\,,
$$

projectors

$$
S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad \stackrel{ab}{[\mathbf{k}]} := \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} + \frac{\mathbf{i}}{\mathbf{k}}\gamma^{\mathbf{a}}\gamma^{\mathbf{b}})\,,
$$

$$
(\stackrel{ab}{(\mathbf{k})})^2 = \mathbf{0}\,, \qquad (\stackrel{ab}{[\mathbf{k}]})^2 = \stackrel{ab}{[\mathbf{k}]}\,,
$$

$$
\stackrel{ab}{(\mathbf{k})}^{\dagger} = \eta^{\mathbf{aa}}\stackrel{ab}{(-\mathbf{k})}\,, \quad \stackrel{ab}{[\mathbf{k}]}^{\dagger} = \stackrel{ab}{[\mathbf{k}]}\,.
$$

$$
\begin{aligned}
\mathbf{S}^{ab}\overset{ab}{(\mathbf{k})} &= \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \mathbf{S}^{ab}\overset{ab}{[\mathbf{k}]} = \frac{k}{2}\overset{ab}{[\mathbf{k}]},\\
\tilde{\mathbf{S}}^{ab}\overset{ab}{(\mathbf{k})} &= \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \tilde{\mathbf{S}}^{ab}\overset{ab}{[\mathbf{k}]} = -frack2\overset{ab}{[\mathbf{k}]}\,.
\end{aligned}
$$

$$
\begin{aligned}
\gamma^{a}\overset{ab}{(\mathbf{k})} &= \eta^{aa}\overset{ab}{[-\mathbf{k}]}\,,\ \gamma^{b}\overset{ab}{(\mathbf{k})} = -ik\overset{ab}{[-\mathbf{k}]}\,,\ \gamma^{a}\overset{ab}{[\mathbf{k}]} = \overset{ab}{(-\mathbf{k})}\,,\ \gamma^{b}\overset{ab}{[\mathbf{k}]} = -ik\eta^{aa}\overset{ab}{(-\mathbf{k})}\,,\\
\tilde{\gamma}^{a}\overset{ab}{(\mathbf{k})} &= -i\eta^{aa}\overset{ab}{[\mathbf{k}]}\,,\ \tilde{\gamma}^{b}\overset{ab}{(\mathbf{k})} = -k\overset{ab}{[\mathbf{k}]}\,,\ \tilde{\gamma}^{a}\overset{ab}{[\mathbf{k}]} = i\overset{ab}{(\mathbf{k})}\,,\ \tilde{\gamma}^{b}\overset{ab}{[\mathbf{k}]} = -k\eta^{aa}\overset{ab}{(\mathbf{k})}\,,\\
\overset{ab}{(\mathbf{k})}\overset{ab}{(-\mathbf{k})} &= \eta^{aa}\overset{ab}{[\mathbf{k}]}\,,\ \overset{ab}{[\mathbf{k}]}\overset{ab}{(\mathbf{k})} = \overset{ab}{(\mathbf{k})}\,,\ \overset{ab}{(\mathbf{k})}\overset{ab}{[-\mathbf{k}]} = \overset{ab}{(\mathbf{k})}\,,\\
\overset{ab}{(\mathbf{k})}\overset{ab}{[\mathbf{k}]} &= \mathbf{0}\,,\ \overset{ab}{[\mathbf{k}]}\overset{ab}{(-\mathbf{k})} = \mathbf{0}\,,\ \overset{ab}{[\mathbf{k}]}\overset{ab}{[-\mathbf{k}]} = \mathbf{0}\,,\\
\overset{ab}{\widetilde{(-\mathbf{k})}}\overset{ab}{(\mathbf{k})} &= -i\eta^{aa}\overset{ab}{[\mathbf{k}]}\,,\ \overset{ab}{\widetilde{[\mathbf{k}]}}\overset{ab}{(\mathbf{k})} = \overset{ab}{(\mathbf{k})}\,,\ \overset{ab}{\widetilde{(\mathbf{k})}}\overset{ab}{[\mathbf{k}]} = i\overset{ab}{(\mathbf{k})}\,,\ \overset{ab}{\widetilde{[-\mathbf{k}]}}\overset{ab}{[\mathbf{k}]} = \overset{ab}{[\mathbf{k}]}\,,\\
\overset{ab}{\widetilde{(\mathbf{k})}}\overset{ab}{(\mathbf{k})} &= \mathbf{0}\,,\ \overset{ab}{\widetilde{[-\mathbf{k}]}}\overset{ab}{(\mathbf{k})} = \mathbf{0}\,,\ \overset{ab}{\widetilde{(\mathbf{k})}}\overset{ab}{[-\mathbf{k}]} = \mathbf{0}\,,\ \overset{ab}{\widetilde{[\mathbf{k}]}}\overset{ab}{[\mathbf{k}]} = \mathbf{0}\,.
\end{aligned}
$$

- γ^a **transforms** $\stackrel{ab}{(k)}$ **into** $\stackrel{ab}{[-k]}$, **never to** $\stackrel{ab}{[k]}$.

- $\tilde{\gamma}^a$ **transforms** $\stackrel{ab}{(k)}$ **into** $\stackrel{ab}{[k]}$, **never to** $\stackrel{ab}{[-k]}$.

- **There are the Clifford odd "basis vector", that is the "basis vector" with an odd number of nilpotents, at least one, the rest are projectors, such "basis vectors" anti commute among themselves.**

- **There are the Clifford even "basis vector", that is the "basis vector" with an even number of nilpotents, the rest are projectors, such "basis vectors" commute among themselves.**

- **Let us see how does one family of the Clifford odd "basis vector", describing internal space of fermions in $d = (13+1)$ look like, if spins in $d = (13+1)$ are analysed with respect to the Standard Model groups.**
- **One irreducible representation** of one **family contains $2^{\frac{(13+1)}{2}-1} = 64$ members which include all the family members, quarks and leptons with the right handed neutrinos included, as well as all the anti-members, antiquarks and antileptons, reachable by either S^{ab} (or by $\mathbb{C}_{\mathcal{N}}$ $\mathcal{P}_{\mathcal{N}}$ on a family member)**.

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J. of Math. Phys. **34**, 3731 (**1993**),
Int. J. of Modern Phys. **A 9**, 1731 (**1994**),
J. of Math. Phys. **44** 4817 (**2003**), hep-th/030322.

S^{ab} generate **all the members of one family**. The **eightplet** (represent. of $SO(7,1)$) **of quarks of a particular colour charge. All are Clifford odd "basis vectors"** .

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^4
		Octet, $\Gamma^{(7,1)}=1,\ \Gamma^{(6)}=-1,$ of quarks							
1	$\mathbf{u_R^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
4	$\mathbf{d_R^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
6	$\mathbf{d_L^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
7	$\mathbf{u_L^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{(-)}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\mathbf{u_R}$ of the 1^{st} row into $\mathbf{u_L}$ of the 7^{th} row, and $\mathbf{d_R}$ of the 4^{rd} row into $\mathbf{d_L}$ of the 6^{th} row, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} generate **all the members of one family with leptons included.** Here is The **eightplet** (represent. of $SO(7,1)$) **of leptons colour chargeless. the $SO(7,1)$ part is identical with the one of quarks.**

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	Q
		Octet, $\Gamma^{(7,1)}=1$, $\Gamma^{(6)}=-1$, of leptons							
1	ν_R	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{(+)} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**1**	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
2	ν_R	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{(+)} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**1**	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
3	e_R	$\overset{03}{(+i)}\overset{12}{(+)} \vert \overset{56}{[-]}\overset{78}{[-]} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**1**	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
4	e_R	$\overset{03}{[-i]}\overset{12}{[-]} \vert \overset{56}{[-]}\overset{78}{[-]} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**1**	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
5	e_L	$\overset{03}{[-i]}\overset{12}{(+)} \vert \overset{56}{[-]}\overset{78}{(+)} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**-1**	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
6	e_L	$\overset{03}{(+i)}\overset{12}{[-]} \vert \overset{56}{[-]}\overset{78}{(+)} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**-1**	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
7	ν_L	$\overset{03}{[-i]}\overset{12}{(+)} \vert \overset{56}{(+)}\overset{78}{[-]} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**-1**	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
8	ν_L	$\overset{03}{(+i)}\overset{12}{[-]} \vert \overset{56}{(+)}\overset{78}{[-]} \vert\vert \overset{9\,10}{(+)}\;\overset{11\,12}{[+]}\;\overset{13\,14}{[+]}$	**-1**	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform ν_R of the 1^{st} line into ν_L of the 7^{th} line, and e_R of the 4^{rd} line into e_L of the 6^{th} line, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} **generate also all the anti-eightplet** (repres. of $SO(7,1)$) of **anti-quarks of the anti-colour charge bellonging to the same family of the Clifford odd basis vectors** .

i		$\|^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^4
		Antioctet, $\Gamma^{(7,1)}=-1$, $\Gamma^{(6)}=1$, of antiquarks							
33	$\mathbf{\bar{d}^{\bar{c1}}_L}$	$\overset{03}{[-i]}\overset{12}{(+)} \mid \overset{56}{(+)}\overset{78}{(+)} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**-1**	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
34	$\bar{d}^{\bar{c1}}_L$	$\overset{03}{(+i)}\overset{12}{[-]} \mid \overset{56}{(+)}\overset{78}{(+)} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**-1**	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
35	$\bar{u}^{\bar{c1}}_L$	$\overset{03}{[-i]}\overset{12}{(+)} \mid \overset{56}{[-]}\overset{78}{[-]} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**-1**	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
36	$\mathbf{\bar{u}^{\bar{c1}}_L}$	$\overset{03}{(+i)}\overset{12}{[-]} \mid \overset{56}{[-]}\overset{78}{[-]} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**- 1**	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
37	$\mathbf{\bar{d}^{\bar{c1}}_R}$	$\overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{(+)}\overset{78}{[-]} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**1**	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
38	$\bar{d}^{\bar{c1}}_R$	$\overset{03}{[-i]}\overset{12}{[-]} \mid \overset{56}{(+)}\overset{78}{[-]} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**1**	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
39	$\bar{u}^{\bar{c1}}_R$	$\overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{[-]}\overset{78}{(+)} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**1**	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
40	$\mathbf{\bar{u}^{\bar{c1}}_R}$	$\overset{03}{[-i]}\overset{12}{[-]} \mid \overset{56}{[-]}\overset{78}{(+)} \mid\mid \overset{9\,10}{[-]}\ \overset{11\,12}{[+]}\ \overset{13\,14}{[+]}$	**1**	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\mathbf{\bar{d}_L}$ of the 1^{st} row into $\mathbf{\bar{d}_R}$ of the 5^{th} row, and $\mathbf{\bar{u}_L}$ of the 4^{rd} row into $\mathbf{\bar{u}_R}$ of the 8^{th} row.

- We discuss so far the internal space of fermions, describing their internal space with Clifford odd "basis vectors".
- A beautiful side of the Clifford algebra is that it offers the description of the internal space of bosons in an equivalent way, describing their internal space with Clifford even "basis vectors".

  This is a new possibility, appearing as a surprise also for me, although I knew all the time that it must be like it came out:
  Internal space of fermions manifesting family members and families,
  Internal space of bosons manifesting NO families, having their Hermitian conjugated partners within the same group, and appearing in two groups, one group, when applying on the Clifford odd "basis vectors", transforms it to the other member of the same family, another group transforms it to the same member of another family.

- **Before we start to discuss Clifford even "basis vectors" describing the internal space of bosons let us write down the action**.
- **Fermions and bosons can exist even if they do not interact**, at least mathematically.
- **Describing their internal space we do not pay attention on their interactions. We treat them as free fields.**
- **Describing the properties of fermions and bosons as we observe, the interaction should be included: A simple and elegant one (this is how I "see nature") demonstrating at low energies all the observed phenomena.**

I have used so far in the spin-charge-family theory a simple action in $d=(13+1)$. **Fermions carry only spins, two kinds of spins (no charges) and interact with the gauge gravitational fields**.

$$\begin{aligned} \mathbf{S} &= \int d^d x\, E\, \mathcal{L}_f + \\ &\quad \int d^d x\, E\, (\alpha\, R + \tilde{\alpha}\, \tilde{R}) \end{aligned}$$

-

$$\begin{aligned} \mathcal{L}_f &= \frac{1}{2}(\bar{\psi}\,\gamma^a p_{0a}\psi) + h.c. \\ p_{0a} &= f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E}\,\{p_\alpha, Ef^{\alpha}{}_a\}_- \\ \mathbf{p}_{0\alpha} &= \mathbf{p}_\alpha - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{S}^{\mathbf{ab}}\omega_{\mathbf{ab}\alpha} - \frac{\mathbf{1}}{\mathbf{2}}\tilde{\mathbf{S}}^{\mathbf{ab}}\tilde{\omega}_{\mathbf{ab}\alpha} \end{aligned}$$

- The Einstein action for a free gravitational field is assumed to be linear in the curvature

$$
\begin{aligned}
\mathcal{L}_{\mathbf{g}} &= \mathbf{E}\,(\alpha\,\mathbf{R} + \tilde{\alpha}\tilde{\mathbf{R}}),\\
\mathbf{R} &= \mathbf{f}^{\alpha[\mathbf{a}}\mathbf{f}^{\beta\mathbf{b}]}\,(\omega_{\mathbf{ab}\alpha,\beta} - \omega_{\mathbf{ca}\alpha}\omega^{\mathbf{c}}{}_{\mathbf{b}\beta}),\\
\tilde{\mathbf{R}} &= \mathbf{f}^{\alpha[\mathbf{a}}\mathbf{f}^{\beta\mathbf{b}]}\,(\tilde{\omega}_{\mathbf{ab}\alpha,\beta} - \tilde{\omega}_{\mathbf{ca}\alpha}\tilde{\omega}^{\mathbf{c}}{}_{\mathbf{b}\beta}),
\end{aligned}
$$

with $E = \det(e^a{}_\alpha)$
and $f^{\alpha[a}f^{\beta b]} = f^{\alpha a}f^{\beta b} - f^{\alpha b}f^{\beta a}$.

- **Describing the interaction of fermions and bosons, the internal spaces of which are described with the Clifford algebra objects, might change the requirements for the boson fields as we shall see later.**

- I shall first discuss the “basis vectors” and correspondingly the creation operators for either the Clifford odd fermion fields or for the Clifford even boson fields in the limit of free fields.

- Let us start to discuss the Clifford even "basis vectors", offering the description of the internal space of bosons , and correspondingly of fermions, within a toy model in $d = (5 + 1)$, pointing out the differences between the "basis vectors" of odd and "basis vectors" of even Clifford algebra elements.

- **Let us repeat:**
  (Progress in Particle and Nuclear Physics,,
  http://doi.org/**10.1016.j.ppnp.2021.103890** , Eq. (14, 16, 28)),
  There are 2^d **Grassmann polynomials of** θ^a**'s and** 2^d **their Hermitian conjugated partners** $\frac{\partial}{\partial \theta_a}$, $(\theta^a)^\dagger = \eta^{aa} \frac{\partial}{\partial \theta_a}$.
- **Let us repeat also that there are** 2^d **Clifford objects, which are products** of γ^a**'s**

$$\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a}),$$

  half of them form Clifford odd "basis vectors" , half of them form Clifford even "basis vectors" .
- **There are** $2^{\frac{d}{2}-1}$ **Clifford odd family members, appearing** $2^{\frac{d}{2}-1}$ **irreducible representations, carrying family quantum numbers, determined with** $\tilde{S}^{ab} = \frac{i}{2} \tilde{\gamma}^a \tilde{\gamma}^b, a \neq b$

$$\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a})$$

- **And there are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ their Hermitian conjugated partners. Together there are 2^{d-1} Clifford odd "basis vectors".**

- **And there are 2^{d-1} Clifford even "basis vectors", appearing in to groups, each with different properties with respect to the Clifford odd "basis vectors".**

In even dimensional spaces fermion and **boson fields demonstrate in $d = (3+1)$ all the properties of the observed quarks and leptons and anti-quarks and anti-leptons and of the observed vector and scalar gauge fields.**

Let us see for small d=(1+1), (2+1), (3+1), (4+1), how do "basis vectors" look like in even and odd dimensional spaces.

- **Even dimensional spaces** have the properties of the **fermion** and **boson** second quantized fields.
- **Half of the Clifford odd** and **Clifford even** objects in $(2n+1)$-dimensional cases can be found by treating the **Clifford odd "basis vectors" and their Hermitian conjugated partners** and the **Clifford even "basis vectors"** in $2(2n'+1)$ (or $4n'$) dimensional part of space.
- **The rest of the "basis vectors" follow by the application of S^{0d}** on the "basis vectors" determining the internal space of **fermion** and **boson** fields in $2(2n'+1)$ (or $4n'$)-dimensional part of space.
  Since S^{0d} are the Clifford even operators, they do not change oddness or evenness of the "basis vectors" or their Hermitian conjugated partners. But they do change their properties:

- **d=(1+1)** **One family only!**

$$\text{Clifford odd}$$

$$\hat{\mathbf{b}}^{1\dagger}_{1} = \stackrel{01}{(+\mathbf{i})}\,, \quad \hat{\mathbf{b}}^{1}_{1} = \stackrel{01}{(-\mathbf{i})}\,,$$

$$\text{Clifford even}$$

$${}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[+\mathbf{i}]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[-\mathbf{i}]}\,.$$

- **d=(2+1)** **One family only!**

$$d = \quad 2+1$$

$$\text{Clifford odd}$$

$$\hat{b}^{1\dagger}_{1} = \stackrel{01}{(+i)}\,, \qquad \hat{b}^{1\dagger}_{2} = \stackrel{01}{[-i]}\;\gamma^{2}\,,$$

$$\hat{b}^{1}_{1} = \stackrel{01}{(-i)}\,, \qquad \hat{b}^{1}_{2} = \stackrel{01}{[+i]}\;\gamma^{2}\,,$$

$$\text{Clifford even}$$

$${}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[+i]}\,, \qquad {}^{I}\mathcal{A}^{1\dagger}_{2} = \stackrel{01}{(-i)}\;\gamma^{2}\,,$$

$${}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{01}{[-i]}\,, \qquad {}^{II}\mathcal{A}^{1\dagger}_{2} = \stackrel{01}{(+i)}\;\gamma^{2}\,.$$

One clearly sees in the case $d = (2+1)$ that the left hand side of the Clifford odd "basis vectors" and the right hand side of the Clifford even "basis vectors", although the first are the Clifford odd objects and the later the Clifford even objects, have similar properties.

- **The case with $d = (3+1)$ is more informative; there are two families.**
  In $d = (3+1)$ there are $16\,(2^{d=4})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and $(\mathcal{S}^{03}, \mathcal{S}^{12})$ of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$.
- Half of them are the Clifford odd "basis vectors", appearing in two families $2^{\frac{4}{2}-1}$, $f = (1,2)$), each with two ($2^{\frac{4}{2}-1}$, $m = (1,2)$), members, $\hat{b}^{m\dagger}_f$.
- There are $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Hermitian conjugated partners $\hat{b}^m_f$ appearing in a separate group (not reachable by S^{ab}).
- There are $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{I}\mathcal{A}^{m\dagger}_f$, which are self adjoint or have their Hermitian conjugated partners within the same group.
  All members of this group are reachable by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ from any starting "basis vector" ${}^{I}\mathcal{A}^{1\dagger}_1$.
- There is another group of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", they are the members of ${}^{II}\mathcal{A}^{m\dagger}_f$, again either Hermitian conjugated to each other or are self adjoint. All are reachable from the starting vector ${}^{II}\mathcal{A}^{1\dagger}_1$ by the application of $\mathcal{S}^{ab}$.

- **d=(3+1)** odd "basis vectors" of two families

$$\begin{array}{cccc}
f=1 & f=2 & & \\
\tilde{S}^{03}=\frac{i}{2},\,\tilde{S}^{12}=-\frac{1}{2} & \tilde{S}^{03}=-\frac{i}{2},\,\tilde{S}^{12}=\frac{1}{2} & S^{03} & S^{12} \\
\hat{b}_1^{1\dagger}=\stackrel{03}{(+i)}\stackrel{12}{[+]} & \hat{b}_2^{1\dagger}=\stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2} \\
\hat{b}_1^{2\dagger}=\stackrel{03}{[-i]}\stackrel{12}{(-)} & \hat{b}_2^{2\dagger}=\stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2}\,,
\end{array} \tag{2}$$

- **d=(3+1)** odd H.C. partners of "basis vectors"

$$\begin{array}{cccc}
S^{03}=-\frac{i}{2},\,S^{12}=\frac{1}{2} & S^{03}=\frac{i}{2},\,S^{12}=-\frac{1}{2} & \tilde{S}^{03} & \tilde{S}^{12} \\
\hat{b}_1^{1}=\stackrel{03}{(-i)}\stackrel{12}{[+]} & \hat{b}_2^{1}=\stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2} \\
\hat{b}_1^{2}=\stackrel{03}{[-i]}\stackrel{12}{(+)} & \hat{b}_2^{2}=\stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2}\,.
\end{array} \tag{3}$$

- **d=(3+1)** even "basis vectors" group I

$$\begin{array}{cccccc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{I}\mathcal{A}_1^{1\dagger}=\stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0 & ,\,{}^{I}\mathcal{A}_2^{1\dagger}=\stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1 \\
{}^{I}\mathcal{A}_1^{2\dagger}=\stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1 & ,\,{}^{I}\mathcal{A}_2^{2\dagger}=\stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0\,,
\end{array} \tag{4}$$

- **d=(3+1)** even "basis vectors" group II

$$\begin{array}{cccccc}
 & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\
{}^{II}\mathcal{A}_1^{1\dagger}=\stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0 & ,\,{}^{II}\mathcal{A}_2^{1\dagger}=\stackrel{03}{(+i)}\stackrel{12}{(-)} & i & -1 \\
{}^{II}\mathcal{A}_1^{2\dagger}=\stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1 & ,\,{}^{II}\mathcal{A}_2^{2\dagger}=\stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0\,.
\end{array} \tag{5}$$

- In even dimensional subspace, $2(2n+1)$ of $d = 2(2n+1)+1$ (or $4n$ of $d = 4n+1$), the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, have $2^{\frac{d-1}{2}-1}$ members, m, in $2^{\frac{d-1}{2}-1}$ families, f, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d-1}{2}-1}$ members in $2^{\frac{d-1}{2}-1}$ families. The Clifford even "basis vectors" appear in two mutually orthogonal groups, each with $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ members.
- The second part of "basis vectors" and their Hermitian conjugated partners, obtained from the first part by the application of S^{0d} with the same number of either the Clifford odd or of the Clifford even objects as the first part, manifest: The Clifford odd "basis vectors" appear in two mutually orthogonal groups, each with $2^{\frac{d-1}{2}-1} \times 2^{\frac{d-1}{2}-1}$ members, self adjoint or with the Hermitian conjugated partners within the same group.

  The Clifford even "basis vectors" appear in $2^{\frac{d-1}{2}-1}$ members, m, in $2^{\frac{d-1}{2}-1}$ families, f, and their Hermitian conjugated partners appear in a separate group of $2^{\frac{d-1}{2}-1}$ members in $2^{\frac{d-1}{2}-1}$ families.

- While $\hat{b}^{m\dagger}_{f}$ have in even dimensional spaces one handedness only (either right or left, depending on the definition of handedness), in odd dimensional spaces, the operator of handedness is a Clifford odd object — the product of an odd number of γ^a's (still commuting with S^{ab}) — transforming the Clifford odd "basis vectors" into Clifford even "basis vectors" and opposite. Correspondingly, the eigenvectors of the operator of handedness are the superposition of the Clifford odd and the Clifford even basis vectors", offering in odd dimensional spaces the right and left handed eigenvectors of the operator of handedness.

- **d=(4+1)**

  In $d=(4+1)$ **there are** $32\,(2^{d=5})$ **"eigenvectors" of the Cartan subalgebra members** (S^{03}, S^{12}) **and** $(\mathcal{S}^{03}, \mathcal{S}^{12})$ **of the Lorentz algebras** S^{ab} **and** $\mathcal{S}^{ab}$**.**

- **Half of the Clifford odd and Clifford even "basis vectors" and their Hermitian conjugated partners can be taken from the case** $3+1$**, the rest half follows by the application of** S^{05} **or** $\tilde{S}^{05}$ **on the first half.**

- **d=(4+1)** Clifford odd

$$
\begin{aligned}
&\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,, \quad \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,, \qquad \hat{b}^{1\dagger}_{3} = \stackrel{03}{[-i]}\stackrel{12}{[+i]}\;\gamma^5\,, \quad \hat{b}^{1\dagger}_{4} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\;\gamma^5\,,\\
&\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,, \quad \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,, \qquad \hat{b}^{2\dagger}_{3} = \stackrel{03}{(+i)}\stackrel{12}{(-)}\;\gamma^5\,, \quad \hat{b}^{2\dagger}_{4} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\;\gamma^5\,,\\
&\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,, \quad \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,, \qquad \hat{b}^{1}_{3} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\;\gamma^5\,, \quad \hat{b}^{1}_{4} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\;\gamma^5\,,\\
&\hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,, \quad \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,, \qquad \hat{b}^{2}_{3} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\;\gamma^5\,, \quad \hat{b}^{2}_{4} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\;\gamma^5\,,
\end{aligned}
$$

- **d=(4+1)** Clifford even

$$
\begin{aligned}
&{}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,, \quad {}^{I}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,, \qquad {}^{I}\mathcal{A}^{1}_{3} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\;\gamma^5\,, \quad {}^{I}\mathcal{A}^{1}_{4} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\;\gamma^5\,,\\
&{}^{I}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-i)}\,, \quad {}^{I}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,, \qquad {}^{I}\mathcal{A}^{2}_{3} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\;\gamma^5\,, \quad {}^{I}\mathcal{A}^{2}_{4} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\;\gamma^5\,,\\[1em]
&{}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,, \qquad {}^{II}\mathcal{A}^{1\dagger}_{3} = \stackrel{03}{(+i)}\stackrel{12}{[+]}\;\gamma^5\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{4} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\;\gamma^5\,,\\
&{}^{II}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,, \quad {}^{II}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,, \qquad {}^{II}\mathcal{A}^{2\dagger}_{3} = \stackrel{03}{[-i]}\stackrel{12}{(-)}\;\gamma^5\,, \quad {}^{II}\mathcal{A}^{2\dagger}_{4} = \stackrel{03}{(-i)}\stackrel{12}{[-]}\;\gamma^5\,.
\end{aligned}
\tag{6}
$$

- **One notices that the right hand side of the Clifford odd "basis vectors" appear in two mutually orthogonal groups, each one with either self-adjoint members or with the Hermitian conjugated partners within the same group.**

- The members of one group

$$\hat{b}^{1\dagger}_{3} = \stackrel{03}{[-i]}\stackrel{12}{[+i]}\,\gamma^5\,,\quad \hat{b}^{1\dagger}_{4} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,\gamma^5\,,\quad \hat{b}^{2\dagger}_{3} = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,\gamma^5\,,\quad \hat{b}^{2\dagger}_{4} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,\gamma^5$$

have the properties, except the commutativity (they are, namely, the Clifford odd objects), as the group of Clifford even objects

$$^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,,\quad ^{II}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,,\quad ^{II}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,,\quad ^{II}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{[-]}\;.$$

The comparable properties also have the Clifford odd members of the group

$$\hat{b}^{1}_{3} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,\gamma^5\,,\quad \hat{b}^{1}_{4} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\,\gamma^5\,,\quad \hat{b}^{2}_{3} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,\gamma^5\,,\quad \hat{b}^{2}_{4} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,\gamma^5\,,$$

and the Clifford even members of the group

$$^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,,\quad ^{I}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,,\quad ^{I}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-i)}\,,\quad ^{I}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\;.$$

The members of both groups have Hermitian conjugated partners within the same group or are self adjoint.

- On the other side, the members of the Clifford even group

$$^{II}\mathcal{A}^{1\dagger}_{3} = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5\,,\quad ^{II}\mathcal{A}^{1\dagger}_{4} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,\gamma^5\,,\quad ^{II}\mathcal{A}^{2\dagger}_{3} = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,\gamma^5\,,\quad ^{II}\mathcal{A}^{2\dagger}_{4} = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,\gamma^5\,,$$

have their Hermitian conjugated partners in a separate group

$$^{I}\mathcal{A}^{1}_{3} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,\gamma^5\quad ^{I}\mathcal{A}^{1}_{4} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,\gamma^5\,,\, ^{I}\mathcal{A}^{2}_{3} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,\gamma^5\,,\, ^{I}\mathcal{A}^{2}_{4} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,\gamma^5\,,$$

just like the Clifford odd objects on the left hand side

$$\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,,\quad \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,,\quad \hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,,\quad \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,,$$

which have their Hermitian conjugated partners in a separate group

$$\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,,\quad \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,,\quad \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,,\quad \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\;.$$

- The “basis vectors” of the right hand side keep oddness if they are partners of the Clifford odd “basis vectors” on left hand side, but demonstrate properties of the Clifford even objects on the left hand side.
- The “basis vectors” of the right hand side keep evenness if they are partners of the Clifford even “basis vectors” on the left hand side, but demonstrate properties of the Clifford odd objects on the left hand side.

- After algebraically application of, for example, ${}^{II}\mathcal{A}^{1\dagger}_{3}(=\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5$ on $\hat{b}^{1\dagger}_{4}=\stackrel{03}{(-i)}\stackrel{12}{(+)}\,\gamma^5$ we are left with $\hat{b}^{1\dagger}_{2}=\stackrel{03}{[+i]}\stackrel{12}{(+)}$.
- The eigenvectors of the operator of handedness in $d=(4+1)$, $\Gamma^{(4+1)}=\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5$, are the superposition of the Clifford odd and Clifford even "basis vectors", for example: $\Gamma^{(4+1)}(\hat{b}^{1\dagger}_{1}[=\stackrel{03}{(+i)}\stackrel{12}{[+]}]\pm{}^{II}\mathcal{A}^{1\dagger}_{3}[=\stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5])=\mp((\hat{b}^{1\dagger}_{1}\pm{}^{II}\mathcal{A}^{1\dagger}_{3})$.

- Let us start now to learn about properties of "basis vectors" constituting the creation operators of boson fields on the case of $d = (5+1)$.
- In $d = (5+1)$ there are $2^{\frac{6}{2}-1}$ members in each of $2^{\frac{6}{2}-1}$ families.
- Clifford odd "basis vectors", $\hat{b}^{m\dagger}_f$, have their Hermitian conjugated partners, $\hat{b}^{m}_f$, in the separate group not reachable either by S^{ab} or by $\tilde{S}^{ab}$. Due to

$$\overset{ab}{(\mathbf{k})}^{\dagger} = \eta^{\mathbf{aa}} \overset{ab}{(-\mathbf{k})}, \overset{ab}{[\mathbf{k}]}^{\dagger} = \overset{ab}{[\mathbf{k}]}.$$

- Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_f$, have their Hermitian conjugated partners, ${}^{I}\hat{\mathcal{A}}^{m}_f$, within the same group reachable by S^{ab} or by $\tilde{S}^{ab}$.

basis vect. $\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$	m $\rightarrow$	$f=1$ $\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}$	$f=2$ $-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2}$	$f=3$ $-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2}$	$f=4$ $\frac{i}{2}, \frac{1}{2}, \frac{1}{2}$	S^{03}	S^{12}	S^{56}
odd I $\hat{b}_f^{m\dagger}$	1	$\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}$	$\overset{03}{[+i]}\overset{12}{[+]}\overset{56}{(+)}$	$\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{[+]}$	$\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	[−i](−)[+]	(−i)(−)(+)	(−i)[−][+]	[−i][−](+)	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	[−i][+](−)	(−i)[+][−]	(−i)(+)(−)	[−i](+)[−]	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	(+i)(−)(−)	[+i](−)[−]	[+i][−](−)	(+i)[−][−]	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
S^{03}, S^{12}, S^{56}	$\rightarrow$	$\underset{03}{-\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	$\underset{03}{\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{-\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
odd II $\hat{b}_f^{m}$	1	(−i)[+][+]	[+i][+](−)	[+i](−)[+]	(−i)(−)(−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	[−i](+)[+]	(+i)(+)(−)	(+i)[−][+]	[−i][−](−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	[−i][+](+)	(+i)[+][−]	(+i)(−)(+)	[−i](−)[−]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	(−i)(+)(+)	[+i](+)[−]	[+i][−](+)	(−i)[−][−]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$	$\rightarrow$	$\underset{03}{-\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{-\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	$\underset{03}{\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	S^{03}	S^{12}	S^{56}
even I ${}^{I}\mathcal{A}_f^{m}$	1	[+i](+)(+)	(+i)[+](+)	[+i][+][+]	(+i)(+)[+]	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(−i)[−](+)	[−i](−)(+)	(−i)(−)[+]	[−i][−][+]	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(−i)(+)[−]	[−i][+][−]	(−i)[+](−)	[−i](+)(−)	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[+i][−][−]	(+i)(−)[−]	[+i](−)(−)	(+i)[−](−)	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}$	$\rightarrow$	$\underset{03}{\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{-\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{\frac{1}{2}}$	$\underset{03}{\frac{i}{2}}, \underset{12}{-\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	$\underset{03}{-\frac{i}{2}}, \underset{12}{\frac{1}{2}}, \underset{56}{-\frac{1}{2}}$	S^{03}	S^{12}	S^{56}
even II ${}^{II}\mathcal{A}_f^{m}$	1	[−i](+)(+)	(−i)[+](+)	[−i][+][+]	(−i)(+)[+]	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	(+i)[−](+)	[+i](−)(+)	(+i)(−)[+]	[+i][−][+]	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	(+i)(+)[−]	[+i][+][−]	(+i)[+](−)	[+i](+)(−)	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	[−i][−][−]	(−i)(−)[−]	[−i](−)(−)	(−i)[−](−)	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

- **Clifford odd "basis vectors"** describing the internal space of **fermions in the case of** $d = (5+1)$ are presented in the table as *odd I* $\hat{b}_f^{m\dagger}$, **having odd numbers of nilpotents**
- $\hat{b}_f^{m}$ **is presented in the same table as** *odd II* $\hat{b}_f^{m}$**.** The two groups are not reachable by either S^{ab} or by $\tilde{S}^{ab}$.
- **Clifford even "basis vectors"** describing the internal space of **bosons in the case of** $d = (5+1)$ are presented in the table as *even I, II* ${}^{I,II}\hat{\mathcal{A}}_f^{m\dagger}$, **having an even numbers of nilpotents.**
- **Their Hermitian conjugated partner appear within the same group of "basis vectors", either I or II, demonstrating correspondingly the properties of the internal space of the gauge fields with respect to the fermion "basis vectors"**.

- **Clifford odd "basis vector"** describing the internal space of **quark** $u^{c1\dagger}_{\uparrow R}$, $\Leftrightarrow b^{1\dagger}_{1} := \overset{03}{(+i)}\overset{12}{[+]} \mid \overset{56}{[+]}\overset{78}{(+)} \mid\mid \overset{9\,10}{(+)}\,\overset{11\,12}{[-]}\,\overset{13\,14}{[-]}$,
  has the Hermitian conjugated partner equal to
  $u^{c1}_{\uparrow R} \Leftrightarrow (\mathbf{b}^{1\dagger}_{1})^{\dagger} = \overset{13\,14}{[-]}\,\overset{11\,12}{[-]}\,\overset{9\,10}{(-)} \mid\mid \overset{78}{(-)}\overset{56}{[+]} \mid \overset{12}{[+]}\overset{03}{(-\mathbf{i})}$, both with an odd number of nilpotents,
  both are the Clifford odd objects, belonging to two group.
- **Quarks "basis vectors"** contain $b^{1\dagger}_{1} = \overset{03}{(+i)}\overset{12}{[+]} \mid \overset{56}{[+]}$ from d=(5+1).
- **Clifford even "basis vectors"**, having an even number of nilpotents, describe the internal space of the corresponding boson field
  ${}^{I}\mathcal{A}^{m}_{f} = \overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{[+]}\overset{78}{(+)} \mid\mid \overset{9\,10}{(+)}\,\overset{11\,12}{[-]}\,\overset{13\,14}{[-]}$,
- it contains ${}^{I}\mathcal{A}^{m}_{f} = \overset{03}{(+i)}\overset{12}{(+)} \mid \overset{56}{[+]}$ from d=(5+1).

Anti-commutation relations for Clifford odd "basis vectors", representing the internal space of fermion fields of quarks and leptons ($i = (u_{R,L}^{c,f,\uparrow,\downarrow}, d_{R,L}^{c,f,\uparrow,\downarrow}, \nu_{R,L}^{f,\uparrow,\downarrow}, e_{R,L}^{f,\uparrow,\downarrow})$), and anti-quarks and anti-leptons, with the family quantum number f.

- $\{\mathbf{b}_{\mathbf{f}}^{\mathbf{m}}, \mathbf{b}_{\mathbf{f}'}^{\mathbf{k}\dagger}\}_{*_{\mathbf{A}}+} |\psi_{\mathbf{o}}> = \delta_{\mathbf{f}\,\mathbf{f}'}\, \delta^{\mathbf{mk}}\, |\psi_{\mathbf{o}}>$,
- $\{\mathbf{b}_{\mathbf{f}}^{\mathbf{m}}, \mathbf{b}_{\mathbf{f}'}^{\mathbf{k}}\}_{*_{\mathbf{A}}+} |\psi_{\mathbf{o}}> = 0 \cdot |\psi_{\mathbf{o}}>$,
- $\{\mathbf{b}_{\mathbf{f}}^{\mathbf{m}\dagger}, \mathbf{b}_{\mathbf{f}'}^{\mathbf{k}\dagger}\}_{*_{\mathbf{A}}+} |\psi_{\mathbf{o}}> = 0 \cdot |\psi_{\mathbf{o}}>$,
- $\mathbf{b}_{\mathbf{f}}^{\mathbf{m}}\, |\psi_{\mathbf{o}}> = 0 \cdot |\psi_{\mathbf{o}}>$,
- $\mathbf{b}_{\mathbf{f}}^{\mathbf{m}\dagger}\, |\psi_{\mathbf{o}}> = |\psi_{\mathbf{f}}^{\mathbf{m}}>$,

  $$|\psi_{\mathbf{o}}> = \overset{03}{[-\mathbf{i}]}\overset{12}{[-]}\overset{56}{[-]} \cdots \overset{13\,14}{[-]}\ |\mathbf{1}>$$

  define the vacuum state for quarks and leptons and antiquarks and antileptons of the family f .

[arXiv:1802.05554v1], [arXiv:1802.05554v4], [arXiv:1902.10628]

Commutation relations for Clifford even "basis vectors", representing the internal space of boson fields of two kinds, ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}\,, i=(I,II)$, which are the gauge fields of the fermion fields

- $$\mathbf{{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f`} \to \left\{ \begin{array}{r} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, \\ \text{or } 0\,, i=(I,II)\,. \end{array} \right.}$$

- $$\mathbf{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} \;=\; 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_{A} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,.}$$

I shall demonstrate the properties of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ as the gauge fields of the corresponding $\hat{b}^{m\dagger}_{f}$ in what follows.

Let us come back to d=(5+1) case and to the properties of the Clifford odd and the Clifford even "basiss vectors"

Let us first treat the properties **of the "basis vectors" for fermion fields in** $d = (5+1)$, then we shall treat properties **of the "basis vectors" for boson fields in** $d = (5+1)$, as well as their mutual interaction.

The "basis vectors" for fermion fields in $d = (5+1)$, **appear in four families, each family is identical with respect to** $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, **distinguishing only in** $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$.

The nilpotents and projectors are chosen to be eigenstates of the Cartan subalgebra of the Lorentz algebra

$$\begin{aligned}
\mathbf{S}^{ab}\overset{ab}{(\mathbf{k})} &= \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \mathbf{S}^{ab}\overset{ab}{[\mathbf{k}]} = \frac{k}{2}\overset{ab}{[\mathbf{k}]},\\
\tilde{\mathbf{S}}^{ab}\overset{ab}{(\mathbf{k})} &= \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \tilde{\mathbf{S}}^{ab}\overset{ab}{[\mathbf{k}]} = -\frac{k}{2}\overset{ab}{[\mathbf{k}]}.
\end{aligned}$$

$\tilde{S}^{01}\ \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]} = -\frac{i}{2}\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{[+]}$,

and the $\hat{b}^{m\dagger}_{f}$ are eigenvectors of all the Cartan subalgebra members.

"Basis vectors" for fermions

f	m	$\hat{b}_f^{m\dagger}$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	N_L^3	N_R^3	τ^3	τ^8	τ	$\tilde{S}^{03}$	$\tilde{S}$
I	1	$\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$-$
	2	$\stackrel{03}{[-i]}\stackrel{12}{(-)} \mid \stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-$
	3	$\stackrel{03}{[-i]}\stackrel{12}{[+]} \mid \stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-$
	4	$\stackrel{03}{(+i)}\stackrel{12}{(-)} \mid \stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-$
II	1	$\stackrel{03}{[+i]}\stackrel{12}{(+)} \mid \stackrel{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	
	2	$\stackrel{03}{(-i)}\stackrel{12}{[-]} \mid \stackrel{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	
	3	$\stackrel{03}{(-i)}\stackrel{12}{(+)} \mid \stackrel{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	
	4	$\stackrel{03}{[+i]}\stackrel{12}{[-]} \mid \stackrel{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	
III	1	$\stackrel{03}{[+i]}\stackrel{12}{[+]} \mid \stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$-$
	2	$\stackrel{03}{(-i)}\stackrel{12}{(-)} \mid \stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-$
	3	$\stackrel{03}{(-i)}\stackrel{12}{[+]} \mid \stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-$
	4	$\stackrel{03}{[+i]}\stackrel{12}{(-)} \mid \stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-$
IV	1	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	
	2	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \mid \stackrel{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	
	3	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	
	4	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \mid \stackrel{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	

Let us demonstrate properties of the internal space of fermions using the odd Clifford subalgebra in two ways:
a. Let us use the superposition of members of Cartan subalgebra for the subgroup $SO(3,1) \times U(1)$: $(N^3_{\pm}, \tau)$

$$N^3_{\pm}(= N^3_{(L,R)}) := \frac{1}{2}(S^{12} \pm iS^{03}), \quad \tau = S^{56},$$

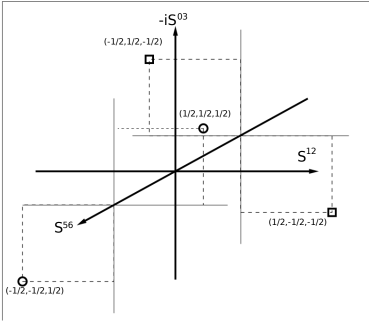
what is meaningful if we understand S^{03} and S^{12} as spins of fermions , S^{56} as their charge,

b. for the subgroup $SU(3) \times U(1)$: (τ', τ^3, τ^8)

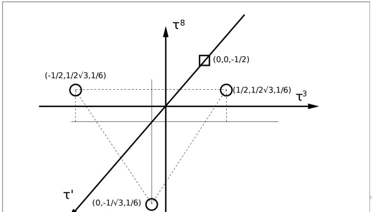
$$\tau^3 := \frac{1}{2}(-S^{1\,2} - iS^{0\,3}), \qquad \tau^8 = \frac{1}{2\sqrt{3}}(-iS^{0\,3} + S^{1\,2} - 2S^{5\,6}),$$
$$\tau' = -\frac{1}{3}(-iS^{0\,3} + S^{1\,2} + S^{5\,6}),$$

if we treat the colour properties for fermions to learn from this toy model as much as we can. The number of commuting operators is three in both cases.

a. **We recognize twice 2 "basis vectors" with charge $\pm\frac{1}{2}$, and with spins up and down.**



b. **We recognize one colour triplet of "basis vectors" with $\tau' = \frac{1}{6}$ and one colour singlet with $\tau' = -\frac{1}{2}$.**



- To see that the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are "the gauge" fields of the Clifford odd "basis vectors", let us algebraically, $*_A$, apply the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f=3}, m=(1,2,3,4)$ on the Clifford odd "basis vectors" .
  Let the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f=3}, m=(1,2,3,4)$ be taken from the third column of *even I*, and $\hat{b}^{m=1\dagger}_{f=1}$, is present as the first Clifford *odd I* "basis vector" on the first and the second table.

- The algebraic application, $*_A$, can easily be evaluated by taking into account

$$\begin{aligned}
\stackrel{ab}{(k)}\stackrel{ab}{(-k)} &= \eta^{aa}\stackrel{ab}{[k]}\,,\;\; \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,,\;\; \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(k)}\stackrel{ab}{[k]} &= 0\,,\; \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,,\;\; \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,
\end{aligned}$$

for any m and f.

We obtain:

- $$
\begin{aligned}
&{}^{I}\hat{\mathcal{A}}^{1\dagger}_{3} (\equiv \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]}) *_{A} \hat{\mathbf{b}}^{1\dagger}_{1} (\equiv \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}) \rightarrow \hat{\mathbf{b}}^{1\dagger}_{1}\,, \text{selfadjoint}\\
&{}^{I}\hat{\mathcal{A}}^{2\dagger}_{3} (\equiv \overset{03}{(-i)}\overset{12}{(-)}\overset{56}{[+]}) *_{A} \hat{\mathbf{b}}^{1\dagger}_{1} \rightarrow \hat{\mathbf{b}}^{2\dagger}_{1} (\equiv \overset{03}{[-i]}\overset{12}{(-)}\overset{56}{[+]})\,,\\
&{}^{I}\hat{\mathcal{A}}^{3\dagger}_{3} (\equiv \overset{03}{(-i)}\overset{12}{[+]}\overset{56}{(-)}) *_{A} \hat{\mathbf{b}}^{1\dagger}_{1} \rightarrow \hat{\mathbf{b}}^{3\dagger}_{1} (\equiv \overset{03}{[-i]}\overset{12}{[+]}\overset{56}{(-)})\,,\\
&{}^{I}\hat{\mathcal{A}}^{4\dagger}_{3} (\equiv \overset{03}{[+i]}\overset{12}{(-)}\overset{56}{(-)}) *_{A} \hat{\mathbf{b}}^{1\dagger}_{1} \rightarrow \hat{\mathbf{b}}^{4\dagger}_{1} (\equiv \overset{03}{(+i)}\overset{12}{(-)}\overset{56}{(-)})\,.
\end{aligned}
$$

Looking at the eigenvalues of the $\hat{b}^{m\dagger}_{1}$ we see that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ obviously carry the integer eigenvalues of $\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}$.

Let us look at the eigenvalues of (τ^3, τ^8, τ') of $\hat{b}_1^{m\dagger}$.

$\hat{b}_1^{1\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (0, 0, -\frac{1}{2})$,
$\hat{b}_1^{2\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (0, -\frac{1}{\sqrt{3}}, \frac{1}{6})$,
$\hat{b}_1^{3\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{6})$,
$\hat{b}_1^{4\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{6})$.

The eigenvalues of (τ^3, τ^8, τ') **of** ${}^{I}\hat{\mathcal{A}}_3^{1\dagger}$ **are obviously**
${}^{I}\hat{\mathcal{A}}_3^{1\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (0, 0, 0)$,
${}^{I}\hat{\mathcal{A}}_3^{2\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (0, -\frac{1}{\sqrt{3}}, \frac{2}{3})$,
${}^{I}\hat{\mathcal{A}}_3^{3\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{3})$,
${}^{I}\hat{\mathcal{A}}_3^{4\dagger}$ **has** $(\tau^3, \tau^8, \tau') = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{3})$,

It can be concluded: $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. **Using this recognition we find the properties of the Clifford even "basis vectors":**

f	m	$*$	$^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	$\mathcal{N}^3_L$	$\mathcal{N}^3_R$	τ^3	τ^8	τ'
I	1	$\star\star$	$\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{(+)}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\triangle$	$\overset{03}{(-i)}\overset{12}{[-]}\overset{56}{(+)}$	$-i$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\ddagger$	$\overset{03}{(-i)}\overset{12}{(+)}\overset{56}{[-]}$	$-i$	1	0	1	0	-1	0	0
	4	$\bigcirc$	$\overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[-]}$	0	0	0	0	0	0	0	0
II	1	$\bullet$	$\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{(+)}$	i	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\otimes$	$\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{(+)}$	0	-1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\overset{03}{[-i]}\overset{12}{[+]}\overset{56}{[-]}$	0	0	0	0	0	0	0	0
	4	$\ddagger$	$\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{[-]}$	i	-1	0	-1	0	1	0	0
III	1	$\bigcirc$	$\overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]}$	0	0	0	0	0	0	0	0
	2	$\odot\odot$	$\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{[+]}$	$-i$	-1	0	0	-1	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
	3	$\bullet$	$\overset{03}{(-i)}\overset{12}{[+]}\overset{56}{(-)}$	$-i$	0	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
	4	$\star\star$	$\overset{03}{[+i]}\overset{12}{(-)}\overset{56}{(-)}$	0	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\odot\odot$	$\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[+]}$	i	1	0	0	1	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bigcirc$	$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[+]}$	0	0	0	0	0	0	0	0
	3	$\otimes$	$\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{(-)}$	0	1	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\triangle$	$\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{(-)}$	i	0	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0

Selfadjoint members are denoted by $\bigcirc$, Hermitian conjugated partners are denoted by the same symbol.

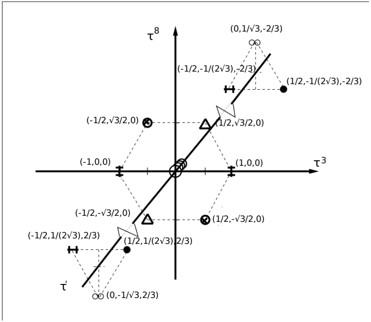
Fig. analyses $^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with respect to Cartan subalgebra members (τ^3, τ^8, τ'). There are

one sextet with $\tau' = 0$,

four singlets with $(\tau^3 = 0, \tau^8 = 0, \tau' = 0)$,

one "anti-triplet" with $\tau' = \frac{2}{3}$ and one "triplet" with $\tau' = -\frac{2}{3}$.

NO FAMILIES!



We learned that the description of the internal spaces of fermions and bosons with the Clifford algebra odd, for fermions, and even, for bosons behave so that they offer:

a. families and all the observed charges of quarks and leptons and anti-quarks and anti-leptons,

b. two kinds of the boson fields, the gauge fields of the corresponding fermion fields, what looks very promising.

Can the Clifford algebra offer more than the spin-charge-family theory?

We now know how to describe the internal space of bosons with "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and fermions with "basis vectors" $\hat{b}^{m'\dagger}_{f'}$.

And we know the action for the case that the boson fields are the gravitational fields only,

$$\mathbf{A} = \int \; d^d x \; E \; \mathcal{L}_f + \int \; d^d x \; E \; (\alpha \, R + \tilde{\alpha} \, \tilde{R}) \, ,$$

defining the interaction between fermions and two kinds of the spin connection fields

$$\mathcal{L}_f - \frac{1}{2} (\bar{\psi} \, \gamma^a p_{0a} \psi) + h.c. p_{0a} = f^{\alpha}{}_a p_{0\alpha} + \frac{1}{2E} \, \{p_{\alpha}, E f^{\alpha}{}_a\}_-$$

$$\mathbf{p_{0\alpha}} = \mathbf{p}_{\alpha} - \frac{\mathbf{1}}{\mathbf{2}} \mathbf{S^{ab}} \omega_{\mathbf{ab}\alpha} - \frac{\mathbf{1}}{\mathbf{2}} \tilde{\mathbf{S}}^{\mathbf{ab}} \tilde{\omega}_{\mathbf{ab}\alpha}$$

- **Let us try to learn more about relations among boson fields and about their relations with the fermion fields.**

- **Let us first recognize that ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$ concern only the internal space of bosons, while in the action it appears beside S^{ab} and $\tilde{S}^{ab}$, which apply on the fermion field , also $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$ which have the vector index α in addition.**

  o o To relate ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\omega_{ab\alpha}$ we must multiply ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ by a vector ${}^{I}\mathcal{C}^{m}_{f\alpha}$.

- **We treat fermions and bosons as free fields, that is as plane waves.**
  Let us relate the application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\ \ {}^{I}\mathcal{C}^{m}_{f\alpha}$ and $\omega_{ab\alpha}$ **by applying both on** $\sum_{m'} \hat{b}^{m'\dagger}_{f'}\ \ \beta^{m'}$

$$\{\sum_{\mathbf{m,f}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} \mathcal{C}^{\mathbf{mf}}_{\alpha}\} *_{\mathbf{A}} \{\sum_{\mathbf{m}'} \hat{\mathbf{b}}^{\mathbf{m}'\dagger}_{\mathbf{f}'} \beta^{\mathbf{m}'}\} = \{\sum_{\mathbf{ab}} \mathbf{S}^{\mathbf{ab}} \omega_{\mathbf{ab}\alpha}\}\ \{\sum_{\mathbf{m}''} \hat{\mathbf{b}}^{\mathbf{m}''\dagger}_{\mathbf{f}'} \beta^{\mathbf{m}''}\}$$

  for a chosen family f'**, the same in** $\{\sum_{\mathbf{m}'} \hat{\mathbf{b}}^{\mathbf{m}'\dagger}_{\mathbf{f}'} \beta^{\mathbf{m}'}\}$ **and in** $\{\sum_{\mathbf{m}''} \hat{\mathbf{b}}^{\mathbf{m}''\dagger}_{\mathbf{f}'} \beta^{\mathbf{m}''}\}$.

- **We relate** $(2^{\frac{d}{2}-1})^2$ **of** ${}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}\, {}^{\mathbf{I}}\mathcal{C}^{\mathbf{m}}_{\mathbf{f}\alpha}$ **with** $\frac{\mathbf{d(d-1)}}{\mathbf{2}}$ **of** $\omega_{\mathbf{ab}\alpha}$**, for a particular** α.

Let us check how it works for $d=(3+1)$ with four $\{{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}\}$ and with six $\{\mathbf{S}^{ab}\,\omega_{ab\alpha}\}$.

For ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{mf}_{\alpha}$ we get from

$$\{{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}(\overset{03}{[+i]}\overset{12}{[+]})\,{}^{I}\mathcal{C}^{1}_{1\alpha}+{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1}(\overset{03}{(-i)}\overset{12}{(-)})\,{}^{I}\mathcal{C}^{2}_{1\alpha}+{}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}(\overset{03}{(+i)}\overset{12}{(+)})\,{}^{I}\mathcal{C}^{1}_{2\alpha}+{}^{I}\hat{\mathcal{A}}^{2\dagger}_{2}(\overset{03}{[-i]}\overset{12}{[-]})\,{}^{I}\mathcal{C}^{2}_{2\alpha}\}$$

$$\{\hat{\mathbf{b}}^{1\dagger}_{1}\beta^{1}_{1}+\hat{\mathbf{b}}^{2\dagger}_{1}\beta^{2}_{1}+\hat{\mathbf{b}}^{3\dagger}_{1}\beta^{3}_{1}+\hat{\mathbf{b}}^{4\dagger}_{1}\beta^{4}_{1}\}$$

$$=\frac{1}{2}\sum_{ab}\mathbf{S}^{ab}\omega_{ab\alpha}\{\hat{\mathbf{b}}^{1\dagger}_{1}\beta^{1}_{1}+\hat{\mathbf{b}}^{2\dagger}_{1}\beta^{2}_{1}+\hat{\mathbf{b}}^{3\dagger}_{1}\beta^{3}_{1}+\hat{\mathbf{b}}^{4\dagger}_{1}\beta^{4}_{1}\}\,.$$

the expressions for four ${}^{I}\mathcal{C}^{mf}_{\alpha}$ in terms of six $\omega_{ab\alpha}$.

$${}^{I}\mathcal{C}^{1}_{1\alpha}=\frac{1}{2}(i\,\omega_{03\alpha}+\omega_{12\alpha})\,,\quad {}^{I}\mathcal{C}^{2}_{2\alpha}=-\frac{1}{2}(i\,\omega_{03\alpha}+\omega_{12\alpha})$$

$${}^{I}\mathcal{C}^{1}_{2\alpha}=i\,\frac{1}{2}(\omega_{01\alpha}-i\,\omega_{02\alpha}-\omega_{31\alpha}+i\,\omega_{32\alpha})$$

$${}^{I}\mathcal{C}^{2}_{1\alpha}=i\,\frac{1}{2}(\omega_{01\alpha}+i\,\omega_{02\alpha}+\omega_{31\alpha}+i\,\omega_{32\alpha})$$

. **For $d>(5+1)$ we get more ${}^{I}\mathcal{C}^{m}_{f\alpha}$, $(2^{\frac{d}{2}-1})^{2}$, than $\omega_{ab\alpha}$, $\frac{d}{2}(d-1)$.**

But they are related.

Let us repeat some general properties of the Clifford even "basis vector" ${}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}$ when they apply on each other.

- Let us denote the self adjoint member in each group of "basis vectors" of particular f as ${}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m_0}\dagger}_{\mathbf{f}}$. We easily see that

$$\begin{aligned}
\{{}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}, {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}'\dagger}_{\mathbf{f}},\}_{-} &= 0\,, \quad \text{if}\,(m,m') \neq m_0\,\text{or}\, m = m_0 = m'\,, \forall f\,,\\
{}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{A}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m_0}\dagger}_{\mathbf{f}} &\rightarrow {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}\,, \quad \forall\, m\,, \forall\, f\,.
\end{aligned}$$

- Two "basis vectors" ${}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}$ and ${}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}'\dagger}_{\mathbf{f}}$ of the same f and of $(m,m') \neq m_0$ are orthogonal.

- $$ {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{A}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}'\dagger}_{\mathbf{f}'} \rightarrow \left\{ \begin{array}{c} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}\,, \\ \text{or zero}\,. \end{array} \right. $$

Looking at the properties of free gravitational fields we can relate also the interaction among ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}_{f\alpha}$ and the interaction among gravitational fields.

We can proceed in equivalent way also when looking for relations between
$\sum_{ab} \tilde{S}^{ab}\, \tilde{\omega}_{ab\alpha}$ and $\sum_{mf}\, {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\;\; {}^{II}\mathcal{C}^{m}_{f\alpha}$

We are then able to replace

$\sum_{ab} S^{ab}\, \omega_{ab\alpha}$ by $\sum_{mf}\, {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\;\; {}^{I}\mathcal{C}^{m}_{f\alpha}$ and
$\sum_{ab} \tilde{S}^{ab}\, \tilde{\omega}_{ab\alpha}$ by $\sum_{mf}\, {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\;\; {}^{II}\mathcal{C}^{m}_{f\alpha}$

in a covariant derivative

$$\mathcal{L}_f - \frac{1}{2}(\bar{\psi}\,\gamma^a p_{0a}\psi) + h.c. \quad \text{with } \mathbf{p_{0a}} = \mathbf{f}^{\alpha}{}_{\mathbf{a}}\mathbf{p}_{\mathbf{0}\alpha} + \frac{\mathbf{1}}{\mathbf{2E}}\,\{p_\alpha, Ef^{\alpha}{}_a\}_-$$

$$\mathbf{p}_{\mathbf{0}\alpha} = \mathbf{p}_\alpha - \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{ab}} \mathbf{S}^{\mathbf{ab}}\omega_{\mathbf{ab}\alpha} - \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{ab}} \tilde{\mathbf{S}}^{\mathbf{ab}}\tilde{\omega}_{\mathbf{ab}\alpha}$$

by

$$\mathbf{p}_{\mathbf{0}\alpha} = \mathbf{p}_\alpha - \sum_{\mathbf{mf}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}\, {}^{\mathbf{I}}\mathcal{C}^{\mathbf{m}}_{\mathbf{f}\alpha} - \sum_{\mathbf{mf}} {}^{\mathbf{II}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}\, {}^{\mathbf{II}}\mathcal{C}^{\mathbf{m}}_{\mathbf{f}\alpha}\,,$$

provided that ${}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\mathcal{C}^{m}_{f\alpha}$ fulfil also the application of both operators on the fermion fields $\sum_{mf} \beta^m\, \hat{b}^{m\dagger}_{f}$ for any β^m and any f.

Although I almost "see" (almost prove) the general relations among

$$^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f} \; ^{I,II}\mathcal{C}^{m}_{f\alpha} \,, \quad \textbf{and}$$

$$S^{ab}\omega_{ab\alpha} \,, \quad \tilde{S}^{ab}\tilde{\omega}_{ab\alpha},$$

for any even d

it still remains to see what new, if any, this new way of second quantization of fermions and bosons brings.

- We have learned that the Clifford algebra offers the description of the internal spaces of fermion and boson fields.
- The "basis vectors" for fermions appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members. Their Hermitian conjugated partners appear in a separate group (of $2^{\frac{d}{2}-1}\times\, 2^{\frac{d}{2}-1}$) number.
- The "basis vectors" for boson fields appear in two orthogonal groups. Each group has $2^{\frac{d}{2}-1}\times\, 2^{\frac{d}{2}-1}$ members, with Hermitian conjugated partners within the same group. They have no families.
  One group transforms, if applying on the $(2^{\frac{d}{2}-1})$ of the Clifford odd "basis vectors" of particular family, members among themselves, keeping the family unchanged.
  The second group, if applying on the particular member of the Clifford odd "basis vectors" of particular family, transforms this member into the same member of another family.

- **The "basis vectors" for boson fields manifest properties of the gauge fields of the corresponding "basis vectors" for fermion fields.**
- **It is the ordinary momentum or coordinate basis which offers the continuously infinite basis.**

Progress in Particle and Nuclear Physics, **http://doi.org/10.1016.j.ppnp.2021.103890**

The second quantization of bosons is newer, partly presented in **Proceedings of the Bled workshop 2021, [arXiv:2112.04378].**

- **Let me introduce the basis in momentum representation** $\{\hat{p}^i\,,\hat{p}^j\}_- = 0$, $\{\hat{x}^k\,,\hat{x}^l\}_- = 0$, $\{\hat{p}^i\,,\hat{x}^j\}_- = i\eta^{ij}$.

$$\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p\,>\,, \quad <\vec{p}\,| =< 0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') =< 0_p\,|\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p}'}|\,0_p\,>\,,\\
&\quad \text{leading to}\\
< 0_p\,|\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{p}'}|\,0_p\,> &= \delta(\vec{p}-\vec{p}')\,,\\
< 0_p\,|\{\hat{b}_{\vec{p}'}\,,\hat{b}^{\dagger}_{\vec{p}}\}_-|\,0_p\,> &= 0\,,
\end{aligned}$$

It follows

$$\begin{aligned}
<\vec{p}\,|\,\vec{x}> &= < 0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}\,>= (< 0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{p}}\,|0_{\vec{p}}\,>)^{\dagger}\\
< 0_p\,|\{\hat{b}^{\dagger}_{\vec{p}},\hat{b}^{\dagger}_{\vec{p}\,'}\}_-|\,0_p\,> &= 0 =< 0_p\,|\{\hat{b}_{\vec{p}},\,\hat{b}_{\vec{p}\,'}\}_-|\,0_p\,>= 0 =< 0_p\,|\{\hat{b}_{\vec{p}},\\
< 0_x\,|\{\hat{b}^{\dagger}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{x}\,'}\}_-|\,0_x\,> &= 0 =< 0_x\,|\{\hat{b}_{\vec{x}},\,\hat{b}_{\vec{x}\,'}\}_-|\,0_x\,>= 0 =< 0_x\,|\{\hat{b}_{\vec{x}},\\
&\quad \text{while}\\
< 0_p\,|\{\hat{b}_{\vec{p}},\,\hat{b}^{\dagger}_{\vec{x}}\}_-|\,0_x\,> &= e^{i\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,,\quad < 0_x\,|\{\hat{b}_{\vec{x}},\,\hat{b}^{\dagger}_{\vec{p}}\}_-|\,0_p\,>=
\end{aligned}$$

$\vec{p}$ determines momentum in ordinary space, $|\psi_o> \,*_T\, |0_{\vec{p}}>$ is the vacuum state for fermions ($|\psi_o>= |\psi_{oc}>$) or for bosons ($|\psi_o>= |\psi_{ob}>$) with the zero momentum, $\hat{b}^{\dagger}_{\vec{p}}$ pushes the

- **For fermions we can write**

$$\{\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) \quad = \quad \sum_{m} c^{sm}{}_{f}\,(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}} *_{T} \;\hat{b}^{m\dagger}_{f}\}\,|\psi_{oc}> \;*_{T}\,|0_{\vec{p}}>,$$

- **For bosons we can write**

$$\{{}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{s}\dagger}_{\mathbf{f}\alpha}(\tilde{\mathbf{p}}) \quad = \quad \sum_{mf} \mathcal{C}^{\mathbf{sm}}{}_{\mathbf{f}\alpha}\,(\tilde{\mathbf{p}})\,\hat{\mathbf{b}}^{\dagger}_{\tilde{\mathbf{p}}} *_{\mathbf{T}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{s}\dagger}_{\mathbf{f}}\}\,|\phi_{ob}> \;*_{T}\,|0_{\vec{p}}> \;.$$

Boson fields need additional space index α, as we have seen and as we can see if require gauge invariance for fermion fields.
I shall try to present the gauge invariance with the boson fields in a separate talk.

While the internal space of fermions if describable by the finite number of the Clifford odd "basis vectors" and the internal space of bosons if describable by the finite of the Clifford even "basis vectors", (for bosons and fermions it is the ordinary space which brings the infinite number of degrees of freedom) the usual second quantization postulates the creation and annihilation operators, anticommuting for fermions on the whole Hilbert space

$$
\begin{aligned}
\{\hat{\mathbf{b}}^{s\dagger}_{f}(\tilde{\mathbf{p}}), \hat{\mathbf{b}}^{s'\dagger}_{f'}(\tilde{\mathbf{p}}\,'))\}_{+}\mathcal{H} &= \mathbf{0}\,,\\
\{\hat{\mathbf{b}}^{s\dagger}_{f}(\tilde{\mathbf{p}}), \hat{\mathbf{b}}^{s'\dagger}_{f'}(\tilde{\mathbf{p}}\,'))\}_{+}\mathcal{H} &= \mathbf{0}\,,\\
\{\hat{\mathbf{b}}^{s\dagger}_{f}(\tilde{\mathbf{p}}), \hat{\mathbf{b}}^{s'\dagger}_{f'}(\tilde{\mathbf{p}}\,'))\}_{+}\mathcal{H} &= \delta^{ss'}\delta_{ff'}\delta(\vec{p}-\vec{p'})\mathcal{H}\,,
\end{aligned}
$$

and commuting for bosons.

The Clifford algebra used in the spin-charge-family theory explains the second postulates of fields.

We have treated so far free fermion fields and boson fields in any even dimensional space. We describe the internal space of fermion fields and boson fields with the odd and even Clifford algebra elements, respectively.

- We learn that all the family members of fermions, they are reachable by S^{ab}, are equivalent, and all the families, they are reachable by $\tilde{S}^{ab}$, are equivalent. We learn that the Hermitian conjugated partners of fermion fields form their own group.
- We learn that the boson fields have their Hermitian conjugated partners within the same group of Clifford even members, and that families play no role for bosons. Boson fields carry in addition the space index.