Can the theory of fermion and boson second quantised fields, with the internal spaces described by "basis vectors", extended to strings, achieve renormalizability?'

N.S. Mankoč Borštnik, University of Ljubljana
H.B. Nielsen, University of Copenhagen

Bled 4-16 of July, 2025 Bled Slovenia

July 4, 2025

We study the extension of the tensor products of the "basis vectors", describing the internal spaces of fermions and boson second quantised fields by the superposition of odd, for fermion, and even, for boson, products of the operators γ^a with the basis in ordinary space-time to strings.

The boson fields gain the space index α, which is equal to $\mu = (0, 1, 2, 3)$ for vector fields and $\sigma \geq 5$ for scalar fields.

For any symmetry $SO(d-1,1)$, d even, of the internal spaces, the point representing the central position of the second quantised either fermion or boson field in ordinary space — with the momenta non zero only in $d = (3+1)$ — is extended to strings.

In the lowest energy level, the extended fields manifest as the point fields,

with the number 2^{d-1} of fermion fields — they appear in families and have their Hermitian conjugated partners in a separate group;

equal to the number 2^{d-1} of boson fields — appearing in two orthogonal groups, both carrying the space index α —

manifesting a kind of supersymmetry.

The Spin-Charge-Family theory,

assuming the description of the internal spaces of fermions and bosons with the "basis vectors", which are superposition of products of

- **odd number of γ^a for fermions and**
- **even number of γ^a for bosons,**

offers an unique description of boson and fermion second quantized fields.

If the internal space involved in creating our universe has $d \geq (13+1)$
and **the ordinary space is active in** $d = (3+1)$ **and no symmetry is broken**,
then both "basis vectors" have the same number of elements.
The same number of fermion and boson second quantized fields, **manifests a kind of supersymmetry**.

- **Making a choice that all "basis vectors" are eigenvectors of the chosen Cartan subalgebra members,**

$$S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\;d},$$
$$\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\;d},$$
$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}.$$

and **arrange the "basis vectors" to be products**

nilpotents,

$$S^{ab}\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{k}{2}\,\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \quad \overset{ab}{(\mathbf{k})} := \mathbf{\frac{1}{2}}(\gamma^{\mathbf{a}} + \frac{\eta^{\mathbf{aa}}}{\mathbf{ik}}\gamma^{\mathbf{b}})$$

and

projectors

$$S^{ab}\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\,\frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \quad \overset{ab}{[\mathbf{k}]} := \mathbf{\frac{1}{2}}(\mathbf{1} + \frac{\mathbf{i}}{\mathbf{k}}\gamma^{\mathbf{a}}\gamma^{\mathbf{b}}),$$

$$(\overset{ab}{(\mathbf{k})})^2 = \mathbf{0}, \qquad (\overset{ab}{[\mathbf{k}]})^2 = \overset{ab}{[\mathbf{k}]},$$

$$\overset{ab}{(\mathbf{k})}^{\dagger} = \eta^{\mathbf{aa}}\,\overset{ab}{(-\mathbf{k})}, \quad \overset{ab}{[\mathbf{k}]}^{\dagger} = \overset{ab}{[\mathbf{k}]},$$

it follows

- the “basis vectors” of fermions have an odd number of nilpotents , the “basis vectors” of bosons have an even number of nilpotents .

- **There are two kinds of the Clifford algebra objects in any d. I recognized that in Grassmann space.**

  J. of Math. Phys. **34** (1993) 3731

- The **Dirac** γ^a (recognized 90 years ago in $d = (3+1)$).

- The **second one:** $\tilde{\gamma}^a$,

  References can be found in
  Progress in Particle and Nuclear Physics,
  http://doi.org/**10.1016.j.ppnp.2021.103890** .

- **The two kinds of the Clifford algebra objects anticommute as follows**

$$\{\gamma^{a},\gamma^{b}\}_{+} = 2\eta^{ab} = \{\tilde{\gamma}^{a},\tilde{\gamma}^{b}\}_{+},$$
$$\{\gamma^{a},\tilde{\gamma}^{b}\}_{+} = 0,$$

- **the postulate**

$$(\tilde{\gamma}^{a}B = i(-)^{n_B}B\gamma^{a})\,|\psi_0>,$$
$$(B = a_0 + a_a\gamma^a + a_{ab}\gamma^a\gamma^b + \cdots + a_{a_1\cdots a_d}\gamma^{a_1}\ldots\gamma^{a_d})|\psi_o>$$

with $(-)^{n_B} = +1, -1$, if B has a Clifford even or odd character, respectively,
$|\psi_o>$ is a vacuum state on which the operators γ^a **apply**,
reduces the Clifford space for fermions and bosons for the factor of two, from 2×2^d **to** 2^d;

- **Consequently, the operators $\tilde{\gamma}^a\tilde{\gamma}^b = -2i\tilde{S}^{ab}$ define the family quantum numbers.**

- These description is elegant and simple to use.
- Analysing the "basis vectors" with respect to the symmetry of the standard model groups $SO(3,1), SU(2), SU(2), SU(3), U(1)$, they manifest in $d = (3+1)$ all the second quantized boson fields observed in $d = (3+1)$, and all the second quantized fermion fields observed in $d = (3+1)$ with the second quantized graviton fields included.
- There are additional fields:
  The additional $SU(2)$ vector boson fields,
  The additional scalar boson fields,
  The additional boson fields, responsible for masses of quarks and leptons and antiquarks and antileptons and $SU(2)$ vector (and scalar) gauge fields
  The right handed neutrinos and left handed antineutrinos
  The fourth family to the observed three families, The additional four families at higher energies.

- **Choosing the simplest action for fermions and bosons, we can describe all the properties of the observed fields.**

The application of S^{ab} and $\tilde{S}^{ab}$ gives:

$$\mathbf{S}^{ab}\overset{ab}{(\mathbf{k})} = \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \mathbf{S}^{ab}\overset{ab}{[\mathbf{k}]} = \frac{k}{2}\overset{ab}{[\mathbf{k}]},$$

$$\tilde{\mathbf{S}}^{ab}\overset{ab}{(\mathbf{k})} = \frac{k}{2}\overset{ab}{(\mathbf{k})}, \quad \tilde{\mathbf{S}}^{ab}\overset{ab}{[\mathbf{k}]} = -\frac{k}{2}\overset{ab}{[\mathbf{k}]}.$$

$$\gamma^{a}\overset{ab}{(\mathbf{k})} = \eta^{aa}\overset{ab}{[-\mathbf{k}]}, \; \gamma^{b}\overset{ab}{(\mathbf{k})} = -ik\overset{ab}{[-\mathbf{k}]}, \; \gamma^{a}\overset{ab}{[\mathbf{k}]} = \overset{ab}{(-\mathbf{k})}, \; \gamma^{b}\overset{ab}{[\mathbf{k}]} = -ik\eta^{aa}\overset{ab}{(-\mathbf{k})},$$

$$\tilde{\gamma}^{a}\overset{ab}{(\mathbf{k})} = -i\eta^{aa}\overset{ab}{[\mathbf{k}]}, \; \tilde{\gamma}^{b}\overset{ab}{(\mathbf{k})} = -k\overset{ab}{[\mathbf{k}]}, \; \tilde{\gamma}^{a}\overset{ab}{[\mathbf{k}]} = i\overset{ab}{(\mathbf{k})}, \; \tilde{\gamma}^{b}\overset{ab}{[\mathbf{k}]} = -k\eta^{aa}\overset{ab}{(\mathbf{k})},$$

$$\overset{ab}{(\mathbf{k})}\overset{ab}{(-\mathbf{k})} = \eta^{aa}\overset{ab}{[\mathbf{k}]}, \; \overset{ab}{[\mathbf{k}]}\overset{ab}{(\mathbf{k})} = \overset{ab}{(\mathbf{k})}, \; \overset{ab}{(\mathbf{k})}\overset{ab}{[-\mathbf{k}]} = \overset{ab}{(\mathbf{k})}, \; **$$

$$\overset{ab}{(\mathbf{k})}\overset{ab}{[\mathbf{k}]} = \mathbf{0}, \; \overset{ab}{[\mathbf{k}]}\overset{ab}{(-\mathbf{k})} = \mathbf{0}, \; \overset{ab}{[\mathbf{k}]}\overset{ab}{[-\mathbf{k}]} = \mathbf{0}, \; **$$

- There are the $2^{\frac{d}{2}-1}$ **Clifford odd "basis vectors"** appearing in $2^{\frac{d}{2}-1}$ **families** and the same number of their **Hermitian conjugated partners**; $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$.
- There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ **Clifford even "basis vectors"** appearing in **two orthogonal groups**.

- Let us see how does one family of odd "basis vector" in $d=(13+1)$ look like, if spins in $d=(13+1)$ are analysed with respect to the Standard Model groups: $SO(3,1)\times SU(2)\times SU(2)\times SU(3)\times U(1)$.
- One irreducible representation of one family contains $2^{\frac{(13+1)}{2}-1}=64$ members which include all the family members, quarks and leptons with the right handed neutrinos included, as well as all the antimembers, antiquarks and antileptons, reachable by either S^{ab} (or by $\mathbb{C}_{\mathcal{N}}\,\mathcal{P}_{\mathcal{N}}$ on a family member).

Jour. of High Energy Phys. **04** (**2014**) 165
J. of Math. Phys. **34**, 3731 (**1993**),
Int. J. of Modern Phys. **A 9**, 1731 (**1994**),
J. of Math. Phys. **44** 4817 (**2003**), hep-th/030322.

S^{ab} generate **all the members of one family**. The **eightplet** (represent. of $SO(7,1)$) **of quarks of a particular colour charge. All are Clifford odd "basis vectors"** , with $SU(3) \times U(1)$ **part** **(**$\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$**, and** $\tau^{41} = 1/6$**)**

i		$\vert {}^a\psi_i >$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^4
		Octet, $\Gamma^{(7,1)} = 1$, $\Gamma^{(6)} = -1$, of quarks							
1	$\mathbf{u_R^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
4	$\mathbf{d_R^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
6	$\mathbf{d_L^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
7	$\mathbf{u_L^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{(-)}\;\stackrel{13\,14}{(-)}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\mathbf{u_R}$ of the 1^{st} row into $\mathbf{u_L}$ of the 7^{th} row, and $\mathbf{d_R}$ of the 4^{rd} row into $\mathbf{d_L}$ of the 6^{th} row, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} generate **all the members** of **one family of quarks, leptonsantiquarks, antileptons**. **Here is the eightplet** (represent. of $SO(7,1)$) **of the colour chargeless leptons. The** $SO(7,1)$ **part is identical with the one of quarks, while the** $SU(3)\times U(1)$ **part is:** $\tau^{33}=0$, $\tau^{38}=0$, $\tau^{41}=-\frac{1}{2}$.

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	Q
		Octet, $\Gamma^{(7,1)}=1$, $\Gamma^{(6)}=-1$, of leptons							
1	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
2	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
3	e_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
4	e_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
5	e_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
6	e_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
7	ν_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
8	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform ν_R of the 1^{st} line into ν_L of the 7^{th} line, and e_R of the 4^{rd} line into e_L of the 6^{th} line, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} **generate also all the anti-eightplet** (repres. of $SO(7,1)$) of **anti-quarks of the anti-colour charge belonging to the same family of the Clifford odd basis vectors** . **(**$\tau^{33} = -1/2$**,** $\tau^{38} = -1/(2\sqrt{3})$**,** $\tau^{41} = -1/6$**).**

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^4
		Antioctet, $\Gamma^{(7,1)} = -1$, $\Gamma^{(6)} = 1$, of antiquarks							
33	$\bar{\mathbf{d}}_{\mathbf{L}}^{\bar{\mathbf{c}}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
34	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
35	$\bar{u}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
36	$\bar{\mathbf{u}}_{\mathbf{L}}^{\bar{\mathbf{c}}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	- 1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
37	$\bar{\mathbf{d}}_{\mathbf{R}}^{\bar{\mathbf{c}}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
38	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
39	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
40	$\bar{\mathbf{u}}_{\mathbf{R}}^{\bar{\mathbf{c}}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{[-]}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\bar{\mathbf{d}}_{\mathbf{L}}$ of the 1^{st} line into $\bar{\mathbf{d}}_{\mathbf{R}}$ of the 5^{th} line, and $\bar{\mathbf{u}}_{\mathbf{L}}$ of the 4^{rd} line into $\bar{\mathbf{u}}_{\mathbf{R}}$ of the 8^{th} line.

- The **Clifford odd "basis vector"** describing the internal space of **quark** $u^{c1\dagger}_{\uparrow R}$, $\Leftrightarrow b^{1\dagger}_{1} :=$
$\stackrel{03}{(+i)}\stackrel{12}{[+]} \mid \stackrel{56}{[+]}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}$,
has the Hermitian conjugated partner equal to
$u^{c1}_{\uparrow R} \Leftrightarrow (\mathbf{b}^{1\dagger}_{1})^{\dagger} = \stackrel{13\,14}{[-]}\;\stackrel{11\,12}{[-]}\;\stackrel{9\,10}{(-)} \mid\mid \stackrel{78}{(-)}\stackrel{56}{[+]} \mid \stackrel{12}{[+]}\stackrel{03}{(-\mathbf{i})}$, both with an odd number of nilpotents,
both are the Clifford odd objects — forming two separate groups.

Anticommutation relations for Clifford odd "basis vectors", representing the internal space of fermion fields of quarks and leptons ($i = (u^{c,f,\uparrow,\downarrow}_{R,L}, d^{c,f,\uparrow,\downarrow}_{R,L}, \nu^{f,\uparrow,\downarrow}_{R,L}, e^{f,\uparrow,\downarrow}_{R,L})$), and anti-quarks and anti-leptons, with the family quantum number f.

- $\{\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}\dagger}_{\mathbf{f'}}\}_{*_{\mathbf{A}}+}|\psi_{\mathbf{o}}> = \delta_{\mathbf{f}\,\mathbf{f'}}\,\delta^{\mathbf{mk}}\,|\psi_{\mathbf{o}}>$,
- $\{\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}}_{\mathbf{f'}}\}_{*_{\mathbf{A}}+}|\psi_{\mathbf{o}}> = 0\cdot|\psi_{\mathbf{o}}>$,
- $\{\mathbf{b}^{\mathbf{m}\dagger}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}\dagger}_{\mathbf{f'}}\}_{*_{\mathbf{A}}+}|\psi_{\mathbf{o}}> = 0\cdot|\psi_{\mathbf{o}}>$,
- $\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}\,|\psi_{\mathbf{o}}> = 0\cdot|\psi_{\mathbf{o}}>$,
- $\mathbf{b}^{\mathbf{m}\dagger}_{\mathbf{f}}\,|\psi_{\mathbf{o}}> = |\psi^{\mathbf{m}}_{\mathbf{f}}>$,

$$|\psi_{\mathbf{o}}> = \stackrel{03}{[-\mathbf{i}]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{13\,14}{[-]}\;|\mathbf{1}>$$

define the vacuum state for quarks and leptons and antiquarks and antileptons of the family f .

- **Clifford even "basis vectors"**, **having an even number of nilpotents, describe the internal space of the corresponding boson field. The gluon field, for example,** ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c1}\to u_R^{c2}}$, **which transforms the** u_R^{c1} **into** u_R^{c2} **looks like:** ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c1}\to u_R^{c2}}$ $(\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\stackrel{9\,10}{(-)}\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]})$.
  If it algebraically multiplies on u_R^{c1} $(\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]})$ **it follows**

$$
{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c1}\to u_R^{c2}}(\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\;\stackrel{\mathbf{9\,10}}{(-)}\stackrel{\mathbf{11\,12}}{(+)}\;\stackrel{13\,14}{[-]}) *_A
$$

$$
u_R^{c1\dagger}, (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[-]}\;\stackrel{13\,14}{[-]}) \rightarrow
$$

$$
u_R^{c2\dagger}, (\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\;\stackrel{13\,14}{[-]}),
$$

$$
\mathbf{{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c1}\to u_R^{c2}}} = \mathbf{u_R^{c2\dagger}} \;\mathbf{*_A}\; \mathbf{(u_R^{c1\dagger})^{\dagger}}\,,
$$

$$
{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c2}\to u_R^{c1}}(\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\;\stackrel{\mathbf{9\,10}}{(+)}\stackrel{\mathbf{11\,12}}{(-)}\;\stackrel{13\,14}{[-]}) *_A u_R^{c2\dagger} \rightarrow u_R^{c1\dagger}\,,
$$

$$
\mathbf{{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u_R^{c2}\to u_R^{c1}}} = \mathbf{u_R^{c1\dagger}} \;\mathbf{*_A}\; \mathbf{(u_R^{c2\dagger})^{\dagger}}\,.
$$

- The gluons ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{ci} \to u_R^{cj}} = u_R^{cj\dagger} *_A (u_R^{ci\dagger})^{\dagger}$ transforming quarks of a particular colour charge to quarks of all the rest colour charges,
  they all are expressed as the algebraic products of a family member and one of the Hermitian conjugated partner.
- The the weak boson ${}^{I}\hat{\mathcal{A}}^{\dagger}_{weak\, u_R^{ci} \to d_R^{ci}} = d_R^{ci\dagger} *_A (u_R^{ci\dagger})^{\dagger}$ transform quarks of a particular weak charge to quarks of another weak charge (keeping the colour charges unchanged).

The second kind of the Clifford even "basis vectors", we call them ${}^{II}\hat{\mathcal{A}}_{\mathbf{f}}^{\dagger m}$, having as well an even number of nilpotents, and consequently commute, describe the internal space of boson fields; they are orthogonal to all ${}^{I}\hat{\mathcal{A}}_{\mathbf{f}}^{\dagger m}$.

${}^{II}\hat{\mathcal{A}}_{\mathbf{f}}^{\dagger m}$ transform a family member of a particular family of fermions to the same family member of all the rest families of quarks and leptons and antiquarks and antileptons.

We namely find

$$e^{-\dagger}_{L\uparrow f=2}\;(\equiv \stackrel{03}{(-)}\stackrel{12}{(+)}\stackrel{56}{(-)}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)})\;*_A\;{}^{II}\hat{\mathcal{A}}_{\mathbf{f}}^{\dagger m}$$

$$(\equiv \stackrel{03}{(+)}\stackrel{12}{(-)}\stackrel{56}{[+]}\stackrel{78}{[-]}\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]})\rightarrow e^{-\dagger}_{L\uparrow f=1}$$

$$(\equiv \stackrel{03}{[-i]}\stackrel{12}{[+]}\stackrel{56}{(-)}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}).$$

Also $"\hat{\mathcal{A}}^{\dagger m}_{\mathbf{f}}$, **like all boson fields,**
can be expressed as algebraic products of the Hermitian conjugate fermion fields and **one of the Clifford odd "basis vector'.**

- **photon** $"\hat{\mathcal{A}}^{\dagger}_{phe^{-\dagger}e^{-}} = (e_L^{-\dagger})^{\dagger} *_A e_L^{-\dagger} =$
  photon $"\hat{\mathcal{A}}^{\dagger}_{phe^{+\dagger}e^{+}} = (e_R^{+\dagger})^{\dagger} *_A e_R^{+\dagger}$

Let be recognized again:

- All bosons "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ (describing internal spaces of boson fields) are expressible as algebraic products of fermion "basis vectors" and their Hermitian conjugated partners, that is as $\hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`'})^{\dagger}$ or as $(\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m''\dagger}_{f``}$.
- Knowing "basis vectors" of fermions appearing in families we know all the boson fields as well.

The even "basis vectors" belonging to two different groups are orthogonal.

$$^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} \;=\; 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}{}_{.f} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,.$$

The members of each of these two groups have the property

$$^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \begin{cases} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f`}\,, i=(I,II) \\ \qquad \text{or zero}\,. \end{cases}$$

The algebraic application, $*_A$**, of even "basis vectors"** ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ **on odd "basis vectors"** $\hat{b}^{m'\dagger}_{f`}$ **and the odd "basis vectors"** $\hat{b}^{m\dagger}_{f}$ **on** ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, gives

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \; *_A \; \hat{b}^{m'\dagger}_{f`} \rightarrow \left\{ \begin{array}{r} \hat{b}^{m\dagger}_{f`} \,, \\ \text{or zero} \,, \end{array} \right.
$$

$$
\hat{b}^{m\dagger}_{f} \; *_A \; {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \left\{ \begin{array}{r} \hat{b}^{m\dagger}_{f``} \,, \\ \text{or zero} \,, \end{array} \right.
$$

Let us point out:

The odd "basis vectors" — with odd number of nilpotents — and even "basis vectors" — with even number of nilpotents differ essentially in their properties:

- The odd "basis vectors" in even dimensional spaces appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members, and have their Hermitian conjugated partners in a separate group, with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ contributions.

  The even "basis vectors" in even dimensional spaces appear in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, having the Hermitian conjugated partners within the same group. They have no families.

- The odd "basis vectors" in even dimensional spaces carry the eigenvalues of the Cartan subalgebra members $\pm\frac{i}{2}$ or $\pm\frac{1}{2}$.

  The even "basis vectors" in even dimensional spaces carry the eigenvalues of the Cartan subalgebra members $(\pm i, 0)$ or $(\pm 1, 0)$.

- There are two kinds of even "basis vectors" and correspondingly two kinds of vector bosons and two kinds of scalar bosons.

  There are $2^{\frac{d}{2}-1}$ families with the same number of members.

- Fermions anticommute and bosons commute without potulates.

If the spin-charge-family theory offers the right way to describe the second quantized fermion and boson fields,

can the extension of points in ordinary space-time, representing either fermions or bosons to strings, help to achieve renormalizability of the proposed spin-charge-family theory?

To extend the points in ordinary space-time to strings we must define the "basis vectors" on a string with coordinates (σ, τ).

We have, in this case, two odd and two even "basis vectors" the eigenvectors of the Cartan subalgebra members

$$S^{01}, \tilde{S}^{01}, \quad \mathcal{S}^{ab} = (S^{01} + \tilde{S}^{01}).$$

Clifford odd

$$\hat{b}^{1\dagger}_{1s} \;=\; \stackrel{01}{(+i)}_s\,, \quad \hat{b}^{1}_{1s} = \stackrel{01}{(-i)}_s\,,$$

Clifford even

$${}^{I}\mathcal{A}^{1\dagger}_{1s} \;=\; \stackrel{01}{[+i]}_s\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{1s} = \stackrel{01}{[-i]}_s\;.$$

The two nilpotent "basis vectors" are Hermitian conjugate to each other. Making a choice that $\hat{b}^{1\dagger}_{1} = \stackrel{01}{(+i)}_s$ is the 'basis vector", the second odd object is then its Hermitian conjugated partner.

There is only one family $(2^{\frac{d}{2}-1}=1)$ **with one member.**

The vacuum state is for this choice equal to

$|\psi_{oc_s}>=\overset{01}{[-i]}_s\,|1>= ((\overset{01}{(+i)}_s)^{\dagger} *_A \overset{01}{(+i)}_s\,|1>.$

There is only one family $(2^{\frac{d}{2}-1}=1)$ **with one member** $(2^{\frac{d}{2}-1}=1)$.

The eigenvalue S^{01} **of** $\hat{b}^{1\dagger}_{1s}(=\overset{01}{(+i)}_s)$ **is** $\frac{i}{2}$.

Each of the two even "basis vectors" is self adjoint
$(({}^{I,II}\mathcal{A}^{1\dagger}_{1s})^{\dagger}={}^{I,II}\mathcal{A}^{1\dagger}_{1s})$,
with the eigenvalues $\mathcal{S}^{01}=(S^{01}+\tilde{S}^{01})$ **equal to** 0, since
$S^{01}\,\overset{01}{[\pm i]}_s=\pm i\,\overset{01}{[\pm i]}_s$ and $\tilde{S}^{01}\,\overset{01}{[\pm i]}_s=\mp i\,\overset{01}{[\pm i]}_s$.

It follows that

$${}^{I}\mathcal{A}^{1\dagger}_{1s}=\hat{\mathbf{b}}^{1\dagger}_{1s} *_A (\hat{\mathbf{b}}^{1\dagger}_{1s})^{\dagger}\,,\qquad {}^{II}\mathcal{A}^{1\dagger}_{1s}=(\hat{\mathbf{b}}^{1\dagger}_{1s})^{\dagger} *_A \hat{\mathbf{b}}^{1\dagger}_{1s}.$$

To find the "basis vectors" for second quantized fermion and boson fields extended to strings, we need to make a tensor product, $*_{T'}$, of "basis vectors" of internal space in $d = 2(2n+1)$ and the "basis vectors" on a string.

We can define the "basis vector" of a gravitino as a tensor product, $*_{T^{\text{`}}}$, of a photon "basis vector"

$$ {}^{I}\hat{\mathcal{A}}^{\dagger}_{phu^{c1}_{L}\to u^{c1}_{L}} (\equiv \stackrel{03}{[-i]}\,\stackrel{12}{[+]}\,|\,\stackrel{56}{[+]}\,\stackrel{78}{[-]}\,||\,\stackrel{9\,10}{[+]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}\,) $$

(having spins and charges in internal space equal to zero), with $\hat{b}^{1\dagger}_{1s} (\equiv \stackrel{01}{(+i)}_{s})$ on a string:

$$ \hat{b}^{1\dagger}_{1gravitino}\;\; (\equiv \stackrel{03}{[-i]}\,\stackrel{12}{[+]}\,|\,\stackrel{56}{[+]}\,\stackrel{78}{[-]}\,||\,\stackrel{9\,10}{[+]}\;\;\stackrel{11\,12}{[-]}\;\;\stackrel{13\,14}{[-]}\;\; *_{T^{\text{`}}}\;\; \stackrel{01}{(+i)}_{s}). $$

This is an anti-commuting object and manifests gravitino if the photon "basis vector" ${}^{I}\hat{\mathcal{A}}^{\dagger}_{phu^{c1}_{L}\to u^{c1}_{L}}$ is in a tensor product with basis in ordinary space-time, carrying the space index $\mu = (0,1,2,3)$.

The extensions of all the other "basis vectors" — either the ones with an odd number of nilpotents describing in $d = 2(2n+1)$ the internal spaces of fermions, or with an even number of nilpotents describing the internal spaces of bosons

— by the tensor product, $*_{T'}$, with the two commuting self adjoint "basis vectors" describing the internal space on the string, ${}^{i}\mathcal{A}^{1\dagger}_{1s}, i = (I, II)$, do not change commutation properties of "building blocs":

The extended "basis vectors" keep commutation properties of the "basis vectors" of anticommuting fermions and commuting bosons .

The extensions of "basis vectors" describing fermions and bosons by the tensor product, $*_{T'}$, with the nilpotent $\hat{b}^{1\dagger}_{1s}$ do change the commutation relations:The commuting ones become anti-commuting, the anti-commuting become commuting.

Let us try to see general properties of tensor products, $*_{T'}$,

of the "basis vectors" with an odd number of nilpotents — describing the internal spaces of the second quantized fermion fields — $\hat{b}^{m\dagger}_{f}$,

and of the "basis vectors" with an even number of nilpotents — describing the internal spaces of the second quantized boson fields — ${}^{I,II}\mathcal{A}^{m\dagger}_{f}$

with the "basis vectors" of a string.

There are four possibilities:

i.

$$\hat{\mathbf{b}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{T}'} {}^{\mathbf{I}}\mathcal{A}^{\mathbf{1}\dagger}_{\mathbf{1s}}$$

represents the anti-commuting "basis vectors" — representing the internal space of fermions — extended with a bosonic string offering the description of the internal spaces of fermions in $d = 2(2n+1)$.

ii.

$${}^{\mathbf{I,II}}\mathcal{A}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{T}'} {}^{\mathbf{I}}\mathcal{A}^{\mathbf{1}\dagger}_{\mathbf{1s}}$$

represents the commuting "basis vectors" — representing the internal space of bosons — extended with a bosonic string, offering the description of the internal spaces of bosons in $d = 2(2n+1)$.

Since ${}^{II}\mathcal{A}^{1\dagger}_{1s}$ defines the vacuum state $|\psi_{ocs}>=[\overset{01}{-i}]_s\,|\,1>$ for $\hat{b}^{1\dagger}_{1s}$, only ${}^{I}\mathcal{A}^{1\dagger}_{1s}$ is used in a tensor product $*_{T'}$ with the string.

iii.

$$ {}^{\mathbf{I,II}}\mathcal{A}_{\mathbf{f}}^{\mathbf{m}\dagger} *_{\mathbf{T'}} \hat{\mathbf{b}}_{\mathbf{1s}}^{\mathbf{1}\dagger} $$

represents the internal space of boson "basis vectors" in d=2(2n+1), extended by the anti-commuting "basis vectors" of a string, offering to the anti-commuting objects — ${}^{I,II}\mathcal{A}_{f}^{m\dagger} *_{T'} \hat{b}_{1s}^{1\dagger}$ the description of the internal spaces with the quantum numbers of bosons in $d = 2(2n+1)$.

iv.

$$ \hat{\mathbf{b}}_{\mathbf{f}}^{\mathbf{m}\dagger} *_{\mathbf{T'}} \hat{\mathbf{b}}_{\mathbf{1s}}^{\mathbf{1}\dagger} $$

represents the commuting "basis vectors" extended with a string offering the description of the internal spaces of bosons with the quantum numbers of fermions in $d = 2(2n+1)$..

We recognize the supersymmetry:

Each ${}^{\mathbf{I,II}}\mathcal{A}^{\mathbf{m}\dagger}_{\mathbf{f}} \; *_{\mathbf{T'}} \; {}^{\mathbf{I}}\mathcal{A}^{\mathbf{1}\dagger}_{\mathbf{1s}}$ and each $\hat{\mathbf{b}}^{\mathbf{m}\dagger}_{\mathbf{f}} \; *_{T'} \; {}^{I}\mathcal{A}^{1\dagger}_{1s}$

has a supersymmetric partner in either
${}^{I,II}\mathcal{A}^{m\dagger}_{f} \; *_{T'} \; \hat{b}^{1\dagger}_{1s}$ or in $\hat{b}^{m\dagger}_{f} \; *_{T'} \; \hat{b}^{1\dagger}_{1s}$.

The extension of the $2^{\frac{d}{2}-1}$ "basis vectors" with an odd number of nilpotents appearing in $2^{\frac{d}{2}-1}$ families with their Hermitian conjugated partners in a separate group, and of the "basis vectors" of an even number of nilpotents appearing in two orthogonal groups, in a tensor extension by $\hat{b}^{1\dagger}_{1s}$ needs further studies to be understood.

Questions for Norma and Holger:

Does the bosonic extension of fermion “basis vector” to a string need that in the tensor product of both with the basis in ordinary space, the bosonic part achieves the string index σ?

The extending object represents in $d = 2(2n+1)$ a fermion, extending to a string.

Does the bosonic extension of boson “basis vector” to a string need that in the tensor product of both with the basis in ordinary space, the boson “basis vector” describing the internal space of bosons achieve the space index α while the string part achieve the string index σ?

In this case the tensor product $*_T$ of a boson “basis vector” with the basis vectors in ordinary space-time requires the space index for a boson while both achieve the dependence on momenta or coordinates in both spaces, the ordinary and strings.

Now I need to write the action for ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}, i = (I, II)$, and for fermion part, both extended by strings, as described above. Wuat differs from the ordinary strings is that my internal apace is different from the known approaches, that supersymmetry is different from the known approaches and that I wont that the ordinary space has only $d = (3 + 1)$.

and each $\mathbf{\hat{b}^{m\dagger}_{f}}\ \mathbf{*_{T'}}\ \mathbf{{}^{I}\mathcal{A}^{1\dagger}_{1s}}$

and in **have a supersymmetric partner** ${}^{I,II}\mathcal{A}^{m\dagger}_{f}\ *_{T'}\ \hat{b}^{1\dagger}_{1s}$.