Understanding the internal spaces of fermion and boson fields in an equivalent way?

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The talk studies the Poincare transformation of fermion and boson fields in the *spin-charge-family* theory, in which the internal spaces of massless fermion and boson second quantised fields can be presented by the "basis vectors" which are superposition of odd (for fermions) and even (for bosons) products of the operators γ^a.

We shall compare the **usual way of studying the Poincare transformations with our way.**

Let us point out that in our way, knowing the “basis vectors” of fermion fields, the theory enables us to express the “basis vectors” for boson fields as algebraic products of fermion “basis vectors” and their Hermitian conjugated partners.

We assume that fermion and boson fields have non-zero momenta only in $d = (3+1)$ ordinary space-time, while the internal spaces experience $d = (13+1)$ dimensions.

In the Kaluza-Klein theories, the internal and external space can have the same number of dimensions.

In our case, there are two kinds of rotations in the internal space of fermions and, consequently, of bosons.

The usual Lorentz rotations consider only one of the two: the one that rotates within one family states of fermions. In our case, there are also the rotations among the families.

The talk demonstrates **both kinds of Lorentz rotations** $d = (13+1)$, **assuming that the fermion and boson fields have non-zero momenta only in** $d = (3+1)$ **ordinary space-time** .

In several papers we demonstrate the situation,when the internal space is $d = (5+1)$ and also $d = (7+1)$.

Our rotations influence both, fermion and boson internal spaces in an equivalent way.

We demonstrate the difference between the usual description of internal spaces and our description.

Let us start with description of internal spaces of fermion and boson fields in the *spin-charge-family* theory.

- **There are two kinds of the Clifford algebra objects in any d . I recognized that in Grassmann space.**

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θ^a**'s and** p^{θ}_a**'s**, $p^{\theta}_a = \frac{\partial}{\partial \theta_a}$
with the property
$(\theta^a)^{\dagger} = \eta^{aa} \frac{\partial}{\partial \theta_a}$.

$$\{\theta^a, \theta^b\}_{+} = 0\,, \qquad \{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_{+} = 0\,,$$

$$\{\theta_a, \frac{\partial}{\partial \theta_b}\}_{+} = \delta_{ab}\,,\ (a,b) = (0,1,2,3,5,\cdots,d)\,.$$

i. The **Dirac** γ^a (recognized 90 years ago in $d = (3+1)$).
ii. The **second one:** $\tilde{\gamma}^a$,

$$\gamma^a = (\theta^a - i\, p^{\theta a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a + i\, p^{\theta a})\,,$$

References can be found in
Progress in Particle and Nuclear Physics,
http://doi.org/**10.1016.j.ppnp.2021.103890** .

- **The two kinds of the Clifford algebra objects anticommute as follows**

$$\{\gamma^{\mathbf{a}},\gamma^{\mathbf{b}}\}_{+} = 2\eta^{\mathbf{ab}} = \{\tilde{\gamma}^{\mathbf{a}},\tilde{\gamma}^{\mathbf{b}}\}_{+},$$
$$\{\gamma^{\mathbf{a}},\tilde{\gamma}^{\mathbf{b}}\}_{+} = 0.$$

- **There are so far observed one kind of quarks and leptons and antiquarks and antileptons and one kind of their vector gauge fields.**

- **The postulate**

$$(\tilde{\gamma}^{\mathbf{a}}\mathbf{B} = \mathbf{i}(-)^{\mathbf{n_B}}\mathbf{B}\gamma^{\mathbf{a}})\,|\psi_0>,$$
$$(\mathbf{B} = a_0 + a_a\gamma^a + a_{ab}\gamma^a\gamma^b + \cdots + a_{a_1\cdots a_d}\gamma^{a_1}\ldots\gamma^{a_d})|\psi_o>$$

with $(-)^{n_B} = +1, -1$, if B has a Clifford even or odd character, respectively, $|\psi_o>$ is a vacuum state on which the operators γ^a **apply**, **reduces the Clifford space for fermions for the factor of two, from** 2×2^d **to** 2^d, while the operators $\tilde{\gamma}^a\tilde{\gamma}^b = -2i\tilde{S}^{ab}$ **define the family quantum numbers.**

- I found in 1990 that it is the Clifford algebra — the algebra of the superposition of products of γ^a's — offering the equivalent procedure for both kinds of the second quantized fields.

  The Clifford odd algebra — the superposition of odd products of γ^a's — offers the description of the internal space of fermion and antifermion second quantized fields.

  The Clifford even algebra — the superposition of even products of γ^a's — offers the description of the internal space of boson second quantized fields .

- **It is convenient to make a choice of the Cartan subalgebra of the infinitesimal generators of the Lorentz algebra of the Lorentz group for any even d, so that we can arrange;**

  the superposition of the odd products of γ^a's, we call them odd "basis vectors";

  and of the even products of γ^a's, we call them even "basis vectors"

  to be eigenstates of all the Cartan subalgebra members:

$$
\begin{aligned}
S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\, d}\,,\\
\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\, d}\,,\\
\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}\,.
\end{aligned}
$$

- It is convenient to write all the **"basis vectors" describing the internal space of either fermion fields or boson fields as products of nilpotents and projectors, which are eigenvectors of the chosen Cartan subalgebra**

nilpotents

$$S^{ab}\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b) = \frac{k}{2}\,\frac{1}{2}(\gamma^a+\frac{\eta^{aa}}{ik}\gamma^b)\,, \quad \overset{ab}{(\mathbf{k})} := \frac{\mathbf{1}}{\mathbf{2}}(\gamma^{\mathbf{a}}+\frac{\eta^{\mathbf{aa}}}{\mathbf{ik}}\gamma^{\mathbf{b}})$$

projectors

$$S^{ab}\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b) = \frac{k}{2}\,\frac{1}{2}(1+\frac{i}{k}\gamma^a\gamma^b)\,, \quad \overset{ab}{[\mathbf{k}]} := \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\frac{\mathbf{i}}{\mathbf{k}}\gamma^{\mathbf{a}}\gamma^{\mathbf{b}})\,,$$

$$(\overset{ab}{(\mathbf{k})})^2 = \mathbf{0}\,, \qquad (\overset{ab}{[\mathbf{k}]})^2 = \overset{ab}{[\mathbf{k}]}\,,$$

$$\overset{ab}{(\mathbf{k})}^{\dagger} = \eta^{\mathbf{aa}}\,\overset{ab}{(-\mathbf{k})}\,, \quad \overset{ab}{[\mathbf{k}]}^{\dagger} = \overset{ab}{[\mathbf{k}]}\,.$$

- **Odd "basis vectors"**, which are eigenvectors of all Cartan subalgebra members $S^{ab} = \frac{i}{2}\gamma^a\gamma^b, a \neq b$, $(S^{03}, S^{12}, S^{56}, \ldots.)$, ar products of **odd number of nilpotents, the rest of projectors**.

  o Appearing in $2^{\frac{d}{2}-1}$ irreducible representations — **families** — each irreducible representation with $2^{\frac{d}{2}-1}$ members (which include **particles and antiparticles**) the **"basis vectors"** describe the internal space of **fermions**.

  o Their **Hermitian conjugated partners**, with again $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members, appear in a separate group.

Let us add

$$\mathbf{S}^{ab} \stackrel{ab}{(\mathbf{k})} = \frac{k}{2}\stackrel{ab}{(\mathbf{k})}, \quad \mathbf{S}^{ab} \stackrel{ab}{[\mathbf{k}]} = \frac{k}{2}\stackrel{ab}{[\mathbf{k}]},$$

$$\tilde{\mathbf{S}}^{ab} \stackrel{ab}{(\mathbf{k})} = \frac{k}{2}\stackrel{ab}{(\mathbf{k})}, \quad \tilde{\mathbf{S}}^{ab} \stackrel{ab}{[\mathbf{k}]} = -\frac{k}{2}\stackrel{ab}{[\mathbf{k}]}.$$

- **Even "basis vectors"** have

  o **even numbers of nilpotents, the rest are projectors**,

  o **nilpotents and projectors are eigenvectors of all the Cartan subalgebra members of** $\mathcal{S}^{ab} = \frac{i}{2}\gamma^a\gamma^b, a \neq b$ $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}, ...)$, **describe** the internal space of **bosons**.

  o Appearing in two groups, with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members each, having their Hermitian conjugated partners within the same group, the even "basis vectors" describe the internal space of bosons.

We shall see that both, fermion "basis vectors" and bosons "basis vectors" demonstrates all the quantum numbers of the observed fields, respectively.

We discussed so far only internal spaces of fermion and boson fields.

The states of fermion and boson fields are tensor products of "basis vectors" and basis in ordinary space-time.

Boson fields carry in addition the space index α, $\alpha = \mu$ for vector bosons, and $\alpha = \sigma$ for scalar bosons.

- **All odd "basis vector" in any even dimension, with $d=(13+1)$ included, has an odd number of nilpotents.**
- **Let us see how does one family of the Clifford odd "basis vector" in $d=(13+1)$ internal space look like, if spins in $d=(13+1)$ are analysed with respect to the Standard Model groups: $SO(3,1)\times$ $SU(2)\times SU(2)\times$ $SU(3)\times U(1)$.**
- **One irreducible representation** of one **family contains $2^{\frac{(13+1)}{2}-1}=64$ members which include all the family members, quarks and leptons with the right handed neutrinos included**, **as well as all the antimembers, antiquarks and antileptons, reachable by either S^{ab} (or by $\mathbb{C}_{\mathcal{N}}$ $\mathcal{P}_{\mathcal{N}}$ on a family member)**.
- **All irreducible representation, all families, are reachable by $\tilde{S}^{ab}$.**

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S^{ab} generate **all the members of one family**. The **eightplet** (represent. of $SO(7,1)$) **of quarks of a particular colour charge. All are odd "basis vectors"** , with $SU(3)\times U(1)$ **part** $(\tau^{33}=1/2$, $\tau^{38}=1/(2\sqrt{3})$, **and** $\tau^{41}=1/6)$, $\gamma^a \stackrel{ab}{(k)} = \eta^{aa}\stackrel{ab}{[-k]}$, $\gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}$, $\gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}$, $\gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}$,

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^4
		Octet, $\Gamma^{(7,1)}=1$, $\Gamma^{(6)}=-1$, of quarks							
1	$\mathbf{u_R^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
4	$\mathbf{d_R^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
6	$\mathbf{d_L^{c1}}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
7	$\mathbf{u_L^{c1}}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \Vert \stackrel{9\;10}{(+)}\stackrel{11\;12}{(-)}\stackrel{13\;14}{(-)}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\mathbf{u_R}$ of the 1^{st} row into $\mathbf{u_L}$ of the 7^{th} row, and $\mathbf{d_R}$ of the 4^{rd} row into $\mathbf{d_L}$ of the 6^{th} row, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} generate **all the members** of **one family of quarks, leptons, antiquarks, antileptons**. **Here is the eightplet** (**represent. of** $SO(7,1)$) **of the colour chargeless leptons. The** $SO(7,1)$ **part is identical with the one of quarks, while the** $SU(3)\times U(1)$ **part is:** $\tau^{33}=0$, $\tau^{38}=0$, $\tau^{41}=-\frac{1}{2}$.

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	Q
		Octet, $\Gamma^{(7,1)}=1$, $\Gamma^{(6)}=-1$, of leptons							
1	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
2	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0
3	e_R	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
4	e_R	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	-1	-1
5	e_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
6	e_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{[-]}\stackrel{78}{(+)} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1
7	ν_L	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
8	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \vert \stackrel{56}{(+)}\stackrel{78}{[-]} \vert\vert \stackrel{9\,10}{(+)}\;\stackrel{11\,12}{[+]}\;\stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform ν_R of the 1^{st} line into ν_L of the 7^{th} line, and e_R of the 4^{rd} line into e_L of the 6^{th} line, doing what the Higgs scalars and γ^0 do in the *standard model*.

S^{ab} **generate also all the anti-eightplet** (repres. of $SO(7,1)$) of **anti-quarks of the anti-colour charge belonging to the same family of the Clifford odd basis vectors** . $(\tau^{33} = -1/2,\ \tau^{38} = -1/(2\sqrt{3}),\ \tau^{41} = -1/6)$.

i		$\vert^{a}\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	$\Gamma^{(4)}$	τ^{13}	τ^{23}	Y	τ^{4}
		Antioctet, $\Gamma^{(7,1)} = -1$, $\Gamma^{(6)} = 1$, of antiquarks							
33	$\bar{d}^{\bar{c}1}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
34	$\bar{d}^{\bar{c}1}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \mid \stackrel{56}{(+)}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	-1	$-\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{6}$
35	$\bar{u}^{\bar{c}1}_{L}$	$\stackrel{03}{[-i]}\stackrel{12}{(+)} \mid \stackrel{56}{[-]}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	-1	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
36	$\bar{u}^{\bar{c}1}_{L}$	$\stackrel{03}{(+i)}\stackrel{12}{[-]} \mid \stackrel{56}{[-]}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	- 1	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{6}$
37	$\bar{d}^{\bar{c}1}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{(+)}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
38	$\bar{d}^{\bar{c}1}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \mid \stackrel{56}{(+)}\stackrel{78}{[-]} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
39	$\bar{u}^{\bar{c}1}_{R}$	$\stackrel{03}{(+i)}\stackrel{12}{(+)} \mid \stackrel{56}{[-]}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	1	$\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$
40	$\bar{u}^{\bar{c}1}_{R}$	$\stackrel{03}{[-i]}\stackrel{12}{[-]} \mid \stackrel{56}{[-]}\stackrel{78}{(+)} \mid\mid \stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[+]}\ \stackrel{13\,14}{[+]}$	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$-\frac{1}{6}$

$\gamma^0\gamma^7$ and $\gamma^0\gamma^8$ transform $\bar{d}_L$ of the 1^{st} line into $\bar{d}_R$ of the 5^{th} line, and $\bar{u}_L$ of the 4^{rd} line into $\bar{u}_R$ of the 8^{th} line.

- **Clifford odd "basis vector"** describing the internal space of **quark** $u^{c1\dagger}_{\uparrow R}$, $\Leftrightarrow b^{1\dagger}_{1} := \overset{03}{(+i)}\overset{12}{[+]} \,|\, \overset{56}{[+]}\overset{78}{(+)} \,||\, \overset{9\,10}{(+)}\;\overset{11\,12}{[-]}\;\overset{13\,14}{[-]}$,
  has the Hermitian conjugated partner equal to
  $u^{c1}_{\uparrow R} \Leftrightarrow (\mathbf{b}^{1\dagger}_{1})^{\dagger} = \overset{13\,14}{[-]}\;\overset{11\,12}{[-]}\;\overset{9\,10}{(-)} \,||\, \overset{78}{(-)}\overset{56}{[+]} \,|\, \overset{12}{[+]}\overset{03}{(-\mathbf{i})}$, **both with an odd number of nilpotents, both are the odd objects — forming two separate groups.**
- **The products of odd numbers of nilpotents, the rest of projectors obey the Dirac's anti-commutation postulates, explaining them.**
- $\{\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}\dagger}_{\mathbf{f}'}\}_{*_\mathbf{A}+}|\psi_\mathbf{o}> = \delta_{\mathbf{f}\,\mathbf{f}'}\,\delta^{\mathbf{mk}}\;|\psi_\mathbf{o}>$,
- $\{\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}}_{\mathbf{f}'}\}_{*_\mathbf{A}+}|\psi_\mathbf{o}> = 0\cdot|\psi_\mathbf{o}>$,
- $\{\mathbf{b}^{\mathbf{m}\dagger}_{\mathbf{f}}, \mathbf{b}^{\mathbf{k}\dagger}_{\mathbf{f}'}\}_{*_\mathbf{A}+}|\psi_\mathbf{o}> = 0\cdot|\psi_\mathbf{o}>$,
- $\mathbf{b}^{\mathbf{m}}_{\mathbf{f}}\;|\psi_\mathbf{o}> = 0\cdot|\psi_\mathbf{o}>$,
- $\mathbf{b}^{\mathbf{m}\dagger}_{\mathbf{f}}\;|\psi_\mathbf{o}> = |\psi^{\mathbf{m}}_{\mathbf{f}}>$,

  $|\psi_\mathbf{o}> = \overset{03}{[-\mathbf{i}]}\overset{12}{[-]}\overset{56}{[-]}\cdots\overset{13\,14}{[-]}\;|\mathbf{1}>$
  define the vacuum state for quarks and leptons and antiquarks and antileptons of the family f.

- Even "basis vectors", having an even number of nilpotents, describe the internal space of the corresponding boson field.
  There are two kinds of even "basis vectors', fulfilling the commutation relations of Dirac for boson fields, explaining them.
-

$$ {}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{A}} {}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}'\dagger}_{\mathbf{f}'} \rightarrow \begin{cases} {}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}'} \,, \\ \text{or } 0 \,, \mathbf{i} = (\mathbf{I}, \mathbf{II}) \,. \end{cases} $$

-

$$ {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{A}} {}^{\mathbf{II}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} = \mathbf{0} = {}^{\mathbf{II}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} *_{\mathbf{A}} {}^{\mathbf{I}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} \,. $$

  ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, i=(I,II) must carry the space index α:
  ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} {}^{i}\mathcal{C}^{m\dagger}_{f}$ i=(I,II)
  (in order to represent the gauge fields of the corresponding fermion fields).

- There are 2^{d-1} **Clifford even "basis vectors"** of two kinds:

  $^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ **and** $^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.

  Both are expressible as algebraic products of the odd "basis vectors" and their Hermitian conjugated partners as

  $^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \rightarrow \hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f`'})^{\dagger}$,

  transforming family members among themselves,

  or as

  $^{II}\hat{\mathcal{A}}^{m\dagger}_{f} \rightarrow (\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m''\dagger}_{f``}$,

  transforming a family member to the same family member of another family.

- The gluon field, for example, ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1} \to u_R^{c2}}$, which transforms the u_R^{c1} into u_R^{c2} looks like: ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1} \to u_R^{c2}}$ $(\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\stackrel{9\,10}{(-)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]})$.
  If it algebraically multiplies on u_R^{c1} $(\equiv \stackrel{03}{(+i)}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]})$ it follows

$$
\begin{aligned}
&{}^{\mathbf{I}}\hat{\mathcal{A}}^{\dagger}_{\mathbf{gl}\, \mathbf{u_R^{c1}} \to \mathbf{u_R^{c2}}} (\equiv \stackrel{03}{[+\mathbf{i}]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\stackrel{9\,10}{(-)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}) *_{\mathbf{A}} \\
&\mathbf{u_R^{c1\dagger}}, (\equiv \stackrel{03}{(+\mathbf{i})}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}) \rightarrow \\
&\mathbf{u_R^{c2\dagger}}, (\equiv \stackrel{03}{(+\mathbf{i})}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{(+)}\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}), \\
&{}^{\mathbf{I}}\hat{\mathcal{A}}^{\dagger}_{\mathbf{gl}\, \mathbf{u_R^{c1}} \to \mathbf{u_R^{c2}}} = \mathbf{u_R^{c2\dagger}} *_{\mathbf{A}} (\mathbf{u_R^{c1\dagger}})^{\dagger}, \\
&{}^{\mathbf{I}}\hat{\mathcal{A}}^{\dagger}_{\mathbf{gl}\, \mathbf{u_R^{c2}} \to \mathbf{u_R^{c1}}} (\equiv \stackrel{03}{[+\mathbf{i}]}\stackrel{12}{[+]}\stackrel{56}{[+]}\stackrel{78}{[+]}\stackrel{9\,10}{(+)}\stackrel{11\,12}{(-)}\stackrel{13\,14}{[-]}) *_{\mathbf{A}} \mathbf{u_R^{c2\dagger}} \rightarrow \mathbf{u_R^{c1\dagger}}, \\
&{}^{\mathbf{I}}\hat{\mathcal{A}}^{\dagger}_{\mathbf{gl}\, \mathbf{u_R^{c2}} \to \mathbf{u_R^{c1}}} = \mathbf{u_R^{c1\dagger}} *_{\mathbf{A}} (\mathbf{u_R^{c2\dagger}})^{\dagger}.
\end{aligned}
$$

  The two gluon fields are Hermitian conjugated to each other.

Let us start to compare usual Poincare transformations and Poincare transformations in our case.

Coordinate transformations which keep the metric tensor $\eta^{mn} = (1, -1, -1, -1, ... - 1)$ unchanged define the isometry group, the group of global Poincare transformations:

$$x'^a = \Lambda^a{}_b x^b + a^a,$$

for the infinitesimal small Lorentz transformations and translations we have

$$\delta x'^a = \omega^a{}_b x^b + a^a,$$
$$\omega^{ab} + \omega^{ba} = 0\,, \forall a^a\,.$$

The Lie algebra of the Poincare group, concerning only space-time

$$\begin{aligned}
M^{ab} &= x^a p^b - x^b p^p = L^{ab}, \quad p^a - i\frac{\partial}{\partial x_a}\,,\\
\{M^{ab}, M^{cd}\}_- &= i(\eta^{bc} M^{ad} - \eta^{bd} M^{ac} - \eta^{ac} M^{bd} + \eta^{ad} M^{bc})\,,\\
\{M^{ab}, p^c\}_- &= i(\eta^{bc} p^a - \eta^{ac} p^b)\,,\\
\{p^a, p^b\}_- &= 0\,.
\end{aligned}$$

Usual Poincare transformations

- The application of the infinitesimal Lorentz transformations on **fermion fields** $\Psi(x)$ can be written as

$$\mathbf{\Psi}'(\mathbf{x}') = (1 + \frac{1}{2}\omega^{ab}\mathbf{S}^{\mathbf{ab}})\,\mathbf{\Psi}(\mathbf{x}),$$

$$\mathbf{S}^{\mathbf{ab}} = \frac{i}{4}\{\gamma^{\mathbf{a}}, \gamma^{\mathbf{b}}\}_{-}\,.$$

$$\mathbf{M}^{\mathbf{ab}} = \mathbf{L}^{\mathbf{ab}} + \mathbf{S}^{\mathbf{ab}}\,,$$

S^{ab} **rotate the internal space of fermions.**

- The application of the infinitesimal Lorentz transformations on **boson fields** $V^a(x)$ can be written as

$$\mathbf{V}'^{\mathbf{a}}(\mathbf{x}') = (\delta^a_b + \omega^a{}_b)\,\mathbf{V}^{\mathbf{b}}(\mathbf{x})\,.$$

transform only the space index of vectors.

The Poincare transformations in our case; only $d=(3+1)$ is active.

- The application of the infinitesimal Lorentz transformations for **fermion fields** we have

$$\begin{aligned} \mathbf{\Psi}'(\mathbf{x}') &= (1+\frac{1}{2}\omega_{ab}(\mathbf{S^{ab}}+\mathbf{\tilde{S}^{ab}}))\,\mathbf{\Psi}(\mathbf{x})\,, \\ \mathbf{S^{ab}} &= \frac{i}{4}\{\gamma^{\mathbf{a}},\gamma^{\mathbf{b}}\}_-\,, \quad \mathbf{\tilde{S}^{ab}}=\frac{i}{4}\{\tilde{\gamma}^{\mathbf{a}},\tilde{\gamma}^{\mathbf{b}}\}_-\,, \end{aligned}$$

$(S^{ab}+\tilde{S}^{ab})$ **rotate the internal space of fermion fields,** S^{ab}**, within the families,** $\tilde{S}^{ab}$**, among the families.**

- **For boson fields we have**

$$\mathbf{V'^a}(\mathbf{x}')=(\delta^{\mathbf{a}}_{\mathbf{b}}+\omega^{\mathbf{a}}{}_{\mathbf{b}})\,(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{2}}\omega_{\mathbf{cd}}(\mathbf{S^{cd}}+\mathbf{\tilde{S}^{cd}}))\mathbf{V^b}(\mathbf{x})\,,$$

transforming the space index and the internal space for both of the two kinds of vectors.

In our case the creation operators for fermion fields are the tensor product, $*_T$, of the odd "basis vectors" and the basis in ordinary space

$$\hat{\mathbf{b}}^{\mathbf{m}\dagger}_{\mathbf{f}}(\mathbf{x}) = \hat{\mathbf{b}}^{\mathbf{m}\dagger}_{\mathbf{f}} \;*_{\mathbf{T}}\; \hat{\mathbf{b}}^{\dagger}_{\mathbf{x}}\,,$$

The creation operators for boson fields are the tensor product, $*_T$, of the even "basis vectors" and the basis in ordinary space, with the space index α added

$$ {}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}\alpha}(\mathbf{x}) \;=\; {}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}} \;*_{\mathbf{T}}\; {}^{\mathbf{i}}\mathcal{C}^{\mathbf{m}}{}_{\mathbf{f}\alpha}\, \hat{\mathbf{b}}^{\dagger}_{\mathbf{x}}\,,\; i=(I,II)\,. \qquad (1)$$

Knowing fermion "basis vectors" we can easily find boson "basis vectors"

$${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \to \hat{b}^{m'\dagger}_{f`} *_A (\hat{b}^{m''\dagger}_{f``})^{\dagger},$$

transforming family members among themselves,

or as

$${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} \to (\hat{b}^{m'\dagger}_{f`})^{\dagger} *_A \hat{b}^{m''\dagger}_{f``},$$

transforming a family member to the same family member of another family.

M^{ab} **determine Lorentz rotations in ordinary space and in internal spaces of fermions and bosons.**

$$\mathbf{M}^{ab} = \mathbf{L}^{ab} + (\mathbf{S}^{ab} + \tilde{\mathbf{S}}^{ab})\,,$$

- **The gluon field, for example,** ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1}\to u_R^{c2}}$, **which transforms the** u_R^{c1} **into** u_R^{c2} **looks like:** ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1}\to u_R^{c2}}$
  $(\equiv \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{[+]}\overset{9\,10}{(-)}\overset{11\,12}{(+)}\overset{13\,14}{[-]})$.
  If it algebraically multiplies on u_R^{c1}
  $(\equiv \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{(+)}\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]})$ **it follows**

$$\mathbf{{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1}\to u_R^{c2}}}(\equiv \overset{03}{[+\mathbf{i}]}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{[+]}\overset{9\,10}{(-)}\overset{11\,12}{(+)}\overset{13\,14}{[-]}) *_{\mathbf{A}}$$

$$\mathbf{u}_{\mathbf{R}}^{\mathbf{c1}\dagger}, (\equiv \overset{03}{(+\mathbf{i})}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{(+)}\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]}) \rightarrow$$

$$\mathbf{u}_{\mathbf{R}}^{\mathbf{c2}\dagger}, (\equiv \overset{03}{(+\mathbf{i})}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{(+)}\overset{9\,10}{[-]}\overset{11\,12}{(+)}\overset{13\,14}{[-]})\,,$$

$$\mathbf{{}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\, u_R^{c1}\to u_R^{c2}}} = \mathbf{u}_{\mathbf{R}}^{\mathbf{c2}\dagger} *_{\mathbf{A}} (\mathbf{u}_{\mathbf{R}}^{\mathbf{c1}\dagger})^{\dagger}\,,$$

One can perform the infinitesimal global Lorentz transformations either on ${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u^{c1}_{R} \to u^{c2}_{R}\; a}$:

$${}^{I}\hat{\mathcal{A}}'^{\dagger}_{gl\; u^{c1}_{R} \to u^{c2}_{R}\; a} =$$
$$(\mathbf{1} + (S^{cd} + \tilde{S}^{cd})\; \omega_{cd})\;\; {}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u^{c1}_{R} \to u^{c2}_{R}\; b}$$

or on

$$(\mathbf{1} + (S^{cd} + \tilde{S}^{cd})\; \omega_{cd})\; \hat{u}^{c2}_{R} *_{A} ((\mathbf{1} + (\mathbf{S^{cd}} + \mathbf{\tilde{S}^{cd}})\omega_{\mathbf{cd}})\hat{\mathbf{u}}^{\mathbf{c1}}_{\mathbf{R}})^{\dagger},$$

since

$${}^{I}\hat{\mathcal{A}}^{\dagger}_{gl\; u^{c1}_{R} \to u^{c2}_{R}} = u^{c2\dagger}_{R} *_{\mathbf{A}} (u^{c1\dagger}_{R})^{\dagger}.$$

Let us now look at the Lorentz transformation on any fermion field ψ. We only pay attention on $d = (3+1)$, leaving all the rest of the internal space of fermions unchanged.

(In ordinary way the rest of charges are expressed with Dirac matrices, in our case the rest of charges are expressed with **nilpotents and projectors, which is for one of quarks equal to** $u_R^{c1\dagger}, (\equiv \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}\overset{78}{(+)}\overset{9\,10}{(+)}\overset{11\,12}{[-]}\overset{13\,14}{[-]})$.)

$$\psi'(\mathbf{x}') = \boldsymbol{\Lambda}\psi(\mathbf{x})\,,$$

with

$$\boldsymbol{\Lambda} = \mathbf{e}^{\mathbf{i}\omega_{\mathbf{ij}}\mathbf{M}^{\mathbf{ij}} + \mathbf{i}\omega_{\mathbf{0i}}\mathbf{M}^{\mathbf{0i}}}$$

and in infinitesimal order we find

$$\boldsymbol{\Psi}'(\mathbf{x}') = (1 + \frac{1}{2}\omega^{ab}\mathbf{S}^{\mathbf{ab}})\,\boldsymbol{\Psi}(\mathbf{x}),$$

$$\mathbf{S}^{\mathbf{ab}} = \frac{i}{4}\{\gamma^{\mathbf{a}}, \gamma^{\mathbf{b}}\}_-\,.$$

Let us evaluate
looking transformations in the first order,

$$((1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i})\psi')^\dagger \, \mathcal{O} \, (1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i})\psi'.$$

It can be shown for $\mathcal{O}$ is identity, 1, that we get

$$\psi'^\dagger \psi',$$

provided that $\omega_{ij}^* = \omega_{ij}$, while $\omega_{0i}^* = -\omega_{0i}$.

Can we show for Lagrange density of the Dirac's equation that it is Lorentz invariant? We can do this if we rewrite the Dirac equation

$$\frac{1}{2}\{(\gamma^0\gamma^m p_m\,\psi(x))^\dagger\,\psi(x)$$
$$+\varepsilon[\psi^\dagger\,\gamma^0\gamma^m p_m\,\psi(x)]\}.$$

$$\frac{1}{2}\{(\gamma^0\gamma^m p_m\,(1+i\omega_{ij}S^{ij}+i\omega_{0i}S^{0i})\psi')^\dagger\,(1+i\omega_{ij}S^{ij}+i\omega_{0i}S^{0i})\psi'+$$
$$\varepsilon[(1+i\omega_{ij}S^{ij}+i\omega_{0i}S^{0i})\psi')^\dagger\,\gamma^0\gamma^m p_m\,(1+i\omega_{ij}S^{ij}+i\omega_{0i}S^{0i})\psi']\}\,,$$

with ε is -1 and $\omega^*_{ij}=\omega_{ij}$, while $\omega^*_{0i}=-\omega_{0i}$.
Then we get

$$\frac{1}{2}\{(\gamma^0\gamma^m p_m\,\psi(x))^\dagger\,\psi(x)$$
$$-[\psi^\dagger\,\gamma^0\gamma^m p_m\,\psi(x)]\}.$$

We should in our case take

$$\mathbf{M^{ab}} = \mathbf{L^{ab}} + (\mathbf{S^{ab}} + \tilde{\mathbf{S}}^{\mathbf{ab}})\,,$$

This we show only for Lorentz transformation within one family, as Dirac assumed. For several families case this would go equivalently since $\tilde{S}^{ab}$ commute with $\gamma^0\gamma^m$. But we should treat this case next time.

Knowing the "basis vectors" for fermions, we know fermion and boson states