The stochastic gradient flow approach to out-of-equilibrium complex systems in particle and gravitational physics, and its phenomenological implications

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Stochastic quantization

Symmetries in GR

- **Generators of gauge symmetries** found from the **Hamiltonian** [1]

$$\mathcal{L}_{\rm ADM} = N\sqrt{h}\left[K^2 - K^{ij}K_{ij} - \bar{R}\right] \qquad \mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_{\rm ADM},$$

$$\boxed{\Pi = \frac{\partial\mathcal{L}_{\rm ADM}}{\partial\dot{N}} = 0, \qquad \Pi_i = \frac{\partial\mathcal{L}_{\rm ADM}}{\partial\dot{N}^i} = 0,}$$

$$[\Pi, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0, \qquad [\Pi_i, \mathcal{H}_{\rm ADM}]_{\rm PB} = 0,$$

$$\mathcal{H} = G_{ijkl}\Pi^{ij}\Pi^{kl} - \sqrt{h}\bar{R} = 0, \qquad \mathcal{H}_i = 2h_{ij}\nabla_k^{(3)}\Pi^{kj} = 0,$$

Super-Hamiltonian **Super-Momentum**

Secondary constraints: eq. of motion

$$R^{0i} - \frac{1}{2}g^{0i}R = \frac{N\mathcal{H}^i + N^i\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

$$R^{00} - \frac{1}{2}g^{00}R = -\frac{\mathcal{H}}{2N^2\sqrt{h}} = 0,$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

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Super-Hamiltonian **Super-Momentum**

Generators of gauge transformation: time, space diffeomorphisms

$$\xi\,[\mathcal{H}, h_{ij}]_{\rm PB} = \delta_t h_{ij} = \mathcal{L}_{\xi\mathbf{n}} h_{ij}$$

$$\xi^k\,[\mathcal{H}_k, h_{ij}]_{\rm PB} = \delta_{\boldsymbol{\xi}} h_{ij} = \mathcal{L}_{\boldsymbol{\xi}} h_{ij}$$

$$\mathcal{H}_{\rm ADM} = \Pi\dot{N} + \Pi_i\dot{N}^i + N\mathcal{H} + N^i\mathcal{H}_i = 0,$$ **The Hamiltonian vanishes**

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Symmetries in GR

Path-integral Quantization [1]:

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \quad (16.5)$$

Delta on the constraints

We are asking the path integral not to fluctuate around the constraints
In GR constraints generate **external** symmetries

Symmetries are imposed in the quantization of GR rather than being **allowed to emerge**

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods

All methods start from a classical description

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**

Difficulties in handling Symmetries

Dynamics limited to a hypersurface in phase-space — Constraints Hypersurface $\chi_i = 0$

- **1. Dirac Prescription**
  define *physical states* as
  $$\chi_i|\psi\rangle = 0$$

- **2. Gribov copies - Faddeev Popov determinant**

Path integral

$$= \int [Dx^\mu] \prod_{t,\alpha} \delta(\chi_\alpha) \prod_t (\text{sdet}\,[\chi_\alpha, \chi_\beta])^{1/2} \exp iS[x^\mu(t)]. \qquad (16.5)$$

[1] Henneaux, Teitelboim, Quantization of gauge systems, Princeton U. Press

Quantization Methods

1. **Canonical Quantization - Dynamic Perspective**
2. **Path-Integral Quantization - Ensemble Average Perspective**
3. **Stochastic Quantization**

Stochastic Dynamic Relaxation of the Ensemble Average

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PERTURBATION THEORY WITHOUT GAUGE FIXING

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ABSTRACT

We propose to formulate the perturbative expansion for field theory starting from the Langevin equation which describes the approach to equilibrium. We show that this formulation can be applied to gauge theories to compute gauge invariant quantities without fixing the gauge. A very simple example is worked out in detail. We also discuss the speed of approaching to equilibrium of the solution of the Langevin equation in the framework of perturbation theory.

Stochastic Quantization I

Introduce a "stochastic time" variable s and noise - Langevin dynamics

$$\frac{\partial}{\partial s}\phi_A\left(x^\mu, s\right) = -\frac{\delta S\left[\phi\right]}{\delta\phi_A} + \eta_A\left(x^\mu, s\right),$$

The noise is **additive** and **Gaussian**: higher-order (even) correlations are functions of $\langle\eta_{\mathrm{A}}\eta_{\mathrm{B}}\rangle$

$$\langle\eta_A\left(x, s\right)\eta_B\left(x', s'\right)\rangle = 2\alpha\delta_{AB}\delta\left(x - x'\right)\delta\left(s - s'\right) \qquad \langle\eta_{\mathrm{A}}\rangle = 0$$

$$p\left(\eta\right) \sim \left[D\eta\right]\exp - \frac{1}{4\alpha}\int \mathrm{d}^4x\,\mathrm{d}s\,\left[\eta\left(x, s\right)\right]^2$$

Stochastic Quantization II

Expectation values: (i) with respect to the noise, (ii) with respect to $P\left(\phi, s\right)$

Average over histories

$$\langle\phi_{\mathrm{A}_1,\eta}\left(x_1, s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N, s\right)\rangle = \frac{\int\left[D\eta\right]\exp\left[-\frac{1}{4}\int\mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2\left(x,\sigma\right)\right]\phi_{\mathrm{A}_1,\eta}\left(x_1, s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N, s\right)}{\int\left[D\eta\right]\exp\left[-\frac{1}{4}\int\mathrm{d}^4x\,\mathrm{d}\sigma\,\eta^2\left(x,\sigma\right)\right]}$$

Average over dynamic measure

$$\langle\phi_{\mathrm{A}_1,\eta}\left(x_1, s\right)\cdots\phi_{\mathrm{A}_N,\eta}\left(x_N, s\right)\rangle = \int\left[D\phi_{\mathrm{B}}\right]\,P\left(\phi, s\right)\,\phi_{\mathrm{A}_1}\left(x_1\right)\cdots\phi_{\mathrm{A}_N}\left(x_N\right)$$

The Fokker-Planck Eq., associated to the Langevin, dictates the dynamics of $P\left(\phi, s\right)$

Stochastic Quantization III

The associated Fokker-Planck ensures the correct "equilibrium" limit of $P\left(\phi, s\right)$

$$\frac{\partial F}{\partial s} = -\frac{\partial F}{\partial \phi_{\mathrm{A}}}\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}} + \frac{1}{2}\frac{\partial^2 F}{\partial \phi_{\mathrm{A}}^2} + \frac{\partial F}{\partial \phi_{\mathrm{A}}}\eta_{\mathrm{A}}$$

Stochastic Calculus

$$\frac{\partial P\left(\phi, s\right)}{\partial s} = \frac{\partial}{\partial \phi_{\mathrm{A}}}\left[P\left(\phi, s\right)\frac{\delta S[\phi]}{\delta \phi_{\mathrm{A}}}\right] + \frac{1}{2}\frac{\partial^2 P\left(\phi, s\right)}{\partial \phi_{\mathrm{A}}^2}$$

Fokker-Planck

$$\frac{\partial P\left(\phi, s\right)}{\partial s} = 0 \longrightarrow \lim_{s\to\infty} P\left(\phi, s\right) = P\left(\phi\right) \sim \exp\left[-S\right]$$

The Euclidean Path-Integral measure is recovered at equilibrium

$$\lim_{s\to+\infty}\langle \phi_{\mathrm{A}}\left(x, s\right)\phi_{\mathrm{B}}\left(x', s\right)\rangle_{p(s)} = \langle \phi_{\mathrm{A}}\left(x\right)\phi_{\mathrm{B}}\left(x'\right)\rangle_{\mathrm{E}}$$

Stochastic quantization applied to gravity

Stochastic Quantization of General Relativity à la Ricci Flow

- Let's consider the presence of matter $S=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial}{\partial s}g_{\mu\nu}\left(x,s\right)=\imath\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda=-1\right)\frac{\delta S}{\delta g_{\alpha\beta}}+g_{\mu\nu}\left(x,s\right)e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}\left(x,s\right)$$

$$\left\langle\tilde{\eta}\left(x,s\right)\tilde{\eta}\left(x',s'\right)\right\rangle=\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- Starting point: seminal Rumpf work [2] with additive tensorial noise

$$\frac{\partial}{\partial s}g_{\mu\nu}\left(x,s\right)=\imath\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)\frac{\delta S}{\delta g_{\alpha\beta}}+\eta_{\mu\nu}\left(x,s\right)$$

$$\left\langle\eta_{\alpha\beta}\left(x,s\right)\eta_{\mu\nu}\left(x',s'\right)\right\rangle=\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- DeWitt Supermetric: special choice [2] $\lambda=-1$
- Strong link to Horava-Lifshitz gravity

$$\mathcal{G}^{\alpha\beta\mu\nu}\left(\lambda\right)=\frac{\sqrt{-g}}{2\kappa}\left[g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}-\lambda g^{\alpha\beta}g^{\mu\nu}\right]$$

$$\mathcal{G}_{\alpha\beta\mu\nu}\left(\lambda\right)=\frac{2\kappa}{\sqrt{-g}}\left[g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\beta\mu}-\frac{\lambda}{2\lambda+1}g_{\alpha\beta}g_{\mu\nu}\right]$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Scalar multiplicative noise

- Let's consider the presence of matter $S=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}R+\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{\mathrm{M}},$

We propose the Langevin equation [1] with scalar multiplicative noise

$$\frac{\partial g_{\mu\nu}}{\partial s}=-2\imath\left[R_{\mu\nu}-\kappa\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)\right]+g_{\mu\nu}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta}$$

$$\langle\tilde{\eta}\left(x,s\right)\tilde{\eta}\left(x',s'\right)\rangle=\delta\left(s-s'\right)\delta^{(4)}\left(x-x'\right)$$

- Zero-noise limit: looks like a Ricci-flow with a target fixed point
- Noise amplitude to be determined at the saddle-point of the equilibrium

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

[2] H. Rumpf, Phys. Rev. D, 33, 4, 1986

Langevin equation and the Itô calculus

$$\boxed{\frac{\partial g_{\mu\nu}}{\partial s} = -2\imath\left[R_{\mu\nu} - \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\right] + g_{\mu\nu}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\tilde{\eta},}$$

- We need to "interpret" the Langevin equation: Itô calculus [1]
- Different variables coupled to the same noise
- Need to compute Itô rules for this case: for a generic F

$$\boxed{\frac{\partial F}{\partial s} = -2\imath\frac{\partial F}{\partial g_{\mu\nu}}\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right] + \alpha e^{\imath\gamma}g_{\mu\nu}g_{\alpha\beta}\,\frac{\partial^2 F}{\partial g_{\mu\nu}\partial g_{\alpha\beta}} + e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\frac{\partial F}{\partial g_{\mu\nu}}g_{\mu\nu}\tilde{\eta}}$$

- The associated Fokker-Planck reads

$$\boxed{\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}}\left(-2\imath\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right]p\right) + \alpha e^{\imath\gamma}\frac{\partial^2}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\left(g_{\mu\nu}g_{\alpha\beta}\,p\right)}$$

[1] H. Rumpf, Phys. Rev. D, 33, 4, 1986

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Emergent Cosmological Constant

- We can look for the steady state solution of the Fokker-Planck

$$\partial_s p = -\frac{\partial}{\partial g_{\mu\nu}}\left(-2\imath\left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right]p\right) + \alpha e^{\imath\gamma}\frac{\partial^2}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\left(g_{\mu\nu}g_{\alpha\beta}\,p\right)$$

- After setting $\partial_s p = 0$ one obtains

Number of independent metric components

$$p\left(g_{\mu\nu}\right) \simeq \mathcal{C}\exp\left\{\frac{1}{D\alpha e^{i\gamma}}\int^{g^{\mu\nu}}\mathcal{D}\bar{g}^{\mu\nu}\left[-2\imath\left(R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right) - \alpha e^{\imath\gamma}\left(1+n_g\right)\bar{g}_{\mu\nu}\right]\right\}$$

- The saddle-point $\frac{\delta p}{\delta g^{\mu\nu}} = 0$ condition reads $R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} - \frac{1}{2}\alpha\left(1+n_g\right)e^{\imath\left(\gamma+\frac{\pi}{2}\right)}g_{\mu\nu} = 0$
- After setting $\frac{1}{2}\alpha\left(1+n_g\right) = \Lambda, \qquad \alpha = \frac{2\Lambda}{1+n_g},$

The cosmological constant
appears as a consequence of the noise at the saddle point at equilibrium (!)

$$\boxed{R_{\mu\nu} - \Lambda e^{\imath\left(\gamma-\frac{\pi}{2}\right)}g_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

ADM variables and Itô calculus

Let us go from the metric tensor to the ADM variables

Multiplicative choice of the noise source entails additivity in the Hamiltonian ADM picture

$$\eta_{\mu\nu} = \eta\, g_{\mu\nu}$$

We need to follow the Itô transformation rule

$$\frac{\partial F}{\partial s} = -2\imath \frac{\partial F}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu}\right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 F}{\partial g_{\mu\nu}\partial g_{\alpha\beta}} + e^{\imath\frac{\gamma}{2}} \sqrt{2\alpha} \frac{\partial F}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Hamiltonian analysis of the stochastic geometry flow

$$\boxed{\frac{\partial N}{\partial s} = -\frac{N}{2}\left[\imath\left(\frac{\mathcal{H}}{\sqrt{h}} + R\right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}\right]}$$

$$\boxed{\frac{\partial N^k}{\partial s} = \frac{\imath N\mathcal{H}^k}{\sqrt{h}}}$$

$$\boxed{\frac{\partial h_{ij}}{\partial s} = \frac{1}{N}\mathcal{L}_{Nn}\left[\mathcal{H}, h_{ij}\right] + \left[\mathcal{H}, \left[\mathcal{H}, h_{ij}\right]\right] + \frac{h_{ij}\mathcal{H}}{2\sqrt{h}} - h_{ij}e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Shift-vector and Navier-Stokes

- Let's consider the shift-vector equation

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

- **The noise term cancels out**

- The fixed point solution automatically implements super-Momentum constraint

$$\mathcal{H}_i = 2 h_{ij} \nabla^{(3)}_k \Pi^{kj} = 0,$$

- In the non-relativistic limit the super-momentum reduces to Navier-Stokes [1]

$$\Pi^{ij} = \sqrt{h}(K h^{ij} - K^{ij}) = \frac{1}{2}\sqrt{h} T^{ij} \longrightarrow \boxed{\text{Brown-York stress tensor}}$$

Its divergence yields Navier-Stokes $r_c^{3/2} \partial^k T_{ki} = \partial_t v_i - \zeta \partial^2 v_i + \partial_i P + v^k \partial_k v_i = 0$

Incompressibility $r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$

[1] I. Bredberg, C. Keeler, V. Lysov, A. Strominger, JHEP, 2012, 146 (2012)

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Navier-Stokes and stochastic force

- It reduces to a forced incompressible Navier-Stokes equation (for Euclidean signature)

$$\frac{\partial N^k}{\partial s} = \frac{\imath N \mathcal{H}^k}{\sqrt{h}}$$

$$\downarrow$$

$$\boxed{\partial_t v^i - \zeta \partial^2 v^i + \partial^i P + v^k \partial_k v^i = -\frac{\imath}{N} \frac{\partial N^i}{\partial s}}$$

$$r_c^{3/2} \partial_k T^{kt} = \partial_k v^k = 0$$

- **The noise term cancels out**
- The fields can enter a turbulent regime
- Turbulent fluctuations are **intermittent**
- Possibility to investigate the multi-fractal hypothesis [1] in GR

The only equation that loses noise responds with intermittency!

Possible role of turbulence in cosmological first-order phase transitions (!)

[1] U. Frisch, G. Parisi; R Benzi, G Paladin, G Parisi, A Vulpiani Journal of Physics A: Mathematical and General 17 (18), 3521

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Out-of-equilibrium dynamics of Black Holes

- Let us consider a "static" spherical symmetric metric $\mathrm{d}s^2 = -e^{\nu(r)}\mathrm{d}t^2 + e^{\mu(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$
- After performing Itô transformation and taking the **quasi-equilibrium limit** $\mu = -\nu$

$$\frac{\partial \nu}{\partial s} = \frac{\mathrm{d}\nu}{\mathrm{d}g_{00}}\frac{\partial g_{00}}{\partial s} + e^{\imath\gamma}\alpha\, g_{00}^2\frac{\mathrm{d}^2\nu}{\mathrm{d}g_{00}^2}\,,$$

$$\frac{\partial \nu}{\partial s} = -\,\imath\left[\frac{\mathrm{d}^2\nu}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2\right]e^{\nu} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\,\left(\tilde{\eta} + e^{\imath\frac{\gamma}{2}}\sqrt{2\alpha}\right)$$

- Let's normalise the equation to 2α

$$\nu = \nu_0(r) + \bar{\nu}(r,s) \qquad \frac{\bar{\nu}}{\sqrt{2\alpha}} = \bar{h} \qquad s\sqrt{2\alpha} = t \qquad \bar{\eta} = -\frac{1}{2}\frac{e^{\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}}\tilde{\eta} \qquad h = \bar{h} + \frac{1}{2}e^{\imath\frac{\gamma}{2}}t$$

$$\frac{1}{2\alpha}\frac{\partial\bar{\nu}}{\partial s} = \frac{\partial\bar{h}}{\partial t} = \imath\left[\frac{e^{\nu_0}}{\sqrt{2\alpha}}\left(\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right) + e^{\nu_0}\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}r}\right)^2\right] + \bar{\eta} - \frac{1}{2}e^{\imath\frac{\gamma}{2}}$$

$$\frac{\partial h}{\partial t} = \imath\left[\nu_{\mathrm{KPZ}}(r,\alpha)\,\nabla^2 h(r,s) + \lambda_{\mathrm{KPZ}}(r)\left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2\right] + \bar{\eta}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

$$\nu_{\mathrm{KPZ}}(r,\alpha) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}(r) = e^{\nu_0(r)}$$

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

[2] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

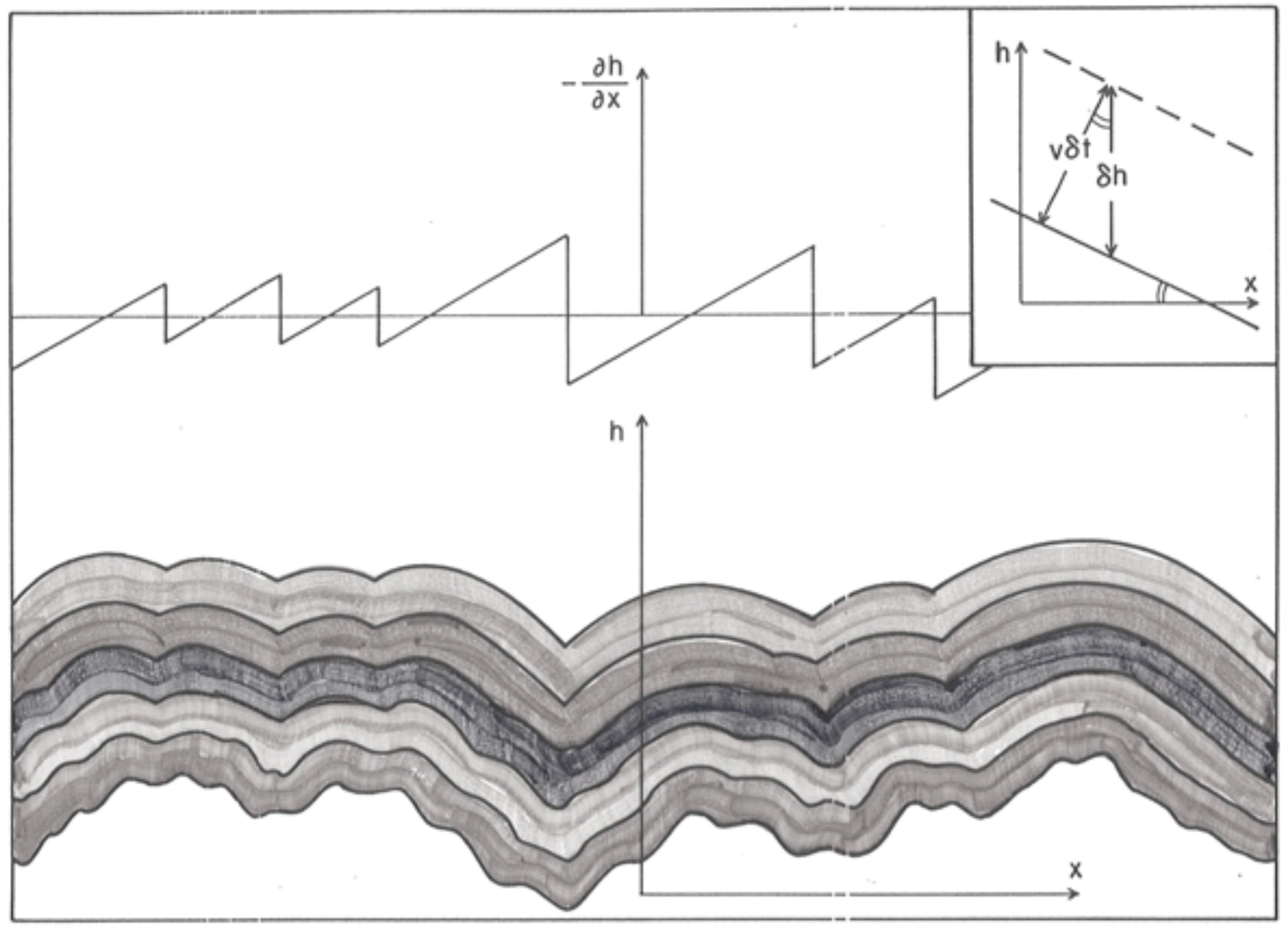
Black Holes and Kardar-Parisi-Zhang

$$\frac{\partial h}{\partial t} = \imath \left[\nu_{\mathrm{KPZ}}\left(r,\alpha\right) \nabla^2 h\left(r,s\right) + \lambda_{\mathrm{KPZ}}\left(r\right) \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2 \right] + \bar{\eta}$$

$$\nu_{\mathrm{KPZ}}\left(r,\alpha\right) = \frac{e^{\nu_0(r)}}{\sqrt{2\alpha}} \qquad \lambda_{\mathrm{KPZ}}\left(r\right) = e^{\nu_0(r)}$$

Kardar-Parisi-Zhang Eq. [1]
for a spherical surface

- KPZ defines its own **dynamic universality class** $z = 3/2$
- It models **surface growth** by deposition: **surface height** h
- It is related to Burgers' equation via a change of variable



$$u = -\lambda \frac{\partial h}{\partial x} \longrightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

- Both models exhibit **intermittency** for small dissipation
- The small dissipation limit is obtained near the horizon

$$\lim_{r\to r_s} \nu_{\mathrm{KPZ}}\left(r,\alpha\right) = \lim_{r\to r_s} \lambda_{\mathrm{KPZ}}\left(r\right) \propto \lim_{r\to r_s} g_{00} = \lim_{r\to r_s} \left(1 - \frac{r_s}{r}\right) = 0$$

[1] M. Kardar, G. Parisi, Y.C. Zhang, PRL 56, 9, 889–892 (1986)

Topology change and out-of-equilibrium DOFs

Breakdown of the conformal symmetry and the scalaron

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + \partial_\mu a^{\perp}_\nu + \partial_\nu a^{\perp}_\mu + \left(\partial_\mu\partial_\nu - \frac{1}{4}\eta_{\mu\nu}\Box\right) a + \frac{1}{4}\eta_{\mu\nu}\varphi$$

$$\Phi = \varphi - \bar{\Box} a$$

Einstein-Hilbert expanded on dS or AdS backgrounds

$$\mathcal{S}^{(2)}_{\mathrm{EH}} = \frac{c^3}{16\pi G}\int d^4x\sqrt{-\bar{g}}\left[\frac{1}{4}h^{\mu\nu}_{\perp}\left(\bar{\Box} - \frac{\bar{R}}{6}\right)h^{\perp}_{\mu\nu} - \frac{3}{32}\Phi\left(\bar{\Box} + \frac{\bar{R}}{3}\right)\Phi\right]$$

Residual gauge transformation: conformal Killing vector and disappearance of the ghost

$$h^{\perp}_{\mu\nu} \to h^{\perp}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu\,, \qquad \Phi \to \Phi + 2\nabla^\mu k_\mu$$

Projective connection & out-of-equilibrium torsional DOFs

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + \mathcal{C}^{\gamma}_{\alpha\beta}$$

$$\mathcal{C}^{\gamma}_{\alpha\beta} = \lambda_1 \delta^{\gamma}_{\alpha} u_{\beta} + \lambda_2 u_{\alpha} \delta^{\gamma}_{\beta} + \lambda_3 w_{\alpha\beta} u^{\gamma} + \lambda_4 u_{\alpha} u_{\beta} u^{\gamma}$$

$$\sqrt{-g}\bar{R} = \sqrt{-g}R + \sqrt{-g} g^{\beta\delta} \left(\mathcal{C}^{\mu}_{\beta\delta} \mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha} \mathcal{C}^{\alpha}_{\mu\delta} \right)$$

Cosmological term induced in the action

$$\begin{aligned} &\sqrt{-g} g^{\beta\delta} \left(\mathcal{C}^{\mu}_{\beta\delta} \mathcal{C}^{\alpha}_{\mu\alpha} - \mathcal{C}^{\mu}_{\beta\alpha} \mathcal{C}^{\alpha}_{\mu\delta} \right) \\ &\qquad = \sqrt{-g} \left[\left(\lambda_2^2 + \lambda_3^2 \right) (D-1)\, u_{\mu} u^{\mu} \right] \end{aligned}$$

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

Out-of-equilibrium torsional DOFs and topology changes

E_0 trivial $SU(3)$ bundle and E_1 non-trivial $SU(3)$ bundle

$$a : E_0 \to E_1$$

Map between trivial and non-trivial bundles induces topology changes

$$c_2(E_1) = \frac{1}{8\pi^2} \int_M tr(F \wedge F)\,, \qquad c_2(E_0) = 0$$

Singularities of the maps produce non-trivial instantons $F = \pm * F$

map a induces singular gauge transformation g_a

$$\oint_{\gamma=\partial S} A = \int_S dA = \int_S d\left(g_a^{-1} dg_a\right) = \int_S F$$

Stokes theorem and area-law for the holonomy

Stochastic Ricci RG flow of Λ

$$ds^2 = -N^2 dt^2 + a^2(t)\left[\frac{(dr)^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

FLRW background

$$S = 6\int d^4x N a^3 R + \int d^4x N a^3 (D-1)\lambda_2^2 \epsilon(\lambda) \qquad R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}^2}{a^2}\right) + \frac{k}{a^2}\right)$$

$$\frac{\partial a}{\partial s} = -\frac{2\imath}{N^2}\left(a\dot{H} + 3aH^2 + \varepsilon N^2 \lambda_2^2\right) + a\eta\,,$$

$$\frac{\partial N}{\partial s} = -2\imath\left(\frac{3}{2N}(\dot{H} + H^2) + \frac{1}{16}N\left(\Lambda_0 + 8\lambda_2^2\right)\right)$$
$$-N\eta\,,$$

$$\frac{\partial \lambda_2}{\partial s} = \imath\left(-2\varepsilon - \imath\eta\right)\lambda_2\,,$$

Stochastic geometry (Ricci) flow equations

Hubble tension: a macroscopic QG effect?

$$\langle \lambda_2^k(s) \rangle = \exp\left[\left(i(-2\varepsilon) + \frac{\Lambda_0}{2}\right)s\right] \langle \lambda_2^k(0) \rangle$$

Thermal time oriented as the proper time implies mild increase of Λ

[1] M. Lulli, **A. Marciano**, X. Shan, arXiv:2112.01490 (2021)

[2] M. Lulli, **A. Marciano**, L. Visinelli, in preparation

Cosmological measurements $67.4 \pm 1.4 \quad (\mathrm{km/s})/\mathrm{Mpc}$

Astronomical measurements $74.03 \pm 1.42 \quad (\mathrm{km/s})/\mathrm{Mpc}$

Relaxation properties and wave-function collapse

Diosi-Penrose model from Stochastic Gradient Flow

Diosi-Penrose quantum collapse master equation as a
the non-relativistic limit of the stochastic geometry (Ricci) flow

Running of the lapse function in a stochastic thermal time

Strategy: describe a discrete system of masses undergoing gravitational interaction

First principle discussion and RG flow induced by a stochastic gradient (Ricci) flow

M. Lulli, **A. Marciano** & X. Shan, arXiv:2112.01490v2

Derivation of the Diosi Penrose quantum collapse master equation from the stochastic geometry (Ricci) flow and discussion of the parameter space

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

Assumptions to recover Diosi-Penrose

DP quantum collapse master equation as out-of-equilibrium relaxation described through the stochastic geometry (Ricci) flow

Similarity: both realizations have multiplicative noise

New task: explain the role of the lapse, recalling that N=1+V+…

Tools: use the Ito calculus to account for the variation of the lapse

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

Non-relativistic limit of the Stochastic Gradient Flow

Consider a discrete system of gravitationally interacting bodies

Ideal gas approximation for the energy-momentum tensor

Use the non-relativistic semi-classical limit of

$$\frac{\partial N}{\partial s} = -\frac{N}{2}\left[\imath\left(\frac{\mathcal{H}}{\sqrt{h}} + R\right) + \frac{\Lambda e^{-\imath\gamma}}{1+n_g} + e^{\imath\frac{\gamma}{2}}\sqrt{\frac{4\Lambda}{1+n_g}}\tilde{\eta}\right]$$

$$d|\psi_t\rangle = \Bigg[-\frac{i}{\hbar}H\,dt + \sqrt{\frac{G}{\hbar}}\int d\mathbf{x}\,\big(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\big)\,dW(\mathbf{x},t)$$
$$-\frac{G}{2\hbar}\int d\mathbf{x}\int d\mathbf{y}\,\frac{\big(\mathcal{M}(\mathbf{x}) - \langle\mathcal{M}(\mathbf{x})\rangle_t\big)\big(\mathcal{M}(\mathbf{y}) - \langle\mathcal{M}(\mathbf{y})\rangle_t\big)}{|\mathbf{x}-\mathbf{y}|}\,dt\Bigg]|\psi_t\rangle,$$

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

Langevin equation and perfect gas model

Langevin equation for the shift

$$\frac{\partial N}{\partial s} = -2\imath \frac{\partial N}{\partial g_{\mu\nu}} \left[R_{\mu\nu} - R^{\mathrm{T}}_{\mu\nu} \right] + \alpha e^{\imath\gamma} g_{\mu\nu} g_{\alpha\beta} \frac{\partial^2 N}{\partial g_{\mu\nu} \partial g_{\alpha\beta}} + e^{\imath\gamma/2} \sqrt{2\alpha} \frac{\partial N}{\partial g_{\mu\nu}} g_{\mu\nu} \tilde{\eta}$$

$$\frac{1}{2}\alpha \left(1 + n_g\right) = \Lambda, \qquad \alpha = \frac{2\Lambda}{1 + n_g},$$

with matter Ricci target

$$R^{\mathrm{T}}_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

$$T^{\mu}_{\nu} = \mathrm{diag} \left\{ -\rho, \rho\sigma T, \rho\sigma T, \rho\sigma T \right\}$$

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

The role of the Itô calculus

Using the Ito calculus

$$\mathrm{d}N = \left\{-\frac{\imath}{2N}\left[\nabla^2 N^2 + \kappa N^2 \rho\left(3\sigma T + 1\right)\right] - \frac{\alpha}{4} e^{\imath\gamma} N\right\} \mathrm{d}s + e^{\imath\gamma/2}\sqrt{\frac{\alpha}{2}} N \mathrm{d}W$$

$$\mathrm{d}N = a\mathrm{d}s + b\mathrm{d}W$$

$$df = \int \mathrm{d}^4 x\, m_x \left[\frac{\delta f}{\delta N_x} a_x + \frac{1}{2}\int \mathrm{d}^4 y\, n_y\, b_x b_y\, k_\varepsilon\left(x, y\right) \frac{\delta}{\delta N_x}\frac{\delta}{\delta N_y} f\right] \mathrm{d}s + \int \mathrm{d}^4 x\, m_x \frac{\delta f}{\delta N_x} b_x \mathrm{d}W_x$$

for a functional f=f[N]

$$\langle \frac{\mathrm{d}f}{\mathrm{d}s} \rangle = -\int \mathrm{d}^4 x\, \mathrm{d}N_x\, m_x\, f \frac{\delta}{\delta N_x}\left[\left(a_x p\left[N_x\right]\right) - \frac{1}{2}\int \mathrm{d}^4 y \mathrm{d}^4 z\, n_y \frac{\delta}{\delta N_y}\left(b_x b_z p\left[N_x\right]\right)\, k_\varepsilon\left(x, y\right) k_\varepsilon\left(y, z\right)\right]$$

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

Stationarized Fokker-Planck and WKB approximation

Stationary Fokker-Planck

$$a_x p\left[N_x\right] - \frac{1}{2}\int \mathrm{d}^4y\mathrm{d}^4z\, n_y \frac{\delta}{\delta N_y}\left(b_x b_z p\left[N_x\right]\right)\, k_{\varepsilon}\left(x,y\right) k_{\varepsilon'}\left(y,z\right) = 0$$

$$p\left[N_x\right] = B\exp\left\{\int^{N_x}\prod_y \mathrm{d}N_y \frac{4}{e^{\imath\gamma}\alpha N_y^2}\left[a_y - \frac{e^{\imath\gamma}\alpha}{2}\,N_y\right]\right\} = B\exp\left[\frac{\imath}{\hbar}S_0\right]$$

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

GKLS equation and non-cocommutativity in stochastic time

Non-unitary but trace-preserving and positive

$$\begin{aligned}\mathrm{d}\Psi = &\left\{\int \mathrm{d}^4x\, m_x \left[2\imath\kappa\rho_x\,(2+3\sigma T)+\frac{\alpha}{4}e^{\imath\gamma}\right]\mathrm{d}s\right\}\Psi\\ &-\left\{\frac{e^{-\imath\frac{\gamma}{2}}}{\sqrt{2\alpha}}\int \mathrm{d}^4x\, m_x[2\imath\kappa\rho_x\,(3\sigma T+2)+3\alpha e^{\imath\gamma}]\mathrm{d}W_x\right\}\Psi\\ &+\left\{\frac{1}{2}\int \mathrm{d}^4x\, m_x \int \mathrm{d}^4y\, n_y\,\frac{1}{2}\,k_\varepsilon\,(x,y)\,\frac{1}{e^{\imath\gamma}\alpha}\mathrm{d}s\,\times\right.\\ &\left.[2\imath\kappa\rho_x\,(2+3\sigma T)+3\alpha e^{\imath\gamma}]\,[2\imath\kappa\rho_y\,(2+3\sigma T)+3\alpha e^{\imath\gamma}]\right\}\Psi\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\rho_\Psi \equiv &\left\langle\frac{\mathrm{d}}{\mathrm{d}s}\rho_\Psi\right\rangle_{\tilde\eta} = \imath\int d^4x\, m_x\,[\mathcal{H}_x,\rho_\Psi]\\ &+\int d^4x\, m_x\int d^4y\, n_y\,\widetilde{k}_\varepsilon\,(x,y)\;\mathcal{L}_x\rho_\Psi\mathcal{L}_y^\dagger\\ &-\frac{1}{2}\int d^4x\, m_x\int d^4y\, n_y\,\widetilde{k}_\varepsilon\,(x,y)\;\{\mathcal{L}_x^\dagger\mathcal{L}_y,\rho_\Psi\}\end{aligned}$$

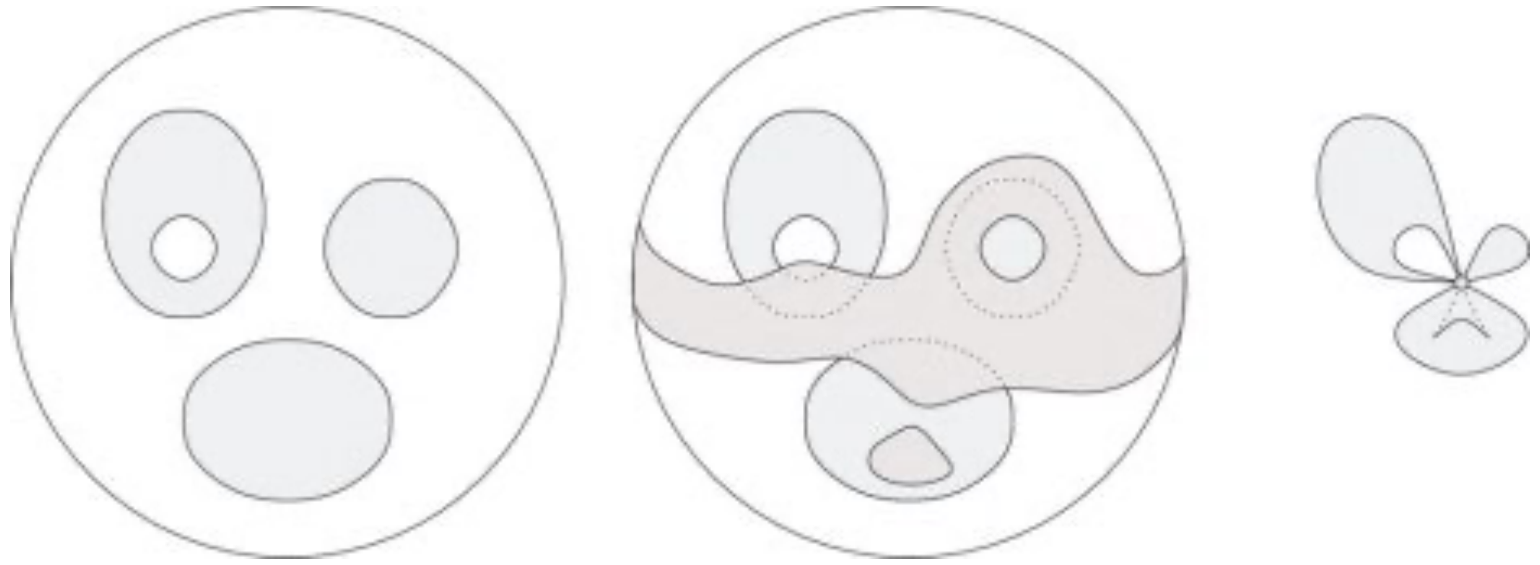
Generalization of GKLS Equation

M. Lulli, **A. Marciano** & K. Piscicchia, arXiv:2307.10136

Berry phase description of confinement

Berry phase and stochastic gradient flows

The gauge-geometry flow allows for topology changes (fluctuations) out of equilibrium



Geometrical interpretation of ground-states

$$\frac{\partial}{\partial s} g_{\mu\nu} = -2\left[R_{\mu\nu} - R^T_{\mu\nu}\right]$$

$$R^T_{\mu\nu} = \frac{8\pi G}{c^4}\left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right]$$

$$= -2\left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu}\right] - g_{\mu\nu}\left(R - T\right)$$

Changes of topologies through defects are induced by singularities in the stochastic gradient flow

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Quark confinement and center symmetry

Elitzur theorem and necessity of breakdown of global symmetry to distinguish phases of a gauge theory

$$
\begin{aligned}
G(x) &= \exp\left(\frac{\imath}{2}\Xi^a(x)\sigma^a\right) , \\
\Xi^a(x) &= \epsilon^a_\mu x^\mu - g\frac{1}{\partial^2}(A_\mu \times \epsilon^\mu)^a + \mathcal{O}(g^2)
\end{aligned}
$$

Occurrence of N-ality classes for SU(N)

$$R(zg) = z^k\, R(g)$$ for z element of $\mathbb{Z}_N$

For holonomies and related (Wilson) loops

$$
\begin{gathered}
H_{\gamma(\tau)}(A) \rightarrow G_{\gamma(\tau_s)} H_{\gamma(\tau_0)}(A) G^\dagger_{\gamma(\tau_t)} \\
G_{\circ\gamma} = z^* G_{\circ\gamma}
\end{gathered}
$$

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Wilson loops and quark interactions

Parallel transport along closed paths of a principal bundle

$$W_\alpha = \mathcal{P}\ \exp\left[\imath g \oint dx^\mu A_\mu(x)\right]$$

Particle pairs interacting through strings

$$\mathcal{C}(T_E) = \psi^\dagger(0, T_E) H_{\gamma_R} \psi^{\cdot}(R, T_E)$$

Wilson loops on Euclidean rectangular path

$$\langle \mathcal{C}(T_E)^\dagger \mathcal{C}(0) \rangle \sim \mathcal{W}_\alpha(R, T_E)$$

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

’t Hooft loop and smeared phase space

Creation of thin cents vortices at the base points of loops

$$B_\alpha W_{\alpha'} = z W_{\alpha'} B_\alpha$$ l=1 linked loops

Smeared conjugated variables

Magnetic order $\mathcal{W}_\alpha \sim e^{-aA_\alpha} \quad \leftrightarrow \quad \langle B_\alpha \rangle \sim e^{-bP_\alpha}$ Confinement

Magnetic disorder $\mathcal{W}_\alpha \sim e^{-a'P_\alpha} \quad \leftrightarrow \quad \langle B_\alpha \rangle \sim e^{-b'A_\alpha}$ SSB of center symmetry

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Geometric phase and quark confinement

Intuition from notable example of U(1)

$$W_{\circ\gamma} = e^{\imath e \Phi_B}$$

Non-vanishing magnetic fluxes recovered from singular gauge transformations applied to vanishing fields

$$W_{\circ\gamma} \to e^{\pm \imath e \Phi_B} W_{\circ\gamma}$$

Holonomy defined as circuitation of gauge connection along path (Wilson loop when paths are closed)

$$W_\alpha = \mathcal{P} \exp\left[\imath g \oint dx^\mu A_\mu(x)\right]$$

Action of center symmetry

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} \qquad W_{\circ\gamma} \to (z^*)^l W_{\circ\gamma}$$

$\circ\gamma$ space-like loop and l linking number

For SU(N) the geometric phase is replaced by a center symmetry group element of $\mathbb{Z}_N$

$$G_{\gamma(\tau_s)} = z G_{\gamma(\tau_t)} \, . \qquad W_{\circ\gamma} \to (z^*)^l W_{\circ\gamma}$$

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Stochastic gauge-geometry flow

Consider the geometry back-reaction (fluctuation of topologies)

$$\begin{aligned}\mathcal{S}_{\mathrm{EYM}} = \quad & +\frac{1}{2\kappa}\int dx^0_M d^{D-1}x\sqrt{-g}R[g_{\mu\nu}] \\ & -\frac{1}{4g_s^2}\int dx^0_M d^{D-1}x\sqrt{-g}F^a_{\mu\nu}F^{a\,\mu\nu}\end{aligned}$$

Stochastic flow complemented with noise multiplicative ansatz

$$\begin{aligned}\frac{\partial A^a_\mu(x,\tau)}{\partial\tau} &= \quad \imath\nabla^\nu F^a_{\nu\mu}(x,\tau) + \xi^a_\mu(x,\tau) \\ \frac{\partial g_{\mu\nu}(x,\tau)}{\partial\tau} &= -\imath\frac{\sqrt{-g}}{2\kappa}\Big[R_{\mu\nu}(x,\tau) - \frac{1}{2}R(x,\tau)g_{\mu\nu}(x,\tau) \;\; -\kappa T_{\mu\nu}(x,\tau)\Big] + \xi_{\mu\nu}(x,\tau)\end{aligned}$$

$$\xi^a_\mu(x,\tau) \equiv \xi(x,\tau)A^a_\mu(x,\tau)$$

$$\xi_{\mu\nu}(x,\tau) \equiv \xi'(x,\tau)g_{\mu\nu}(x,\tau)$$

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Topology changes

Stochastic noise induce fluctuations of the manifold's (hadronic ground states) topology

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\gamma} U_{\beta}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g_{\mu\nu}(\tau) = -2n^{\alpha}(\tau) U_{\alpha}(\tau) g_{\mu\nu}(\tau) = -2\lambda(\tau)\, g_{\mu\nu}(\tau)$$

Geometric flow and appearance of singularities at the geometry level

$\updownarrow$

Singularities of the gauge transformations and appearance of thin vortices

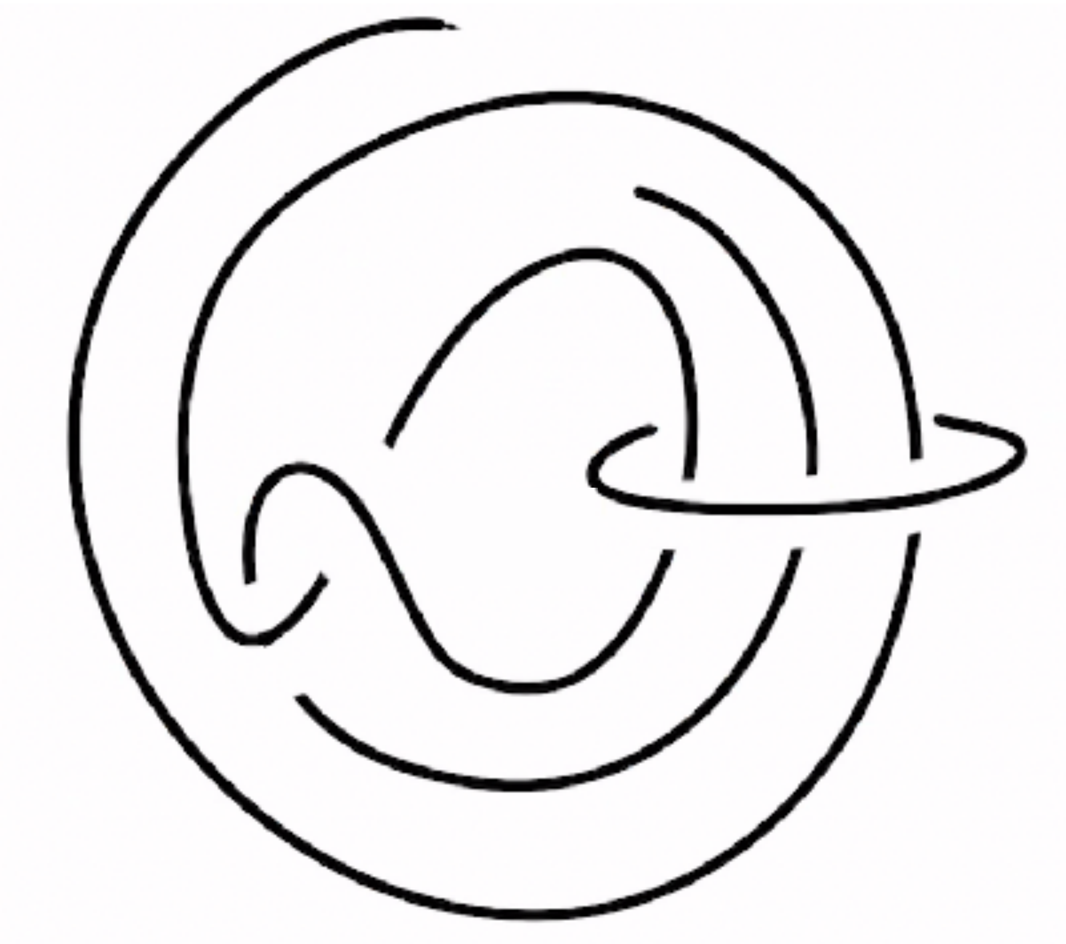
Dual fluxes are created dynamically in the stochastic time and source the area-law

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Quark Confinement

$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A) \rangle = \quad \frac{1}{Z} \int \mathcal{D}A \, \mathrm{Tr}\left[\mathcal{P} e^{\imath \oint_{\gamma_1} A_\mu dx^\mu}\right] \mathrm{Tr}\left[\mathcal{P} e^{\imath \oint_{\gamma_2} A_\mu dx^\mu}\right] e^{\imath S_{\mathrm{YM}}[A]}$$

VEV of the product of two Wilson loops



$$S_{\mathrm{YM}} = \int_M F \wedge *F \qquad \oint_{\gamma=\partial\Gamma} A = \int_\Gamma F$$

$$\langle W_{\gamma_1}(A) W_{\gamma_2}(A) \rangle \sim e^{\imath \, \mathrm{Area}(\Gamma_1) + \imath \, \mathrm{Area}(\Gamma_2)} \, e^{2\imath \, lk(\gamma_1, \gamma_2)}$$

Linking number between Wilson and ’t Hooft loops

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Dimensional transmutation

Rescaling between leaf of the bundle with changes of topology

$$ds_T^2 = \frac{da_T^2}{a_{T^2}} = d\left(\frac{t}{L}\right)^2 \quad \longleftrightarrow \quad a_T(t,L) = a_0 \cdot \exp\left(\frac{t}{L}\right)$$

Level fo the Chen-Simons action and rescaling

$$\int_\Sigma tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = 8\pi^2 CS(\Sigma)$$

$$a = L_P \cdot \exp\left(\frac{3}{2 \cdot CS(\Sigma(2,5,7))}\right)$$

QCD scale dynamically generated in the stochastic time

$$a_1 = L_P \cdot \exp\left(\frac{140}{3}\right) \approx 7.5 \cdot 10^{-15} m$$

T. Asselmeyer-Maluga, M. Lulli, **A. Marciano**, R. Pasechnik, E. Zappala, arXiv:2408.15986

Pre-geometry a la Wilczek

Wilczek pre-geometry

Internal symmetry extended to SO(4,1)/SO(3,2) — see e.g. [1]

$$\mathscr{L}_{\mathrm{W}} = k_{\mathrm{W}} \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} F^{AB}_{\mu\nu} \nabla_\rho \phi^C \nabla_\sigma \phi^D \phi^E$$

$$\mathscr{L}_{\mathrm{W}} \xrightarrow{SSB} \mathscr{L}_{\mathrm{EH}} - M_{\mathrm{P}}^2 \Lambda e$$

$$M_{\mathrm{P}}^2 \equiv -8 k_{\mathrm{W}} v^3 m^2$$

$$\Lambda \equiv \pm 6 m^2 = \mp \frac{3 M_{\mathrm{P}}^2}{4 k_{\mathrm{W}} v^3}$$

$$k_{\mathrm{W}} \sim -1 \left[\phi\right]^{-3} \Rightarrow v \sim 10^{40} \left[\phi\right]$$

$$\mathscr{L}_{\mathrm{MM}} = k_{\mathrm{MM}} \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} F^{AB}_{\mu\nu} F^{CD}_{\rho\sigma} \phi^E$$

$$\mathscr{L}_{\mathrm{MM}} \xrightarrow{SSB} \mathscr{L}_{\mathrm{EH}} - M_{\mathrm{P}}^2 \Lambda e + \lambda e \mathscr{G}$$

$$M_{\mathrm{P}}^2 \equiv \pm 32 k_{\mathrm{MM}} v m^2$$

$$\Lambda \equiv \pm 3 m^2 = \frac{3 M_{\mathrm{P}}^2}{32 k_{\mathrm{MM}} v}$$

$$\lambda \equiv -4 k_{\mathrm{MM}} v$$

$$k_{\mathrm{MM}} \sim \pm 1 \left[\phi\right]^{-1} \Rightarrow v \sim 10^{119} \left[\phi\right]$$

[1] A. Addazi, S. Capozziello, **A. Marciano** and G. Meluccio, arXiv: 2409.02200

Canonical analysis of Wilczek pre-geometry

$$\mathcal{L}_{\mathrm{W}} = k_{\mathrm{W}} \epsilon_{ABCDE} \epsilon^{\mu\nu\rho\sigma} F^{AB}_{\mu\nu} \nabla_\rho \phi^C \nabla_\sigma \phi^D \phi^E$$

$$\mathcal{L}_{\mathrm{SSB}} = -k_{\mathrm{SSB}} v^{-4} |J| (\phi^2 \mp v^2)^2$$

$$\downarrow$$

Conjugated momenta

$$\Pi_E = 2\epsilon_{ABCDE}\epsilon^{0ijk}\nabla_k\phi^C\phi^D \times \\ \left[k_{\mathrm{W}} F^{AB}_{ij} - 2\mathrm{sgn}(J) k_{\mathrm{SSB}} v^{-4} \nabla_i \phi^A \nabla_j \phi^B (\phi^2 \mp v^2)^2\right]$$

$$\Pi^i_{AB} = 2k_{\mathrm{W}}\epsilon_{ABCDE}\epsilon^{0ijk}\nabla_j\phi^C\nabla_k\phi^D\phi^E, \quad \Pi^0_{AB} = 0$$

$$\downarrow \text{SSB}$$

$$\Pi^i_{ab} = -\frac{M_P^2}{4}\epsilon_{abcd}\epsilon^{0ijk} e^c_j e^d_k, \qquad \Pi^0_{ab} = 0$$

Hamiltonian recovered via SSB matches ADM gravity in time gauge n0 = 1, fixing e00= −1/N

[1] A. Addazi, S. Capozziello, JL. Liu, **A. Marciano,** G. Meluccio and XL Su, aXiv:2505.01272;

DOFs of Wilczek pre-geometry

Hamiltonian analysis à la Dirac

$$Z_A, Z^i_{AB}, Z^0_{AB} \approx 0$$ **Primary constraints**

$$\dot{Z}^0_{AB} \approx 0$$ **Secondary constraints**

$$\mathcal{H} = \lambda^A Z_A + \lambda_i^{AB} Z^i_{AB} + \lambda_0^{AB} Z^0_{AB} + \tilde{\lambda}_0^{AB} \dot{Z}^0_{AB}$$

Total Hamiltonian

$$\downarrow$$

90 dynamical variables, 20 gauge choices, 10 first-class constraints, 44 second-class constraints

$$\downarrow$$

Three physical DOFs: one extra conformal mode appears

[1] A. Addazi, S. Capozziello, JL. Liu, **A. Marciano,** G. Meluccio and XL Su, aXiv:2505.01272;

Plebanski formulation of Wilczek pre-geometry

Plebanski form in Wilczek pre-geometry

$$B_{AB}^{(\mathrm{W})} = -2k_{\mathrm{W}}\epsilon_{ABCDE}\nabla\phi^C \wedge \nabla\phi^D \phi^E$$

Plebanski action for Wilczek pre-geometry

$$S_{\text{P-W}} = \int (B_{AB} \wedge F^{AB} + \epsilon_{ABCDE} B^{AB} \wedge B^{CD} \phi^E)$$

$$B_{\mathrm{Holst}}^{AB} \propto \pm\nabla\phi^A \wedge \nabla\phi^B \qquad B_{\mathrm{EH}}^{AB} \propto \pm\, \epsilon^{AB}{}_{CDE}\nabla\phi^C \wedge \nabla\phi^D \phi^E$$

[1] A. Addazi, S. Capozziello, JL. Liu, **A. Marciano,** G. Meluccio and XL Su, aXiv:2505.01272;

Stochastic gradient flow of the Plebanski formulation

Geometric RG flow from TQFT to infrared theories of gravity

$$\mathcal{S} = \int B^{AB} \wedge F^{AB} + \phi^A B^{BC} \wedge B^{DE} \epsilon_{ABCDE}$$

$$SO(4,1) \to SO(3,1)$$

Einstein-Hilbert as the symmetry broken phase of Wilczek gravity

Stochastic gradient low breakdown of symmetries to gravity

$$\begin{aligned}
\frac{\partial A}{\partial s}^{AB} &= -i\mathcal{D}B^{AB} + \xi_g A^{AB}\,, \\
\frac{\partial B}{\partial s}^{AB} &= -iF^{AB} - i\epsilon^{AB}{}_{CDE}\,B^{CD}\phi^E + \xi_f B^{AB}\,, \\
\frac{\partial \phi}{\partial s}^{A} &= -i\epsilon^{A}{}_{BCDE}\,B^{BC} \wedge B^{DE} + \xi_s \phi^A\,,
\end{aligned}$$

[1] A. Addazi, S. Capozziello, JL. Liu, **A. Marciano,** G. Meluccio and XL Su, aXiv:2505.01272;

RG flowing from a TQFT to a theory of gravity

$$\frac{\partial \phi^A}{\partial s} - \xi_s \phi^A \approx 0 \qquad \longrightarrow \qquad \phi^A(s) \approx e^{-\frac{\sigma^2_{\xi_s}}{2}(s-s_0)} \phi^A_0(s)$$

Fixed point and UV flows toward TQFT

$$S_{\mathrm{BF}} = \int \mathrm{tr}(B \wedge F)$$

Decoupling between matter and pre-geometry

$$\downarrow$$

Topological BF with SO(3,2)/SO(4,1) gauge symmetry

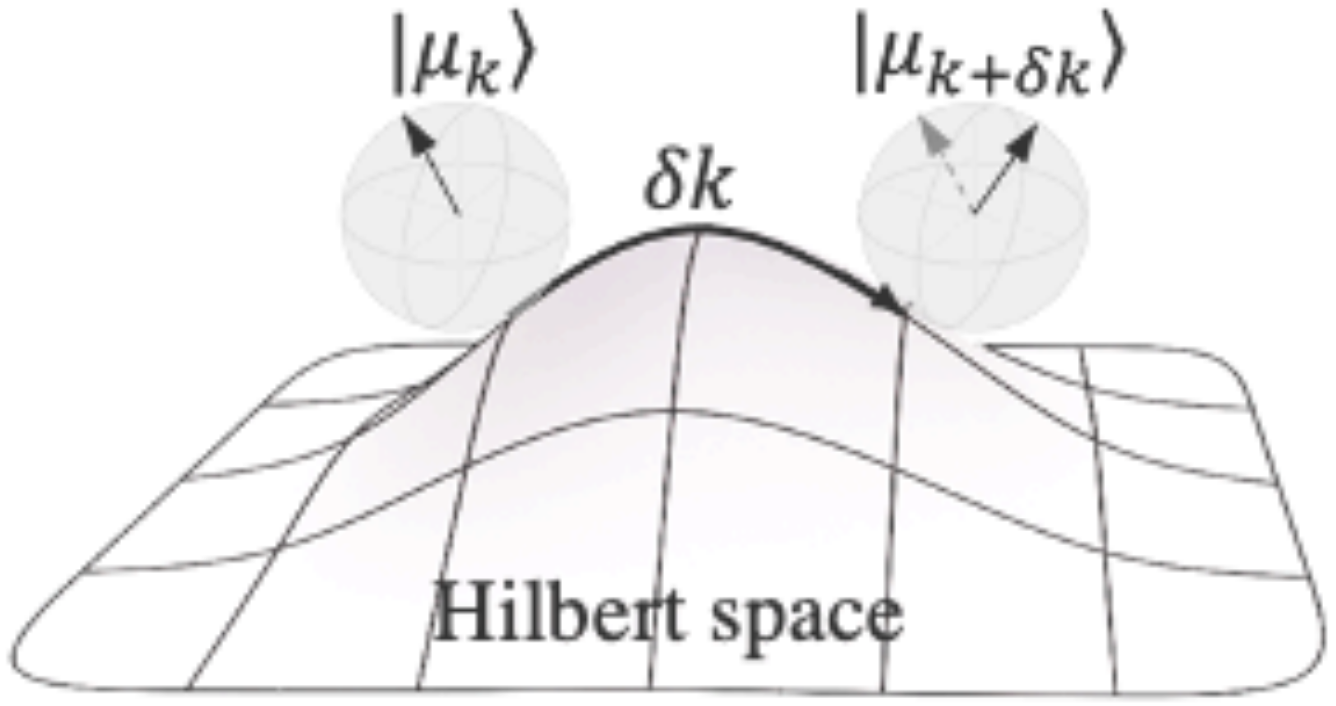
[1] A. Frenkel, P. Horava and S. Randall, arXiv: 2010.15369;

[2] G. Mistretta and T. Prokopec, arXiv: 2310.14827;

[1] A. Addazi, S. Capozziello, JL. Liu, **A. Marciano,** G. Meluccio and XL Su, aXiv:2505.01272;

Topology changes and condensed matter

Topology changes in out-of-equilibrium condensed matter



Quantum Geometry: $T = g - \frac{i}{2}\Omega$

P. Törmä, Phys. Rev. Lett. 131, 240001 (2023)

Quantum metric g:
The amplitude distance between $|\mu_k\rangle$ and $|\mu_{k+\delta k}\rangle$

Berry curvature Ω:
The phase distance between $|\mu_k\rangle$ and $|\mu_{k+\delta k}\rangle$

Quantum metric can induce a lot of interesting phenomena:

Intrinsic nonlinear response;
Fractional Chern insulator states ;
Superconductivity and and other phenomena in flat band systems;
Linear displacement current;
……

Y. Gao, et al. Phys. Rev. Lett. 112, 166601 (2014)
C. Wang, et al. Phys. Rev. Lett. 127, 277201 (2021)
T. Liu, et al. Nat. Sci. Rev. nwae334 (2024)
L. Xiang, et al., Phys. Rev. B 109, 115121 (2024)

Anomalous Hall effect

Spin Hall effect

Valley Hall effect

BCD nonlinear Hall

Outlook and conclusions

Stochastic quantisation and out-of-equilibrium breakdown of symmetries

Geometric RG flow complemented with stochastic (multiplicative) noise

Consequences of the SGF: astrophysics, cosmology, particle physics and foundational aspects

Emergent gravity from the pre-geometric Wilczek theory

Symmetry breaking & topology changes in out-of-equilibrium condensed matter systems

谢谢

Thank you!

Grazie!

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