What anomalies can tell about SM and gravity

Loriano Bonora,

SISSA, Trieste

What comes beyond the standard model, Bled 2025

I would like to consider the question:

what happens when we put together gravity and SM as field theories?

A mandatory issue to consider is that of anomalies, so I would like more specifically to answer the question:

what do anomalies have to say about the interaction between gravity and SM? Is there anything useful that we can learn from it?

the answer is: YES

Summary: first half

- Anomalies: chiral and trace
- Consistent and covariant anomalies
- Trace anomalies: odd and even parity
- Basic subdivision: O and NO anomalies
- Obstructions (i.e. non-existence of propagators) and family's index theorem
- NO anomalies (propagators exist)

Partial conclusion of the analysis: the MSM does have O anomalies that need be canceled

Summary: second half

- A L-R symmetric model
- Axial-complex analysis
- Conformal invariance
- Trace anomalies: even parity
- WZ terms
- Renormalization and unitarity
- Connection with cosmology

L.B. *Fermions and anomalies in FTs, Springer 2023*

L.B. and S. Giaccari, Arxiv:2412.07470

How to compute anomalies

- Perturbative methods (Feynman diagrams, ...)
- Non-Perturbative (heat kernel like methods: Seeley-Schingwer-DeWitt, Fujikawa,....)
- Family's index theorem (for odd parity anomalies)

Gauge anomalies in chiral theories

(well-known things…)

What are anomalies?

An anomaly is a quantum effect that violates a classical symmetry.
Examples:

- Covariant gauge anomaly. The lagrangian $L = i\bar{\psi}(\partial\!\!\!/ + V\!\!\!/)\psi$ is invariant under

$$V\!\!\!/ \to V\!\!\!/ + \gamma_5 D\!\!\!\!/\lambda, \qquad \psi \to (1+\gamma_5\lambda)\psi \qquad \lambda = \lambda(x)^a T^a, \qquad D_\mu = \partial_\mu + V_\mu$$

It follows that the current $j^a_{\mu 5} = \frac{i}{2}\bar{\psi}\gamma_\mu\gamma_5 T^a\psi$ is classically conserved, but, at one loop,

$$[D^\mu j_{\mu 5}]^a = \frac{\hbar}{16\pi^2}\epsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\left(T^a F^{\mu\nu}F^{\lambda\rho}\right), \qquad F = dV + \frac{1}{2}[V,V]$$

- Consistent gauge anomaly. The lagrangian $L = i\bar{\psi}_L(\partial\!\!\!/ + A\!\!\!/)\psi_L$, where $\psi_L = \frac{1+\gamma_5}{2}\psi$ is invariant under

$$A_\mu \to A_\mu + \partial_\mu\lambda + [A_\mu, \lambda], \qquad \psi_L \to \psi_L - \lambda\psi_L$$

The current $j^a_{\mu L} = \frac{i}{2}\bar{\psi}_L\gamma_\mu T^a\psi_L$ is classically conserved, but

$$[D^\mu j_{\mu L}]^a = \frac{\hbar}{24\pi^2}\epsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left(T^a\partial^\mu(A^\nu\partial^\lambda A^\rho + \frac{1}{2}A^\nu A^\lambda A^\rho)\right) \equiv \mathcal{A}^a$$

WZ consistency conditions

Given the effective action

$$W[V] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{n!} \int \prod_{i=1}^{n} d^{\mathsf{d}}x_i \, V_{\mu_i}(x_i) \langle 0|\mathcal{T} j_{\mu_1}(x_1) \ldots j_{\mu_n}(x_n)|0\rangle_c$$

invariance is expressed via the functional operator $X^a(x)$ defined by

$$X^a(x) = \partial_\mu \frac{\delta}{\delta V_\mu^a(x)} + f^{abc} V_\mu^b(x) \frac{\delta}{\delta V_\mu^c(x)},$$

as follows

$$X^a(x) W[V] = 0$$

In a number of cases this WI is violated

$$X^a(x) W[V] = \mathcal{A}^a(x)$$

Applying $X^b(y)$ to both sides and then inverting the two operations, we find a remarkable relation of group-theoretical nature

$$X^a(x)\mathcal{A}^b(y) - X^b(y)\mathcal{A}^a(x) + f^{abc}\mathcal{A}^c(x)\delta(x-y) = 0,$$

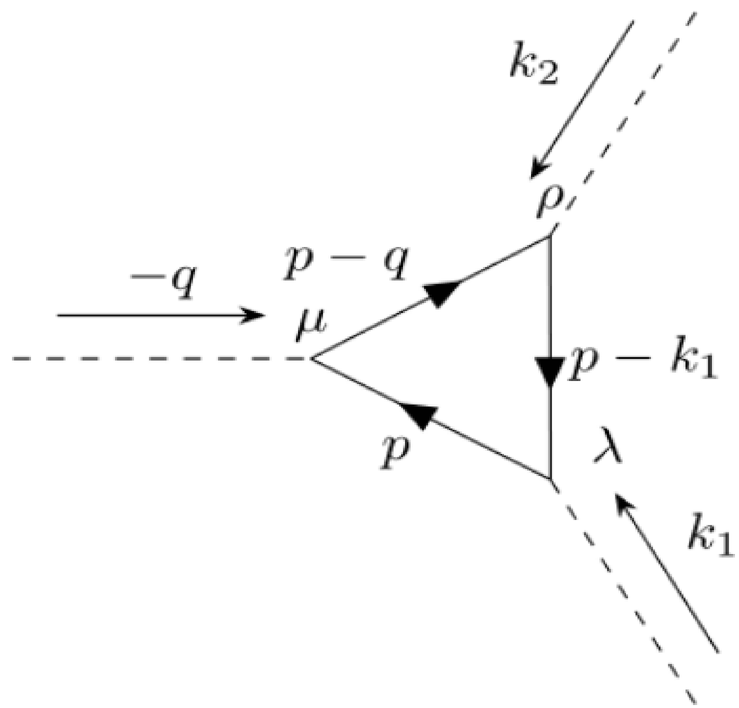
that the anomaly must satisfy. These are the Wess-Zumino (WZ) consistency conditions.

The triangle diagram

The fermion propagator is $\frac{i}{\not p}$ and the vertex $i\gamma_\mu P_L T^a$. The Fourier transform of the three currents amplitude $\langle j_L\, j_L\, j_L\rangle$ is given by

$$\begin{aligned}
\widetilde{F}^{(L)abc}_{\mu\lambda\rho}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left\{\frac{1}{\not p}\gamma_\lambda\frac{1-\gamma_5}{2}T^b\frac{1}{\not p-\not k_1}\gamma_\rho\frac{1-\gamma_5}{2}T^c\frac{1}{\not p-\not q}\gamma_\mu\frac{1-\gamma_5}{2}T^a\right\} \\
&\equiv \mathrm{Tr}(T^aT^bT^c)\widetilde{F}^{(L)}_{\mu\lambda\rho}(k_1,k_2)
\end{aligned}$$

where $q = k_1+k_2$. This is to be contracted with q^μ. It is UV divergent and needs to be regularized. The relevant Feynman diagram is



There is also a divergent quadrangle diagram. The pentagon diagram is UV convergent.

WZ consistency conditions and cohomology

Introduce the FP ghosts $c(x) = c^a(x)T^a$: the gauge trasformations become the BRST transformations

$$\mathfrak{s}V_\mu = D_\mu c, \qquad \mathfrak{s}c = -\frac{1}{2}[c, c],$$

This operation is nilpotent and the functional operator that generate them

$$\mathfrak{s} = \int d^{\mathsf{d}}x \left(\mathfrak{s}V_\mu^a(x)\frac{\partial}{\partial V_\mu^a(x)} + \mathfrak{s}c^a(x)\frac{\partial}{\partial c^a(x)} \right).$$

it is a nilpotent operator: $\mathfrak{s}^2 = 0$. Let us define the integrated anomaly

$$\mathcal{A}_c = \int d^4x c^a(x)\mathcal{A}^a(x)$$

The previous formulae are condensed as follows

$$\mathfrak{s}W[V] = \mathcal{A}_c$$

and the WZ c.c. can be written

$$\mathfrak{s}\mathcal{A}_c = 0$$

$\mathfrak{s}$ is a coboundary operator, $\mathcal{A}_c$ is a cocycle, it represent a cohomology class.

Descent equations and anomalies

There is a time honored formalism to construct all possible solutions of the consistency conditions. Start from an order n symmetric polynomial in some representation of the Lie algebra, $P_n(T^{a_1}, ..., T^{a_n})$, invariant under the adjoint transformations. In many cases these polynomials are symmetric traces of the generators in the corresponding representation

$$P_n(T^{a_1}, ..., T^{a_n}) = Str(T^{a_1} ... T^{a_n}) = d^{a_1 \ldots a_n}$$

Let $V = V_\mu dx^\mu = V_\mu^a T^a dx^\mu$ and $F = dV + V \wedge V = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, Then one can construct the 2n-form

$$\Delta_{2n}(V) = P_n(F, F, \ldots F)$$

Then

$$P_n(F, F, \ldots F) = d\left(n \int_0^1 dt\, P_n(V, F_t, \ldots, F_t)\right) = d\Delta^{(0)}_{2n-1}(V)$$

where $V_t = tV$ and its curvature $F_t = dV_t + \frac{1}{2}[V_t, V_t]$. This is the first of a sequence of equations that can be shown to hold

$$\begin{aligned}
&\Delta_{2n}(V) - d\Delta^{(0)}_{2n-1}(V) = 0 \\
&\mathfrak{s}\Delta^{(0)}_{2n-1}(V) - d\Delta^{(1)}_{2n-2}(V, c) = 0 \\
&\mathfrak{s}\Delta^{(1)}_{2n-2}(V, c) - d\Delta^{(2)}_{2n-3}(V, c) = 0 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Anomaly cancellation

First mechanism:

Both covariant and consistent anomalies in 4d are proportional to the tensor

$$t^{abc} = Str(T^a T^b T^c)$$

For antisymmetric T^a this tensor vanishes identically. This is true, for example, for the Lorentz group $SO(4)$.

Second mechanism:

The anomaly coefficiemts for the various species sum up to zero.

Third mechanism:

Introducing WZ terms (which requires new fields) or by means of the Green-Schwarz mechanism.

Minimal Standard Model

Three families of massless quarks and leptons

$\mathsf{G}/fields$	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$\frac{1}{6}$
u_R	3	1	$\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$-\frac{1}{2}$
e_R	1	1	-1

The second column specifies the representations of $SU(3)$, the third the ones of $SU(2)$ and the last is the list of $U(1)$ representations, denoted by the corresponding hypercharge eigenvalue. The hypercharge is defined by

$$Y = Q - T_3, \qquad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

here Q is the electromagnetic charge and T_3 the third generator of $SU(2)$ in the doublet representation.

MSM: gauge anomalies

- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)}$: there are two left-handed and two right-handed triplet, whose anomalies cancel one another.

- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)}$, which vanishes because the tensor d^{abc} vanishes in general for the Lie algebra $\mathfrak{su}(2)$.

- $T^{\mathfrak{su}(2)} \times T^{\mathfrak{su}(2)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(2)$ generators in two doublet representations. These traces are non-vanishing because $\mathrm{tr}(T^a T^b) \sim \delta^{ab}$, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charges, whose total value is $6\left(\frac{1}{6}\right) - 2(\frac{1}{2}) = 0$.

- $T^{\mathfrak{su}(3)} \times T^{\mathfrak{su}(3)} \times T^{\mathfrak{u}(1)}$, in which case we have the trace of two $\mathfrak{su}(3)$ left triplet generators and two right triplet generators. These traces are again non-vanishing, but they are multiplied by the corresponding $\mathfrak{u}(1)$ charge, whose total value is $3\left(2\left(\frac{1}{6}\right) - \frac{2}{3} + \frac{1}{3}\right) = 0$.

- $T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)} \times T^{\mathfrak{u}(1)}$, in this case the tensor is proportional to the overall sum of the charge products: $6\left(\frac{1}{6}\right)^3 - 3\left(\frac{2}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^3 + 2\left(-\frac{1}{2}\right)^3 - (-1)^3 = 0$.

Conjugate chiral spinors

A frequent alternative notation is to use the Lorentz covariant conjugates $(u_R)^c$, $(d_R)^c$ and $(e_R)^c$ instead of u_R, d_R and e_R, in order to collect all the fields in a unique left-handed multiplet.

The symbol $(\psi_R)^c$ (for instance u_R) can be rewritten as

$$(\psi_R)^c = \gamma^0 C \psi_R^* = \gamma^0 C P_R^* \psi^* = P_L \gamma^0 C \psi^* = P_L \psi^c = (\psi^c)_L.$$

Inserted into the kinetic term, it gives

$$\sqrt{g}\,\overline{(\psi^c)_L}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)(\psi^c)_L = \sqrt{g}\,\overline{(\psi_R)^c}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)(\psi_R)^c = \sqrt{g}\,\overline{\psi_R}\,\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu)\psi_R$$

In this case one has to reverse the signs of the $U(1)$ charges and replace the representation 3 of $SU(3)$ of u_R, d_R with the $\bar{3}$ of $(u_R)^c, (d_R)^c$.

Trace anomalies

When a metric is involved

When a metric is present at least two symmetries are involved: diffeomorphisms

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

and Weyl transformations

$$\delta_\omega g_{\mu\nu} = 2\,\omega\, g_{\mu\nu}$$

The effective action is

$$W[g] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{2^n n!} \int \prod_{i=1}^{n} d^{\mathsf{d}} x_i \sqrt{g(x_i)}\, h^{\mu_i \nu_i}(x_i) \langle 0 | \mathcal{T} T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n) | 0 \rangle$$

and the Ward identity for diffeomorphims takes the form

$$\delta_\xi\, W[g] = 0$$

This relation may be violated at one-loop,

$$\delta_\xi\, W[g] = \mathcal{A}_\xi$$

The term in the RHS is linear in ξ and, since $\delta_\xi^2 = 0$, it satisfies the consistency condition

$$\delta_\xi\, \mathcal{A}_\xi = 0$$

Weyl transformation and trace anomalies

The invariance under Weyl transformations, or conformal invariance, is expressed by

$$0 = \delta_\omega W[g] = \int d^{\mathsf{d}}x\, \frac{\delta W[g]}{\delta g^{\mu\nu}(x)} \delta_\omega g^{\mu\nu}(x) = -\int d^{\mathsf{d}}x\, \omega(x) \langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle g^{\mu\nu}(x)$$

Since ω is a generic infinitesimal function, this implies

$$\langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle g^{\mu\nu}(x) = 0$$

where

$$\langle\!\langle T_{\mu\nu}(x) \rangle\!\rangle = \sum_{n=0}^{\infty} \frac{i^n}{2^n n!} \int \prod_{i=1}^{n} d^{\mathsf{d}}x_i \sqrt{g(x_i)}\, h^{\mu_i\nu_i}(x_i) \langle 0 | \mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) \ldots T_{\mu_n\nu_n}(x_n) | 0 \rangle$$

But we may have violations of this classical invariance

$$\delta_\omega W[g] = \mathcal{A}_\omega$$

Due to the nilpotence of δ_ω the consistency condition

$$\delta_\omega \mathcal{A}_\omega = 0$$

must be satisfied.

Possible trace anomalies

The possible non-trivial cocycles of δ_ω with vanishing diffeomorphism partner in 4d are well-known, they take the form

$$\Delta[g,\omega] = \int d^4x\,\sqrt{g}\,\omega\,T[g], \qquad \delta_\omega \Delta[g,\omega] = 0$$

where the density $T[g](x)$ can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}\right).$$

Other possible cocycles have densities

$$T_e[V] = F_{\mu\nu}F^{\mu\nu}, \qquad T_o[V] = \varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}.$$

Weyl fermion coupled to a metric

Consider again

$$S=\int d^4x\,\sqrt{g}\,i\overline{\psi_R}\gamma^\mu\left(\partial_\mu+\frac{1}{2}\omega_\mu\right)\psi_R$$

where $\psi_R = P_R\psi$, $P_R=\frac{1+\gamma_5}{2}$. The action can be rewritten as

$$S=\int d^4x\,\sqrt{g}\,\left[\frac{i}{2}\overline{\psi_R}\gamma^\mu\overset{\leftrightarrow}{\partial}_\mu\psi_R-\frac{1}{4}\epsilon^{\mu abc}\omega_{\mu ab}\overline{\psi_R}\gamma_c\gamma_5\psi_R\right]$$

where it is understood that the derivative applies to ψ_L and $\overline{\psi_L}$ only. We have used the relation $\{\gamma^a,\Sigma^{bc}\}=i\,\epsilon^{abcd}\gamma_d\gamma_5$.

The classical e.m. tensor is

$$T_{\mu\nu}=\frac{i}{4}\overline{\psi_R}\gamma_\mu\overset{\leftrightarrow}{\nabla}_\nu\psi_R+\{\mu\leftrightarrow\nu\}$$

This theory is invariant under diffeomorphisms $\delta_\xi g_{\mu\nu}=\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu$ and Weyl transformations $\delta_\omega g_{\mu\nu}=2\omega g_{\mu\nu}$. As a consequence, classically,

$$\nabla^\mu T_{\mu\nu}(x)=0,\qquad T^\mu_\mu(x)=0$$

on shell.

Weyl fermion *cont.*

The fermion propagator is

$$\frac{i}{\not{p} + i\epsilon}$$

and there is only one graviton-fermion-fermion (V_{ffg}) vertex given by

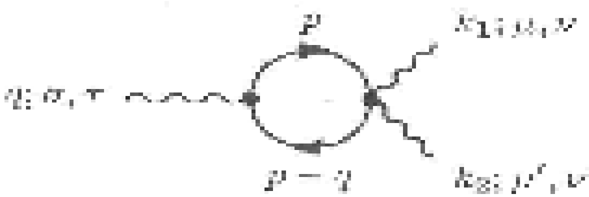
$$\frac{i}{8}\left[(p+p')_\mu \gamma_\nu + (p+p')_\nu \gamma_\mu\right] \frac{1+\gamma_*}{2}$$

where p and p' are the two graviton momenta, and a two-fermion-two-graviton vertex (V^ϵ_{ffhh}) is

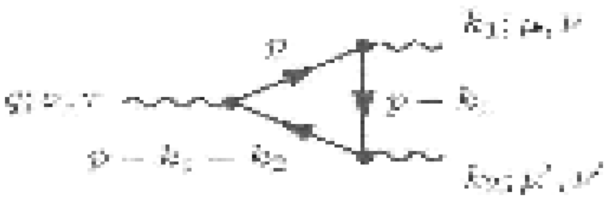
$$V^\epsilon_{ffhh}: \qquad \frac{1}{64} t_{\mu\nu\mu'\nu'\kappa\lambda}(k-k')^\lambda \gamma^\kappa \frac{1+\gamma_5}{2}$$

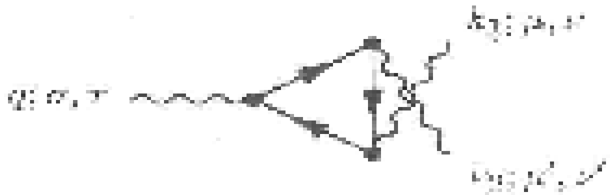
where

$$t_{\mu\nu\mu'\nu'\kappa\lambda} = \eta_{\mu\mu'}\epsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\kappa\lambda}$$



Relevant diagrams





The triangle diagram for the e. m. trace

$$\begin{aligned}\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big\{\frac{i}{8}\left[(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right]\left(\frac{1+\gamma_5}{2}\right)\frac{i}{(\not p-\not k_1)} \\ &\times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right]\left(\frac{1+\gamma_5}{2}\right) \\ &\times \frac{i}{(\not p-\not k_1-\not k_2)}(2\not p-\not k_1-\not k_2)\left(\frac{1+\gamma_5}{2}\right)\frac{i}{\not p}\Big\}.\end{aligned}$$

After regularization it becomes

$$\begin{aligned}\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) &= -\frac{1}{256}\int \frac{d^4p}{(2\pi)^4}\int\frac{d^{n-4}\ell}{(2\pi)^{n-4}}\mathrm{Tr}\Big\{\left[\frac{\not p+\not\ell}{p^2-\ell^2}(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right] \\ &\times \frac{(\not p+\not\ell-\not k_1)}{(p-k_1)^2-\ell^2}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right] \\ &\times \frac{(\not p+\not\ell-\not k_1-\not k_2)}{(p-k_1-k_2)^2-\ell^2}(2\not p+2\not\ell-\not k_1-\not k_2)\left(\frac{1+\gamma_5}{2}\right)\Big\}\end{aligned}$$

The result

We use the expansion $g_{\mu\nu}(x) \approx \eta_{\mu\nu} + h_{\mu\nu}(x)$. The triangle diagrams eventually give

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial_\tau h^\sigma_\rho - \partial_\mu\partial_\sigma h^\tau_\nu\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right)$$

The corresponding covariant expression is

the trace anomaly

$$\langle T^{\mu}_{\mu}(x)\rangle = \frac{i}{768\pi^2}\,\frac{1}{2}\,\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau}$$

Therefore

$$e = \frac{1}{1536\pi^2}$$

... other calculations are possible!

The previous result corresponds, at the lowest order, to

$$\langle 0|\mathcal{T} T^{\mu}_{\mu}(x) T_{\mu'\nu'}(y) T_{\alpha\beta}(z)|0\rangle^{(odd)}$$

One can also compute

$$\eta^{\mu\nu}\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu'\nu'}(y) T_{\alpha\beta}(z)|0\rangle^{(odd)}$$

and get

$$\eta^{\mu\nu}\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu'\nu'}(y) T_{\alpha\beta}(z)|0\rangle^{(odd)} = 0$$

This is because

$$\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu'\nu'}(y) T_{\alpha\beta}(z)|0\rangle^{(odd)} = 0$$

for algebraic reasons!

So, what is the right result?

Ambiguities in trace anomaly calculations

The true reason for these contradictory results is due to the ambiguities in the (perturbative) calculation of the trace anomaly

There are four sources of ambiguity:

1) The divergent integrals
2) The very definition of trace anomaly
3) The cohomological ambiguity (Weyl + diffeomorphisms)
4) The `new' ambiguity

Ambiguities

1. Loop integration for a Feynman diagram is UV divergent $\longrightarrow$ Solution: choose a regularization scheme

2. Ambiguous definition of trace anomaly $\longrightarrow$ Solution: Define it as:

$$T(x) = g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle T^{\mu}_{\mu}(x)\rangle\rangle$$

3. Cohomological ambiguity: $\longrightarrow$ Solution: Diffeomorphims must be conserved

4. 'New' ambiguity: $\longrightarrow$ Solution: Go to higher loops or use a non-perturbative method

Trace anomaly: definition

The (gravity) effective action is defined by

$$W[h] = W[0] + \sum_{n=1}^{\infty} \frac{i^{n-1}}{2^n n!} \int \prod_{i=1}^{n} d^4x_i \sqrt{g(x_i)} h^{\mu_i\nu_i}(x_i) \langle 0|\mathcal{T} T_{\mu_1\nu_1}(x_1) \dots T_{\mu_n\nu_n}(x_n)|0\rangle$$

The one-loop one-point function is defined by

$$\langle\langle T_{\mu\nu}(x)\rangle\rangle = \sum_{n=0}^{\infty} \frac{i^{n}}{2^n n!} \int \prod_{i=1}^{n} d^4x_i \sqrt{g(x_i)} h^{\mu_i\nu_i}(x_i) \langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) \dots T_{\mu_n\nu_n}(x_n)|0\rangle$$

with $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$. The trace anomaly is

$$T(x) = g^{\mu\nu}(x)\langle\langle T_{\mu\nu}(x)\rangle\rangle - \langle\langle T_{\mu}^{\mu}(x)\rangle\rangle$$

To the lowest order

$$T^{(odd)}(x) = \eta^{\mu\nu}\langle 0|\mathcal{T} T_{\mu\nu}(x) T_{\mu_1\nu_1}(x_1) T_{\mu_2\nu_2}(x_2)|0\rangle - \langle 0|\mathcal{T} T_{\mu}^{\mu}(x) T_{\mu_1\nu_1}(x_1) T_{\mu_2\nu_2}(x_2)|0\rangle + \dots$$

Since $T_{\mu}^{\mu}(x) = 0$ on shell, this is the quantization of 0.

The 'new' ambiguity

Let us apply the same procedure to the odd-parity trace anomaly in 4D. The two amplitudes in question are

$$
\begin{aligned}
\widetilde{T}_{\mu\nu\mu'\nu'}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big\{\frac{i}{8}\left[(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right]\left(\frac{1+\gamma_5}{2}\right)\frac{i}{(\not p - \not k_1)} \\
&\times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right]\left(\frac{1+\gamma_5}{2}\right) \\
&\times \frac{i}{(\not p - \not k_1 - \not k_2)}(2\not p - \not k_1 - \not k_2)\left(\frac{1+\gamma_5}{2}\right)\frac{i}{\not p}\Big\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{T}'_{\mu\nu\mu'\nu'}(k_1,k_2) &= \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\Big\{\frac{i}{8}\left[(2p-k_1)_\mu\gamma_\nu + (\mu\leftrightarrow\nu)\right]\frac{i}{(\not p - \not k_1)} \\
&\times \frac{i}{8}\left[(2p-2k_1-k_2)_{\mu'}\gamma_{\nu'} + (\mu'\leftrightarrow\nu')\right] \\
&\times \frac{i}{(\not p - \not k_1 - \not k_2)}(2\not p - \not k_1 - \not k_2)\left(\frac{1+\gamma_5}{2}\right)\frac{i}{\not p}\Big\}.
\end{aligned}
$$

Once regularized they lead to different results (like in the previous case).

cont.

The conservation of the e.m. tensor is determined by the amplitudes

$$
\begin{aligned}
q^\mu \widetilde{\mathfrak{T}}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) \;=\;& -\frac{1}{512}\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not p}((2p-k_1)_\lambda\gamma_\rho + (\lambda\leftrightarrow\rho))\left(\frac{1+\gamma_5}{2}\right)\frac{1}{\not p-\not k_1}\right.\right.\\
&\times((2p-2k_1-k_2)_\alpha\gamma_\beta + (\alpha\leftrightarrow\beta))\left(\frac{1+\gamma_5}{2}\right)\\
&\left.\left.\times\;\frac{1}{\not p-\not q}((2p-q)\cdot q\,\gamma_\nu + (2p-q)_\nu \not q)\right)\left(\frac{1+\gamma_5}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
q^\mu \widetilde{\mathfrak{T}}^{(1)}_{\mu\nu\lambda\rho\alpha\beta}(k_1,k_2) = -\frac{1}{512}\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not p}((2p-k_1)_\lambda\gamma_\rho + (\lambda\leftrightarrow\rho))\;\frac{1}{\not p-\not k_1}\right.\right.\\
\left.\left.\times((2p-2k_1-k_2)_\alpha\gamma_\beta + (\alpha\leftrightarrow\beta))\;\frac{1}{\not p-\not q}((2p-q)\cdot q\,\gamma_\nu + (2p-q)_\nu \not q)\right)\left(\frac{1+\gamma_5}{2}\right)\right]
\end{aligned}
$$

Once regularized they both vanish.

Conclusion: therefore there is no way to repair the trace anomaly!

Solution for the 'new' ambiguity

Does it mean that we cannot compute the trace anomaly? No, it only means that this problem cannot be solved at the lowest perturbative order.

Possible solutions:

- go to higher order (four-point, five-point, ... amplitudes)
- use non-perturbative methods (heat kernel methods)

Gauge-induced trace anomalies

Consider the action of a Dirac fermion coupled to a metric and an Abelian vector field

$$S = \int d^4x \sqrt{g}\, i\overline{\psi}\gamma^\mu \left(D_\mu + \frac{1}{2}\omega_\mu - iV_\mu\right)\psi$$

with the usual notation. The vector current is $j_\mu = \overline{\psi}\gamma_\mu\psi$ and the stress-energy tensor

$$T_{\mu\nu} = \frac{i}{4}\overline{\psi}\gamma_\mu \overset{\leftrightarrow}{\nabla}_\nu \psi + \{\mu \leftrightarrow \nu\}, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu - iV_\mu$$

With the same methods (both pertubative and non-perturbative) we get

$$g^{\mu\nu}\langle\langle T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd} - \langle\langle g^{\mu\nu} T_{R\mu\nu}(x)\rangle\rangle\Big|_{odd} = -\frac{i}{96\pi^2}\varepsilon_{\mu\nu\lambda\rho}\,\partial^\mu V^\nu(x)\partial^\lambda V^\rho(x)$$

The non-Abelian version of this result is

$$\mathcal{A}_\omega^{(odd,R)} = -\frac{i}{384\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}(F^{\mu\nu}F^{\lambda\rho})$$

Trace anomaly in chiral theories with non-pertubative methods.

Idea: use Bardeen’s method also for trace anomalies

Bardeen's method for chiral anomaly

Consider a theory of Dirac fermions coupled a vector V_μ and an axial A_μ gauge potentials, both valued in a Lie algebra with T^a. The action is

$$S[V,A] = i\int d^4x\,\overline{\psi}\left(\partial\!\!\!/ + V\!\!\!/ + \gamma_5 A\!\!\!/\right)\psi$$

It is invariant under two sets of gauge transformations

$$\begin{cases} V_\mu \longrightarrow V_\mu + D_{V\mu}\alpha \\ A_\mu \longrightarrow A_\mu + [A_\mu,\alpha], \\ \psi \longrightarrow (1-\alpha)\psi \end{cases} \qquad\qquad \begin{cases} V_\mu \longrightarrow V_\mu + [A_\mu,\beta] \\ A_\mu \longrightarrow A_\mu + D_{V\mu}\beta \\ \psi \longrightarrow (1+\gamma_5\beta)\psi \end{cases}$$

where $D_{V\mu} = \partial_\mu + [V_\mu,\ \cdot\ ]$ and $\alpha = \alpha^a(x)T^a, \beta = \beta^a(x)T^a$.

As a consequence there are two covariantly conserved currents, $j_\mu = j^a_\mu T^a$ and $j_{5\mu} = j^a_{5\mu}T^a$, where $j^a_\mu = \overline{\psi}\gamma_\mu T^a\psi$ and $j^a_{5\mu} = \overline{\psi}\gamma_\mu\gamma_5 T^a\psi$.

After quantization one finds:

$$[D^\mu_V j_\mu]^a + [A^\mu, j_{5\mu}]^a = 0$$

while the axial conservation becomes anomalous:

$$\begin{aligned}[D^\mu_V j_{5\mu}]^a + [A^\mu, j_\mu]^a \;&=\; \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\Big[T^a\Big(\frac{1}{4}F^{\mu\nu}_V F^{\lambda\rho}_V + \frac{1}{12}F^{\mu\nu}_A F^{\lambda\rho}_A - \frac{1}{6}F^{\mu\nu}_V A^\lambda A^\rho \\ &\quad - \frac{1}{6}A^\mu A^\nu F^{\lambda\rho}_V - \frac{2}{3}A^\mu F^{\nu\lambda}_A A^\rho - \frac{1}{3}A^\mu A^\nu A^\lambda A^\rho\Big)\Big]\end{aligned}$$

where $F^{\mu\nu}_V = \partial^\mu V^\nu - \partial^\nu V^\mu + [V^\mu, V^\nu]$, and $F^{\mu\nu}_A = \partial^\mu A^\nu - \partial^\nu A^\mu + [V^\mu, A^\nu] + [A^\mu, V^\nu]$.

Bardeen's method, *cont.*

From this expression we can derive two results in particular. Setting $A_\mu = 0$ we get the covariant anomaly

$$[D_V^\mu j_{5\mu}]^a = \frac{1}{16\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left(T^a F_V^{\mu\nu} F_V^{\lambda\rho}\right)$$

which is the covariant anomaly.

Taking the chiral limit $V \to \frac{V}{2}, A \to \frac{V}{2}$ we get

$$[D_{V\mu} j_L^\mu]^a = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}\left[T^a \partial^\mu \left(V^\nu \partial^\lambda V^\rho + \frac{1}{2} V^\nu V^\lambda V^\rho\right)\right]$$

where $j^a_{L\mu} = \overline{\psi}_L \gamma_\mu T^a \psi_L$, here $\psi_L = \frac{1+\gamma_5}{2}\psi$, which is the consistent non-Abelian gauge anomaly.

The advantage of this method is that we work with Dirac fermions (as opposed to Weyl fermions).

Axial-complex analysis

Axial-complex numbers are defined by

$$\hat{a} = a_1 + \gamma_5 a_2$$

where a_1 and a_2 are real numbers. Arithmetic is defined in the obvious way. Introducing the chiral projectors $P_\pm = \frac{1\pm\gamma_5}{2}$, we can also write

$$\hat{a} = a_+ P_+ + a_- P_-, \qquad a_\pm = a_1 \pm a_2$$

I consider functions $\hat{f}(\hat{x})$ of the axial-complex variable

$$\widehat{x} = x_1 + \gamma_5 x_2$$

defined via their Taylor expansions.

Derivatives are defined in the obvious way:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu}\right)$$

For axial-analytic functions

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}},$$

Integral of $\widehat{f}(\hat{x})$

$$\int d\hat{x}\, \widehat{f}(\hat{x})$$

is the rapidly decreasing primitive $\widehat{g}(\hat{x})$ of $\widehat{f}(\hat{x})$. And so on....

MAT (metric-axial-tensor) gravity

Introduce

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = k_{\mu\nu}$$

together with generalized vierbeins

$$\widehat{e}_\mu^a = e_\mu^a + \gamma_5 c_\mu^a, \qquad \widehat{e}_a^\mu = e_a^\mu + \gamma_5 c_a^\mu$$

One can define Christoffel symbols

$$\widehat{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^{(1)\lambda} + \gamma_5 \Gamma_{\mu\nu}^{(2)\lambda}$$

and Riemann tensor

$$\widehat{\mathcal{R}}_{\mu\nu\lambda}{}^\rho = \mathcal{R}^{(1)}_{\mu\nu\lambda}{}^\rho + \gamma_5 \mathcal{R}^{(2)}_{\mu\nu\lambda}{}^\rho$$

MAT (cont.)

–

Generalized Weyl transformation

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2(\omega+\gamma_5\eta)}\widehat{g}_{\mu\nu}.$$

The volume density is

$$\begin{aligned}\sqrt{\widehat{g}} &= \sqrt{\det(\widehat{g})}\\ &= \frac{1}{2}\left(\sqrt{\det(g+f)}+\sqrt{\det(g-f)}\right)+\frac{\gamma_5}{2}\left(\sqrt{\det(g+f)}-\sqrt{\det(g-f)}\right)\end{aligned}$$

and transforms as

$$\sqrt{\widehat{g}} \rightarrow e^{4(\omega+\gamma_5\eta)}\sqrt{\widehat{g}}$$

and

$$\frac{1}{\sqrt{\widehat{g}}}\partial_\nu\sqrt{\widehat{g}} = \frac{1}{2}\widehat{g}^{\mu\lambda}\partial_\nu\widehat{g}_{\mu\lambda} = \widehat{\Gamma}^{\mu}_{\mu\nu}$$

Fermions in MAT

The action of a fermion in a background of MAT gravity is

$$
\begin{aligned}
S &= \int d^4x\, i\overline{\psi}\gamma^a \widehat{e}_a^m \left(\partial_m + \frac{1}{2}\widehat{\Omega}_m\right)\sqrt{\widehat{g}}\,\psi \\
&= \int d^4x\, i\overline{\psi}\gamma^a (e_a^m + \gamma_5 c_a^m) \left(\partial_m + \frac{1}{2}\left(\Omega_m^{(1)} + \gamma_5 \Omega_m^{(2)}\right)\right)\sqrt{\widehat{g}}\,\psi
\end{aligned}
$$

It is classically invariant under generalized Weyl transformations. One can define a generalized e.m. tensor

$$
\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{\widehat{g}}} \frac{\overleftarrow{\delta}\, S}{\delta \widehat{g}_{\mu\nu}}
$$

Differentiating with respect to ω and η one obtains two classical Ward identities

$$
\begin{aligned}
T^{\mu\nu} g_{\mu\nu} + T_5^{\mu\nu} f_{\mu\nu} &= 0, \\
T^{\mu\nu} f_{\mu\nu} + T_5^{\mu\nu} g_{\mu\nu} &= 0,
\end{aligned}
$$

The Schwinger-DeWitt method

To represent the determinant of the Dirac operator one can use the DeWitt method based of the Schwinger proper time. One introduces the vacuum-to-vacuum amplitude

$$\langle x, s | x', 0\rangle = \langle x | e^{i\mathcal{F}s} | x'\rangle,$$

where $\mathcal{F}_x$ is a differential quadratic operator, in the ordinary case the Dirac square

$$\mathcal{F}_x = \nabla_\mu g^{\mu\nu} \nabla_\nu - \frac{1}{4} R + \mathcal{V},$$

where $\mathcal{V} = \Sigma^{ab} e_a^\mu e_b^\nu \left(\partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]\right)$. It follows that the functional determinant W can be represented as

$$W = -\frac{1}{4} \int_0^\infty \frac{ds}{is} \mathsf{Tr}\left(e^{i\mathcal{F}s}\right) + \mathrm{const} \equiv \mathsf{L} + \mathrm{const},$$

where L is the relevant effective action

$$\mathsf{L} = \int d^{\mathsf{d}}x \, L(x) \equiv \int d^{\mathsf{d}}x \, \langle x | \mathsf{L} | x\rangle,$$

The trick is to use the (heat kernel) differential equation

$$i \frac{\partial}{\partial s} \langle x, s | x', 0\rangle = -\mathcal{F}_x \langle x, s | x', 0\rangle,$$

The DWS method, *cont.*

insert in it the ansatz

$$\langle x, s | x', 0 \rangle = -\lim_{m \to 0} \frac{i}{(4\pi s)^{\frac{d}{2}}} \sqrt{\mathfrak{D}(x, x')} e^{i\left(\frac{\sigma(x,x')}{2s} - m^2 s\right)} \Phi(x, x', s)$$

It follows that $\Phi(x, x', s)$ must satisfy the differential equation

$$i\frac{\partial \Phi}{\partial s} + \frac{i}{s}\nabla^{\mu}\Phi\nabla_{\mu}\sigma + \frac{1}{\sqrt{\mathfrak{D}}}\nabla^{\mu}\nabla_{\mu}\left(\sqrt{\mathfrak{D}}\Phi\right) - \left(\frac{1}{4}R - \mathcal{V} - m^2\right)\Phi = 0$$

Expanding

$$\Phi(x, x', s) = \sum_{n=0}^{\infty} a_n(x, x')(is)^n$$

one finds that the a_n must satisfy the recursive relations:

$$(n+1)a_{n+1} + \nabla^{\mu}a_{n+1}\nabla_{\mu}\sigma - \frac{1}{\sqrt{\mathfrak{D}}}\nabla^{\mu}\nabla_{\mu}\left(\sqrt{\mathfrak{D}}a_n\right) + \left(\frac{1}{4}R - \mathcal{V} - m^2\right)a_n = 0$$

Using these equations one can compute the coincidence limits

$$[a_n](x) = \lim_{x' \to x} a_n(x, x')$$

The DWS method, *cont.*

The appropriate elliptic operator in the MAT case is

$$\widehat{\mathfrak{F}} = \overline{\widehat{\gamma}}^{\mu}\overline{\widehat{\nabla}}_{\mu}\widehat{\gamma}^{\nu}\widehat{\nabla}_{\nu}$$

where $\gamma^{\mu}\widehat{\nabla}_{\mu} = \gamma^{a}(\tilde{e}_{a}^{\mu} + \gamma_5\tilde{c}_{a}^{\mu})\left(\partial_{\mu} + \frac{1}{2}\left(\Omega_{\mu}^{(1)} + \gamma_5\Omega_{\mu}^{(2)}\right)\right)$.

In 4D the relevant coefficient is $[a_2]$

$$\begin{aligned}[\widehat{a}_2] \;&=\; \frac{1}{2}m^4 - \frac{1}{12}m^2\widehat{R} + \frac{1}{288}\widehat{R}^2 - \frac{1}{120}\widehat{R}_{;\mu}{}^{\mu} - \frac{1}{180}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{180}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho}\\ &\quad+\frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\end{aligned}$$

where $\widehat{\mathcal{R}}_{\mu\nu} = \widehat{R}_{\mu\nu}{}^{ab}\Sigma_{ab}$. and the effective action is given by

$$\begin{aligned}\widehat{L}(\widehat{x}) \;&\approx\; \frac{1}{32\pi^2}\left(\frac{1}{\mathsf{d}-4} - \frac{3}{4}\right)\mathrm{tr}\left(m^4 - 2m^2[\widehat{a}_1] + 2[\widehat{a}_2]\right)\sqrt{\widehat{g}}\\ &\quad+\frac{i}{64\pi^2}\mathrm{tr}\int_0^{\infty} d\widehat{s}\,\ln(4\pi i\mu^2\widehat{s})\sqrt{\widehat{g}}\frac{\partial^3}{\partial(i\widehat{s})^3}\left(e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})]\right)\end{aligned}$$

Taking the variation with respect to $\widehat{\omega}$ one finds

$$\delta_{\widehat{\omega}}\widehat{L} = -\frac{1}{768\pi^2}\int d^4x\,\mathrm{tr}\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\Big|_{\mathrm{odd}} = \int d^{\mathsf{d}}\widehat{x}\,\mathrm{tr}\sqrt{\widehat{g}}\,\widehat{\omega}\,\widehat{\mathfrak{T}}$$

where $\widehat{\mathfrak{T}} = \widehat{g}^{\mu\nu}\widehat{\Theta}_{\mu\nu} = \widehat{g}^{\mu\nu}\langle\langle\widehat{T}_{\mu\nu}\rangle\rangle$.

The chiral limit

The collapsing limit is defined by $h_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, k_{\mu\nu} \to \frac{h_{\mu\nu}}{2}$. What happens in this limit?

$$\Gamma^{(1)\lambda}_{\mu\nu} \to \frac{1}{2}\gamma^{\lambda}_{\mu\nu}, \qquad \Gamma^{(2)\lambda}_{\mu\nu} \to \frac{1}{2}\gamma^{\lambda}_{\mu\nu}$$

Moreover

$$\Omega^{(1)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu}, \qquad \Omega^{(2)ab}_{\mu} \to \frac{1}{2}\omega^{ab}_{\mu} \tag{1}$$

and

$$\mathcal{R}^{(1)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \qquad \mathcal{R}^{(2)}_{\mu\nu\lambda}{}^{\rho} \to \frac{1}{2}R_{\mu\nu\lambda}{}^{\rho}, \tag{2}$$

where $\gamma^{\lambda}_{\mu\nu}, \omega^{ab}_{\mu}, R_{\mu\nu\lambda}{}^{\rho}$ are the ordinary Christoffel, spin connection and Riemann tensor.

The chiral limit (*cont.*)

And the fermion action becomes

$$S' = \int d^4x \left[i\overline{\Psi}\gamma^a \frac{1+\gamma_5}{2}\partial_a \Psi + i\overline{\Psi}\gamma^a \hat{e}_a^m \left(\partial_m + \frac{1}{2}\omega_m\right) \frac{1-\gamma_5}{2}\Psi \right]$$

and as for the e.m. tensor, both $T^{\mu\nu}$ and $T_5^{\mu\nu}$ become

$$T'^{\mu\nu}(x) = 4\frac{\delta S'}{\delta h_{\mu\nu}(x)}$$

As a consequence the traces also collapse to the same

$$\mathcal{T}(x) \to \langle\langle T'^{\mu\nu} \rangle\rangle g_{\mu\nu} \equiv \mathcal{T}'(x), \qquad \mathcal{T}_5(x) \to \langle\langle T'^{\mu\nu} \rangle\rangle g_{\mu\nu} \equiv \mathcal{T}'(x)$$

Precisely to the Pontryagin density

$$\mathcal{T}'(x) = \frac{N}{4}\,\epsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}$$

The DWS method: the results

In conclusion the Pontryagin Weyl anomaly for left-handed Weyl fermion

$$\mathcal{T}'(x) = \frac{i}{1536\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}.$$

We can easily compute also the ABJ-like anomaly, by taking the limit the (vector) limit $g_{\mu\nu} \to g_{\mu\nu}, f_{\mu\nu} \to 0$. The result is the anomaly

$$\mathcal{T}_5(x) = \frac{i}{768\pi^2}\epsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta}.$$

which coincides with the perturbative result found above.

There are plenty of different anomalies: gauge and trace, even- and odd-parity, consistent and covariant.

- Are they all on the same footing and of the same importance?

- Why some are dangerous and other are not?

- Do they have to be cancelled?

The methods considered so far do not allow us to answer these questions!
We need the family’s index theorem

The Dirac operator and its index

Consider a gauge field theory on a *Euclidean* spacetime M. The Dirac operator

$$\not{\mathcal{D}} = \not{D} + i\not{V}$$

acts on the tensor product of a spinor bundle $S^{\pm}_{\mathbb{C}}$ with a vector bundle E corresponding to a representation ρ of the structure group G of $\mathsf{P}(X, \mathsf{G})$: $S^{\pm}_{\mathbb{C}} \otimes E$.

The relevant connection is the spin connection plus a gauge connection $V = V_\mu dx^\mu$ valued in the representation ρ of the Lie algebra of G with antihermiten generators. The Dirac operator acts on the space of sections of $S^{\pm}_{\mathbb{C}}(E) \equiv S^{\pm}_{\mathbb{C}} \otimes E$ (i.e. on the spinor fields) and maps it to itself. Accordingly it splits into

$$\not{\mathcal{D}} = \begin{pmatrix} 0 & \not{\mathcal{D}}^- \\ \not{\mathcal{D}}^+ & 0 \end{pmatrix}$$

The index of $\not{\mathcal{D}}^+$ is defined by

$$Ind\,\not{\mathcal{D}}^+ = dim(ker\,\not{\mathcal{D}}^+) - dim(ker\,\not{\mathcal{D}}^-)$$

Atiyah and Singer showed that

$$ch\left(ind(\not{\mathcal{D}}^+)\right) = \int_{\mathsf{M}} ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q})$$

Here $\mathcal{Q} \equiv \frac{\mathcal{A}}{\mathcal{G}}$ is the orbit space of connections,

The family's index theorem

The Atiyah-Singer family's index theorem says that the index of the Dirac operator for a Weyl fermion is given

$$c_1\left(ind(\not{D}^+)\right) = \int_M ch(\mathcal{V}) \cdot \hat{A}(T\mathcal{Q})\Big|_{\mathsf{d},2}$$

where $\mathcal{V}$ is the gauge bundle and $T\mathcal{Q}$ is the tangent bundle to the moduli space of the theory. ch is the Chern character

$$ch(\mathcal{V}) = r + \frac{i}{2\pi}\mathrm{tr}\, F + \frac{i^2}{2(2\pi)^2}\mathrm{tr}\, F^2 + \frac{i^3}{3!(2\pi)^3}\mathrm{tr}\, F^3 + \ldots$$

and $\widehat{A}$ is the $\widehat{A}$-genus

$$\widehat{A}(X) \;\; = \;\; 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\,\mathrm{tr}\, R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}\,\mathrm{tr}\,(R^2)^2 + \frac{1}{360}\,\mathrm{tr}\, R^4\right]$$

$c_1\left(ind(\not{D}^+)\right)$ represents an obstruction to the invertibility of $\not{D}^+$. I.e. $c_1\left(ind(\not{D}^+)\right) \neq 0$ the Weyl fermion propagator does not exist!

The Pontryagin class $\mathrm{tr}R^2$ and Chern class $\mathrm{tr}F^2$ are obstructions to the existence of the fermion propagator.

Here is the conundrum: anomalies are dangerous when they prevent the existence of the fermion propagators.

I call type O the anomalies whose family's index is nonvanishing

I call type NO all the others

The family's index of a self-adjoint operator vanishes identically.

Examples: the Dirac operator for Dirac fermions; the Maxwell operator (after gauge fixing)….

Anomaly recap

- Where do they appear: $\begin{cases} \text{in divergence of current or of e.m. tensor} \\ \text{in trace of e.m. tensor} \end{cases}$

- Local Anomalies are of two types: $\begin{cases} \text{type O} & \begin{cases} \text{prevent existence of propagators,} \\ \text{dangerous : must be canceled} \end{cases} \\ \text{type NO} & \begin{cases} \text{no obstruction for propagators,} \\ \text{need not be canceled} \end{cases} \end{cases}$

- In what theories: $\begin{cases} \text{type O} & \text{only in chiral theories} \\ \text{type NO} & \text{in any theory} \end{cases}$

- Cancellation: $\begin{cases} \text{type O} & \begin{cases} \text{A : group theoretical (unavailable for trace anomalies)} \\ \text{B : coefficient matching (unlikely for trace anomalies)} \\ \text{C : Wess}-\text{Zumino terms or Green}-\text{Schwarz mechanisms} \end{cases} \\ \text{type NO} & \begin{cases} \text{in general not required} \\ \text{for even trace anomalies} \begin{cases} \text{B : coefficient matching unlikely} \\ \text{C : with Wess}-\text{Zumino terms} \end{cases} \end{cases} \end{cases}$

Thus we have to find all the type O anomalies in the SM coupled to gravity and make sure that they vanish.

They are:

- chiral gauge anomalies (they vanish)
- gravitational anomalies (absent)
- mixed gauge-gravity anomalies
- chiral trace anomalies

Mixed gauge-gravity anomaly

The pure gravitational anomalies in the divergence of the e.m. tensor vanish identically, but there is a mixed gauge-gravity chiral anomaly! Consider a right-handed fermion ψ_R coupled both to a metric and to an Abelian gauge field V_μ.

The relevant current $j_\mu = \bar{\psi}_R \gamma_\mu \psi_R$ is classically conserved, but after quantization

$$\partial^\mu j_{R\mu} = \frac{1}{1536\pi^2}\, \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}.$$

Its integrated form is $\sim \Delta_G(\lambda, g) = \int d^4x \lambda\, \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}$, which is a diffeomorphism-invariant (trivially) consistent Abelian gauge cocycle. This cocycle can take different forms, for instnace $\Delta_G(\lambda, g)$ is equivalent to

$$\Delta_{\mathsf{d}}(\xi, g, V) = \int d^4x \sqrt{g}\, \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr}\left(\partial_\mu \Xi\, \Gamma_\nu\right) F_{\lambda\rho} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, Ξ represents the matrix $\Xi_\tau{}^\sigma = \partial_\tau \xi^\sigma$ and Γ_μ represents the matrix $\Gamma_{\mu\sigma}^\tau$. Nevertheless this anomaly vanishes in the MSM coupled to a metric:

- $\Sigma \times \Sigma \times T^{\mathrm{u}(1)}$, the trace $\mathrm{tr}\left(\Sigma^{ab}\Sigma^{cd}\right)$ is non-vanishing, but it is multiplied by the total $U(1)$ charge: $6\left(\frac{1}{6}\right) - 3\left(\frac{2}{3}\right) - 3\left(-\frac{1}{3}\right) + 2\left(-\frac{1}{2}\right) + 1 = 0$.

The addition of sterile neutrinos does not alter this conclusion.

Trace anomalies in the SM

MSM trace-gravity

- The SM multiplet, when coupled to gravity, produces an overall non-vanishing (imaginary) coefficient for the Pontryagin density in the trace anomaly.

This breakdown is naturally avoided if we add to the above MSM multiplet a right-handed sterile neutrino.

MSM trace-gauge

- We have six units of the anomaly $\sim \mathrm{tr}F^2$ with curvature $F \equiv F^{\mathfrak{su}(3)}$ and six units with opposite sign. Therefore the MSM multiplet is free of these anomalies.
- We have instead 4 units of the same anomaly with gauge field $F \equiv F^{\mathfrak{su}(2)}$ and positive sign, computed in the doublet representation of $\mathfrak{su}(2)$.
- Finally we have a $U(1)$ gauge-induced trace anomaly with vanishing total coefficient: $6\left(\frac{1}{6}\right)^2 - 3\left(\frac{2}{3}\right)^2 - 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - (-1)^2 = 0$

The addition of sterile neutrinos does not change these conclusions.
The $SU(2)$ gauge-induced odd trace anomalies do not cancel in the MSM.

Summary: second half

- A L-R symmetric model
- Axial-complex analysis
- Conformal invariance
- Trace anomalies: even parity
- WZ terms
- Renormalization and unitarity
- Connection with cosmology

A chirally symmetric model

The left-handed multiplet is

$\mathsf{G}/fields$	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2	$\frac{1}{6}$
$(u_R)^c$	$\bar{3}$	1	$-\frac{2}{3}$
$(d_R)^c$	$\bar{3}$	1	$\frac{1}{3}$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2	$-\frac{1}{2}$
$(e_R)^c$	1	1	1
$(\nu_R)^c$	1	1	0

This multiplet couples to a left gravitational metric and connection, and to the $SU(3)_L \times SU(2) \times U(1)_L$ gauge fields.

A chirally symmetric model

The right-handed multiplet is

$\mathsf{G}/fields$	$SU(3)$	$SU(2)$	$U(1)$
$\begin{pmatrix} u' \\ d' \end{pmatrix}_R$	3	2	$\frac{1}{6}$
$(u'_L)^c$	$\bar{3}$	1	$-\frac{2}{3}$
$(d'_L)^c$	$\bar{3}$	1	$\frac{1}{3}$
$\begin{pmatrix} \nu'_e \\ e' \end{pmatrix}_R$	1	2	$-\frac{1}{2}$
$(e'_L)^c$	1	1	1
$(\nu'_L)^c$	1	1	0

coupled to a right gravitational metric and connection. This multiplet couples to the $SU(3)_R \times SU(2) \times U(1)_R$ gauge fields.

All O-type anomalies cancel.

Axial-complex analysis

Axial-complex numbers are defined by

$$\hat{a} = a_1 + \gamma_5 a_2$$

where a_1 and a_2 are real numbers. Arithmetic is defined in the obvious way. Introducing the chiral projectors $P_\pm = \frac{1 \pm \gamma_5}{2}$, we can also write

$$\hat{a} = a_+ P_+ + a_- P_-, \qquad a_\pm = a_1 \pm a_2$$

I consider functions $\hat{f}(\hat{x})$ of the axial-complex variable

$$\widehat{x} = x_1 + \gamma_5 x_2$$

defined via their Taylor expansions.

Derivatives are defined in the obvious way:

$$\frac{\partial}{\partial \hat{x}^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x_1^\mu} + \gamma_5 \frac{\partial}{\partial x_2^\mu} \right)$$

For axial-analytic functions

$$\frac{d}{d\hat{x}} = \frac{\partial}{\partial x_1} \equiv \frac{\partial}{\partial \hat{x}},$$

Integral of $\widehat{f}(\hat{x})$

$$\int d\hat{x}\, \widehat{f}(\hat{x})$$

is the rapidly decreasing primitive $\widehat{g}(\hat{x})$ of $\widehat{f}(\hat{x})$. And so on....

MAT (metric-axial-tensor) gravity

Introduce

$$\widehat{g}_{\mu\nu} = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$$

with

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad f_{\mu\nu} = k_{\mu\nu}$$

together with generalized vierbeins

$$\widehat{e}_\mu^a = e_\mu^a + \gamma_5 c_\mu^a, \qquad \widehat{e}_a^\mu = e_a^\mu + \gamma_5 c_a^\mu$$

One can define Christoffel symbols

$$\widehat{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^{(1)\lambda} + \gamma_5 \Gamma_{\mu\nu}^{(2)\lambda}$$

and Riemann tensor

$$\widehat{\mathcal{R}}_{\mu\nu\lambda}{}^\rho = \mathcal{R}^{(1)}_{\mu\nu\lambda}{}^\rho + \gamma_5 \mathcal{R}^{(2)}_{\mu\nu\lambda}{}^\rho$$

MAT (cont.)

–

Generalized Weyl transformation

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2(\omega+\gamma_5\eta)}\widehat{g}_{\mu\nu}.$$

The volume density is

$$\begin{aligned}\sqrt{\widehat{g}} &= \sqrt{\det(\widehat{g})} \\ &= \frac{1}{2}\left(\sqrt{\det(g+f)}+\sqrt{\det(g-f)}\right)+\frac{\gamma_5}{2}\left(\sqrt{\det(g+f)}-\sqrt{\det(g-f)}\right)\end{aligned}$$

and transforms as

$$\sqrt{\widehat{g}} \rightarrow e^{4(\omega+\gamma_5\eta)}\sqrt{\widehat{g}}$$

and

$$\frac{1}{\sqrt{\widehat{g}}}\partial_\nu\sqrt{\widehat{g}} = \frac{1}{2}\widehat{g}^{\mu\lambda}\partial_\nu\widehat{g}_{\mu\lambda} = \widehat{\Gamma}^{\mu}_{\mu\nu}$$

A L-R symmetric model: the fermion action

$$\begin{aligned}\widehat{S}_f &= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a\widehat{e}_a^\mu\left(\mathcal{D}_\mu+\frac{1}{2}\widehat{\Omega}_\mu\right)\psi\right)(\widehat{x})\\ &= \int d^4\widehat{x}\left(i\overline{\psi}\sqrt{\overline{\widehat{g}}}\gamma^a(\tilde{e}_a^\mu+\gamma_5\tilde{c}_a^\mu)\left(\mathcal{D}_\mu+\frac{1}{2}\left(\Omega_\mu^{(1)}+\gamma_5\Omega_\mu^{(2)}\right)\right)\psi\right)(\widehat{x})\end{aligned}$$

It splits as $\widehat{S}_f = S_f^{(+)} + S_f^{(-)}$ where

$$S_f^{(\pm)} = \int d^4\widehat{x}\left(\sqrt{g_\pm}\, i\overline{\psi}P_\mp\gamma^a e_a^{(\pm)\mu}P_\pm\left(\mathcal{D}_\mu^{(\pm)}+\frac{1}{2}\Omega_\mu^{(\pm)}P_\pm\right)\psi\right)(\widehat{x})$$

More in detail,

$$S_f^{(+)} \equiv S_{fR} = \int d^4\widehat{x}\left(\sqrt{g_+}\, i\overline{\psi'}_R\gamma^a e_a^{(+)\mu}\left(\mathcal{D}_\mu^{(+)}+\frac{1}{2}\Omega_\mu^{(+)}\right)\psi'_R\right)(\widehat{x})$$

where ψ'_R represents the right-handed multiplet, and

$$\mathcal{D}_\mu^{(+)} = \partial_\mu + \mathsf{g}_X^+ X_\mu^{(+)} + \mathsf{g}_W W_\mu + \mathsf{g}_B^+ B_\mu^{(+)}$$

A L-R symmetric model: the gauge action

The action for the $SU(2)$ gauge fields is

$$\widehat{S}_g = -\frac{1}{8}\int d^4\widehat{x}\,\mathrm{Tr}\left(\frac{1}{\widehat{\mathrm{g}}^2}\sqrt{\widehat{g}}\,\widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}\right), \qquad F_{\mu\nu} = F^a_{\mu\nu}T^a$$

where Tr denotes the trace over all matrices, including γ_5. The coupling splits

$$\frac{1}{\widehat{\mathrm{g}}^2} = \frac{1}{\mathrm{g}_+^2}P_+ + \frac{1}{\mathrm{g}_-^2}P_-$$

Thus the action splits as $S_g = S_g^{(+)} + S_g^{(-)}$

$$S_g^{(\pm)} = -\frac{1}{4\mathrm{g}_\pm^2}\int d^4\widehat{x}\,\sqrt{g_\pm}\,\mathrm{tr}\left(g_\pm^{\mu\mu'}g_\pm^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'}\right)$$

A L-R symmetric model: the gauge action

The action for the $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$ gauge fields is

$$\widehat{S}_{aeg} = -\frac{1}{8}\int d^4\widehat{x}\,\mathrm{Tr}\left(\frac{1}{\widehat{\mathrm{g}}^2}\sqrt{\widehat{g}}\,\widehat{g}^{\mu\mu'}\widehat{g}^{\nu\nu'}\widehat{F}_{\mu\nu}\widehat{F}_{\mu'\nu'}\right), \qquad \widehat{F}_{\mu\nu} = \widehat{F}^a_{\mu\nu}T^a$$

where $\widehat{F} = \widehat{d}\widehat{V} + \frac{1}{2}[\widehat{V}, \widehat{V}]$, and $\widehat{V}_\mu = V_\mu + \gamma_5 A_\mu$. T^a are the generators of $SU(3) \times U(1)$. The action splits as

$$S^{(\pm)}_{aeg} = -\frac{1}{4\mathrm{g}^2_\pm}\int d^4\widehat{x}\,\sqrt{g_\pm}\,\mathrm{tr}\left(g_\pm^{\mu\mu'}g_\pm^{\nu\nu'}F^{(\pm)}_{\mu\nu}F^{(\pm)}_{\mu'\nu'}\right)$$

where $F^{(\pm)}_{\mu\nu} = \hat{d}V^{(\pm)} + \frac{1}{2}[V^{(\pm)}, V^{(\pm)}]$ and $V^{(\pm)}_\mu = V_\mu \pm A_\mu$, and $\widehat{\mathrm{g}} = \mathrm{g} + \gamma_5 \mathrm{h} = \mathrm{g}_+ P_+ + \mathrm{g}_- P_-$.

AE scalar doublet action

$$\widehat{S}_{aed} = \int d^4\widehat{x}\,\mathrm{Tr}\left[\sqrt{\widehat{g}}\left(\widehat{g}^{\mu\nu}\mathcal{D}_\mu\widehat{H}^\dagger\mathcal{D}_\nu\widehat{H} - \widehat{M}^2\widehat{H}^\dagger\widehat{H} - \frac{\widehat{\lambda}}{4}\left(\widehat{H}^\dagger\widehat{H}\right)^2\right)\right]$$

where $\mathcal{D}_\mu = \partial_\mu - i\mathsf{g}W_\mu$, and W_μ is an $SU(2)$ gauge field (notice that W_μ is not AE). We can decompose $\widehat{H}$ as $\widehat{H} = H_+P_+ + H_-P_-$ where $H_\pm = \mathsf{h} \pm \mathsf{k}$, as well as $\widehat{M}^2 = M_+^2P_+ + M_-^2P_-$ and $\widehat{\lambda} = \lambda_+P_+ + \lambda_-P_-$. Then $\widehat{S}_{aed}$ splits as $\widehat{S}_{aed} = S_{aed}^{(+)} + S_{aed}^{(-)}$ with

$$S_{aed}^{(\pm)} = \int d^4\widehat{x}\,\sqrt{g_\pm}\left[g_\pm^{\mu\nu}\mathcal{D}_\mu H_\pm^\dagger\mathcal{D}_\nu H_\pm - M_\pm^2 H_\pm^\dagger H_\pm - \frac{\lambda_\pm}{4}\left(H_\pm^\dagger H_\pm\right)^2\right]$$

$S_{aes}^{(+)} + S_{aes}^{(-)}$ and $S_{aed}^{(+)} + S_{aed}^{(-)}$ are parity-invariant.

AE real scalar action

Action for an axially-extended (AE) real scalar field $\widehat{\Phi} = \phi + \gamma_5 \pi$, where ϕ is an ordinary real and π a psudoreal scalar field

$$\widehat{S}_{aes} = \int d^4\widehat{x}\,\mathrm{Tr}\left[\sqrt{\widehat{g}}\left(\widehat{g}^{\mu\nu}\partial_\mu\widehat{\Phi}\partial_\nu\widehat{\Phi} - \widehat{m}^2\widehat{\Phi}^2 - \frac{\widehat{\lambda}}{4}\widehat{\Phi}^4\right)\right]$$

We decompose $\widehat{\Phi}$ as $\widehat{\Phi} = \Phi_+ P_+ + \Phi_- P_-$ where $\Phi_\pm = \phi \pm \pi$, as well as $\widehat{m}^2 = m_+^2 P_+ + m_-^2 P_-$ and $\widehat{\lambda} = \lambda_+ P_+ + \lambda_- P_-$. Then we have the splitting $\widehat{S}_{aes} = S_{aes}^{(+)} + S_{aes}^{(-)}$, with

$$S_{aes}^{(\pm)} = \int d^4\widehat{x}\,\sqrt{g_\pm}\left(g_\pm^{\mu\nu}\partial_\mu\Phi_\pm\partial_\nu\Phi_\pm - m_\pm^2\Phi_\pm^2 - \frac{\lambda_\pm}{4}\Phi_\pm^4\right)$$

EH-like action

The Einstein-Hilbert-like action in this context takes the form

$$\widehat{S}_{EH} = \frac{1}{4}\int d^4\widehat{x}\,\mathrm{Tr}\left(\frac{1}{\widehat{\kappa}}\sqrt{\widehat{g}}\,\widehat{R}\right),$$

where $\widehat{R}$ is the Ricci scalar

$$\widehat{R} = \widehat{g}^{\mu\lambda}\widehat{R}_{\mu\lambda} = \widehat{g}^{\mu\lambda}\left(\widehat{R}^{(1)}_{\mu\lambda} + \gamma_5 R^{(2)}_{\mu\lambda}\right) = R^{(+)}P_+ + R^{(-)}P_-$$

where $R^{(\pm)} = g_{\pm}^{\mu\nu}R^{(\pm)}_{\mu\lambda}$. Moreover $\frac{1}{\widehat{\kappa}} = \frac{1}{\kappa_+}P_+ + \frac{1}{\kappa_-}P_-$.

The action splits as $\widehat{S}_{EH} = S^{(+)}_{EH} + S^{(-)}_{EH}$ with

$$S^{(\pm)}_{EH} = \frac{1}{2\kappa_\pm}\int d^4\widehat{x}\,\sqrt{g_\pm}R^{(\pm)}$$

Remark. The previous formalism works even if there is only one metric, that is $g_+ = g_-$.

The $\mathcal{T}$ theory

We define left and right Yukawa couplings

$$S_{YdL} = \frac{y^{-}_{H_d}}{2} \int d^4\widehat{x}\, \sqrt{g_-}\, (\overline{\psi_{dL}}\, H_{d-}\chi_{sR}) + h.c.$$

where ψ_{dL} is a left-handed $SU(2)$ doublet, $\widehat{H}_{d-}$ is also an $SU(2)$ doublet, while χ_{sR} is a right-handed singlet, all of them belonging to the left-handed multiplet. Similarly,

$$S_{YdR} = \frac{y^{+}_{H_d}}{2} \int d^4\widehat{x}\, \sqrt{g_+}\, (\overline{\chi'_{dR}}\, H_{d+}\psi'_{sL}) + h.c.$$

The total $\mathcal{T}$ action is

$$\widehat{S}_{ch-sym} = \widehat{S}_f + \widehat{S}_g + \widehat{S}_{aeg} + (S_Y + S_{Yd} + h.c.) + \widehat{S}_{aes} + \widehat{S}_{aed} + \widehat{S}_{EH}$$

It splits as

$$\mathcal{T} = \mathcal{T}_L \cup \mathcal{T}_R$$

Standard model and Weyl invariance

In the above theory the fermion action, the gauge actions and the Yukawa couplings are conformal invariant.

The scalar actions and the EH action are not conformal invariant.

It is possible to render the whole action conformal invariant by introducing a scalar field, φ, the dilaton.

Weyl geometry

In an ordinary gravitational background geometry the Weyl transformation is given by

$$g_{\mu\nu} \to e^{2\omega} g_{\mu\nu}$$

The Christoffel symbols transform as

$$\Gamma^{\lambda}_{\mu\nu} \to \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu}\,\partial_\nu\omega + \delta^{\lambda}_{\nu}\,\partial_\mu\omega - g_{\mu\nu}g^{\lambda\rho}\partial_\rho\omega$$

We can construct Weyl-invariant Christoffel symbols as follows

$$\widetilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \left(\delta^{\lambda}_{\mu}\,\partial_\nu\varphi + \delta^{\lambda}_{\nu}\,\partial_\mu\varphi - g_{\mu\nu}g^{\lambda\rho}\partial_\rho\varphi\right)$$

where the field φ (a dilaton) under Weyl transforms as

$$\varphi \to \varphi + \omega$$

We can construct Weyl invariant Ricci tensor

$$\widetilde{R}_{\mu\nu} = R_{\mu\nu} + 3D_\nu S_\mu - D_\mu S_\nu + g_{\mu\nu} D \cdot S + 2S_\mu S_\nu - 2g_{\mu\nu} S \cdot S$$

where $S_\mu = \partial_\mu\varphi$. Important: φ is dimensionless !

Weyl geometrization

One possible Weyl invariant action for gravity and a scalar field is

$$S^{(c)}_{EH+s} = \frac{1}{2\kappa}\int d^4x\sqrt{g}\,\left(e^{-2\varphi}+\zeta\Phi^2\right)\left(\widetilde{R}+\mathfrak{c}\,e^{-2\varphi}\right) + \frac{1}{2}\int d^4x\,\sqrt{g}\,\left[g^{\mu\nu}\mathsf{D}_\mu\Phi\mathsf{D}_\nu\Phi - m^2e^{-2\varphi}\Phi^2 - \frac{\lambda}{4}\Phi^4\right]$$

where

$$\widetilde{R} = R + 6\,\left(D\cdot\partial\varphi - \partial\varphi\cdot\partial\varphi\right)$$

and

$$\mathsf{D}_\mu\Phi = \left(\partial_\mu + \partial_\mu\varphi\right)\Phi$$

Now this has to be done for $\mathcal{T}$ and transform it into $\mathcal{TW}$.

Embedding $\mathcal{T}$ in Weyl geometry

Action terms that do not need Weyl embedding

$$\widehat{S}_f^{(c)} = \widehat{S}_f, \qquad \widehat{S}_g^{(c)} = \widehat{S}_g, \qquad \widehat{S}_{aeg}^{(c)} = \widehat{S}_{aeg}, \qquad S_Y{}^{(c)} = S_Y, \qquad S_{Yd}^{(c)} = S_{Yd}$$

For the remaining terms a very general Weyl invariant form is

$$\begin{aligned}
\widehat{S}^{(c)}_{EH+aes+aed+C} \;=\;& \frac{1}{2}\int d^4\widehat{x}\,\mathrm{tr}\left[\frac{1}{\widehat{\kappa}}\sqrt{\widehat{g}}\left(e^{-2\widehat{\varphi}} + \widehat{\zeta}_h \widehat{H}^\dagger \widehat{H} + \widehat{\zeta}\widehat{\Phi}^2\right)\left(\widetilde{\widehat{R}} + \widehat{\mathfrak{c}}\, e^{-2\widehat{\varphi}}\right)\right] \\
&+ \int d^4\widehat{x}\,\mathrm{tr}\left\{\sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu}\left(\widehat{\mathsf{D}}_\mu \widehat{H}\right)^\dagger\left(\widehat{\mathsf{D}}_\nu \widehat{H}\right) + \widehat{M}^2 \widehat{H}^\dagger \widehat{H} - \frac{\widehat{\lambda}_h}{4}\left(\widehat{H}^\dagger \widehat{H}\right)^2\right]\right\} \\
&+ \frac{1}{2}\int d^4\widehat{x}\,\mathrm{tr}\left\{\sqrt{\widehat{g}}\left[\widehat{g}^{\mu\nu}\widehat{\mathsf{D}}_\mu \widehat{\Phi}\widehat{\mathsf{D}}_\nu \widehat{\Phi} + \widehat{m}^2 e^{-2\widehat{\varphi}}\widehat{\Phi}^2 - \frac{\widehat{\lambda}}{4}\widehat{\Phi}^4\right]\right\} \\
&- \int d^4x\,\mathrm{tr}\left[\frac{1}{\widehat{\eta}}\sqrt{\widehat{g}}\,\widehat{C}_{\mu\nu\lambda\rho}\widehat{C}^{\mu\nu\lambda\rho}\right],
\end{aligned}$$

where $\widehat{C}_{\mu\nu\lambda\rho}$ is the Weyl tensor,

$$\widetilde{\widehat{R}} = \widehat{R} + 6\,\widehat{g}^{\mu\nu}\left(\widehat{D}_\mu \widehat{\partial}_\nu \widehat{\varphi} - \widehat{\partial}_\mu \widehat{\varphi}\widehat{\partial}_\nu \widehat{\varphi}\right)$$

and $\widehat{\mathsf{D}}_\mu = \mathsf{D}^+_\mu P_+ + \mathsf{D}^-_\mu P_-$, $\mathsf{D}^\pm_\mu = \partial_\mu + \partial_\mu \varphi^{(\pm)} - i\mathsf{g}W_\mu$.

Conformal invariance can be implemented in a classical field theory by simply adding a scalar field.

At quantum level we meet the issue of conformal (trace) anomalies. This raises two problems:

- Finding all of them
- Cancelling them

- We know that odd-parity anomalies do cancel
- Even parity anomalies are not dangerous, but if we want to preserve Weyl invariance we must cancel them.

Possible trace anomalies

The possible non-trivial cocycles of δ_ω with vanishing diffeomorphism partner in 4d are well-known, they take the form

$$\Delta[g,\omega] = \int d^4x \sqrt{g}\,\omega\, T[g], \qquad \delta_\omega \Delta[g,\omega] = 0$$

where the density $T[g](x)$ can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,$$

the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}\right).$$

Other possible cocycles have densities

$$T_e[V] = F_{\mu\nu}F^{\mu\nu}, \qquad T_o[V] = \varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}.$$

Other even trace anomalies

The action of a scalar field is not automatically conformal invariant, but it can be made so. The improved energy momentum tensor for a real scalar is

$$T^{(s)}_{\mu\nu} = \partial_\mu \Phi \, \partial_\nu \Phi - \frac{1}{2}\eta_{\mu\nu} \, \partial_\lambda \Phi \, \partial^\lambda \Phi + \frac{1}{6} \left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) \Phi^2$$

which is conserved and traceless on-shell.

For instance, in a quantum theory of a Dirac fermion interacting with Φ, a non-vanishing trace pops up

$$\mathcal{A}^{(f,s)}_\omega = \frac{1}{48\pi^2} \int d^4x \, \omega \left(8\, \Phi \, \Box \Phi - \partial_\mu \Phi \, \partial^\mu \Phi - 9R\, \Phi^2\right)$$

This comes from the triangle diagram with three fermion propagators and two exiting scalars via the interaction $\bar{\psi}\Phi\psi$.

In the $\mathcal{T}$ theory there are several anomalies like this.

How do we cancel even trace anomalies?

- The first mechanism (anomaly polynomial vanishes identically) is not available

- The second mechanism (vanishing overlap of various species) is extremely unlikely

- We are left with the WZ terms.

WZ terms

Any trace anomaly can be written in the form

$$\mathcal{A}_\omega[g,f] = \int d^{\mathsf{d}}x\,\sqrt{g}\,\omega\,F[g,f]$$

where $g = \{g_{\mu\nu}\}$ is the metric, ω is the Weyl transformation parameter $\delta_\omega g_{\mu\nu} = 2\omega\, g_{\mu\nu}$, f denotes any other field and F is a local function of g and f

The corresponding WZ term is defined as follows

$$\mathcal{W}_{WZ}[\sigma,g,f] = \int_0^1 dt \int d^{\mathsf{d}}x\,\sqrt{g(t)}F[g(t),f(t)]\,\sigma$$

in terms of the dimensionless field σ, with $\delta_\omega\sigma = -\omega$. Moreover $g_{\mu\nu}(t) = e^{2\sigma t}g_{\mu\nu}$ and $f(t) = e^{-y\,t\,\sigma}f$,, and $y=0$ a gauge field, $y = \frac{\mathsf{d}-2}{2}$ for a scalar field.

We have

$$\delta_\omega\mathcal{W}_{WZ}[\sigma,g,f] = -\mathcal{A}_\omega[g,f]$$

For instance the WZ term for the anomaly with density $\sim F_{\mu\nu}F^{\mu\nu}$ takes the simple form

$$\mathcal{W}_{WZ}[\sigma,g,V] \sim \int d^4x\sqrt{g}\,\sigma\,F_{\mu\nu}F^{\mu\nu}$$

Therefore in any classically conformal invariant theory, where Weyl invariance is broken at one-loop by trace anomalies, we can restore invariance by adding suitable WZ terms.

But for our TW theory there may be problems with unitarity: the spectrum very likely contains ghost particles (negative norm states)

Pottel-Sibold (2023-2024), Oda-Saake (2020), Oda (2022-2024)

I personally interpret this as a signal that the T theory is UV incomplete, i.e. an effective field theory

Evolution of $\mathcal{T} = \mathcal{T}_{\mathcal{R}} \cup \mathcal{T}_{\mathcal{L}}$... *basic idea*

We first simplify the model by setting $H_+ = H_-$, then we fix the Weyl symmetry in the R sector by choosing $\varphi_+ = 0$. This operation produces masses only for H and Φ_+,

$$M_h^2 = M_\pm^2 + \frac{\mathfrak{c}_\pm \zeta_{h\pm}}{\kappa_\pm} \qquad \text{and} \qquad M_{\Phi+}^2 = m_+^2 + \frac{\mathfrak{c}_+ \zeta_+}{\kappa_+},$$

respectively. Then we postulate an early breakdown of the L-R symmetry by setting $\Phi_+ = v_+ + \rho_+$, where ρ_+ is a real field and $v_+ = \sqrt{\frac{2M_+^2}{\lambda_+}}$, a real number, denotes the minimum of the potential, we have

$$D_\mu \Phi_+ D_\mu \Phi_+ \;\; = \;\; \partial_\mu (v_+ + \rho_+) \partial^\mu (v_+ + \rho_+)$$

The quadratic terms in $\frac{1}{2} D_\mu \Phi D_\mu \Phi - \mathsf{V}$ (V is the potential) are

$$\frac{1}{2} \partial_\mu \rho_+ \partial^\mu \rho_+ - \frac{1}{2} \lambda v_+^2 \rho_+^2$$

The result is a massive scalar ρ_+.

Finally, the Yukawa couplings $\sqrt{g_+}\, \frac{y_+}{2}\, \overline{\psi}'_R \Phi_+ (\psi'_R)^c + h.c.$, when $\langle \Phi_+ \rangle = v_+$, produces a mass term proportional to v_+. Here ψ'_R denotes the right multiplet. Since all these masses, as well as the mass of ρ_+, are proportional to v_+, they can be made arbitrarily large.

Evolution of $\mathcal{T} = \mathcal{T}_\mathcal{R} \cup \mathcal{T}_\mathcal{L}$... *cont.*

After this breaking (supposed to be at very high energy) the two halves evolve almost independently, except for the common weak interaction. The symmetry $SU(3)_L \times SU(3)_R \times SU(2) \times U(1)_R \times U(1)_R$ remains unbroken.

$\mathcal{T}_R$ is a theory of very heavy quarks and leptons, for which nucleation is not possible. They are subject to the right gravity and electromagnetism, and interact weakly with the left sector.

$\mathcal{T}_L$ evolves as a conformal invariant theory until we fix the gauge $\varphi_- = 0$. When the energy is low enough there is room for a breakdown of the electroweak symmetry $SU(2) \times U(1)_L$, that generate masses for the three $SU(2)$ gauge bosons and for H.

The masses of the gauge bosons and the Higgs field are of course the same on both sides $M_W^2 = \mathrm{g}_+^2 v_+^2 = \mathrm{g}_-^2 v_-^2$ ($\mathrm{g}_\pm$ are the $SU(2)$ gauge couplings in the two sides) and $M_h^2 = M_\pm^2 + \frac{\varsigma_\pm \zeta_{h\pm}}{\kappa_\pm}$. In this process both $SU(3)$ and the $U(1)_R$ symmetries remain unbroken.

It goes without saying that, if the above makes any sense, the R model lends itself as a candidate for the black matter sector and the whole $\mathcal{T}$ theory may harbor the ambition to unveil the mystery of baryon asymmetry.

THANKS