On CP-violation and the interdependence of quark masses

A. Kleppe, SACT, Oslo

The Standard Model does not predict families.

It just predicts fermions.

But the families are there, with their enigmatic mass hierarchies.

To begin with, experimentalists realized that there were heavier siblings of the up- and down-quarks.

It also became clear that there were more than one type of neutrino:

$\Rightarrow$ the fundamental fermions (d, u, e, ν_e) have the heavier siblings (s, c, μ, ν_μ).

And with the arrival of the third family (b, t, τ, ν_τ), there was also CP-violation.

The mass Lagrangian reads:

$$\mathcal{L}_{mass} = \bar{f}\,Mf + \bar{f}'M'f' = \bar{\psi}D\psi + \bar{\psi}'D'\psi',$$

where f, f' are quark flavour states with charge 2/3 and -1/3, respectively, and ψ, ψ' are the corresponding mass states whose masses are the eigenvalues of the matrix M.

Mass states are states with well defined mass, flavour states are states in the weak basis that do not have well defined masses. Mass states are also referred to as "physical particles", since these are the particles that take part in electromagnetic and strong interactions.

But neutrinos, that only interact weakly, never appear as anything but flavour states, so also flavour states deserve to be called "physical".

Many attempts have been made to find a mass matrix of a form that would shed some light on the mystery of the fermion masses.

An ansatz is an educated guess based on some assumptions.

The guess concerns what kind of structure the mass matrix of a given charge sector has in the weak basis.

Our assumption is that the fermionic mass matrices have an underlying democratic texture, and our zeroth ansatz is the democratic matrix:

$$M_0 = \frac{T}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

In the weak basis the democratic matrix M_0 is totally flavour symmetric, in the sense that the (weak) states f_j of a given charge are indistinguishible (``absolute democracy'').

This is contrary to experimental data, but its eigenvalues (0,0,T) reflect the actual particle spectra with two very light and one very heavy state.
It's therefore reasonable to assume that actual mass matrices that represent physical particles, have some kind of modified democratic texture.

Modified mass matrices

One natural first step is to modify the diagonal matrix elements,

$$M = \begin{pmatrix} L & K & K \\ K & L & K \\ K & K & L \end{pmatrix},$$

which gives a matrix that indeed has three non-zero mass eigenstates, $(L-K, L-K, L+2K)$, but two of the masses are degenerate.

In order to get three different mass eigenstates, more modifications are needed, e.g.

$$M = \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix}$$

We now have three different mass eigenstates

This corresponds to three families.

Both mixing and CP-violation are constraints on the structure of the mass matrices

The usual "mathematical reason" for CP-violation is that the 3×3 weak mixing matrix has a phase that cannot be reformulated away.

But in the 1980-ies Cecilia Jarlskog discovered that a signum of CP-violation is that the commutator of the mass matrices is nonzero:

$$det[M_u, M_d] \neq 0$$

She formulated a direct measure of weak CP-violation, the Jarlskog invariant:

$$J_{CP} = -idet[M_u, M_d]/2P_u P_d = Im(V_{us} V_{cb} V_{kj}^* V_{il}^*)$$

where M_u and M_d are the mass matrices for the up- and down-sectors,

V_{xy} are matrix elements of the weak mixing (Cabbibo-Kobayashi-Maskawa) matrix,

and $P_u = (m_u - m_c)(m_c - m_t)(m_t - m_u)$ and $P_d = (m_d - m_s)(m_s - m_b)(m_b - m_d)$

To calculate the Jarlskog invariant J_{CP}, we can use the Wolfenstein parametrization of the weak mixing matrix,

$$V_{Wolf} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

where $\lambda = 0.2245$, $A = 0.836$, $\rho = 0.122$, $\eta = 0.355$.

Inserting the mixing matrix elements for these values in the expression $J_{CP} = Im(V_{us}V_{cb}V^*_{kj}V^*_{il})$, we get

$$J_{CP} = 3.096 \times 10^{-5}$$

in agreement with the value given by the Particle Data group, $J_{CP} = (3.18 \pm 0.15) \times 10^{-5}$.

In order to get physically realistic mass matrices, we must take into account constraints from data,

like from the observed CP-violation.

$\Rightarrow$ the commutator $[M_u, M_d] \neq 0$

And at least one of the mass matrices must have complex entries.

As a demonstration we consider two simple mass matrices

$$M = \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} L & X & Y \\ X & L & Y \\ Y & Y & L \end{pmatrix}$$

where the matrix elements K, A, B, L, X, Y all have dimension mass. We immediately see that their commutator

$$MN - NM = \begin{pmatrix} 0 & 0 & AY - BX \\ 0 & 0 & AY - BX \\ BX - AY & BX - AY & 0 \end{pmatrix}$$

has determinant zero, so they clearly do not fulfil the requirements for quark mass matrices corresponding to physical particles.

Complex mass matrices

In order to obtain more realistic mass matrices, we introduce complexification,

$$M = \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} L & X+iG & Y+iF \\ X-iG & L & Y+iF \\ Y-iF & Y-iF & L \end{pmatrix}$$

and now the determinant for the commutator is non-vanishing,

$$det[M,N] = 8iBFG^2(A^2 - B^2),$$

thus

$$J_{CP} = -idet[M,N]/2P_uP_d = 4BFG^2(A^2 - B^2)/P_uP_d$$

In order to diminish the number of parameters, we try different versions of complexification, ending up with this simple choice,

$$M_u = \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix} \quad \text{and} \quad M_d = \begin{pmatrix} L & Y & Y - iF \\ Y & L & Y \\ Y + iF & Y & L \end{pmatrix}$$

It has with six parameters K, A, B, L, Y, F, and has a non-vanishing determinant for the commutator:

$$det(M_u M_d - M_d M_u) = 2iBF^3(A^2 - B^2),$$

The matrix M_u for the up-quarks, which only has real matrix elements, is flavour symmetric in the first two families. This can be seen by spelling out the mass Lagrangian in flavour space:

$$L_{mass} = \bar{\phi} M_u \phi =$$

$$= (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} =$$

$$= K(\bar{\phi}_1\phi_1 + \bar{\phi}_2\phi_2 + \bar{\phi}_3\phi_3) + A(\bar{\phi}_1\phi_2 + \bar{\phi}_2\phi_1) + B[(\bar{\phi}_1 + \bar{\phi}_2)\phi_3 + \bar{\phi}_3(\phi_1 + \phi_2)],$$

where ϕ_j are flavour states with charge 2/3. The flavour symmetry in the first two families means that in the mass Lagrangian, the states ϕ_1 and ϕ_2 are exchangeable.

The corresponding flavour symmetry in down-quark mass matrix M_d, is broken by the presence of complex matrix elements.

The choice of letting one of the mass matrices being completely real facilitates the calculation, since M_u has explicit, easily calculated eigenvalues

$$m_1 = K - A$$

$$m_2 = (2K + A - \sqrt{8B^2 + A^2})/2$$

$$m_3 = (2K + A + \sqrt{8B^2 + A^2})/2,$$

and matrix elements

$$K = trace(M_u)/3$$

$$A = K - m_u$$

$$B = \sqrt{(K^2 + KA - m_c m_t)/2},$$

Using the following mass values for the up- and down-sectors

$$m_u(M_Z) = 1.24\,MeV, \quad m_c(M_Z) = 624\,MeV, \quad m_t(M_Z) = 171550\,MeV$$
$$m_d(M_Z) = 2.69\,MeV, \quad m_s(M_Z) = 53.8\,MeV, \quad m_b(M_Z) = 2850\,MeV$$

we get numerical values for the matrix elements in the up-sector

$$K = 57391.75,\ A = 57390.5,\ B = 56923.2,$$

and the mass matrix for the up-quarks reads

$$M_u(M_Z) = \begin{pmatrix} 57391.75 & 57390.5 & 56923.22 \\ 57390.5 & 57391.75 & 56923.22 \\ 56923.22 & 56923.22 & 57391.75 \end{pmatrix} = 56923.22 \begin{pmatrix} 1.00823 & 1.00820 & 1 \\ 1.00820 & 1.00823 & 1 \\ 1 & 1 & 1.00823 \end{pmatrix}$$

which is a nearly democratic matrix.

Now we can numerically calculate the determinant for the commutator:

$$det(M_u M_d - M_d M_u) = 2iBF^3(A^2 - B^2),$$

which we insert into J_{CP} to calculate the numerical value of F,

$$J_{CP} = -idet[M_u, M_d]/2P_u P_d = BF^3(A^2 - B^2)/P_u P_d = 0.00003096,$$

$$\Rightarrow F^3 = 0.00003096 \times P_u P_d/(B(A^2 - B^2)),$$

$$\Rightarrow F = 42.295.$$

The matrix elements of M_d

Now we know F, but we also want to calculate the other matrix elements of M_d.

$$M_d = \begin{pmatrix} L & Y & Y - iF \\ Y & L & Y \\ Y + iF & Y & L \end{pmatrix}$$

To do the calculation, we use matrix invariants.

The cleanest way to express the matrix invariants of a 3 $\times$ 3 matrix M, is in terms of traces:

1. $trace(M) = m_1 + m_2 + m_3$

2. $C_2(M) = m_1 m_2 + m_1 m_3 + m_3 m_2 = \frac{1}{2}[(trace(M))^2 - trace(M^2)]$

3. $det(M) = m_1 m_2 m_3 = \frac{1}{6}[trace(M))^3 + 2trace(M^3) - 3trace(M)trace(M^2)]$

In terms of the matrix elements, these invariants are

1. $trace(M_d) = 3L$
2. $C_2(M_d) = 3L^2 - 3Y^2 - F^2$
3. $det(M_d) = L^3 + 2Y^3 - L(3Y^2 + F^2)$

From relation 2., we see that $3Y^2 + F^2 = 3L^2 - C_2(M_d)$, thus

$$det(M_d) = L^3 + 2Y^3 - L(3L^2 - C_2(M_d)) \Rightarrow 2Y^3 = det(M_d) + 2L^3 - LC_2(M_d)$$

thus

$$Y = \left[\frac{det(M_d) + 2L^3 - LC_2(M_d)}{2}\right]^{1/3}$$

Inserting the numerical values of the invariants, we get $Y = 940.4$.

The resulting numerical mass matrices read

$$M_u(M_Z) = 56923.22 \begin{pmatrix} 1.00823 & 1.00820 & 1 \\ 1.00820 & 1.00823 & 1 \\ 1 & 1 & 1.00823 \end{pmatrix}$$

and

$$M_d(M_Z) = 940.35 \begin{pmatrix} 1.03 & 1 & 1 - i\,0.045 \\ 1 & 1.03 & 1 \\ 1 + i\,0.045 & 1 & 1.03 \end{pmatrix}$$

which both have a democratic texture and satisfy the requirements for CP-violation.

As a check, we insert the determinant of their commutator in the expression for J_{CP}, and get

$J_{CP} = 460273644675702800/2(m_u - m_c)(m_c - m_t)(m_t - m_u)(m_d - m_s).. = 0.000030971.$

The eigenvalues of M_d

The eigenvalues of the up-quark matrix M_u were easily found,

$m_1, m_2, m_3 = K - A, (2K + A - \sqrt{8B^2 + A^2})/2, (2K + A + \sqrt{8B^2 + A^2})/2,$

In order to find the eigenvalues of M_d, we have to solve

$$det \begin{pmatrix} L-\lambda & Y & Y-iF \\ Y & L-\lambda & Y \\ Y+iF & Y & L-\lambda \end{pmatrix} = 0$$

That is,

$$(L-\lambda)^3 + 2Y^3 - (L-\lambda)(3Y^2 + F^2) = 0$$

In

$$(L-\lambda)^3 + 2Y^3 - (L-\lambda)(3Y^2+F^2) = 0$$

We substitute $\lambda = L + w$, which gives the cubic equation

$$w^3 - w(3Y^2+F^2) - 2Y^3 = 0$$

We make the ansatz

$$w = u\cos\theta,$$

where

$$u = 2\sqrt{\frac{3Y^2+F^2}{3}}$$

and

$$\cos(3\theta) = Y^3\left(\frac{3}{3Y^2+F^2}\right)^{3/2}$$

This gives the down-quark mass eigenvalues

$$m_j = L + 2\sqrt{\frac{3Y^2 + F^2}{3}} \cos\left[\arccos[Y^3(\frac{3}{3Y^2 + F^2})^{3/2}] - \frac{2\pi j}{3}\right]$$

where $j = d, s, b$

The mass eigenvalues of M_d thus depend on F, which in its turn depends on the matrix elements A and B in M_u.

We can express the up quark matrix elements in terms of the up-quark masses:

$$K = (m_u + m_c + m_t)/3$$

$$A = (m_c + m_t - 2m_u)/3$$

$$B = \frac{1}{3}\sqrt{(m_t - 2m_c + m_u)(2m_t - m_c - m_u)/2}$$

Likewise, the down sector has matrix elements

$$L = (m_d + m_s + m_t)/3$$

$$Y = \left[\frac{det(M_d) + 2L^3 - LC_2(M_d)}{2}\right]^{1/3}$$

$$F = \left[0.00003096 \times P_u P_d/(B(A^2 - B^2))\right]^{1/3}$$

i.e. the matrix element F depends on the up-quark matrix elements A and B.

The other way round, using the expression for the Jarlskog invariant

$$J_{CP} = -idet[M,N]/2P_uP_d = BF^3(A^2 - B^2)/P_uP_d = 0.00003096$$

we get that

$$A^2 - B^2 = 0.00003096\frac{P_uP_d}{F^3B}, \text{ i.e.}$$

$\Rightarrow$

$$A = \sqrt{\frac{(BF)^3 + 0.00003096P_uP_d}{F^3B}}$$

Our two mass matrices are now defined by five parameters, K, B, L, Y, F.

Conclusion

We have examined the structure of the simple, nearly democratic quark mass matrices

$$M = \begin{pmatrix} K & A & B \\ A & K & B \\ B & B & K \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} L & Y & Y - iF \\ Y & L & Y \\ Y + iF & Y & L \end{pmatrix}$$

taking into account the constraint coming from CP-violation, specifically the Jarlskog invariant, which implies that

at least one of the mass matrices must be complex, and that their commutator is nonzero.

The constraint moreover reduces the number of parameters from six to five.

This means that the mass values of the up-quarks are intertwined.

This is explicitely demonstrated in our approach, for example by expressing A, which is a matrix element in the up-sector matrix M_u, as

$$A = \sqrt{\frac{(BF)^3 + 0.00003096 P_u P_d}{F^3 B}}$$

where B and F are matrix elements is the up-sector and down-sector matrices, respectively.

This result does not depend on a given ansatz or parametrization, but is true for any quark mass matrices, since they always have to satisfy the Jarlskog invariant.