The twenty-eighth workshop "What Comes Beyond the Standard Models?"

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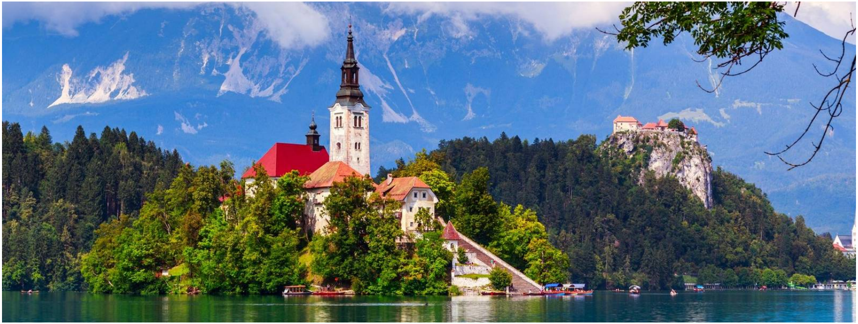

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The renormalization group invariants and exact results for various supersymmetric theories

Investigating of quantum corrections can shed a light to the structure of the surrounding world. For instance, the very precise agreement of the theoretical prediction of the electron anomalous magnetic moment with the experimental data tells us that the nature is described by quantum field theory.
The unification of running couplings and absence of divergent quantum corrections to the Higgs boson mass can be considered as indirect indications to the existence of supersymmetry and Grand Unification.

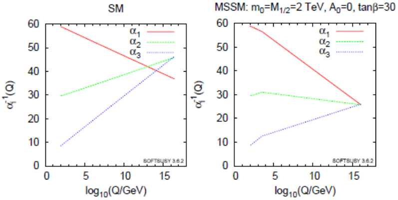


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

Some important information about new physics can be obtained from the detailed analysis of quantum corrections to (the lightest) Higgs boson in supersymmetric theories, anomalous magnetic moment of muon, etc.

The renormalization group invariants (RGI) are the scale independent values. Some of them are approximate, but sometimes it is possible to construct the expressions that are RGI in all orders.

For instance, in the SM/MSSM it is possible to construct the approximate RGI from the masses of down quarks and charged leptons

$$\frac{d}{d\ln\mu}\Big(\frac{m_e m_s}{m_d m_\mu}\Big) \approx 0; \qquad \frac{m_e m_s}{m_d m_\mu} \approx \frac{1}{9}\cdot 0.866 \approx \frac{1}{9}.$$

This expression is almost protected from quantum corrections and, therefore, at the unification scale it is impossible to reconcile this result with the prediction of the simplest $SU(5)$ GUT

$$m_d = m_e; \qquad m_s = m_\mu; \qquad m_b = m_\tau.$$

A way to solve this problem is to consider more complicated models leading, for example, to the Georgi and Jarlskog textures

H. Georgi, C. Jarlskog, Phys. Lett. B **86** (1979), 297.

If the Yukawa matrices for the down quarks and charged leptons are chosen in the form

$$Y_d = \begin{pmatrix} 0 & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{pmatrix}; \qquad Y_e = \begin{pmatrix} 0 & B & 0 \\ B & -3A & 0 \\ 0 & 0 & C \end{pmatrix},$$

then for $B \ll A$ we obtain

$$\frac{m_e m_s}{m_d m_\mu} \approx \frac{1}{9}.$$

The factor -3 can be obtained either from from the Higgs superfield coming from the representation 45_h of the group $SU(5)$

$$5 \times \overline{10} \times 45_h,$$

or with the help of the nonrenormalizable interaction

$$\frac{1}{M} \cdot 5 \times \overline{10} \times 5_h \times 75_H,$$

where 75_H acquires vev breaking $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$, see

S. Raby, Lect. Notes Phys. **939** (2017), 1-308 Springer, 2017.

Superspace and superfields

In what follows we will deal with supersymmetric theories. It is convenient to formulate them in superpace, because in this case supersymmetry is manifest.

$\mathcal{N}=1$ superspace is a space with the coordinates (x^μ,θ), where θ is an auxiliary anticommuting Majorana spinor, $\bar\theta=\theta^T C$. Superfields are the functions defined on the superspace. For instance, the $\mathcal{N}=1$ chiral scalar superfield by definition satisfies the constraint

$$(1-\gamma_5)D_a\phi \equiv (1-\gamma_5)_a{}^b D_b\phi = 0,$$

where

$$D_a \equiv \frac{\partial}{\partial\bar\theta^a} - i(\gamma^\mu\theta)_a\partial_\mu,$$

is the supersymmetric covariant derivative.

The solution can be written with the help of the chiral coordinates

$$y^\mu = x^\mu + \frac{i}{2}\bar\theta\gamma^\mu\gamma_5\theta$$

and has the form

$$\phi(y^\mu,(1+\gamma_5)\theta) = \varphi(y^\mu) + \bar\theta(1+\gamma_5)\psi(y^\mu) + \frac{1}{2}\bar\theta(1+\gamma_5)\theta\, f(y^\mu).$$

where φ and f are complex scalar fields, and ψ is a Majorana spinor.

Constructing supersymmertic actions with the help of superfields

For the components of a chiral superfield the supersymmetry transformations are written as

$$\delta\varphi = \bar\varepsilon(1+\gamma_5)\psi;$$
$$\delta\psi = (\mathrm{Re}\, f + i\gamma_5 \mathrm{Im}\, f)\varepsilon - i\partial_\mu(\mathrm{Re}\,\varphi + i\gamma_5\mathrm{Im}\,\varphi)\gamma^\mu\varepsilon;$$
$$\delta f = \partial_\mu\Big[-i\bar\varepsilon\gamma^\mu(1+\gamma_5)\psi\Big]$$

and mix the bose and fermi fields.

Supersymmetric actions can be constructed integrating the real and/or chiral superfields over the superspace,

$$S = \int d^4x\, d^4\theta\, \boldsymbol{V} + \Big(\int d^4x\, d^2\theta\, \boldsymbol{\phi} + \mbox{c.c.}\Big),$$

where $\boldsymbol{V} = \boldsymbol{V}^*$ is an arbitrary real superfield and $\boldsymbol{\phi}$ is an arbitrary chiral superfield, $(1-\gamma_5)D_a\boldsymbol{\phi} = 0$. Any such structure is invariant under supersymmetry transformations, where the integrals over the anticommuting variables are defined as

$$\int d^4\theta \equiv \frac{1}{8}\Big(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}\Big)^2;\quad \int d^2\theta \equiv \frac{1}{4}\frac{\partial}{\partial\theta}(1+\gamma_5)\frac{\partial}{\partial\bar\theta};\quad \int d^2\bar\theta \equiv \frac{1}{4}\frac{\partial}{\partial\theta}(1-\gamma_5)\frac{\partial}{\partial\bar\theta}.$$

Wess–Zumino model with mass and interaction

Choosing $\boldsymbol{V} = \phi^*\phi/4$ and $\boldsymbol{\phi} = m\phi^2/4 + \lambda\phi^3/6$, where m is a constant with dimension of mass and λ is a dimensionless constant, we obtain the model

$$S = \frac{1}{4}\int d^4x\, d^4\theta\, \phi^*\phi + \Big(\int d^4x\, d^2\theta\, \Big[\frac{1}{4}m\phi^2 + \frac{1}{6}\lambda\phi^3\Big] + \text{c.c.}\Big).$$

In components (for real m and λ)

$$S = \int d^4x\, \Big(\partial_\mu\varphi^*\partial^\mu\varphi + i\bar\psi\gamma^\mu\partial_\mu\psi + f^*f - m\bar\psi\psi + m\varphi f + \lambda\varphi^2 f$$
$$+m\varphi^* f^* + \lambda(\varphi^*)^2 f^* - \lambda\varphi\bar\psi(1+\gamma_5)\psi - \lambda\varphi^*\bar\psi(1-\gamma_5)\psi\Big).$$

This action is quadratic in the auxiliary fields f and f^*, which can be eliminated on shell,

$$S = \int d^4x\, \Big(\partial_\mu\varphi^*\partial^\mu\varphi + i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi - |m\varphi + \lambda\varphi^2|^2$$
$$-\lambda\varphi\bar\psi(1+\gamma_5)\psi - \lambda\varphi^*\bar\psi(1-\gamma_5)\psi\Big).$$

We see that the masses of the superparners ar the same and the scalar potential is positive.

Supersymmetric gauge field and its gauge field strength

The gauge field is a component of the real gauge superfield $V(x,\theta) = V^*(x,\theta)$. In the Wess–Zumino gauge it can be written as

$$V(x,\theta) = -\frac{1}{2}\bar\theta\gamma^\mu\gamma_5\theta A_\mu(x) + i\sqrt{2}(\bar\theta\theta)\bar\theta\gamma_5\lambda(x) + \frac{1}{4}(\bar\theta\theta)^2 D(x).$$

The supersymmetric analog of the gauge field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

is the (right) spinor chiral superfield

$$W_a \equiv \frac{1}{32}\bar D(1-\gamma_5)D\Big(e^{-2V}(1+\gamma_5)D_a e^{2V}\Big) \equiv \frac{1}{16}\bar D^2\Big(e^{-2V}(1+\gamma_5)D_a e^{2V}\Big).$$

One of its components is the usual gauge field strength $F_{\mu\nu}$. In particular, in the Wess-Zumino gauge this superfield is given by the expression

$$W_a = \frac{1}{2}(1+\gamma_5)\Big\{-i\sqrt{2}\lambda_a(y) - \theta_a D(y) + \frac{i}{2}\gamma^{\mu\nu}\theta_a F_{\mu\nu}(y) - \frac{1}{\sqrt{2}}\bar\theta(1+\gamma_5)\theta\,\gamma^\mu D_\mu\lambda_a(y)\Big\},$$

where $D_\mu\lambda = \partial_\mu\lambda + i[A_\mu,\lambda]$ is the covariant derivative of the gaugino field.

$\mathcal{N}=1$ supersymmetric QCD (SQCD)

The action for (the simplified version of) quantum chromodynamics (QCD) is given by

$$S=\int d^4x\,\Big(-\frac{1}{2e^2}\mbox{tr}\,F_{\mu\nu}^2+i\bar\Psi\gamma^\mu D_\mu\Psi-m\bar\Psi\Psi\Big),$$

where Ψ is the Dirac spinor in a certain representation R of the gauge group G. The Dirac spinor can be constructed from two Majorana spinors ψ and $\widetilde\psi$,

$$\Psi=\frac{1}{\sqrt{2}}\Big((1+\gamma_5)\psi+(1-\gamma_5)\widetilde\psi\Big).$$

Therefore, for constructing a theory with a Dirac spinor, we need 2 chiral scalar superfields ϕ and $\widetilde\phi$ in the representations R and $\bar R$ of the group G. Then, the action of $\mathcal{N}=1$ SQCD in the superfield formulation can be presented in the form

$$S=\frac{1}{2e^2}\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi^+e^{2V}\phi+\widetilde\phi^+e^{-2V^T}\widetilde\phi\Big)$$
$$+\Big(\frac{1}{2}m\int d^4x\,d^2\theta\,\widetilde\phi^T\phi+\mbox{c.c.}\Big),$$

where we took into account that the generators of the conjugated representations differ in sign and transposing.

$\mathcal{N}=1$ SQCD

The theory is invariant under the gauge transformations, which can be written in the superfield form,

$$\phi \to e^{i\Lambda}\phi; \qquad \widetilde{\phi} \to e^{-i\Lambda^T}\widetilde{\phi}; \qquad e^{2V} \to e^{i\Lambda^+}e^{2V}e^{-i\Lambda}; \qquad W_a \to e^{i\Lambda}W_a e^{-i\Lambda},$$

and are parametrized by the chiral superfield $\Lambda = e\Lambda^A T^A$ (or $\Lambda = e\Lambda^A t^A$). After eliminating the auxiliary fields f, $\widetilde{f}$ and D the expression for the action in components takes the form

$$S = \int d^4x \Big\{ -\frac{1}{4}(F_{\mu\nu}^A)^2 + i\bar{\Psi}\gamma^\mu D_\mu\Psi - m\bar{\Psi}\Psi + i\lambda^A\gamma^\mu D_\mu\lambda^A + D_\mu\varphi^+$$
$$\times D^\mu\varphi + D_\mu\widetilde{\varphi}^+ D^\mu\widetilde{\varphi} - m^2\varphi^+\varphi - m^2\widetilde{\varphi}^+\widetilde{\varphi} - \frac{e^2}{2}\Big(\varphi^+T^A\varphi - \widetilde{\varphi}^T T^A\widetilde{\varphi}^*\Big)^2$$
$$+ i\bar{\Psi}(1-\gamma_5)\lambda\varphi - i\varphi^+\bar{\lambda}(1+\gamma_5)\Psi - i\widetilde{\varphi}^T\bar{\lambda}(1-\gamma_5)\Psi + i\bar{\Psi}(1+\gamma_5)\lambda\widetilde{\varphi}^*\Big\}.$$

We see that the masses of superpartners coincide, and the scalar potential is positive. All this facts are typical features of supersymmetric theories.

Supersymmetric gauge theories

RGIs can exist in certain supersymmetric theories. In $\mathcal{N}=1$ superspace these theories are described by the action

$$S=\frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mathrm{c.c.}\Big\}.$$

Here V is the gauge superfield, ϕ_i are the chiral matter superfields in the representation R of the gauge group G, and

$$W_a=\frac{1}{16}\bar{D}^2\Big(e^{-2V}(1+\gamma_5)D_a e^{2V}\Big)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im}(T^A)_m{}^j+m_0^{mj}(T^A)_m{}^i=0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k+\lambda_0^{imk}(T^A)_m{}^j+\lambda_0^{mjk}(T^A)_m{}^i=0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group G in the representation R.

It is important that the renormalizability requires that the superpotential

$$W = \frac{1}{2}m_0^{ij}\phi_i\phi_j + \frac{1}{3}\lambda_0^{ijk}\phi_i\phi_j\phi_k$$

should not contain chiral superfields in a more than the third power.

Supersymmetric theories can be quantized in terms of superfields. Analyzing the corresponding Feynman rules

M. T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. B **159** (1979), 429.

one can derive

The nonrenormalization theorem: The superpotential does not receive divergent quantum corrections. (In other words, the superpotential is RGI).

Consequently, mass and Yukawa coupling renormalizations are related to the renormalization of the chiral matter superfields. Namely, if $\phi_i = (\sqrt{Z})_i{}^j\phi_{R,j}$, then

$$m^{ij} = m_0^{kl}(\sqrt{Z})_k{}^i(\sqrt{Z})_l{}^j; \qquad \lambda^{ijk} = \lambda_0^{mnp}(\sqrt{Z})_m{}^i(\sqrt{Z})_n{}^j(\sqrt{Z})_p{}^k.$$

NSVZ β-function for $\mathcal{N}=1$ supersymmetric theories

It is also possible to construct RGIs in supersymmetric theories using the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456

which relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories.

For a general $\mathcal{N}=1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

For instance, for the pure $\mathcal{N}=1$ SYM theory

$$S = \frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, W^a W_a$$

from the NSVZ β-function we obtain the equation

$$\frac{1}{\alpha^2}\Big(1-\frac{C_2\alpha}{2\pi}\Big)\frac{d\alpha}{d\ln\mu} = -\frac{3C_2}{2\pi}.$$

Integrating it we obtain the all-loop RGI

$$\Big(\frac{\mu^3}{\alpha}\Big)^{C_2}\exp\Big(-\frac{2\pi}{\alpha}\Big) = \mathrm{RGI}.$$

Such expressions appear in calculating the instanton contributions to the effective action, and the NSVZ β-function was first obtained by requiring their renormalization group invariance, see

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381.

For theories with chiral matter superfields the analogous invariants contain the renormalization constants for the matter superfields or masses.

The so-called $P = \frac{1}{3}Q$ theories by definition satisfy the constraint

$$\lambda^*_{imn}\lambda^{jmn} - 4\pi\alpha C(R)_i{}^j = \frac{2\pi\alpha}{3} Q\delta_i^j,$$

where $Q \equiv T(R) - 3C_2$. It was demonstrated

I.Jack, D.R.T.Jones, C.G.North, Nucl. Phys. B **473** (1996), 308

that in these theories in the first two orders of the perturbation theory the ratio of the Yukawa couplings to the gauge coupling is RG invariant,

$$\frac{d}{d\ln\mu}\Big(\frac{\lambda^{ijk}}{e}\Big) = 0.$$

similarly to $\mathcal{N} = 2$ supersymmetric theories. If this relation was exact, then it would presumably allow to reduce a number of couplings if we set $\lambda^{ijk} = ec^{ijk}$, where c^{ijk} are certain constants, see

S. Heinemeyer, M. Mondragon, N. Tracas, G. Zoupanos, Phys. Rept. **814** (2019) 1

for more details. However, in the three-loop approximation the above relation is not valid.

RGIs for theories with softly broken supersymmetry

It is known that the renormalization of soft term in theories with softly broken supersymmetry can be related to the renormalization of the rigid theory

J. Hisano, M. A. Shifman, Phys. Rev. D **56** (1997), 5475;
I. Jack, D. R. T. Jones, Phys. Lett. B **415** (1997) 383;
L. V. Avdeev, D. I. Kazakov, I. N. Kondrashuk, Nucl. Phys. B **510** (1998) 289.

For instance, the renormalization of the gaugino mass in the softly broken $\mathcal{N}=1$ SYM theory

$$S = \frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\,(1+2m\theta^2)W^a W_a$$

is described by the RGI

$$\frac{\alpha m}{\beta(\alpha)} = \mathrm{RGI}.$$

Differentiating this equation with respect to $\ln\mu$ and substituting the NSVZ expression for the β-function it is possible to obtain the all-order expression for the anomalous dimension of the gaugino mass.

The generalizations for theories containing the chiral matter superfields are also possible.

Scheme dependence of the NSVZ equation

Nevertheless, it is necessary to remember that the NSVZ equation is valid only for certain renormalization prescriptions. Therefore, the all-loop renormalization group invariance of the above expressions does not hold for a general subtraction scheme.

Note that in the $\overline{\text{DR}}$-scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the β-function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138; Nucl.Phys. B **486** (1997) 479;
R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP **0612** (2006) 024.

However, in this case it is possible to make a special redefinition of the coupling constant which restores the NSVZ relation.

The all-loop NSVZ schemes have been constructed with the help of the higher covariant derivative regularization

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

The higher covariant derivative regularization

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\,d^2\theta\,W^a\left(e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}\right)_{Adj}R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\left(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\right)_{Adj}W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a;\qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\bar D_{\dot a}e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\,d^4\theta\,\boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

It is also necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0)=F(0)=K(0)=1$.

The Pauli–Villars determinants

For regularizing the residual one-loop divergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \text{Det}(PV, M_\varphi)^{-1} \text{Det}(PV, M)^c \times \exp\Big\{ i\Big(S_{\text{reg}} + S_{\text{gf}} + S_{\text{FP}} + S_{\text{NK}} + S_{\text{sources}} \Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\text{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\text{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

(Here we use chiral commuting Pauli–Villars superfields.)
The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lie in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\text{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

The β-function as an integral of double total derivatives

The NSVZ equation naturally appears with the higher derivative regularization because in this case the integrals giving the β-function defined in terms of the bare couplings

$$\beta(\alpha_0, \lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}$$

are integrals of double total derivatives with respect to the loop momenta

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445;
K.S., Nucl.Phys. **B 852** (2011) 71; JHEP **10** (2019) 011.

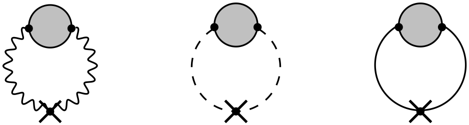
This can be seen even in the one-loop approximation

S. S. Aleshin, A. E. Kazantsev, M. B. Skoptsov, K.S., JHEP **05** (2016), 014.

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = \int \frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big\{-\frac{\pi C_2}{q^2}\Big[\ln\Big(1+\frac{M_\varphi^2}{q^2 R^2(q^2/\Lambda^2)}\Big)$$
$$+2\ln\Big(1+\frac{M_\varphi^2}{q^2}\Big)\Big] + \frac{\pi T(R)}{q^2}\ln\Big(1+\frac{M^2}{q^2 F^2(q^2/\Lambda^2)}\Big)\Big\} + O(\alpha_0,\lambda_0^2),$$

where a small vicinity of the singular point $q^\mu = 0$ is excluded from the integration region.

The double total derivatives effectively cut internal lines in the supergraphs and reduce a number of loop integrations by 1.



Then the NSVZ equation for the renormalization group functions (RGFs) defined in terms of the bare couplings is obtained by summing singular contributions in all orders

K.S., Eur. Phys. J. C **80** (2020) no.10, 911.

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

Note that qualitatively this result can be obtained with the help of a certain modification of the one-loop calculation, namely, by replacing tree propagators with the exact propagators.

The original NSVZ equation is reproduced after taking into account the non-renormalization of the triple gauge-ghost vertices

K.S., Nucl.Phys. **B909** (2016) 316.

This nonrenormalization theorem produces the equation

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2} Z_c Z_V) = 0,$$

which allows to express the β-function in terms of the anomalous dimensions of quantum superfields,

$$\beta(\alpha_0,\lambda_0) = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big).$$

Using this relation we obtain the NSVZ β-function in the original form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The NSVZ scheme for the standard RGFs

For the standard RGFs

$$\widetilde\beta(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\mbox{const}};\qquad \widetilde\gamma(\alpha,\lambda) \equiv \frac{d\ln Z}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\mbox{const}}$$

the all-loop NSVZ scheme turns out to be the HD+MSL scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms, because in this case

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459

$$\widetilde\beta(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda);$$

$$\widetilde\gamma(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \gamma(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Therefore, for the pure $\mathcal{N}=1$ SYM theory the RGI

$$\Big(\frac{\mu^3}{\alpha}\Big)^{C_2}\exp\Big(-\frac{2\pi}{\alpha}\Big) = \mbox{RGI}.$$

is valid in the HD+MSL scheme and is not valid in the $\overline{\mbox{DR}}$ scheme.

The NSVZ relation for $\mathcal{N}=1$ SQED

Let us demonstrate the scheme dependence of the NSVZ equation in the simplest case of $\mathcal{N}=1$ SQED with N_f flavors

$$S = \frac{1}{4e^2}\mbox{Re}\int d^4x\, d^2\theta\, W^a W_a + \sum_{\alpha=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\phi_\alpha^* e^{2V}\phi_\alpha + \widetilde\phi_\alpha^* e^{-2V}\widetilde\phi_\alpha\Big).$$

For this theory the NSVZ β-function takes the form

$$\beta(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, JETP Lett. **42** (1985) 224; Phys. Lett. B **166** (1986) 334.

Expressions for the three-loop β-function and the two-loop anomalous dimension of the matter superfields for $\mathcal{N}=1$ SQED can be found in

A. L. Kataev and K.S., Phys. Lett. B **730** (2014) 184; Theor. Math. Phys. **181** (2014) 1531.

The HD+MSL-scheme

$$\widetilde{\gamma}_{\text{HD+MSL}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{HD+MSL}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3)\Big).$$

The MOM-scheme (The result is the same for dimensional reduction and for the higher derivative regularization.)

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

The $\overline{\text{DR}}$-scheme

I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. **B386** (1996) 138.

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

Gauge theories with multiple gauge couplings

Let us investigate a possibility of constructing RGIs for some gauge theories with multiple gauge couplings. In this case the gauge group is a direct product

$$G = G_1 \times G_2 \times \ldots \times G_n,$$

where any G_i is either a simple group or $U(1)$. In this case there are n gauge coupling constants α_1, α_2, ..., α_n.
Such theories can be interesting for phenomenology because they include

- QCD+QED
- The Standard Model
- The MSSM
- Some Grand Unified Theories, e.g., the flipped $SU(5)$ theory.

Following

A. L. Kataev, K.S., JETP Lett. **121** (2025) no.5, 315;
D. Rystsov, K.S., Phys. Rev. D **111** (2025) no.1, 016012,

we argue that in some $\mathcal{N} = 1$ supersymmetric theories with multiple gauge couplings one can construct all-loop RGIs from the gauge and Yukawa couplings. We will also discuss under what renormalization prescriptions the renormalization group invariance is valid in all orders.

The simplest example of a theory with two gauge coupling constants $\alpha_s \equiv g^2/4\pi$ and $\alpha = e^2/4\pi$ is QCD+QED. In the massless limit this theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}^2 - \frac{1}{4e^2}\boldsymbol{F}_{\mu\nu}^2 + \sum_{\mathsf{a}=1}^{N_f} i\bar\Psi_{\mathsf{a}}\gamma^\mu \mathcal{D}_\mu \Psi_{\mathsf{a}},$$

which is invariant under the transformations of the gauge group $G \times U(1)$. The Dirac spinors Ψ_{a} (where the subscript a numerates flavors) lie in a certain irreducible representation R of the group G and have the electromagnetic charges q_{a}. In this case the covariant derivatives are written in the form

$$\mathcal{D}_\mu\Psi_{\mathsf{a}} = \partial_\mu\Psi_{\mathsf{a}} + A_\mu\Psi_{\mathsf{a}} + iq_{\mathsf{a}}\boldsymbol{A}_\mu\Psi_{\mathsf{a}},$$

where A_μ and $\boldsymbol{A}_\mu$ are the non-Abelian and Abelian gauge fields, respectively. The corresponding gauge field strengths are given by the expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \qquad \boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu.$$

In quantum field theory the couplings α_s and α depend on scale,

$$\frac{d\alpha}{d\ln\mu} = \beta(\alpha,\alpha_s); \qquad \frac{d\alpha_s}{d\ln\mu} = \beta_s(\alpha_s,\alpha).$$

$\mathcal{N} = 1$ SQCD+SQED

It is convenient to formulate the supersymmetric version of the above model in terms of superfields

$$S = \frac{1}{2g^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^a W_a + \frac{1}{4e^2}\,\mathrm{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathrm{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathrm{a}}^+ e^{2V+2q_{\mathrm{a}}\boldsymbol{V}}\phi_{\mathrm{a}} + \widetilde{\phi}_{\mathrm{a}}^+ e^{-2V^T-2q_{\mathrm{a}}\boldsymbol{V}}\widetilde{\phi}_{\mathrm{a}}\Big),$$

because in this case $\mathcal{N} = 1$ supersymmetry is manifest.

Here V and $\boldsymbol{V}$ are the gauge superfields corresponding to the subgroups G and $U(1)$, respectively. The chiral matter superfields ϕ_{a} and $\widetilde{\phi}_{\mathrm{a}}$ belong to the (conjugated) representations R and $\overline{R}$, respectively, and have opposite $U(1)$ charges.

Two supersymmetric gauge superfield strengths are written in the form

$$W_a = \frac{1}{16}\overline{D}^2\Big(e^{-2V}(1+\gamma_5)D_a e^{2V}\Big);\qquad \boldsymbol{W}_a = \frac{1}{8}\overline{D}^2\Big((1+\gamma_5)D_a\boldsymbol{V}\Big).$$

Is it possible to relate running of two gauge coupling constants in this model?

The NSVZ equations can also be written for theories with multiple gauge couplings,

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046.

In the particular case $q_\text{a} = 1$ for $\mathcal{N} = 1$ SQCD+SQED they take the form

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1 - C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1 - \gamma(\alpha_s,\alpha)\Big)\Big];$$
$$\frac{\beta(\alpha,\alpha_s)}{\alpha^2} = \frac{1}{\pi}\,\text{dim}\,R\,N_f\Big(1 - \gamma(\alpha_s,\alpha)\Big).$$

Here we took into account that if the representation for the matter superfields is irreducible, then

$$\gamma(\alpha_s,\alpha)_i{}^j = \gamma(\alpha_s,\alpha)\cdot\delta_i^j,$$

where i and j include both the indices numerating chiral matter superfields ϕ_a and $\widetilde\phi_\text{a}$ and the indices corresponding to the representation R (or $\overline{R}$).

Comparing the above expressions for the β-functions we see that the anomalous dimension of the matter superfeilds can be eliminated.

After eliminating the anomalous dimension of the matter superfields we obtain that the β-functions satisfy the all-order exact equation

$$\Big(1-\frac{C_2\alpha_s}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\dim R}\cdot\frac{\beta(\alpha,\alpha_s)}{\alpha^2}.$$

Evidently, this equation is valid in the HD+MSL scheme, because the original NSVZ equations are satisfied for this renormalization prescription.

Taking into account the boundary conditions for the HD+MSL scheme it is possible to integrate the relation between the β-functions over μ. Then we obtain the equation which relates running of the strong and electromagnetic couplings in the theory under consideration.

$$\frac{1}{\alpha_s} - \frac{1}{\alpha_{s0}} + \frac{C_2}{2\pi}\ln\frac{\alpha_s}{\alpha_{0s}} = -\frac{3C_2}{2\pi}\ln\frac{\Lambda}{\mu} + \frac{T(R)}{\dim R}\Big(\frac{1}{\alpha}-\frac{1}{\alpha_0}\Big).$$

This in particular implies that the expression

$$\Big(\frac{\alpha_s}{\mu^3}\Big)^{C_2}\exp\Big(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\dim R}\cdot\frac{2\pi}{\alpha}\Big) = \text{RGI}$$

is the renormalization group invariant, i.e. the expression which vanishes afer differentiating with respect to $\ln\mu$.

With the higher covariant derivative regularization, the three-loop β-functions for the $\mathcal{N}=1$ SQCD+SQED have been calculated in

O. Haneychuk and K.S., Eur. Phys. J. C **85** (2025) no.5, 540

In the three-loop approximation the scheme dependence becomes essential, and the result depends on both regularization parameters

$$A \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{R(x)};\qquad a_\varphi \equiv \frac{M_\varphi}{\Lambda};\qquad a_G \equiv \frac{M_G}{\Lambda};\qquad a_1 \equiv \frac{M_1}{\Lambda}.$$

and renormalization parameters defined by the equations

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f\, \textbf{dim}\, R}{\pi}\Big(\ln\frac{\Lambda}{\mu} + d_1\Big) - \frac{\alpha_s}{\pi^2} N_f C(R)\, \textbf{dim}\, R\Big(\ln\frac{\Lambda}{\mu} + d_2\Big) - \frac{\alpha}{\pi^2} N_f\, \textbf{dim}\, R$$
$$\times\Big(\ln\frac{\Lambda}{\mu} + \widetilde{d}_2\Big) + O(\alpha_s^2, \alpha_s\alpha, \alpha^2);$$

$$\frac{1}{\alpha_{s0}} = \frac{1}{\alpha_s} + \frac{3C_2}{2\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{11}\Big) - \frac{N_f T(R)}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{12}\Big) + \frac{3\alpha_s}{4\pi^2}(C_2)^2\Big(\ln\frac{\Lambda}{\mu} + b_{21}\Big)$$
$$-\frac{\alpha_s}{2\pi^2} N_f C_2 T(R)\Big(\ln\frac{\Lambda}{\mu} + b_{22}\Big) - \frac{\alpha_s}{\pi^2} N_f C(R) T(R)\Big(\ln\frac{\Lambda}{\mu} + b_{23}\Big) - \frac{\alpha}{\pi^2} N_f T(R)\Big(\ln\frac{\Lambda}{\mu}$$
$$+\widetilde{b}_{21}\Big) + O(\alpha_s^2, \alpha_s\alpha, \alpha^2).$$

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2} = \frac{N_f\,\textbf{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2-\frac{\alpha^2}{\pi^2}N_f\,\textbf{dim}\,R\Big(\ln a_1$$
$$+1+\frac{A}{2}+\widetilde{d}_2-d_1\Big)+\frac{3\alpha_s^2}{2\pi^2}C_2C(R)\Big(\ln a_\varphi+1+\frac{A}{2}+d_2-b_{11}\Big)-\frac{\alpha_s^2}{\pi^2}N_fC(R)T(R)$$
$$\times\Big(\ln a_G+1+\frac{A}{2}+d_2-b_{12}\Big)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi}\Big(3C_2-2N_fT(R)\Big)+\frac{\alpha}{\pi^2}N_fT(R)+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2+2N_fC_2T(R)$$
$$+4N_fC(R)T(R)\Big)-\frac{\alpha^2}{\pi^3}(N_f)^2T(R)\,\textbf{dim}\,R\Big(\ln a_1+1+\frac{A}{2}+\widetilde{b}_{21}-d_1\Big)-\frac{1}{2\pi^3}N_fT(R)$$
$$\times\Big(\alpha+\alpha_sC(R)\Big)^2+\frac{\alpha\alpha_s}{2\pi^3}N_fC_2T(R)-\frac{3\alpha_s^2}{8\pi^3}(C_2)^3\Big(1+3b_{21}-3b_{11}\Big)+\frac{\alpha_s^2}{4\pi^3}N_f(C_2)^2$$
$$\times T(R)\Big(1+3b_{21}-3b_{11}+3b_{22}-3b_{12}\Big)+\frac{3\alpha_s^2}{2\pi^3}N_fC_2C(R)T(R)\Big(\ln a_\varphi+\frac{4}{3}+\frac{A}{2}$$
$$+b_{23}-b_{11}\Big)-\frac{\alpha_s^2}{2\pi^3}(N_f)^2C_2T(R)^2(b_{22}-b_{12})-\frac{\alpha_s^2}{\pi^3}(N_f)^2C(R)T(R)^2\Big(\ln a_G+1$$
$$+\frac{A}{2}+b_{23}-b_{12}\Big)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

The HD+MSL scheme is obtained if

$$b_{11}=b_{12}=b_{21}=b_{22}=b_{23}=\widetilde{b}_{23}=\widetilde{b}_{21}=0;\qquad d_1=d_2=\widetilde{d}_2=0$$

The equation relating the β-function is satisfied if

$$\widetilde{b}_{21}=\widetilde{d}_2;\qquad b_{21}=b_{11};\qquad b_{22}=b_{12};\qquad b_{23}=d_2.$$

In particular, it is valid in the HD+MSL scheme independently of values of the regularization parameters a_φ, a_G, a_1, and A.

One can try to find a (minimal) scheme in which RGFs have the simplest form and the exact relation between β-function is still valid. This is achieved if the (remaining) parameters satisfy the equations

$$b_{12}=b_{11}+\ln\frac{a_G}{a_\varphi};\qquad d_2=b_{11}-\ln a_\varphi-1-\frac{A}{2};\qquad \widetilde{d}_2=d_1-\ln a_1-1-\frac{A}{2}.$$

In this case the expressions for the three-loop β-functions take the simplest form

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}=\frac{N_f\,\mbox{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}=-\frac{1}{2\pi}\Big(3C_2-2N_fT(R)\Big)+\Big(1+\frac{\alpha_s C_2}{2\pi}\Big)\Big[\frac{\alpha}{\pi^2}N_fT(R)+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2+2N_f$$

$$\times C_2T(R)+4N_fC(R)T(R)\Big)\Big]-\frac{1}{2\pi^3}N_fT(R)\Big(\alpha+\alpha_sC(R)\Big)^2+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

The three-loop verification, $\overline{\text{DR}}$ scheme

The three-loop RGFs in the $\overline{\text{DR}}$ scheme are given by the expressions

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}\Big|_{\overline{\text{DR}}} = \frac{N_f \dim R}{\pi}\Big\{1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi}C(R) - \frac{1}{2\pi^2}\Big(\alpha + \alpha_s C(R)\Big)^2 - \frac{3\alpha^2}{4\pi^2}N_f \dim R$$
$$+\frac{9\alpha_s^2}{8\pi^2}C_2 C(R) - \frac{3\alpha_s^2}{4\pi^2}N_f C(R)T(R) + O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}\Big|_{\overline{\text{DR}}} = -\frac{1}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha}{\pi^2}N_f T(R) + \frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2 + 2N_f C_2 T(R)$$
$$+4N_f C(R)T(R)\Big) - \frac{3\alpha^2}{4\pi^3}(N_f)^2 T(R)\dim R - \frac{1}{2\pi^3}N_f T(R)\Big(\alpha + \alpha_s C(R)\Big)^2 + \frac{\alpha\alpha_s}{2\pi^3}N_f C_2$$
$$\times T(R) - \frac{21\alpha_s^2}{32\pi^3}(C_2)^3 + \frac{5\alpha_s^2}{8\pi^3}N_f (C_2)^2 T(R) + \frac{13\alpha_s^2}{8\pi^3}N_f C_2 C(R)T(R) - \frac{\alpha_s^2}{8\pi^3}(N_f)^2 C_2 T(R)^2$$
$$-\frac{3\alpha_s^2}{4\pi^3}(N_f)^2 C(R)T(R)^2 + O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

Substituting these expressions we obtain

$$\Big(1 - \frac{\alpha_s C_2}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} + \frac{3C_2}{2\pi} - \frac{T(R)}{\dim R}\cdot\frac{\beta(\alpha_s,\alpha)}{\alpha^2}$$
$$= -\frac{9\alpha_s^2}{32\pi^3}(C_2)^3 + \frac{3\alpha_s^2}{8\pi^3}(C_2)^2 N_f T(R) + O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3) \ne O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

Thus, the expression under consideration is not RGI in the $\overline{\text{DR}}$ scheme starting from the three loops, where the scheme dependence manifests itself.

Next, let us consider the theory in which the matter superfields have different $U(1)$ charges q_{a},

$$S=\frac{1}{2g^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4e^2}\,\mathrm{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathrm{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathrm{a}}^+e^{2V+2q_{\mathrm{a}}\boldsymbol{V}}\phi_{\mathrm{a}}+\widetilde{\phi}_{\mathrm{a}}^+e^{-2V-2q_{\mathrm{a}}\boldsymbol{V}}\widetilde{\phi}_{\mathrm{a}}\Big)$$

and investigate the limit $\alpha=e^2/4\pi\to 0$. In this case the renormalization group running of the strong coupling constant α_s is exactly the same as in usual $\mathcal{N}=1$ SQCD with the gauge group G and N_f flavors. The running of the electromagnetic coupling constant is described by the Adler D-function

S. L. Adler, Phys. Rev. D **10** (1974), 3714,

which is related to the β-function for the coupling constant α in the limit $\alpha\to 0$,

$$D(\alpha_s)=\frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

The NSVZ-like expression for the Adler D-function

In the limit $\alpha \to 0$ the anomalous dimensions of the matter superfields do not depend on α and, therefore, on q_{a}. This implies that in this case all anomalous dimensions of the chiral matter superfields are the same,

$$\lim_{\alpha\to 0} \gamma_{\mathsf{a}}(\alpha_s,\alpha) = \gamma(\alpha_s).$$

Then the NSVZ β-function for $\mathcal{N}=1$ SQCD takes the form

$$\frac{\beta_s(\alpha_s)}{\alpha_s^2} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s)\Big)\Big].$$

The exact NSVZ-like expression for the Adler D-function in the theory under consideration has been derived in

M. Shifman and K.S., Phys. Rev. Lett. **114** (2015) 051601;
Phys. Rev. D **91** (2015), 105008.

$$D(\alpha_s) = \frac{3}{2}\,\mbox{dim}\,R\sum_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2\Big(1-\gamma(\alpha_s)\Big) \equiv \frac{3}{2}\,\boldsymbol{q^2}\,\mbox{dim}\,R\Big(1-\gamma(\alpha_s)\Big),$$

where $\boldsymbol{q^2} \equiv \sum\limits_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2.$

$\mathcal{N} = 1$ SQCD+SQED with different $U(1)$ changes

Therefore, the β-function of $\mathcal{N} = 1$ SQCD can be expressed in terms of the Adler D-function by the all-loop equation

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi(1 - C_2\alpha_s/2\pi)}\Big[3C_2 - \frac{4\,T(R)N_f D(\alpha_s)}{3\,\boldsymbol{q^2}\,\mbox{dim}\,R}\Big],$$

which relates the renormalization group running of the strong and electromagnetic coupling constants in the limit $\alpha \to 0$. Evidently, this equation is valid in the HD+MSL scheme in all orders.

Thus, from the NSVZ equation we see that

1. If all $U(1)$ charges q_{a} are the same, then in the $\mathcal{N} = 1$ SQCD+SQED, which is a theory with two gauge couplings, it is possible to relate their running.

2. If the charges q_{a} are different, then it is possible to relate the β-function of $\mathcal{N} = 1$ SQCD to the Adler D-function. Actually, in this case the exact relation exists only in the limit $\alpha \to 0$.

3. These exact relations are valid in the HD+MSL scheme, but do not hold in the $\overline{\text{DR}}$ scheme.

The MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU_3\times SU_2\times U_1$ and softly broken supersymmetry. Consequently, there are 3 gauge coupling constants e_3, e_2, and e_1 in the MSSM (their number is equal to the number of factors in the gauge group). Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

Superfield	SU_3	SU_2	$U_1\ (Y)$	Superfield	SU_3	SU_2	$U_1\ (Y)$
$3\times Q$	$\bar 3$	2	$-1/6$	$3\times N$	1	1	0
$3\times U$	3	1	$2/3$	$3\times E$	1	1	-1
$3\times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3\times L$	1	2	$1/2$	H_u	1	2	$-1/2$

where for the superfields which include left quarks and leptons we use the brief notations

$$Q=\left(\begin{array}{c}\widetilde U\\ \widetilde D\end{array}\right);\qquad L=\left(\begin{array}{c}\widetilde N\\ \widetilde E\end{array}\right).$$

The MSSM contains three gauge couplings

$$\alpha_3 = \frac{e_3^2}{4\pi};\qquad \alpha_2 = \frac{e_2^2}{4\pi};\qquad \alpha_1 = \frac{5}{3}\cdot\frac{e_1^2}{4\pi}$$

corresponding to the subgroups $SU(3)$, $SU(2)$, and $U(1)$, respectively. (The factor $5/3$ in the coupling constant α_1 is introduced in order that the unification of couplings has the form $\alpha_1 = \alpha_2 = \alpha_3$.) There are also dimensionless Yukawa couplings $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$ (which are 3×3 matrices) inside the superpotential

$$\begin{aligned} W = (Y_U)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} H_{u1}\\ H_{u2}\end{pmatrix} U_{aJ}\\ +(Y_D)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} H_{d1}\\ H_{d2}\end{pmatrix} D_{aJ}\\ +(Y_E)_{IJ}\left(\widetilde{N}\ \widetilde{E}\right)_I \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} H_{d1}\\ H_{d2}\end{pmatrix} E_{J}\\ +\boldsymbol{\mu}\left(H_{u1}\ H_{u2}\right)\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix} H_{d1}\\ H_{d2}\end{pmatrix}. \end{aligned}$$

Moreover, the superpotential includes a term with the parameter $\boldsymbol{\mu}$, which has the dimension of mass.

The renormalization group running of the gauge couplings in the MSSM can be described exactly in all loops with the help of the NSVZ β-functions

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

$$\frac{\beta_1}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$

$$\frac{\beta_2}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$

$$\frac{\beta_3}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big)\Big].$$

They relate three gauge β-functions of the theory to the anomalous dimensions of the chiral matter superfields. The renormalization group functions (RGFs) are defined by the equations

$$\beta_i(\alpha, Y) = \frac{d\alpha_i}{d\ln\mu}\Big|_{\alpha_0, Y_0=\mbox{const}}; \qquad \gamma_i(\alpha, Y) = \frac{d\ln Z_i}{d\ln\mu}\Big|_{\alpha_0, Y_0=\mbox{const}},$$

where the subscript 0 denotes the bare values.

The exact equations describing the renormalization of the MSSM Yukawa couplings

RGFs describing the renormalization of the Yukawa couplings and of the parameter $\boldsymbol{\mu}$ can also be related to the anomalous dimensions of the matter superfields due to the nonrenormalization of the superpotential

M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

$$\frac{dY_U}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} Y_U + (\gamma_Q)^T Y_U + Y_U \gamma_U\Big);$$
$$\frac{dY_D}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_D + (\gamma_Q)^T Y_D + Y_D \gamma_D\Big);$$
$$\frac{dY_E}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_E + (\gamma_L)^T Y_E + Y_E \gamma_E\Big);$$
$$\frac{d\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big)\boldsymbol{\mu}.$$

It is important that these equations are valid in the HD+MSL scheme because in this scheme all renormalization constants contain only powers of $\ln\Lambda/\mu$, where Λ is the dimensionful regularization parameter.

The renormalization group equations for the Yukawa couplings can be multiplied by the corresponding inverse matrices. After that, it is possible to calculate traces of the resulting equations using the formula

$$\mathrm{tr}\Big[M^{-1}\frac{dM}{d\ln\mu}\Big] = \frac{d}{d\ln\mu}\mathrm{tr}\ln M = \frac{d}{d\ln\mu}\ln\det M,$$

Then (taking into account that the indices numerating generators range from 1 to 3) we see that the equations describing how the determinants of the Yukawa matrices depend on the renormalization point μ are written as

$$\gamma_{\det Y_U} \equiv \frac{d\ln\det Y_U}{d\ln\mu} = \mathrm{tr}\Big[(Y_U)^{-1}\frac{dY_U}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_u} + \mathrm{tr}\big(\gamma_Q + \gamma_U\big)\Big);$$

$$\gamma_{\det Y_D} \equiv \frac{d\ln\det Y_D}{d\ln\mu} = \mathrm{tr}\Big[(Y_D)^{-1}\frac{dY_D}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_Q + \gamma_D\big)\Big);$$

$$\gamma_{\det Y_E} \equiv \frac{d\ln\det Y_E}{d\ln\mu} = \mathrm{tr}\Big[(Y_E)^{-1}\frac{dY_E}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_L + \gamma_E\big)\Big).$$

They can be solved together with the NSVZ equations and the equation describing the renormalization of the parameter $\boldsymbol{\mu}$.

Collecting the above equations we obtain the system of differential equations describing the renormalization of the MSSM parameters exactly in all orders

$$\frac{d}{d\ln\mu}\Big(\frac{5}{3}\cdot\frac{2\pi}{\alpha_1}\Big) = -11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2} + 2\ln\alpha_2\Big) = -1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3\Big) = 3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big);$$

$$\frac{d\ln\det Y_U}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_u} + \mbox{tr}(\gamma_Q + \gamma_U)\Big);$$

$$\frac{d\ln\det Y_D}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}(\gamma_Q + \gamma_D)\Big);$$

$$\frac{d\ln\det Y_E}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}(\gamma_L + \gamma_E)\Big);$$

$$\frac{d\ln\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big).$$

The anomalous dimensions of the chiral matter superfields and $\boldsymbol{\mu}$ can be eliminated, thereby obtaining a differential equation which contains only derivatives of the gauge and Yukawa couplings.

Eliminating the anomalous dimensions of the matter superfields

First, we eliminate $\mbox{tr}(\gamma_L)$, $\mbox{tr}(\gamma_E)$, and $\gamma_{H_u}+\gamma_{H_d}$. The resulting equations contain γ_Q, γ_D, and γ_U only in the combination $\mbox{tr}(2\gamma_Q+\gamma_U+\gamma_D)$,

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3}+3\ln\alpha_3\Big)=3+\frac{1}{2}\mbox{tr}\Big(2\gamma_Q+\gamma_U+\gamma_D\Big);$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2}+2\ln\alpha_2+\frac{5}{3}\cdot\frac{2\pi}{\alpha_1}-2\ln\boldsymbol{\mu}-2\ln\mbox{det}\,Y_E+2\ln\mbox{det}\,Y_D\Big)=-12+\frac{4}{3}\mbox{tr}\Big(2\gamma_Q+\gamma_U+\gamma_D\Big);$$

$$\frac{d}{d\ln\mu}\Big(\ln\mbox{det}\,Y_D+\ln\mbox{det}\,Y_U-3\ln\boldsymbol{\mu}\Big)=\frac{1}{2}\mbox{tr}\Big(2\gamma_Q+\gamma_U+\gamma_D\Big).$$

This allows either eliminating the one-loop constants or eliminating the parameter $\boldsymbol{\mu}$. The resulting equations take the form

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3}+3\ln\alpha_3+\frac{\pi}{2\alpha_2}+\frac{1}{2}\ln\alpha_2+\frac{5\pi}{6\alpha_1}-\frac{1}{2}\ln\mbox{det}\,Y_E-\frac{7}{6}\ln\mbox{det}\,Y_D-\frac{5}{3}\ln\mbox{det}\,Y_U+\frac{9}{2}\ln\boldsymbol{\mu}\Big)=0,$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3}+3\ln\alpha_3-\frac{\pi}{\alpha_2}-\ln\alpha_2-\frac{5\pi}{3\alpha_1}+\ln\mbox{det}\,Y_E-\frac{2}{3}\ln\mbox{det}\,Y_D+\frac{1}{3}\ln\mbox{det}\,Y_U-9\ln\mu\Big)=0,$$

respectively.

Integrating the first equation we obtain the expression RGI_1, which does not explicitly depend on the scale μ, but contains the parameter $\boldsymbol{\mu}$. Integrating the second equation gives the expression RGI_2 independent of $\boldsymbol{\mu}$, but containing the scale μ,

$$\mathrm{RGI}_1 = \frac{\boldsymbol{\mu}^{9/2}\left(\alpha_3\right)^3\left(\alpha_2\right)^{1/2}}{\left(\det Y_E\right)^{1/2}\left(\det Y_U\right)^{5/3}\left(\det Y_D\right)^{7/6}}\exp\Big(\frac{2\pi}{\alpha_3}+\frac{\pi}{2\alpha_2}+\frac{5\pi}{6\alpha_1}\Big);$$

$$\mathrm{RGI}_2 = \frac{\left(\alpha_3\right)^3\det Y_E\left(\det Y_U\right)^{1/3}}{\mu^9\,\alpha_2\left(\det Y_D\right)^{2/3}}\exp\Big(\frac{2\pi}{\alpha_3}-\frac{\pi}{\alpha_2}-\frac{5\pi}{3\alpha_1}\Big).$$

Instead of the renormalization group invariants $(\mathrm{RGI}_1,\ \mathrm{RGI}_2)$ it is possible to use the equivalent set $(\mathrm{RGI}_3,\ \mathrm{RGI}_4)$, where the expressions

$$\mathrm{RGI}_3 \equiv \Big(\frac{\mathrm{RGI}_1}{\mathrm{RGI}_2}\Big)^{2/3} = \frac{\boldsymbol{\mu}^3\,\mu^6\,\alpha_2}{\left(\det Y_E\right)\left(\det Y_U\right)^{4/3}\left(\det Y_D\right)^{1/3}}\exp\Big(\frac{\pi}{\alpha_2}+\frac{5\pi}{3\alpha_1}\Big);$$

$$\mathrm{RGI}_4 \equiv \left(\mathrm{RGI}_1\right)^{2/3}\left(\mathrm{RGI}_2\right)^{1/3} = \frac{\boldsymbol{\mu}^3\left(\alpha_3\right)^3}{\mu^3\det Y_U\det Y_D}\exp\Big(\frac{2\pi}{\alpha_3}\Big)$$

also have a rather simple form.

The three-loop check

Differentiating $\ln(\text{RGI}_3)$ and $\ln(\text{RGI}_4)$ with respect to $\ln\mu$ we obtain the equations

$$0 = \Big(\frac{1}{\alpha_2} - \frac{\pi}{\alpha_2^2}\Big)\beta_2 - \frac{5\pi}{3{\alpha_1}^2}\beta_1 + 6 + 3\gamma_{\boldsymbol{\mu}} - \gamma_{\det Y_E} - \frac{4}{3}\gamma_{\det Y_U} - \frac{1}{3}\gamma_{\det Y_D};$$

$$0 = \Big(\frac{3}{\alpha_3} - \frac{2\pi}{\alpha_3^2}\Big)\beta_3 - 3 + 3\gamma_{\boldsymbol{\mu}} - \gamma_{\det Y_U} - \gamma_{\det Y_D}.$$

The scheme dependence of these equations becomes essential starting from the order $O(\alpha^2, \alpha Y^2, Y^4)$ corresponding to the three-loop approximation for the β-functions and to the two-loop approximation for the anomalous dimensions.

In the HD+MSL scheme they should be satisfied in all orders independently of the regularization parameters

$$A \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{R(x)}; \qquad a_{\varphi,3} \equiv \frac{M_{\varphi,3}}{\Lambda}; \qquad a_{\varphi,2} \equiv \frac{M_{\varphi,2}}{\Lambda};$$

$$B \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{F^2(x)}; \qquad a_3 \equiv \frac{M_3}{\Lambda}; \qquad a_2 \equiv \frac{M_2}{\Lambda}; \qquad a_1 \equiv \frac{M_1}{\Lambda},$$

where $R(x)$ and $F(x)$ are the higher derivative regulator functions, and M_i are the Pauli–Villars masses.

The three-loop β-functions for the MSSM in the HD+MSL scheme have been calculated in

O. Haneychuk, V. Shirokova, K.S., JHEP **09** (2022), 189.

$$\frac{\beta_1(\alpha,Y)}{\alpha_1^2} = -\frac{1}{2\pi}\cdot\frac{3}{5}\Big\{ -11 - \frac{199\alpha_1}{60\pi} - \frac{9\alpha_2}{4\pi} - \frac{22\alpha_3}{3\pi} + \frac{1}{8\pi^2}\mbox{tr}\Big(\frac{13}{3}Y_U Y_U^+ + \frac{7}{3} Y_D Y_D^+ + 3 Y_E Y_E^+\Big)$$
$$+\frac{1}{2\pi^2}\Big[\frac{5131\alpha_1^2}{3600} + \frac{27\alpha_2^2}{16} + \frac{88\alpha_3^2}{9} + \frac{23\alpha_1\alpha_2}{40} + \frac{137\alpha_1\alpha_3}{45} + \alpha_2\alpha_3 + \frac{2189\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2}\Big)$$
$$+\frac{9\alpha_2^2}{4}\Big(7\ln a_2 - 6\ln a_{\varphi,2} + 1 + \frac{A}{2}\Big) - 22\alpha_3^2\Big(3\ln a_{\varphi,3} - 2\ln a_3 + 1 + \frac{A}{2}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_U Y_U^+\Big)$$
$$\times\Big[2\alpha_2 + 2\alpha_3 + (B-A)\Big(\frac{169\alpha_1}{180} + \frac{13\alpha_2}{4} + \frac{52\alpha_3}{9}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_D Y_D^+\Big)\Big[\frac{\alpha_2}{2} + 2\alpha_3 + (B-A)$$
$$\times\Big(\frac{49\alpha_1}{180} + \frac{7\alpha_2}{4} + \frac{28\alpha_3}{9}\Big)\Big] + \frac{1}{8\pi^3}\mbox{tr}\Big(Y_E Y_E^+\Big)\Big[\frac{3\alpha_2}{2} + (B-A)\Big(\frac{27\alpha_1}{20} + \frac{9\alpha_2}{4}\Big)\Big] - \frac{1}{(8\pi^2)^2}\Big[\frac{15}{4}$$
$$\times\mbox{tr}\Big((Y_U Y_U^+)^2\Big) + \frac{11}{4}\mbox{tr}\Big((Y_D Y_D^+)^2\Big) + \frac{9}{4}\mbox{tr}\Big((Y_E Y_E^+)^2\Big) + \frac{19}{6}\mbox{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big) + \frac{17}{4}\Big(\mbox{tr}(Y_U Y_U^+)\Big)^2$$
$$+\frac{5}{4}\Big(\mbox{tr}(Y_D Y_D^+)\Big)^2 + \frac{5}{4}\Big(\mbox{tr}(Y_E Y_E^+)\Big)^2 + \frac{25}{6}\mbox{tr}\Big(Y_E Y_E^+\Big)\,\mbox{tr}\Big(Y_D Y_D^+\Big)\Big]\Big\} + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\frac{\beta_2(\alpha,Y)}{\alpha_2^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6); \qquad \frac{\beta_3(\alpha,Y)}{\alpha_3^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6).$$

The three-loop verification, the HD+MSL scheme

The two-loop anomalous dimensions in the HD+MSL scheme can be calculated starting from the anomalous dimensions of the chiral matter superfields

$$\gamma_{\det Y_U}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_u}(\alpha,Y) + \mathrm{tr}\,\gamma_Q(\alpha,Y) + \mathrm{tr}\,\gamma_U(\alpha,Y)\Big)$$

$$= -\frac{13\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{4\alpha_3}{\pi} + \frac{1}{16\pi^2}\,\mathrm{tr}\Big(12\,Y_U Y_U^+ + Y_D Y_D^+\Big) + \frac{1}{2\pi^2}\Big[\frac{169\alpha_1^2}{1200} + \frac{27\alpha_2^2}{16} + \frac{16\alpha_3^2}{3}$$

$$+\frac{3\alpha_1\alpha_2}{8} + \frac{17\alpha_1\alpha_3}{15} + 3\alpha_2\alpha_3 - \frac{27\alpha_2^2}{2}\Big(\ln a_{\varphi,2} + 1 + \frac{A}{2}\Big) - 36\alpha_3^2\Big(\ln a_{\varphi,3} + 1 + \frac{A}{2}\Big)$$

$$+\frac{429\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2}\Big) + \frac{63\alpha_2^2}{4}\Big(\ln a_2 + 1 + \frac{A}{2}\Big) + 24\alpha_3^2\Big(\ln a_3 + 1 + \frac{A}{2}\Big)\Big] + \frac{1}{16\pi^3}$$

$$\times\,\mathrm{tr}(Y_U Y_U^+)\Big[\frac{7\alpha_1}{10} + \frac{3\alpha_2}{2} + 12\alpha_3 + (B-A)\Big(\frac{13\alpha_1}{5} + 9\alpha_2 + 16\alpha_3\Big)\Big] + \frac{1}{16\pi^3}\mathrm{tr}(Y_D Y_D^+)$$

$$\times\Big[\frac{\alpha_1}{10} + (B-A)\Big(\frac{7\alpha_1}{60} + \frac{3\alpha_2}{4} + \frac{4\alpha_3}{3}\Big)\Big] - \frac{1}{(16\pi^2)^2}\Big[31\,\mathrm{tr}\Big((Y_U Y_U^+)^2\Big) + 2\,\mathrm{tr}\Big((Y_D Y_D^+)^2\Big)$$

$$+11\,\mathrm{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big) + 9\Big(\mathrm{tr}(Y_U Y_U^+)\Big)^2 + 3\Big(\mathrm{tr}(Y_D Y_D^+)\Big)^2 + \mathrm{tr}(Y_E Y_E^+)\,\mathrm{tr}(Y_D Y_D^+)\Big]$$

$$+O\Big(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6\Big);$$

$$\gamma_{\det Y_D}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \mathrm{tr}\,\gamma_Q(\alpha,Y) + \mathrm{tr}\,\gamma_D(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$

$$\gamma_{\det Y_E}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \mathrm{tr}\,\gamma_L(\alpha,Y) + \mathrm{tr}\,\gamma_E(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$

$$\gamma_{\mu}(\alpha,Y) = \frac{1}{2}\Big(\gamma_{H_u}(\alpha,Y) + \gamma_{H_d}(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

Substituting the above expressions for RGFs in the HD+MSL scheme we see that in the considered approximation the derivatives of the expressions RGI_3 and RGI_4 vanish independently of the values of the regularization parameters,

$$\Big[\Big(\frac{1}{\alpha_2}-\frac{\pi}{\alpha_2^2}\Big)\beta_2-\frac{5\pi}{3\alpha_1^2}\beta_1+6+3\gamma_{\boldsymbol{\mu}}-\gamma_{\text{det}\,Y_E}$$
$$-\frac{4}{3}\gamma_{\text{det}\,Y_U}-\frac{1}{3}\gamma_{\text{det}\,Y_D}\Big]_{\text{HD+MSL}}=O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big);$$

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_{\boldsymbol{\mu}}-\gamma_{\text{det}\,Y_U}-\gamma_{\text{det}\,Y_D}\Big]_{\text{HD+MSL}}$$
$$=O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big).$$

Therefore, in the considered approximation the expressions RGI_1 and RGI_2 also do not depend on the renormalization point μ in the HD+MSL scheme.

Certainly, this is quite expected because the HD+MSL scheme is NSVZ in all orders.

However, the most popular renormalization prescription in the supersymmetric case is the $\overline{\mbox{DR}}$ scheme. The matter is that dimensional regularization

G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972), 189;
C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12** (1972), 20;
J. F. Ashmore, Lett. Nuovo Cim. **4** (1972), 289;
G. M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972), 329

explicitly breaks supersymmetry, because the numbers of boson and fermion degrees of freedom differently depend on the space-time dimension.
That is why in the supersymmetric case it is more convenient to use its modification called dimensional reduction

W. Siegel, Phys. Lett. B **84** (1979), 193.

In this case the γ-matrices are taken in the integer dimension (usually, $D = 4$), while the loop integrals are calculated in the dimension $D = 4 - \varepsilon$.

The $\overline{\mbox{DR}}$ scheme is obtained if the dimensional reduction is supplemented by modified minimal subtraction.

The three-loop β-functions for the MSSM in the $\overline{\text{DR}}$ scheme have been calculated in

I. Jack, D. R. T. Jones, A. F. Kord, Annals Phys. **316** (2005), 213.

$$\frac{\beta_1(\alpha,Y)}{\alpha_1^2} = -\frac{1}{2\pi}\cdot\frac{3}{5}\Big\{-11 - \frac{199\alpha_1}{60\pi} - \frac{9\alpha_2}{4\pi} - \frac{22\alpha_3}{3\pi} + \frac{1}{8\pi^2}\mathrm{tr}\Big(\frac{13}{3}Y_U Y_U^+ + \frac{7}{3}Y_D Y_D^+ + 3Y_E Y_E^+\Big)$$
$$+\frac{1}{2\pi^2}\Big(\frac{32117\alpha_1^2}{1800} + \frac{27\alpha_2^2}{8} - \frac{121\alpha_3^2}{18} + \frac{23\alpha_1\alpha_2}{40} + \frac{137\alpha_1\alpha_3}{45} + \alpha_2\alpha_3\Big) + \frac{1}{8\pi^3}\mathrm{tr}\Big(Y_U Y_U^+\Big)\Big(\frac{169\alpha_1}{360}$$
$$+\frac{29\alpha_2}{8} + \frac{44\alpha_3}{9}\Big) + \frac{1}{8\pi^3}\mathrm{tr}\Big(Y_D Y_D^+\Big)\Big(\frac{49\alpha_1}{360} + \frac{11\alpha_2}{8} + \frac{32\alpha_3}{9}\Big) + \frac{1}{8\pi^3}\mathrm{tr}\Big(Y_E Y_E^+\Big)\Big(\frac{27\alpha_1}{40} + \frac{21\alpha_2}{8}\Big)$$
$$-\frac{1}{(8\pi^2)^2}\Big[7\,\mathrm{tr}\Big((Y_U Y_U^+)^2\Big) + \frac{9}{2}\mathrm{tr}\Big((Y_D Y_D^+)^2\Big) + \frac{9}{2}\mathrm{tr}\Big((Y_E Y_E^+)^2\Big) + \frac{29}{6}\mathrm{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)$$
$$+\frac{15}{2}\Big(\mathrm{tr}(Y_U Y_U^+)\Big)^2 + 3\Big(\mathrm{tr}(Y_D Y_D^+)\Big)^2 + 2\Big(\mathrm{tr}(Y_E Y_E^+)\Big)^2 + 7\,\mathrm{tr}\Big(Y_E Y_E^+\Big)\,\mathrm{tr}\Big(Y_D Y_D^+\Big)\Big]\Big\}$$
$$+O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$

$$\frac{\beta_2(\alpha,Y)}{\alpha_2^2} = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$
$$\frac{\beta_3(\alpha,Y)}{\alpha_3^2} = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

The three-loop verification, the $\overline{\text{DR}}$ scheme

Again, the two-loop anomalous dimensions in the $\overline{\text{DR}}$ scheme can be calculated starting from the anomalous dimensions of the chiral matter superfields

$$\gamma_{\mathbf{det}\,Y_U}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_u}(\alpha,Y) + \mathbf{tr}\,\gamma_Q(\alpha,Y) + \mathbf{tr}\,\gamma_U(\alpha,Y)\Big)$$

$$= -\frac{13\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{4\alpha_3}{\pi} + \frac{1}{16\pi^2}\,\mathbf{tr}\Big(12\,Y_U Y_U^+ + Y_D Y_D^+\Big) + \frac{1}{2\pi^2}\Big[\frac{2743\alpha_1^2}{1200} + \frac{45\alpha_2^2}{16} - \frac{2\alpha_3^2}{3}$$

$$+\frac{3\alpha_1\alpha_2}{8} + \frac{17\alpha_1\alpha_3}{15} + 3\alpha_2\alpha_3\Big] + \frac{1}{16\pi^3}\,\mathbf{tr}(Y_U Y_U^+)\Big[\frac{7\alpha_1}{10} + \frac{3\alpha_2}{2} + 12\alpha_3\Big] + \frac{1}{16\pi^3}\mathbf{tr}(Y_D Y_D^+)$$

$$\times\frac{\alpha_1}{10} - \frac{1}{(16\pi^2)^2}\Big[31\,\mathbf{tr}\Big((Y_U Y_U^+)^2\Big) + 2\,\mathbf{tr}\Big((Y_D Y_D^+)^2\Big) + 11\,\mathbf{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big) + 9\Big(\mathbf{tr}(Y_U Y_U^+)\Big)^2$$

$$+3\Big(\mathbf{tr}(Y_D Y_D^+)\Big)^2 + \mathbf{tr}(Y_E Y_E^+)\,\mathbf{tr}(Y_D Y_D^+)\Big] + O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big);$$

$$\gamma_{\mathbf{det}\,Y_D}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \mathbf{tr}\,\gamma_Q(\alpha,Y) + \mathbf{tr}\,\gamma_D(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\gamma_{\mathbf{det}\,Y_E}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \mathbf{tr}\,\gamma_L(\alpha,Y) + \mathbf{tr}\,\gamma_E(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\gamma_{\boldsymbol{\mu}}(\alpha,Y) = \frac{1}{2}\Big(\gamma_{H_u}(\alpha,Y) + \gamma_{H_d}(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6).$$

However, in the $\overline{\text{DR}}$ scheme the derivatives of $\ln\text{RGI}_3$ and $\ln\text{RGI}_4$ with respect to $\ln\mu$ do not vanish in that orders where the scheme dependence becomes essential.

The three-loop verification, the $\overline{\text{DR}}$ scheme

$$\Big[\Big(\frac{1}{\alpha_2}-\frac{\pi}{\alpha_2^2}\Big)\beta_2-\frac{5\pi}{3\alpha_1^2}\beta_1+6+3\gamma_\mu-\gamma_{\det Y_E}-\frac{4}{3}\gamma_{\det Y_U}-\frac{1}{3}\gamma_{\det Y_D}\Big]_{\overline{\text{DR}}}$$

$$=\frac{1}{2\pi^2}\Big(\frac{1243\alpha_1^2}{400}+\frac{17\alpha_2^2}{16}-5\alpha_3^2\Big)+\frac{1}{16\pi^3}\text{tr}(Y_U Y_U^+)\Big(\frac{143\alpha_1}{180}+\frac{11\alpha_2}{4}+\frac{44\alpha_3}{9}\Big)$$

$$+\frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)\Big(\frac{14\alpha_1}{45}+2\alpha_2+\frac{32\alpha_3}{9}\Big)+\frac{1}{16\pi^3}\text{tr}(Y_E Y_E^+)\Big(\frac{9\alpha_1}{10}+\frac{3\alpha_2}{2}\Big)$$

$$-\frac{1}{(16\pi^2)^2}\Big[11\,\text{tr}\Big((Y_U Y_U^+)^2\Big)+8\,\text{tr}\Big((Y_D Y_D^+)^2\Big)+6\,\text{tr}\Big((Y_E Y_E^+)^2\Big)+\frac{19}{3}\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)$$

$$+11\Big(\text{tr}(Y_U Y_U^+)\Big)^2+8\Big(\text{tr}(Y_D Y_D^+)\Big)^2+2\Big(\text{tr}(Y_E Y_E^+)\Big)^2+\frac{26}{3}\text{tr}\Big(Y_E Y_E^+\Big)\,\text{tr}\Big(Y_D Y_D^+\Big)\Big]$$

$$+O\Big(\alpha^3,\alpha^2 Y^2,\alpha Y^4,Y^6\Big)\ne O\Big(\alpha^3,\alpha^2 Y^2,\alpha Y^4,Y^6\Big);$$

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_\mu-\gamma_{\det Y_U}-\gamma_{\det Y_D}\Big]_{\overline{\text{DR}}}=\frac{1}{2\pi^2}\Big(\frac{363\alpha_1^2}{400}+\frac{9\alpha_2^2}{16}-\frac{21\alpha_3^2}{8}\Big)$$

$$+\frac{1}{16\pi^3}\text{tr}(Y_U Y_U^+)\Big(\frac{13\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)+\frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)\Big(\frac{7\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)$$

$$-\frac{1}{(16\pi^2)^2}\Big[6\,\text{tr}\Big((Y_U Y_U^+)^2\Big)+6\,\text{tr}\Big((Y_D Y_D^+)^2\Big)+6\Big(\text{tr}(Y_U Y_U^+)\Big)^2+6\Big(\text{tr}(Y_D Y_D^+)\Big)^2$$

$$+2\,\text{tr}\Big(Y_E Y_E^+\Big)\,\text{tr}\Big(Y_D Y_D^+\Big)+4\,\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big]+O\Big(\alpha^3,\alpha^2 Y^2,\alpha Y^4,Y^6\Big)$$

$$\ne O\Big(\alpha^3,\alpha^2 Y^2,\alpha Y^4,Y^6\Big).$$

The parameter μ in the MSSM superpotential should be of the order of the electroweak scale, which is impossible to explain in MSSM. The μ problem can be solved in the Next-to-Minimal Supersymmetric Standard Model (NMSSM)

M. Maniatis, Int. J. Mod. Phys. A **25** (2010), 3505;
U. Ellwanger, C. Hugonie, A. M. Teixeira,Phys. Rept. **496** (2010), 1,

which contains an additional chiral matter superfield S. This superfield is a singlet with respect to $SU(3)\times SU(2)\times U(1)$. Then it is possible to replace the μ term

$$\Delta W_{\mbox{MSSM}} = \mu\,(H_{u1}\ \ H_{u2}) \left(\begin{array}{cc} 0 & 1\\ -1 & 0 \end{array}\right) \left(\begin{array}{c} H_{d1}\\ H_{d2} \end{array}\right)$$

by the gauge invariant expression

$$\Delta W_{\mbox{NMSSM}} = \lambda S\,(H_{u1}\ \ H_{u2}) \left(\begin{array}{cc} 0 & 1\\ -1 & 0 \end{array}\right) \left(\begin{array}{c} H_{d1}\\ H_{d2} \end{array}\right) + \frac{\kappa}{3} S^3,$$

in which λ and κ are new dimensionless couplings. In this case the effective value of μ is equal to the vacuum expectation value of (the lowest component of) S multiplied by λ and can have an order of the electroweak scale.

Due to the nonrenormalization of the superpotential, the anomalous dimensions of λ and κ satisfy the all-loop equations

$$\gamma_\kappa \equiv \frac{d\ln\kappa}{d\ln\mu} = \frac{3}{2}\gamma_S; \qquad \gamma_\lambda \equiv \frac{d\ln\lambda}{d\ln\mu} = \frac{1}{2}\Big(\gamma_S + \gamma_{H_u} + \gamma_{H_d}\Big).$$

Therefore, the sum which for the MSSM gives $\gamma_{\boldsymbol{\mu}}$ can be written as

$$\frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big) = \gamma_\lambda - \frac{1}{3}\gamma_\kappa.$$

The NSVZ relations for NMSSM are the same as for MSSM (although the anomalous dimensions are different). The equations describing the running of the Yukawa couplings Y_E, Y_U, and Y_D also remain unchanged. That is why RGIs for NMSSM can be obtained from the ones for MSSM after the replacement

$$\boldsymbol{\mu}\ \to\ \lambda\kappa^{-1/3}.$$

The expression $\mbox{RGI}_2$ does not depend on $\boldsymbol{\mu}$ and, therefore, is also RGI for NMSSM. The $\mbox{RGI}_1$ after this replacement takes the form

$$\widetilde{\mbox{RGI}}_1 = \frac{\lambda^{9/2}\,(\alpha_3)^3\,(\alpha_2)^{1/2}}{\kappa^{3/2}\,\big(\det Y_E\big)^{1/2}\,\big(\det Y_U\big)^{5/3}\,\big(\det Y_D\big)^{7/6}}\,\exp\Big(\frac{2\pi}{\alpha_3} + \frac{\pi}{2\alpha_2} + \frac{5\pi}{6\alpha_1}\Big).$$

$6D$, $\mathcal{N} = (1,0)$ supersymmetic theories in the harmonic superspace

Usual supersymmetric theories in higher dimensions are not renormalizable, because the degree of divergence increases with the number of loops. However, in this case it is possible to consider theories with higher derivatives. We will consider a model analogous to the $6D$, $\mathcal{N} = (1,0)$ higher derivative supersymmetric theory proposed in

E. A. Ivanov, A. V. Smilga and B. M. Zupnik, Nucl. Phys. B **726** (2005), 131.

It is convenient to formulate it in $6D$, $\mathcal{N} = (1,0)$ harmonic superspace

P. S. Howe, G. Sierra and P. K. Townsend, Nucl. Phys. B **221** (1983), 331;
P. S. Howe, K. S. Stelle and P. C. West, Class. Quant. Grav. **2** (1985), 815;
B. M. Zupnik, Sov. J. Nucl. Phys. **44** (1986), 512;
E. A. Ivanov and A. V. Smilga, Phys. Lett. B **637** (2006), 374;
I. L. Buchbinder and N. G. Pletnev, Nucl. Phys. B **892** (2015), 21,

which is similar to the usual $4D$, $\mathcal{N} = 2$ harmonic superspace

A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984), 469;
A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, "Harmonic superspace", Cambridge, UK: Univ. Pr. (2001) 306 p..

The $6D$, $\mathcal{N}=(1,0)$ harmonic superspace

The harmonic superspace is especially convenient for formulating $6D$, $\mathcal{N}=(1,0)$ supersymmetric theories because it makes $\mathcal{N}=(1,0)$ supersymmetry manifest.

The $6D$, $\mathcal{N}=(1,0)$ harmonic superspace is parametrized by the coordinates $(x^\mu, \theta_i^a, u^{\pm i})$, where $\mu = 0,\ldots,5$, θ_i^a (with $a=1,\ldots,4$ and $i=1,2$) are the anticommuting left-handed spinors, and the harmonic variables $u^{\pm i}$ satisfy the relations

$$u_i^- = (u^{+i})^*, \qquad u^{+i}u_i^- = 1, \qquad u_i^\pm = \varepsilon_{ij}u^{\pm j}.$$

It contains the analytic subspace closed under supersymmetry transformations with coordinates

$$x_A^\mu = x^\mu + \frac{i}{2}\theta^-\gamma^\mu\theta^+, \qquad \theta^{+a} = u_i^+\theta^{ai}, \qquad u_i^\pm.$$

The gauge superfield and the hypermultiplet (in the adjoint representation) are described by the analytic superfields $V^{++} = e_0 V^{++A}t^A$ and $q^+ = e_0 q^{+A}t^A$, respectively,

$$D_a^+ V^{++} = 0; \qquad D_a^+ q^+ = 0,$$

where $D_a^+ = u_i^+ D_a^i$. Also we will need the harmonic derivatives

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}, \qquad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}.$$

Following the paper

I. L. Buchbinder, A. S. Budekhina, E. A. Ivanov and K.S., Phys. Rev. D **111** (2025) no.12, 125014.

we consider the theory which in the $6D$, $\mathcal{N}=(1,0)$ harmonic superspace is described by the action

$$S=\pm\frac{1}{2e_0^2}\mathrm{tr}\int d\zeta^{(-4)}(F^{++})^2-\frac{2}{e_0^2}\,\mathrm{tr}\int d\zeta^{(-4)}\widetilde{q^+}\nabla^{++}q^+.$$

Here $\nabla^{++}q^+\equiv D^{++}q^++i[V^{++},q^+]$, and the integration measure is given by the expression

$$\int d\zeta^{(-4)}\equiv\int d^6x\,d^4\theta^+\,du.$$

In particular, this measure contains the integration over harmonics. The harmonic superspace analog of the gauge field strength is defined by the equations

$$F^{++}\equiv(D^+)^4V^{--},\qquad \text{where}$$

$$V^{--}(z,u)\equiv\sum_{n=1}^{\infty}(-i)^{n+1}\int du_1\dots du_n\frac{V^{++}(z,u_1)V^{++}(z,u_2)\dots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}.$$

The $6D$, $\mathcal{N}=(1,0)$ higher derivative theory

In components this action contains the term with higher derivatives of the gauge field (and its superpartners)

$$S = \mathrm{tr}\int d^6x\,\Big\{\pm\frac{1}{e_0^2}(\mathcal{D}_\mu F^{\mu\nu})^2 + \ldots\Big\}.$$

Due to the higher derivatives the degree of divergence does not increase with a number of loops.

The theory could contain quadratic and logarithmical divergences, but the quadratic divergences cancel each other in the one-loop approximation (and presumably in all loops) due to the presence of the hypermultiplet in the adjoint representation. Moreover, due to the presence of the hypermultiplet, the theory is not anomalous

A. V. Smilga, Phys. Lett. B **647** (2007), 298

and seems to be renormalizable.

The hypermultiplet and ghosts do not receive divergent quantum corrections because the corresponding parts of the action are not given by the integrals over the total harmonic superspace. (This is analogous to the nonrenormalization of the superpotential in the $4D$ case.)

Let us regularize the theory under consideration by higher covariant derivatives. The higher derivative term is constructed with the help of the operator

$$\Box \equiv \frac{1}{2}(D^+)^4(\nabla^{--})^2,$$

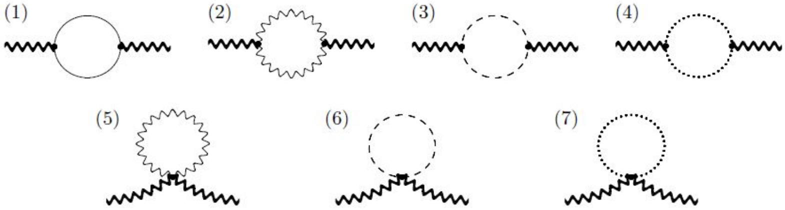
(where $\nabla^{--} = D^{--} + iV^{--}$), which is analogous to the Laplace operator when acting on analytic superfields. The sum of the original action and the higher derivative term can be written in the form

$$S_{\text{reg}} = \pm\frac{1}{2e_0^2}\text{tr}\int d\zeta^{(-4)} F^{++} R\Big(\frac{\Box}{\Lambda^2}\Big) F^{++} - \frac{2}{e_0^2}\,\text{tr}\int d\zeta^{(-4)}\widetilde{q^+}\nabla^{++}q^+,$$

where $R(0) = 1$ and $R(x) \to \infty$ at $x \to \infty$. For regularizing the one-loop divergences it is also necessary to add the Pauli–Villars superfields with the mass $M = a\Lambda$. (For simplicity, we do not present the explicit expression for their action.) Then the generating functional of the regularized theory takes the form

$$Z[\text{sources}] = \int Dv^{++} D\widetilde{q^+} Dq^+ Db\, Dc\, D\varphi\, \text{Det}^{1/2}\Big[\Box^2 R\Big(\frac{\Box}{\Lambda^2}\Big)\Big]$$
$$\times\, \text{Det}(PV, M)\, \exp\Big(iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{FP}} + iS_{\text{NK}} + iS_{\text{sources}}\Big).$$

The renormalization of the coupling constant is determined by the following harmonic supergraphs:



The solid lines correspond to the hypemultiplet;
the wavy lines correspond to the gauge superfield;
the dashed lines denote propagators of the Faddeev–Popov ghosts;
the dotted lines denote propagators of the Nielsen–Kallosh ghosts.

Quadratic divergences in these superdiagrams cancel each other, while the logarithmical divergences determine the β-function $\beta(\alpha_0)$ (defined in terms of the bare coupling constant), where $\alpha_0 \equiv e_0^2/4\pi$.

The one-loop β-function

The result for the β-function obtained after calculating the above superdiagrams can be written in the form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \mp 2\pi C_2 \int \frac{d^6q}{(2\pi)^6} \frac{d}{d\ln\Lambda} \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q^\mu} \left[\frac{1}{q^4} \ln\left(1 + \frac{M^4}{q^4 R(q^2/\Lambda^2)}\right)\right] + O(\alpha_0).$$

We see that exactly as in the $4D$ case the β-function is given by integrals of double total derivatives with respect to the loop momentum. Note that, due to the presence of an arbitrary regulator function $R(x)$, this fact is highly nontrivial. Calculating the integrals we obtain the one-loop result

$$\beta(\alpha_0) = \mp \frac{\alpha_0^2 C_2}{2\pi^2} + O(\alpha_0^3).$$

This expression agrees with the results of the calculations made with dimensional reduction in

E. A. Ivanov, A. V. Smilga and B. M. Zupnik, Nucl. Phys. B **726** (2005), 131;
L. Casarin and A. A. Tseytlin, JHEP **08** (2019), 159;
I. L. Buchbinder, E. A. Ivanov, B. S. Merzlikin and K.S., JHEP **08** (2020), 169; Nucl. Phys. B **961** (2020), 115249

by various methods if one takes into account the contribution of the hypermultiplet in the adjoint representation.

The resemblance in the structure of the one-loop results for $4D$, $\mathcal{N}=1$ supersymmetric Yang–Mills theory and for the considered $6D$, $\mathcal{N}=(1,0)$ higher derivative theory allows to suggest that it may be possible to write down the all-loop exact expression for the β-function. It can be constructed by replacing the tree propagators by the exact ones in the one-loop singular contributions. (For $4D$, $\mathcal{N}=1$ theories the similar procedure gives the NSVZ expression.) In the $6D$ case the result has the form

$$\beta(\alpha_0) = \mp\frac{\alpha_0^2 C_2}{2\pi^2\Big(1\mp\alpha_0 C_2/8\pi^2\Big)}.$$

Certainly, this derivation is not rigourous and should be verified by explicit multiloop calculations. (If possible), it would be also expedient to construct its rigorous all-order proof analogous to the one for the $4D$, $\mathcal{N}=1$ case.

Similarly to the pure $4D$, $\mathcal{N}=1$ SYM theory, it is possible to integrate the renormalization group equation and obtain the expression that does not receive quantum corrections in any order of the perturbation theory,

$$\Big(\frac{\alpha}{\mu^4}\Big)^{C_2}\exp\Big(\pm\frac{8\pi^2}{\alpha}\Big) = \text{RGI}.$$

Conclusion

- In certain $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings one can construct such combinations of various couplings that do not depend on scale in all orders or, in other words, RGIs.
- In particular, in $\mathcal{N}=1$ SQCD interacting with $\mathcal{N}=1$ SQED RGI can be constructed from the strong and electromagnetic coupling constants (if the matter superfields have the same absolute values of the electromagnetic charges). Therefore, in this theory two gauge couplings do not run independently.
- For the MSSM and NMMSM one can construct two independent RGIs from the gauge couplings, Yukawa couplings and the μ parameter. They are scale independent in all orders in the HD+MSL scheme, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms.
- The explicit three-loop calculations confirm the renormalization group invariance of the constructed expressions in the HD+MSL scheme. However, in the $\overline{\text{DR}}$ scheme they start to depend on scale in such an approximation where the scheme dependence becomes essential.
- Presumably, there is an exact NSVZ-like β-function for a certain $6D$, $\mathcal{N}=(1,0)$ supersymmetric theory with higher derivatives. Due to this exact β-function, this theory also possesses an RGI.

Thank you for the attention!