CP-violation and fermion mass matrices

Circulant matrices, the Jarlskog invariant, and the Koide relation

A. Kleppe

SACT, Oslo

Bled Workshop, *What Comes Beyond the Standard Models*, 2026

The question

Last year: the Jarlskog invariant binds the two quark mass matrices tightly together. This talk is a sequel.
A mass matrix ansatz asks: what is the fundamental structure of the mass matrices? The usual question is whether some ansatz *works*.

Here the question is sharper

Which structures are excluded — and why?

The structure under scrutiny: the circulant matrix, the natural home of the democratic scenario.
Punchline: circulancy is *fatal* for quarks, but *fits* the charged leptons — via the Koide relation.

Flavour states and mass states

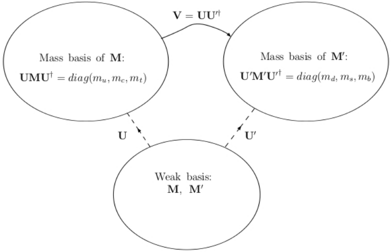
Fermion mass term:

$$\mathcal{L}_{\rm mass} = \bar{f} M f + \bar{f}' M' f' \qquad \longrightarrow \qquad \bar{\psi} D \psi + \bar{\psi}' D' \psi'$$

with $D = {\rm diag}(m_1, m_2, m_3)$, obtained by unitary rotations U, U'.

- Flavour states f: eigenstates of the weak interaction.
- Mass states ψ: the physical particles.

Experiment: $V_{\rm CKM} = UU'^{\dagger} \neq 1$ — the up-sector and down-sector mass bases differ; the mixing matrix bridges them.



Leptons are different: The idea of a flavour basis as something different than the mass bases of the two quark sectors may seem far fetched, but the neutrinos prove the reality of the flavour basis. We never "see" neutrino mass states, since they never appear in interactions: the flavour states ν_e, ν_μ, ν_τ are the neutrinos we "see".

The constraint from CP-violation

With three generations, V_{CKM} has a phase that no field redefinition can remove.

Jarlskog (1985)

CP-violation $\iff$ $\det[M_u, M_d]$ purely imaginary and nonzero, and

$$J_{CP} = \frac{-i\,\det[M_u, M_d]}{2P_uP_d}, \qquad P_u = (m_u - m_c)(m_c - m_t)(m_t - m_u), \ \ldots$$

Equivalent, rephasing-invariant form:

$$J_{CP} = \mathrm{Im}\big(V_{ij}V_{kl}V^*_{kj}V^*_{il}\big), \qquad i \neq k,\ j \neq l.$$

Measured value (PDG): $J_{CP} = (3.18 \pm 0.15) \times 10^{-5}$.
$\Rightarrow$ A meaningful ansatz must supply both matrices, with $\det[M_u, M_d] \neq 0$.

What the constraint really says

The commutator does not measure a physical operation. It measures the incompatibility of two structures sharing the same flavour space — the obstruction to simultaneous diagonalisation.

J_{CP} relates three things:

1. the eigenvalues of M_u,
2. the eigenvalues of M_d,
3. the relative orientation of the two matrices.

J_{CP} is intrinsically *relational*: a two-sector quantity. We cannot change M_u without adjusting M_d — the two charge sectors are locked together.

The democratic scheme

Two arguments meet:

From the mass basis: third-family dominance,

$$M_1 = C \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(rank one).}$$

From the flavour basis: identical Yukawa couplings before flavour symmetry breaking,

$$M_D = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \text{eigenvalues } (0, 0, 3C) \quad \text{(rank one).}$$

Same rank-one skeleton, viewed in two bases.

The democratic matrix M_D is an instance of a circulant matrix:

$$M_{\mathrm{circ}} = \begin{pmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{pmatrix}.$$

Circulant matrices: frozen eigenvectors

Every 3×3 circulant matrix — *regardless of its entries* — has the same eigenvectors:

$$f_j = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ \omega^j \\ \omega^{2j} \end{pmatrix}, \qquad \omega = e^{2\pi i/3}, \qquad F = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

F diagonalises *any* circulant matrix: $F^\dagger M F = \mathrm{diag}(m_1, m_2, m_3)$.

In a sense, all circulant matrices are one and the same

Same eigenspace, same diagonalising matrix — only the spectrum changes.

But a mass matrix must encode two things: the masses *and* the mixing. For a circulant matrix these roles are completely severed: the eigenvalues and the eigenvectors never communicate. A circulant matrix does only half the job.

One circulant quark mass matrix: the ansatz

Two circulants commute $\Rightarrow$ both sectors circulant is immediately excluded. What about *one*?

Circulant hermitian up-sector, $Z = X + iY$:

$$M = \begin{pmatrix} A & Z & Z^* \\ Z^* & A & Z \\ Z & Z^* & A \end{pmatrix}, \qquad \begin{aligned} A &= \tfrac{1}{3}(m_1 + m_2 + m_3) \\ X &= \tfrac{1}{6}(2m_1 - m_2 - m_3) \\ Y &= \pm\tfrac{1}{2\sqrt{3}}(m_3 - m_2) \end{aligned}$$

Non-circulant hermitian down-sector, three parameters, distinct diagonal:

$$N = \begin{pmatrix} L & R & R \\ R & R & R \\ R & R & S \end{pmatrix}, \qquad \det[M, N] = 2iY(R-L)(S-L)(S-R)(Y^2 - 3X^2).$$

The collapse of J_{CP}

Insert X, Y in terms of the up masses:

$$Y(Y^2 - 3X^2) = -\frac{1}{6\sqrt{3}}(m_1 - m_2)(m_2 - m_3)(m_3 - m_1) = -\frac{P_u}{6\sqrt{3}}.$$

The up-sector factor cancels against P_u:

$$J_{CP} = \frac{-i\,\det[M, N]}{2P_u P_d} = -\frac{1}{6\sqrt{3}}\,\frac{(R - L)(S - L)(S - R)}{P_d}.$$

Only down-sector quantities remain. The up sector has vanished from the CP-measure.

Is this an artefact of the texture N — or general?

The general mechanism

For arbitrary hermitian M_u, N_d, rotate to the up-mass basis:

$$[M_u, N_d] \longrightarrow [\mathrm{diag}(m_1, m_2, m_3),\ U_u N_d U_u^\dagger],$$

and U_u depends on the up-quark masses $\Rightarrow$ both sectors enter.

If M_u is circulant

$U_u = F$ — *mass-independent*. The decoupling has happened before the commutator is even computed. Since det of the rotated commutator carries the factor P_u, the up masses cancel:

$$J_{CP} \propto \frac{\mathcal{F}(N_d)}{P_d} \qquad \text{— the up sector drops out entirely.}$$

- General case: J_{CP} depends only on N_d — its eigenvalues *and* its orientation relative to the DFT basis.
- Our texture: the collapse goes all the way to the down *masses* alone.

Why this is fatal for quarks

J_{CP} is the measure of the *relative tilt* between the two sectors. With a circulant M_u:

- The quantity is still numerically nonzero — but it has stopped measuring what it is defined to measure: the up-sector tilt has dropped out of the tilt-measure.
- J_{CP} is intrinsically two-sector; a circulant sector collapses it to a one-sector expression, emptying it of its relational content.

A second, independent obstruction:

$$V_{\mathrm{CKM}} = F\, U_d^\dagger,$$

with every row of F democratic ($|F_{ij}|^2 = 1/3$), must reproduce the strongly *hierarchical* observed $|V_{ij}|$.

Conclusion for the quark sector

A quark mass matrix cannot be circulant. Circulant matrices are simply too rigid for the quark sector.

The lepton sector: a different situation

The same structural argument applies — but the physics differs:

- Leptonic CP-violation is *not established*: hinted at for neutrinos, not even hinted at for charged leptons.
- The PMNS angles are large, almost "democratic" in spirit — precisely what a circulant structure suggests.
- If neutrinos are Majorana: extra CP phases with no quark analogue, invisible to the Jarlskog measure.

A circulant charged-lepton matrix, sourcing all CP-violation from the neutrino sector, remains viable in a way its quark analogue is not.

The Koide relation

Koide (1981–83): the charged-lepton masses satisfy

$$Q = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} \approx \frac{2}{3}.$$

The history is remarkable:

- At the time: $m_\tau \approx 1784^{+3}_{-4}$ MeV $\Rightarrow Q \approx 0.6671$ — a 2σ discrepancy.
- Taken as exact, the relation *predicts* $m_\tau = 1776.97$ MeV — 7 MeV below the accepted value.
- BES (1992): m_τ shifts down to precisely this value.

Today (PDG 2024): $m_\tau = 1776.93(9)$ MeV $\Rightarrow$ $Q = 0.66666446(508)$.

Proposed at 2σ from the data; predicted m_τ to five digits; experiment moved to meet it. Holds to 10^{-5} for pole masses (not exactly RG-invariant: $Q(\mu)$ drifts by $\sim 0.2\%$ by M_Z).

Circulant hermitian M with eigenvalues $= \sqrt{\text{masses}}$:

$$M = \begin{pmatrix} A & Z & Z^* \\ Z^* & A & Z \\ Z & Z^* & A \end{pmatrix}, \qquad (m_1, m_2, m_3) = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}).$$

Then

$$Q = \frac{\mathrm{Tr}(M^2)}{(\mathrm{Tr}\, M)^2} = \frac{3A^2 + 6|Z|^2}{9A^2} = \frac{1}{3} + \frac{2|Z|^2}{3A^2}.$$

One condition on one invariant ratio

$$Q = \frac{2}{3} \quad \Longleftrightarrow \quad |Z| = \frac{A}{\sqrt{2}}.$$

Two free parameters remain; *every* spectrum they generate satisfies $Q = 2/3$ identically.

M^2 is again hermitian circulant $\Rightarrow$ a circulant hermitian charged-lepton mass matrix is entirely natural.

What Koide fixes — and what it leaves free

Eigenvalues of the circulant matrix (Viète form of the cubic):

$$\lambda_k = A + 2|Z|\cos\Big(\theta + \frac{2\pi k}{3}\Big), \qquad \theta = \arg Z.$$

- Koide fixes only the amplitude $|Z|/A = 1/\sqrt{2}$ of the root distribution.
- The phase θ is a genuine free parameter — completely unconstrained by $Q = 2/3$.

Honesty clause: the fit succeeds to 10^{-5} *because* the pole masses independently satisfy Koide to that accuracy — an empirical input, not a consequence of the ansatz. The same fit with $\overline{\mathrm{MS}}$ masses at M_Z would fail by $\sim 0.2\%$.

Crucially: the construction rests on charged-lepton ground alone — three well-measured masses, one algebraic condition, no neutrino input.

Circulant charged leptons and the PMNS matrix

If the charged-lepton matrix is circulant, its diagonaliser is F, with $|F_{ij}|^2 = 1/3$: complete democracy. With a trivial (diagonal) neutrino matrix, $U_{\mathrm{PMNS}} = F^\dagger$ — *trimaximal mixing*:

	circulant/DFT	NuFit-6.0 (NO)	status
$\sin^2\theta_{12}$	1/2	$0.307^{+0.012}_{-0.011}$	badly overshoots
$\sin^2\theta_{23}$	1/2	$0.561^{+0.012}_{-0.015}$	close to maximal
$\sin^2\theta_{13}$	1/3	$0.02195^{+0.00054}_{-0.00058}$	far off

θ_{23} lands close to its 45° prediction; θ_{12} and especially θ_{13} do not.
$\Rightarrow$ A nontrivial neutrino matrix is required to cancel the excess democratic mixing. The circulant charged-lepton sector is a *starting point* for U_{PMNS} — unlike its role in the Koide relation, where it succeeds unaided.

Conclusions

- J_{CP} locks the two quark sectors together: a viable ansatz must supply both matrices, reproducing six masses, the CKM elements and J_{CP}.
- If one quark mass matrix is circulant, its eigenvectors freeze to the DFT basis, the up masses cancel out of J_{CP}, and the invariant collapses to a one-sector expression — emptied of the relational content that defines it. A quark mass matrix cannot be circulant.
- For the charged leptons, precisely a circulant matrix (on the square-root-of-mass level, with $|Z| = A/\sqrt{2}$) reproduces the Koide relation — and with no established CP-violation in that sector, nothing forbids it.
- This would decouple charged leptons from neutrinos, confining leptonic CP-violation to the neutrino sector.
- The quark and lepton sectors may be more different than one usually imagines.

References

- C. Jarlskog, Phys. Rev. Lett. **55** (1985) 1039.
- M. Kobayashi, T. Maskawa, Prog. Theor. Phys. **49** (1973) 652.
- A. Kleppe, arXiv:2508.11081 [hep-ph], Bled Proceedings Vol. 26 (2025).
- Y. Koide, Lett. Nuovo Cimento **34** (1982) 201; Phys. Lett. B **120** (1983) 161; hep-ph/0005137; arXiv:1809.00425.
- P. M. Fishbane, P. Kaus, Phys. Rev. D **49** (1994) 4780.
- H. Fritzsch, Phys. Lett. B **70** (1977) 436; **73** (1978) 317.
- P. J. Davis, *Circulant Matrices*, Wiley (1979).
- J. Z. Bai et al. (BES), Phys. Rev. Lett. **69** (1992) 3021.
- Z.-z. Xing, H. Zhang, hep-ph/0602134; G.-y. Huang, S. Zhou, Phys. Rev. D **103** (2021) 016010.
- S. Navas et al. (PDG), Phys. Rev. D **110** (2024) 030001.
- NuFit-6.0: Esteban et al., JHEP **12** (2024) 216.