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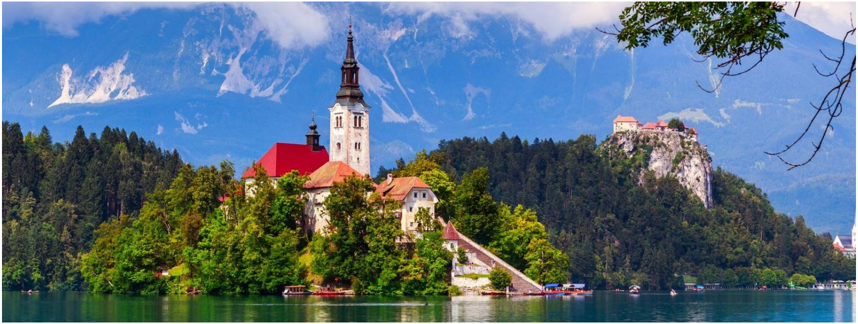

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Application of approximate renormalization group invariants to the study of the Yukawa unification

Some problems of the Standard Model

Although the Standard Model successfully explains all current experimental data, it is obviously not a final theory. For instance,

- The Standard Model contains a lot of fields with various quantum numbers, the origin of which being unclear.
- There are a lot of parameters (like gauge couplings, Weinberg angle, masses etc.) the values of which need to be explained.

The most natural approach to resolving these problems is to view the Standard Model as the low-energy remnant of theories with a larger gauge symmetry, commonly known as Grand Unified Theories. The simplest of them is based on the gauge group $SU(5)$

H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32** (1974), 438.

In this case, it is possible to explain certain features of the Standard Model, but unresolved problems still remain. Namely,

- The mass of the Higgs boson diverge quadratically, so that it should be fine tuned at the Grand Unification scale.
- The running of the gauge couplings does not agree with the prediction of the Grand Unified theories.
- At present, it is impossible to construct a Grand Unified Theory that explains masses and mixing of fermions in the Standard Model in a natural and satisfactory way.
- It is not clear, why there are three families of fermions.

A possibility of supersymmetric high energy physics

Most Grand Unified Theories predict the unification of gauge couplings of the form $\alpha_1 = \alpha_2 = \alpha_3$, where

$$\alpha_3 = \frac{e_3^2}{4\pi}; \qquad \alpha_2 = \frac{e_2^2}{4\pi}; \qquad \alpha_1 = \frac{5}{3}\cdot\frac{e_1^2}{4\pi}$$

correspond to the subgroups $SU(3)$, $SU(2)$, and $U(1)$, respectively. The factor $5/3$ in the definition of the coupling constant α_1 encodes the prediction for the Weinberg angle $\sin^2\theta_W = 3/8$ (at the unification scale).

The unification of the gauge couplings is not satisfied in the Standard Model, but in its supersymmetric extensions the running couplings really meet in a single point,

J. R. Ellis, S. Kelley, D. V. Nanopoulos, Phys. Lett. B **260** (1991), 131;
U. Amaldi, W. de Boer, H. Furstenau, Phys. Lett. B **260** (1991), 447;
P. Langacker, M. X. Luo, Phys. Rev. D **44** (1991), 817.

This fact can be considered as a strong indirect evidence in favor of both supersymmetry and Grand Unification. However, supersymmetry should be broken at low energies. In the simplest models soft supersymmetry breaking terms are introduced for this purpose. They explicitly break supersymmetry, do not introduce quadratic divergences, and can be considered as remnants of the spontaneous supersymmetry breaking.

The Minimal Supersymmetric Standard Model (MSSM)

The MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU(3)\times SU(2)\times U(1)$ and softly broken supersymmetry. Consequently, there are 3 gauge coupling constants e_3, e_2, and e_1 in the MSSM (their number is equal to the number of factors in the gauge group). Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

Superfield	$SU(3)$	$SU(2)$	$U(1)$ (Y)	Superfield	$SU(3)$	$SU(2)$	$U(1)$ (Y)
$3\times Q$	$\bar{3}$	2	$-1/6$	$3\times N$	1	1	0
$3\times U$	3	1	$2/3$	$3\times E$	1	1	-1
$3\times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3\times L$	1	2	$1/2$	H_u	1	2	$-1/2$

where for the superfields which include left quarks and leptons we use the brief notations

$$Q=\left(\begin{array}{c}\widetilde{U}\\ \widetilde{D}\end{array}\right);\qquad L=\left(\begin{array}{c}\widetilde{N}\\ \widetilde{E}\end{array}\right).$$

Unlike SM, its supersymmetric extensions (like the MSSM) necessarily contain at least two doublets of the Higgs superfields H_d and H_u.

The MSSM superpotential and gauge symmetry breaking

The MSSM superpotential contains dimensionless Yukawa couplings $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$, which are 3×3 matrices,

$$\begin{aligned} W = & (Y_U)_{IJ} \Big(\widetilde{U}\ \widetilde{D}\Big)^a_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{u1} \\ H_{u2} \end{pmatrix} U_{aJ} \\ & + (Y_D)_{IJ} \Big(\widetilde{U}\ \widetilde{D}\Big)^a_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} D_{aJ} \\ & + (Y_E)_{IJ} \Big(\widetilde{N}\ \widetilde{E}\Big)_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} E_J \\ & + \boldsymbol{\mu} \left(H_{u1}\ H_{u2}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix}. \end{aligned}$$

This superpotential produces mass matrices for fermions after the gauge symmetry breaking

$$SU(3)\times SU(2)\times U(1) \ \to\ SU(3)\times U(1)_{em}$$

by vacuum expectation values of the lowest scalar components of the Higgs superfields H_u and H_d (denoted by h_u and h_d, respectively),

$$(h_u)_0 = \begin{pmatrix} 0 \\ v_u \end{pmatrix}; \qquad (h_d)_0 = \begin{pmatrix} v_d \\ 0 \end{pmatrix}.$$

The mass matrices for fermions are given by the expressions

$$M_u = v_u Y_U; \qquad M_d = -v_d Y_D; \qquad M_e = -v_d Y_E.$$

The usual masses are equal to the absolute values of the eigenvalues of these matrices and have the hierarchical structure.
Instead of v_u and v_d it is more convenient to use the parameters

$$v \equiv \sqrt{{v_u}^2 + {v_d}^2} = \sqrt{\frac{2m_z^2}{e_1^2 + e_2^2}} \approx 174.\,\text{GeV}; \qquad \text{tg}\,\beta \equiv \frac{v_u}{v_d}.$$

At present, the value of $\text{tg}\,\beta$ is unknown.

Comparing the expressions for fermion masses in the MSSM and in the Standard Model (and neglecting the threshold corrections), we see that the Yukawa couplings are related by the equations

$$Y_U = Y_U^{(\text{SM})}/\sin\beta; \qquad Y_D = Y_D^{(\text{SM})}/\cos\beta; \qquad Y_E = Y_E^{(\text{SM})}/\cos\beta.$$

Scalar superpartners of quarks and leptons have much larger masses because one of the soft terms is the scalar mass (although there are several contributions to their masses). The soft terms also include the gaugino masses.

There are $5 = 2 \cdot 2 \cdot 2 - 3$ Higgs bosons in MSSM (unlike $1 = 2 \cdot 2 - 3$ in the Standard Model). Three of them are neutral, and two ones have the charges $\pm e$. One of the neutral Higgs bosons should be light and is identified with the Higgs boson discovered in 2012. The other Higgs bosons have much larger masses.

Due to supersymmetry, there are no quadratically divergent quantum corrections. Consequently, it is not necessary to make a huge fine tuning of the Higgs mass at the unification scale. Another interesting prediction of supersymmetry is that the mass of the lightest Higgs boson at the tree level should be less than m_Z. However, quantum corrections increase it,

$$m_h^2 \approx m_Z^2 \cos^2(2\beta) + \frac{3m_t^4}{3v^2} \ln \frac{\sqrt{m_{\widetilde{t}_1} m_{\widetilde{t}_2}}}{m_t}.$$

According to this equation, the masses of the t-quark superpartners $m_{\widetilde{t}_{1,2}}$ cannot be too large.

The Higgs mass $m_h \approx 125.2$ GeV can be obtained for reasonable values of the MSSM parameters. Therefore, the prediction of the light Higgs boson mass can also be considered as an indirect evidence in favor of supersymmetry. Comparing its value with the theoretical predictions one can restrict values of some MSSM parameters.

Mass relations in Grand Unified theories

The MSSM can also be considered as a low-energy remnant of a certain (softly broken) supersymmetric Grand Unified theory. An interesting prediction of such theories is the existence of certain equations relating the Yukawa couplings, which, in turn, lead to the relations between elementary particle masses. For instance, in the simplest $SU(5)$ model the invariant $\bar{5}\times 10\times \bar{5}$ produces the relation $Y_E = (Y_D)^T$. This implies that at a certain scale

$$m_d = m_e; \qquad m_s = m_\mu; \qquad m_b = m_\tau.$$

However, this prediction cannot be satisfied. This can be seen with the help of a certain approximate renormalization group invariant (RGI). By definition, approximate RGIs are almost independent of scale. (Sometimes it is possible to construct exact RGIs, which do not depend on scale in all orders.) In the SM/MSSM it is possible to construct the approximate RGI from the masses of down quarks and charged leptons,

$$\frac{d}{d\ln\mu}\Big(\frac{m_d m_\mu}{m_e m_s}\Big) \approx 0.$$

The numeric value of this ratio is

$$\frac{(Y_D)_{11}(Y_E)_{22}}{(Y_E)_{11}(Y_D)_{22}} = \frac{(Y_D^{\text{(SM)}})_{11}(Y_E^{\text{(SM)}})_{22}}{(Y_E^{\text{(SM)}})_{11}(Y_D^{\text{(SM)}})_{22}} \approx 10.58 \pm 0.11 \neq 1.$$

S. Antusch, K. Hinze, S. Saad, Phys. Rev. D **113** (2026) no.9, 095011.

Taking into account that the expression $(Y_D)_{11}(Y_E)_{22}/(Y_E)_{11}(Y_D)_{22}$ remains almost constant throughout the whole evolution from the electroweak scale up to the scale of Grand Unification, we see that the simplest equality $Y_E = (Y_D)^T$ cannot be satisfied for any values of the SM/MSSM parameters.

A way to solve this problem is to consider more complicated models leading, for example, to the Georgi and Jarlskog textures

H. Georgi, C. Jarlskog, Phys. Lett. B **86** (1979), 297.

If the Yukawa matrices for the down quarks and charged leptons are chosen in the form

$$Y_D = \begin{pmatrix} 0 & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{pmatrix}; \qquad Y_E = \begin{pmatrix} 0 & B & 0 \\ B & -3A & 0 \\ 0 & 0 & C \end{pmatrix},$$

then for $B \ll A$ we obtain

$$\frac{m_d m_\mu}{m_e m_s} \approx 9.$$

The factor -3 can be obtained, e.g., if the Higgs superfield comes from the representation 45_h of the group $SU(5)$ and the Yukawa interaction is based on the invariant $5 \times \overline{10} \times 45_h$.

Exact renormalization group equations for MSSM

Thus, it is very interesting to construct expressions protected against the radiative corrections. For this purpose, it necessary to know the renormalization group equations for the theory under consideration.

It is well-known that in supersymmetric theories gauge and Yukawa β-functions are related to the renormalization of the chiral matter superfields by certain all-order equations. Namely,

- The renormalization of the Yukawa couplings can be expressed in terms of the matter renormalization due to the nonrenormalization of the superpotential

  M. T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. B **159** (1979), 429.

- The gauge β-functions are related to the anomalous dimensions of the matter superfields by the NSVZ equation(s)

  V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
  D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
  M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456

- Some similar equations can be used for describing renormalization of the soft breaking parameters

J. Hisano, M. A. Shifman, Phys. Rev. D **56** (1997), 5475;
I. Jack, D. R. T. Jones, Phys. Lett. B **415** (1997) 383;
L. V. Avdeev, D. I. Kazakov, I. N. Kondrashuk, Nucl. Phys. B **510** (1998) 289.

Due to existence of the exact equations describing the renormalization in the supersymmetric case, it is possible to construct some all-loop exact RGIs. For instance, for the pure $\mathcal{N}=1$ SYM theory

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a$$

from the NSVZ β-function we obtain the equation

$$\frac{d\alpha}{d\ln\mu}=-\frac{3C_2\alpha^2}{2\pi\big(1-C_2\alpha/2\pi\big)},$$

where $f^{ACD}f^{BCD}=C_2\delta^{AB}$. Integrating it we obtain the all-loop RGI

$$\Big(\frac{\mu^3}{\alpha}\Big)^{C_2}\exp\Big(-\frac{2\pi}{\alpha}\Big)=\mbox{RGI}.$$

Such expressions appear in calculating the instanton contributions to the effective action, and the NSVZ β-function was first obtained by requiring their renormalization group invariance, see

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381.

The NSVZ equations for the MSSM

The NSVZ equations can be written even for theories with multiple gauge couplings

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046

and, in particular, for the (rigid part of the) MSSM

$$\frac{\beta_1}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$
$$\frac{\beta_2}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$
$$\frac{\beta_3}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big)\Big].$$

They relate three gauge β-functions of the theory to the anomalous dimensions of the chiral matter superfields. The renormalization group functions (RGFs) are defined by the equations

$$\beta_i(\alpha, Y) = \frac{d\alpha_i}{d\ln\mu}\bigg|_{\alpha_0, Y_0=\mbox{const}}; \qquad \gamma_i(\alpha, Y) = 2Z_i^{-1/2}\frac{dZ_i^{1/2}}{d\ln\mu}\bigg|_{\alpha_0, Y_0=\mbox{const}},$$

where the subscript 0 denotes the bare values.

The renormalization of the MSSM Yukawa couplings

RGFs describing the renormalization of the Yukawa couplings and of the parameter $\boldsymbol{\mu}$ can also be related to the anomalous dimensions of the matter superfields due to the nonrenormalization of the superpotential

M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

$$\frac{dY_U}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} Y_U + (\gamma_Q)^T Y_U + Y_U \gamma_U\Big);$$
$$\frac{dY_D}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_D + (\gamma_Q)^T Y_D + Y_D \gamma_D\Big);$$
$$\frac{dY_E}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_E + (\gamma_L)^T Y_E + Y_E \gamma_E\Big);$$
$$\frac{d\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big)\boldsymbol{\mu}.$$

It is known that the exact equations for both gauge and Yukawa couplings are valid in the HD+MSL scheme (when a theory is regularized by higher covariant derivatives

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

and divergences are removed by minimal subtractions of logarithms).

The renormalization group equations for the Yukawa couplings can be multiplied by the corresponding inverse matrices. After that, it is possible to calculate traces of the resulting equations using the formula

$$\mathrm{tr}\Big[M^{-1}\frac{dM}{d\ln\mu}\Big] = \frac{d}{d\ln\mu}\mathrm{tr}\ln M = \frac{d}{d\ln\mu}\ln\det M,$$

Then (taking into account that the indices numerating generators range from 1 to 3) we see that the equations describing how the determinants of the Yukawa matrices depend on the renormalization point μ are written as

$$\gamma_{\det Y_U} \equiv \frac{d\ln\det Y_U}{d\ln\mu} = \mathrm{tr}\Big[(Y_U)^{-1}\frac{dY_U}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_u} + \mathrm{tr}\big(\gamma_Q+\gamma_U\big)\Big);$$

$$\gamma_{\det Y_D} \equiv \frac{d\ln\det Y_D}{d\ln\mu} = \mathrm{tr}\Big[(Y_D)^{-1}\frac{dY_D}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_Q+\gamma_D\big)\Big);$$

$$\gamma_{\det Y_E} \equiv \frac{d\ln\det Y_E}{d\ln\mu} = \mathrm{tr}\Big[(Y_E)^{-1}\frac{dY_E}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_L+\gamma_E\big)\Big).$$

They can be solved together with the NSVZ equations and the equation describing the renormalization of the parameter $\boldsymbol{\mu}$.

Collecting the above equations we obtain the system of differential equations describing the renormalization of the MSSM parameters exactly in all orders

$$\frac{d}{d\ln\mu}\Big(\frac{5}{3}\cdot\frac{2\pi}{\alpha_1}\Big) = -11 + \mathrm{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2} + 2\ln\alpha_2\Big) = -1 + \mathrm{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3\Big) = 3 + \mathrm{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big);$$

$$\frac{d\ln\det Y_U}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_u} + \mathrm{tr}\big(\gamma_Q + \gamma_U\big)\Big);$$

$$\frac{d\ln\det Y_D}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_Q + \gamma_D\big)\Big);$$

$$\frac{d\ln\det Y_E}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mathrm{tr}\big(\gamma_L + \gamma_E\big)\Big);$$

$$\frac{d\ln\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big).$$

The anomalous dimensions of the chiral matter superfields can be eliminated, thereby obtaining differential equations which contain only derivatives of the gauge and Yukawa couplings and of the parameter $\boldsymbol{\mu}$.

Eliminating the anomalous dimensions of the chiral matter superfields gives the equations

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3}+3\ln\alpha_3+\frac{\pi}{2\alpha_2}+\frac{1}{2}\ln\alpha_2+\frac{5\pi}{6\alpha_1} -\frac{1}{2}\ln\det Y_E-\frac{7}{6}\ln\det Y_D-\frac{5}{3}\ln\det Y_U+\frac{9}{2}\ln\boldsymbol{\mu}\Big)=0,$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3}+3\ln\alpha_3-\frac{\pi}{\alpha_2}-\ln\alpha_2-\frac{5\pi}{3\alpha_1} +\ln\det Y_E-\frac{2}{3}\ln\det Y_D+\frac{1}{3}\ln\det Y_U-9\ln\mu\Big)=0.$$

From these equations we obtain that the expressions

$$\mathrm{RGI}_1=\frac{\boldsymbol{\mu}^{9/2}\,(\alpha_3)^3\,(\alpha_2)^{1/2}}{\big(\det Y_E\big)^{1/2}\,\big(\det Y_U\big)^{5/3}\,\big(\det Y_D\big)^{7/6}}\,\exp\Big(\frac{2\pi}{\alpha_3}+\frac{\pi}{2\alpha_2}+\frac{5\pi}{6\alpha_1}\Big);$$

$$\mathrm{RGI}_2=\frac{(\alpha_3)^3\det Y_E\,\big(\det Y_U\big)^{1/3}}{\mu^9\,\alpha_2\,\big(\det Y_D\big)^{2/3}}\,\exp\Big(\frac{2\pi}{\alpha_3}-\frac{\pi}{\alpha_2}-\frac{5\pi}{3\alpha_1}\Big)$$

do not receive quantum corrections in all orders of the perturbation theory and are, therefore, the exact renormalization group invariants (RGIs).

The all-loop RGIs for the MSSM

Instead of the renormalization group invariants (RGI_1, RGI_2) it is possible to use the equivalent set (RGI_3, RGI_4), where the expressions

$$\mathsf{RGI}_3 \equiv \Big(\frac{\mathsf{RGI}_1}{\mathsf{RGI}_2}\Big)^{2/3} = \frac{\boldsymbol{\mu}^3\,\mu^6\,\alpha_2}{(\det Y_E)\,(\det Y_U)^{4/3}\,(\det Y_D)^{1/3}}\,\exp\Big(\frac{\pi}{\alpha_2}+\frac{5\pi}{3\alpha_1}\Big);$$

$$\mathsf{RGI}_4 \equiv (\mathsf{RGI}_1)^{2/3}(\mathsf{RGI}_2)^{1/3} = \frac{\boldsymbol{\mu}^3\,(\alpha_3)^3}{\mu^3\det Y_U\,\det Y_D}\,\exp\Big(\frac{2\pi}{\alpha_3}\Big)$$

also have a rather simple form.

D. Rystsov, K.S., Phys. Rev. D **111** (2025) no.1, 016012.

Similar RGIs can be constructed for other theories, e.g., for the Next-to-Minimal Supersymmetric Standard Model (NMSSM)

M. Maniatis, Int. J. Mod. Phys. A **25** (2010), 3505;
U. Ellwanger, C. Hugonie, A. M. Teixeira,Phys. Rept. **496** (2010), 1.

Nevertheless, although these RGIs remain constant throughout the whole renormalization group evolution in the MSSM, it is very difficult to use them for analyzing some phenomenological consequences. For this purpose, it is better to deal with the approximate RGIs, which receive only relatively small quantum corrections.

One-loop anomalous dimensions of the MSSM chiral superfields

Let us proceed to constructing of some approximate RGIs. For this purpose we will need the renormalization group equations in the lowest orders of the perturbation theory. The one-loop anomalous dimensions of the chiral matter superfields in the MSSM read as

$$\gamma_Q(\alpha,Y)^T = -\frac{\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{8\pi^2}\Big(Y_U Y_U^+ + Y_D Y_D^+\Big) + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_U(\alpha,Y) = -\frac{4\alpha_1}{15\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{4\pi^2} Y_U^+ Y_U + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_D(\alpha,Y) = -\frac{\alpha_1}{15\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{4\pi^2} Y_D^+ Y_D + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_L(\alpha,Y)^T = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{1}{8\pi^2} Y_E Y_E^+ + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_E(\alpha,Y) = -\frac{3\alpha_1}{5\pi} + \frac{1}{4\pi^2} Y_E^+ Y_E + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_{H_u}(\alpha,Y) = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{3}{8\pi^2} Y_U^+ Y_U + O(\alpha^2,\alpha Y^2, Y^4);$$
$$\gamma_{H_d}(\alpha,Y) = -\frac{3\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{1}{8\pi^2}\Big(3Y_D^+ Y_D + Y_E^+ Y_E\Big) + O(\alpha^2,\alpha Y^2, Y^4).$$

Starting from these expressions we are able to construct two-loop β-functions with the help of the NSVZ equation and the anomalous dimensions of the Yukawa couplings using the nonrenormalization of the superpotential.

Two-loop β-functions for MSSM

The corresponding two-loop β-functions can be obtained with the help of the NSVZ equations and are well-known, see, e.g.,

I. Jack, D. R. T. Jones, A. F. Kord, Annals Phys. **316** (2005), 213.

$$\frac{\beta_1(\alpha,Y)}{\alpha_1^2} = -\frac{1}{2\pi}\cdot\frac{3}{5}\Big\{-11-\frac{199\alpha_1}{60\pi}-\frac{9\alpha_2}{4\pi}-\frac{22\alpha_3}{3\pi}$$
$$+\frac{1}{8\pi^2}\mbox{tr}\Big(\frac{13}{3}Y_U Y_U^+ +\frac{7}{3}Y_D Y_D^+ + 3Y_E Y_E^+\Big)\Big\} + O\Big(\alpha^2,\alpha Y^2, Y^4\Big);$$
$$\frac{\beta_2(\alpha,Y)}{\alpha_2^2} = -\frac{1}{2\pi}\Big\{-1-\frac{9\alpha_1}{20\pi}-\frac{25\alpha_2}{4\pi}-\frac{6\alpha_3}{\pi}$$
$$+\frac{1}{8\pi^2}\,\mbox{tr}\Big(3\,Y_U Y_U^+ + 3\,Y_D Y_D^+ + Y_E Y_E^+\Big)\Big\} + O\Big(\alpha^2,\alpha Y^2, Y^4\Big);$$
$$\frac{\beta_3(\alpha,Y)}{\alpha_3^2} = -\frac{1}{2\pi}\Big\{3-\frac{11\alpha_1}{20\pi}-\frac{9\alpha_2}{4\pi}-\frac{7\alpha_3}{2\pi}$$
$$+\frac{1}{8\pi^2}\,\mbox{tr}\Big(2\,Y_U Y_U^+ + 2\,Y_D Y_D^+\Big)\Big\} + O\Big(\alpha^2,\alpha Y^2, Y^4\Big).$$

(Certainly, below the supersymmetric threshold it is necessary to use the renormalization group equations for the Standard Model, which are not presented here.) The β-functions are also modified in presence of extra matter.

One-loop anomalous dimensions of the Yukawa couplings

For the Yukawa couplings we can also take into account that the Yukawa couplings have hierarchical structure. Namely, they are the largest for the third generation and are the smallest for the first one. This allows to neglect small terms containing Yukawa couplings of the first and second generations. Then, for the third and second generations (neglecting mixing between them) we obtain

$$\frac{d\ln (Y_U)_{33}}{d\ln\mu} \approx -\frac{13\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{3}{8\pi^2}(Y_U)^2_{33} + \frac{1}{16\pi^2}(Y_D)^2_{33};$$

$$\frac{d\ln (Y_U)_{22}}{d\ln\mu} \approx -\frac{13\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{3}{16\pi^2}(Y_U)^2_{33};$$

$$\frac{d\ln (Y_D)_{33}}{d\ln\mu} \approx -\frac{7\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{1}{16\pi^2}(Y_U)^2_{33} + \frac{3}{8\pi^2}(Y_D)^2_{33} + \frac{1}{16\pi^2}(Y_E)^2_{33};$$

$$\frac{d\ln (Y_D)_{22}}{d\ln\mu} \approx -\frac{7\alpha_1}{60\pi} - \frac{3\alpha_2}{4\pi} - \frac{4\alpha_3}{3\pi} + \frac{3}{16\pi^2}(Y_D)^2_{33} + \frac{1}{16\pi^2}(Y_E)^2_{33};$$

$$\frac{d\ln (Y_E)_{33}}{d\ln\mu} \approx -\frac{9\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{3}{16\pi^2}(Y_D)^2_{33} + \frac{1}{4\pi^2}(Y_E)^2_{33};$$

$$\frac{d\ln (Y_E)_{22}}{d\ln\mu} \approx -\frac{9\alpha_1}{20\pi} - \frac{3\alpha_2}{4\pi} + \frac{3}{16\pi^2}(Y_D)^2_{33} + \frac{1}{16\pi^2}(Y_E)^2_{33}.$$

Looking at this equations we can try to find such combinations of Yukawa couplings that receive relatively small quantum corrections and are, therefore, approximate RGIs.

As initial conditions we take the values of the Yukawa couplings for the Standard Model at the mass of Z-boson in the $\overline{\mbox{MS}}$ scheme found in

S. Antusch, K. Hinze, S. Saad, Phys. Rev. D **113** (2026) no.9, 095011

on the base of the experimental data presented in

S. Navas *et al.* [Particle Data Group], *Review of particle physics*, Phys. Rev. D **110** (2024) no.3 030001.

Namely, for the third and second generations

$$(Y_U^{(\mathrm{SM})})_{33}(m_Z) = 0.967 \pm 0.004; \qquad (Y_U^{(\mathrm{SM})})_{22}(m_Z) = (3.56 \pm 0.06)\cdot 10^{-3};$$

$$(Y_D^{(\mathrm{SM})})_{33}(m_Z) = (1.630 \pm 0.009)\cdot 10^{-2}; \qquad (Y_D^{(\mathrm{SM})})_{22}(m_Z) = (3.06 \pm 0.04)\cdot 10^{-4};$$

$$(Y_E^{(\mathrm{SM})})_{33}(m_Z) = (0.99378 \pm 0.00014)\cdot 10^{-2};$$

$$(Y_E^{(\mathrm{SM})})_{22}(m_Z) = (5.85042 \pm 0.00075)\cdot 10^{-4}.$$

In what follows, we will consider only Yukawa couplings of the third and second generations and neglect mixing between them.

The aim is to analyze which Yukawa relations could be satisfied for a certain choice of the parameters and which cannot.

For this purpose, we will construct some approximate RGIs containing Yukawa couplings. Using the (approximate) scale independence of these expressions, it is possible
1) to express them in terms of the Standard Model Yukawa couplings, $\mathrm{tg}\,\beta$, etc. at low energies;
2) to guess their values at the unification scale with the help of arguments based on the group theory.

Comparing the results, we will try to estimate values of the unknown parameters

$$\mathrm{tg}\,\beta; \qquad \left|\frac{(Y_U)_{33}}{(Y_D)_{33}}\right|; \qquad \left|\frac{(Y_U)_{22}}{(Y_D)_{22}}\right|, \qquad \text{etc.}$$

Doing this, we will consider only the Yukawa couplings of the third and second generations and neglect mixing.

In general we will assume that the models under consideration can have a different field content compared to the MSSM, but will neglect the Yukawa interaction of the possible exotic superfields.

For simplicity, we assume that masses of all superpartners are equal. The renormalization group equations are taken in the two-loop approximation for the gauge couplings and in the one-loop approximation for the Yukawa couplings both below and above the supersymmetric threshold.

The first approximate RGI

Let us construct such expressions that remain almost constant throughout the whole renormalization group evolution following

K. Krylov, D. Rystsov and K.S., arXiv:2604.23329 [hep-ph].

The first example of such an expression is

$$I_1 = \left|\left(\frac{(Y_U)_{33}}{(Y_D)_{33}}\right)^3\left(\frac{(Y_D)_{22}}{(Y_U)_{22}}\right)^5\right|.$$

The derivative of the logarithm of this expression with respect to $\ln\mu$ does not contain the (large) couplings α_3, $[(Y_U)_{33}]^2$, and $[(Y_D)_{33}]^2$ and is given by

$$\frac{d\ln I_1}{d\ln\mu} = \frac{\alpha_1}{5\pi} + \frac{1}{8\pi^2}[(Y_E)_{33}]^2 + \ldots$$

At low energies (the scale of superpartner masses m_{SUSY}) the expression for I_1 can be estimated as

$$I_1 = \text{tg}^2\beta\left|\left(\frac{(Y_U^{(\text{SM})})_{33}}{(Y_D^{(\text{SM})})_{33}}\right)^3\left(\frac{(Y_D^{(\text{SM})})_{22}}{(Y_U^{(\text{SM})})_{22}}\right)^5\right|.$$

For the scales m_z, $m_{\text{SUSY}} = 5\cdot 10^3$ GeV, and $m_{\text{GUT}} = 2\cdot 10^{16}$ GeV we obtained

$$I_1\Big|_{m_Z} \approx \text{tg}^2\beta\cdot 0.98;\qquad I_1\Big|_{m_{\text{SUSY}}=5\cdot 10^3\text{ GeV}} \approx \text{tg}^2\beta\cdot 1.18;\qquad I_1\Big|_{m_{\text{GUT}}} \approx \text{tg}^2\beta\cdot 1.25.$$

The plot of $I_1/\text{tg}^2\beta$ as a function of scale for the MSSM with $\text{tg}\,\beta = 18$ is presented on the right.

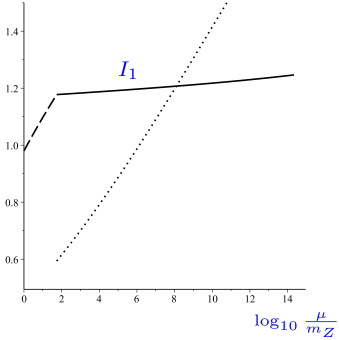
For comparison, by dots we also presented the plot of the function

$$\frac{1}{10\cdot\text{tg}^2\beta}\left|\left(\frac{(Y_U)_{33}}{(Y_D)_{33}}\right)^5\left(\frac{(Y_D)_{22}}{(Y_U)_{22}}\right)^3\right|$$

(where we formally changed $3\leftrightarrow 5$). The plot of the function

$$\left|\left(\frac{(Y_U^{(\text{SM})})_{33}}{(Y_D^{(\text{SM})})_{33}}\right)^3\left(\frac{(Y_D^{(\text{SM})})_{22}}{(Y_U^{(\text{SM})})_{22}}\right)^5\right|$$

is also presented by dashes.



We see that the expression I_3 really remains almost constant throughout the region of supersymmetric evolution.

The second approximate RGI

It is also desirable to construct such an RGI that can be estimated at low energies without involving $\mathrm{tg}\,\beta$. For this purpose let us assume that at the unification scale

$$|(Y_U)_{33}| = x\,|(Y_D)_{33}|\,,$$

where x is a certain real number, and the absolute value of $|(Y_U)_{33}| - x\,|(Y_D)_{33}|$ remains small throughout the whole renormalization group evolution. Under this assumption it is possible to obtain that the expression

$$I_2 = \left|\frac{(Y_U)_{33}}{(Y_D)_{33}}\cdot\frac{(Y_D)_{22}}{(Y_U)_{22}}\cdot\left(\frac{(Y_E)_{33}}{(Y_D)_{33}}\cdot\frac{(Y_D)_{22}}{(Y_E)_{22}}\right)^{\frac{2(x^2-1)}{x^2+3}}\right|$$

slightly depends on scale and is therefore an approximate RGI. At low energies (the scale of superpartner masses m_{SUSY}) the expression for I_2 can be estimated as

$$I_2 = \left|\frac{(Y_U^{(\mathrm{SM})})_{33}}{(Y_D^{(\mathrm{SM})})_{33}}\cdot\frac{(Y_D^{(\mathrm{SM})})_{22}}{(Y_U^{(\mathrm{SM})})_{22}}\cdot\left(\frac{(Y_E^{(\mathrm{SM})})_{33}}{(Y_D^{(\mathrm{SM})})_{33}}\cdot\frac{(Y_D^{(\mathrm{SM})})_{22}}{(Y_E)_{22}^{(\mathrm{SM})}}\right)^{\frac{2(x^2-1)}{x^2+3}}\right|.$$

Note that this expression does not include $\mathrm{tg}\,\beta$. However, the value of the parameter x remains unknown.

The modified second approximate RGI

According to the Georgi–Jarlskog texture, at the scale of Grand Unification

$$(Y_D)_{33} = (Y_E)_{33}; \qquad (Y_D)_{22} = -\frac{1}{3}(Y_E)_{22}.$$

In this case

$$r \equiv \left|\frac{(Y_E)_{33}}{(Y_D)_{33}}\cdot\frac{(Y_D)_{22}}{(Y_E)_{22}}\right|_{m_{\text{GUT}}} = \frac{1}{3}.$$

However, in fact, for MSSM this ratio is slightly different. According to

S. Antusch, K. Hinze, S. Saad, Phys. Rev. D **113** (2026) no.9, 095011.

$\text{tg}\,\beta$	5	10	30	50
r	0.301	0.304	0.304	0.304

(for $m_{SUSY} = 3\cdot 10^3$ GeV). Taking into account that $(0.304)^{-1} \approx 3.29$ it is reasonable to modify the invariant under consideration,

$$I_2 \to \widetilde{I}_2 \equiv \left|\frac{(Y_U)_{33}}{(Y_D)_{33}}\cdot\frac{(Y_D)_{22}}{(Y_U)_{22}}\cdot\left(3.29\cdot\frac{(Y_E)_{33}}{(Y_D)_{33}}\cdot\frac{(Y_D)_{22}}{(Y_E)_{22}}\right)^{\frac{2(x^2-1)}{x^2+3}}\right|.$$

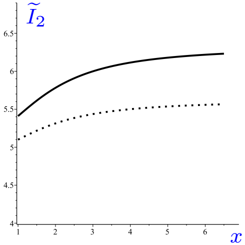
Then the ratio in the round brackets tends to 1 at the scale of Grand Unification.

The dependence of the RGI $\widetilde{I}_2$ on x

As we have already mentioned, it is possible to roughly estimate a value of $\widetilde{I}_2$ at the unification scale using the equation

$$\widetilde{I}_2 = \left|\frac{(Y_U^{(\mathrm{SM})})_{33}}{(Y_D^{(\mathrm{SM})})_{33}}\cdot\frac{(Y_D^{(\mathrm{SM})})_{22}}{(Y_U^{(\mathrm{SM})})_{22}}\cdot\left(3.29\cdot\frac{(Y_E^{(\mathrm{SM})})_{33}}{(Y_D^{(\mathrm{SM})})_{33}}\cdot\frac{(Y_D^{(\mathrm{SM})})_{22}}{(Y_E)_{22}^{(\mathrm{SM})}}\right)^{\frac{2(x^2-1)}{x^2+3}}\right|.$$

The plot of this expression as a function of x is presented in the figure



At this scale we obtain

$$|(Y_U)_{33}| = x\,|(Y_D)_{33}|\,;\qquad |(Y_U)_{22}| \approx \frac{x}{\widetilde{I}_2}\,|(Y_D)_{22}|\,.$$

The dependence of tg β on x

Taking into account that I_1 and I_2 slightly depend on scale and eliminating the ratio $\left|(Y_D)_{22}/(Y_U)_{22}\right|_{m_{\text{GUT}}}$, it is possible to estimate the value of tg β as a function of x,

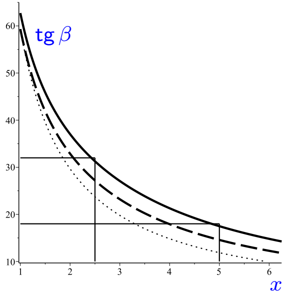
$$\text{tg}\,\beta \approx \frac{1}{x}\cdot\frac{(Y_U^{(\text{SM})})_{33}}{(Y_D^{(\text{SM})})_{33}}\cdot\left(3.29\cdot\frac{(Y_E^{(\text{SM})})_{33}}{(Y_D^{(\text{SM})})_{33}}\cdot\frac{(Y_D^{(\text{SM})})_{22}}{(Y_E)_{22}^{(\text{SM})}}\right)^{\frac{5(x^2-1)}{x^2+3}}.$$

The plot of this function is presented in the figure together with the tree curve $\frac{1}{x}\cdot\frac{m_t}{m_b}$ denoted by dots. The value at m_Z is plotted by dashes, while the value at $m_{\text{SUSY}} = 5\cdot 10^3$ GeV is given by the solid line.

Using this plot one can suggest various options for the Yukawa relations. For instance, $x = 5$ corresponds to tg $\beta \approx 18$ and to the Yukawa relations

$$|(Y_U)_{33}| = 5\,|(Y_D)_{33}|\,;$$
$$|(Y_U)_{22}| = \frac{4}{5}\,|(Y_D)_{22}|$$

at the unification scale.



RG evolution of the Yukawa couplings and of the RGI I_2

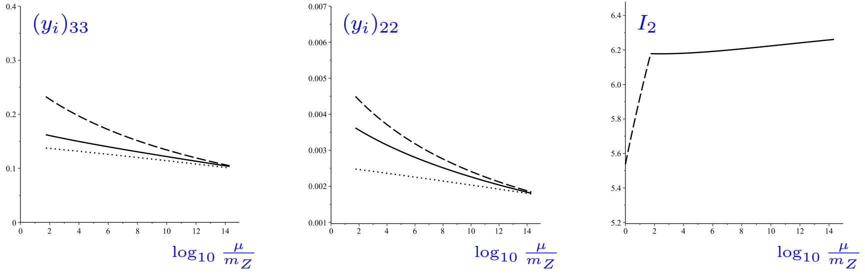
As an example, let us consider $x = 5$ corresponding to $\text{tg}\,\beta = 18$ and $I_2/x \approx 5/4$. The evolution of the Yukawa couplings

$$(y_U)_{33} \equiv |(Y_U)_{33}|/5; \qquad (y_D)_{33} \equiv |(Y_D)_{33}|; \qquad (y_E)_{33} \equiv \frac{3}{4}\cdot|(Y_E)_{33}|$$

are presented in the left figure, and the evolution of the constants

$$(y_U)_{22} \equiv \frac{5}{4}\cdot|(Y_U)_{22}|; \qquad (y_D)_{22} \equiv |(Y_D)_{22}|; \qquad (y_E)_{22} \equiv \frac{4}{3}\cdot 3.29\cdot|(Y_E)_{22}|$$

is presented in the central figure. Solid, dashed, and dotted lines correspond to y_U, y_D, and y_E, respectively. The plot of the RGI I_2 is given in the right fugure. We see that it really remains almost constant.



The third approximate RGI

A value of the ratio $|(Y_E)_{22}/(Y_D)_{22}|$ at the scale of Grand Unification can approximately be estimated with the help of the approximate invariant

$$I_3 \equiv \left|\frac{(Y_E)_{22}}{(Y_D)_{22}}\right| \cdot \Big(\frac{\mu}{m_{\textsf{GUT}}}\Big)^{-4\alpha_3/3\pi}.$$

For a theory with n_R additional superfields forming the representations R and $\overline{R}$ of the group $SU(5)$, the derivative of its logarithm with respect to $\ln\mu$ calculated with the help of the expressions presented above is given by the expression

$$\frac{d\ln I_3}{d\ln\mu} = \frac{\alpha_1}{3\pi} - \frac{2(3-n)}{3\pi^2}(\alpha_3)^2\ln\frac{\mu}{m_{\textsf{GUT}}} + \ldots,$$

where

$$n = \frac{1}{2}n_5 + \frac{3}{2}n_{10} + \frac{7}{2}n_{15} + 5n_{24} + \ldots$$

and dots denote one-loop terms proportional to the Yukawa couplings of the second and first generations and all higher order contributions except for the one of the two-loop terms written explicitly.

We see that the expression for $d\ln I_3/d\ln\mu$ is relatively small, so that I_3 does not receive large quantum corrections and can be used for estimating the value of $|(Y_E)_{22}/(Y_D)_{22}|$.

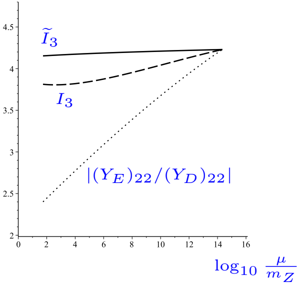
For the MSSM (for which $n=0$) it is possible to modify the expression for I_3 in such a way that its derivative becomes even smaller. Namely, let us consider the expression

$$\widetilde{I}_3 \equiv \left|\frac{(Y_E)_{22}(\mu)}{(Y_D)_{22}(\mu)}\right| \cdot \Big(\frac{\alpha_3(\mu)}{\alpha(m_{\text{GUT}})}\Big)^{8/9}\Big(\frac{\alpha_1(\mu)}{\alpha(m_{\text{GUT}})}\Big)^{10/99}.$$

Note that, due to the equation

$$\big(\alpha_3(\mu)\big)^{8/9} \approx \Big(\frac{1}{\alpha_3(m_{\text{GUT}})}+\frac{3}{2\pi}\ln\frac{\mu}{m_{\text{GUT}}}\Big)^{-8/9} \approx \big(\alpha_3(m_{\text{GUT}})\big)^{8/9}\Big(\frac{\mu}{m_{\text{GUT}}}\Big)^{-4\alpha_3/3\pi},$$

this expression is really close to I_3 constructed on the previous slide. Plots of the functions $|(Y_E)_{22}(\mu)/(Y_D)_{22}(\mu)|$ (dots), I_3 (dashes), and $\widetilde{I}_3$ (solid) are presented on the right for $\text{tg}\,\beta=18$, $m_{\text{SUSY}}=5$ GeV. We see that $\widetilde{I}_3$ really does not almost depend on scale and can be used for estimating the ratio $|(Y_E)_{22}/(Y_D)_{22}|$ at the scale of Grand Unification, which appears more than 3 (about 4.3).



Equations relating the Yukawa couplings appear in theories with wide gauge symmetry. The matter is that the wide gauge invariance requires that the coefficients of various terms in the action should be related to each other. For instance, the $SO(10)$ invariant $\overline{16} \times \overline{16} \times 10$ leads to the relations

$$Y_U = Y_D = Y_E = -Y_\nu.$$

However, this relation evidently is not satisfied, at least, for all generations. However, it is possible to consider more complicated invariants and try to analyze their predictions.

Here we will consider the group E_6

F. Gursey, P. Ramond, P. Sikivie, Phys. Lett. B **60** (1976), 177

and try to construct the Yukawa relations that mostly resemble the ones discussed earlier starting from its certain cubic invariant.
The group E_6 has the dimension 78 and the fundamental representation 27. It is known

R. Slansky, Phys. Rept. **79** (1981), 1

that

$$\overline{27} \times \overline{27} = 27_s + 351_a + 351'_s.$$

Therefore, there is the invariant $\overline{27} \times \overline{27} \times \overline{351}'$ which we will try to investigate.

The group E_6 has the maximal subgroup $SO(10)\times U(1)$, with respect to which

$$27\Big|_{E_6} = 1(4) + 10(-2) + 16(1)\Big|_{SO(10)\times U(1);}$$
$$\overline{27}\Big|_{E_6} = 1(-4) + 10(2) + \overline{16}(-1)\Big|_{SO(10)\times U(1);}$$
$$78\Big|_{E_6} = 1(0) + 16(-3) + \overline{16}(3) + 45(0)\Big|_{SO(10)\times U(1),}$$

where 16 and $\overline{16}$ are the right and left spinor representations of $SO(10)$. Here we use a single spinor index $a=1,\ldots,32$, so that

$$t_{\boldsymbol{A}} = \{t_{ij}, t_a, t\},$$

$$E_6\quad \left\{ \begin{array}{l} [t_{ij}, t_{kl}] = \dfrac{i}{\sqrt{12}}\Big(\delta_{il}t_{jk} - \delta_{jl}t_{ik} - \delta_{ik}t_{jl} + \delta_{jk}t_{il}\Big); \\ [t_{ij}, t] = 0; \qquad [t, t_a] = \dfrac{1}{4}(\Gamma^{(10)}_{11})_a{}^b t_b; \qquad [t_{ij}, t_a] = -\dfrac{i}{2\sqrt{12}}(\Gamma^{(10)}_{ij})_a{}^b t_b; \\ [t_a, t_b] = -\dfrac{i}{4\sqrt{12}}(\Gamma^{(10)}_{ij}C^{(10)})_{ab} t_{ij} + \dfrac{1}{4}(\Gamma^{(10)}_{11}C^{(10)})_{ab} t. \end{array}\right.$$

The E_6 representations in terms of $SO(10)$ (super)fields

First, we construct the invariant $\overline{27}\times\overline{27}\times\overline{351}'$. It is convenient to formulate it in terms of $SO(10)$ fields. Then according to

$$\overline{27}\Big|_{E_6} = 1(-4)+10(2)+\overline{16}(-1)\Big|_{SO_{10}\times U_1};$$
$$\overline{351}'\Big|_{E_6} = 1(8)+10(2)+16(5)+54(-4)+126(2)+\overline{144}(-1)\Big|_{SO(10)\times U(1)},$$

$$\overline{27}\sim\{P,\, P_i\, P^a\};\qquad \overline{351}'\sim\Big\{A,\, A_i,\, A_a,\, A_{ij},\, A_{ijklm},\, A_{ia}\Big\},$$

where A_{ij} is the symmetric traceless tensor, A_{ijklm} is the completely antisymmetric tensor that satisfies the condition

$$A_{i_1i_2i_3i_4i_5} = -\frac{i}{5!}\varepsilon_{i_1i_2i_3i_4i_5j_1j_2j_3j_4j_5}A_{j_1j_2j_3j_4j_5},$$

and A_{ia} satisfies the constraint $(B\Gamma_i)^{ab}A_{ib}=0$. The normalization is fixed by

$$N_{27}\equiv\overline{27}\times 27\equiv P\bar{P}+P_i\bar{P}_i+P^a\bar{P}_a;$$
$$N_{351'}\equiv\overline{351}'\times 351'\equiv A\bar{A}+A_i\bar{A}_i+A_a\bar{A}^a+\frac{1}{2}A_{ij}\bar{A}_{ij}+\frac{1}{5!}A_{ijklm}\bar{A}_{ijklm}+A_{ia}\bar{A}_i^a.$$

The E_6 invariant $\overline{27}\times\overline{27}\times\overline{351}'$

The invariant $\overline{27}\times\overline{27}\times\overline{351}'$ is derived by requiring the manifest $SO(10)\times U(1)$ invariance and the invariance under a certain hidden symmetry corresponding to $16(-3)$ and $\overline{16}(3)$ (with the respective parameters ε_a and ε^a) in the branching rule

$$78\Big|_{E_6} = 1(0) + 16(-3) + \overline{16}(3) + 45(0)\Big|_{SO(10)\times U(1)},$$

while the terms $1(0)$ and $45(0)$ in the right hand side correspond to the subgroups $U(1)$ and $SO(10)$, respectively, for which the invariance is manifest.

In the explicit form, the invariant under consideration is written as

$$\overline{27}\times\overline{27}\times\overline{351}' = P^2 A + \frac{4}{\sqrt{10}}\,PP_iA_i + \sqrt{2}\,P\,P^a A_a + \frac{1}{\sqrt{2}}\,P_iP_jA_{ij}$$
$$+\sqrt{2}\,P_iP^aA_{ia} + \frac{1}{4\cdot 5!}P^a(\Gamma_{ijklm}B)_{ab}P^bA_{ijklm} + \frac{1}{4\sqrt{5}}P^a(\Gamma_iB)_{ab}P^bA_i.$$

The relative coefficients in this expressions are fixed by the invariance under the hidden symmetry transformations (which are presented on the next slide).

Starting from this invariant we will try to obtain Yukawa relations similar to those discussed earlier.

For the representation $\overline{27}$ the transformations of the hidden symmetry are written as

$$\delta P = \varepsilon_a P^a; \qquad \delta P_i = \frac{1}{\sqrt{2}}\varepsilon^a(\Gamma_i B)_{ab}P^b; \qquad \delta P^a = \varepsilon^a P + \frac{1}{\sqrt{2}}\varepsilon_b(B\Gamma_i)^{ab}P_i.$$

The transformations of various parts of the representation $\overline{351}'$ take the form

$$\delta A = -\sqrt{2}\,\varepsilon^a A_a; \qquad \delta A_i = -\frac{\sqrt{10}}{4}\varepsilon_a(B\Gamma_i)^{ab}A_b - \frac{\sqrt{5}}{2}\varepsilon^a A_{ia};$$

$$\delta A_a = -\sqrt{2}\,\varepsilon_a A - \frac{\sqrt{10}}{4}\varepsilon^b(\Gamma_i B)_{ab}A_i - \frac{1}{2\sqrt{2}}\cdot\frac{1}{5!}\varepsilon^b(\Gamma_{ijklm}B)_{ab}A_{ijklm};$$

$$\delta A_{ij} = -\frac{1}{\sqrt{2}}\Big[\varepsilon_a(B\Gamma_i)^{ab}A_{jb} + \varepsilon_a(B\Gamma_j)^{ab}A_{ib}\Big];$$

$$\delta A_{ijklm} = -\frac{1}{2\sqrt{2}}\varepsilon_a(B\Gamma_{ijklm})^{ab}A_b - \frac{1}{2}\Big[\varepsilon^a(\Gamma_{ijkl})_a{}^b A_{mb} + \varepsilon^a(\Gamma_{mijk})_a{}^b A_{lb}$$
$$+\varepsilon^a(\Gamma_{lmij})_a{}^b A_{kb} + \varepsilon^a(\Gamma_{klmi})_a{}^b A_{jb} + \varepsilon^a(\Gamma_{jklm})_a{}^b A_{ib}\Big];$$

$$\delta A_{ia} = -\frac{9}{4\sqrt{5}}\varepsilon_a A_i - \frac{1}{4}\cdot\frac{1}{4!}\varepsilon_b(\Gamma_{jklm})_a{}^b A_{ijklm} - \frac{1}{\sqrt{2}}\varepsilon^b(\Gamma_j B)_{ab}A_{ij}$$
$$+\frac{1}{4\sqrt{5}}\varepsilon_b(\Gamma_{ij})_a{}^b A_j + \frac{1}{4}\cdot\frac{1}{5!}\varepsilon_b(\Gamma_{ijklmn})_a{}^b A_{jklmn}.$$

Next, it is necessary to consider certain parts of the above invariant and write them in terms of $SU(5) \subset SO(10)$ (super)fields. This can be done by the same method, but the overall constant should be found with the help of a different technique. Omitting the technical details we present the results for some $SO(10)$ invariants which will be needed in what follows

$$I_{10} \equiv \overline{16} \times \overline{16} \times 10 = P^a(\Gamma_i B)_{ab} P^b A_i$$
$$= -2\sqrt{2}\Big(\phi\,\phi_\alpha R^\alpha + \phi_\alpha \phi^{\alpha\beta} R_\beta + \frac{1}{8}\varepsilon_{\alpha\beta\mu\nu\sigma}\phi^{\alpha\beta}\phi^{\mu\nu}R^\sigma\Big);$$

$$I_{126} \equiv \overline{16} \times \overline{16} \times 126 = \frac{1}{5!}P^a(\Gamma_{ijklm}B)_{ab}P^b A_{ijklm} = 4\sqrt{2}\Big(\phi^2 K + \sqrt{\frac{3}{2}}\phi\phi_\alpha K^\alpha$$
$$+\frac{1}{\sqrt{2}}\phi\phi^{\alpha\beta}K_{\alpha\beta} + \frac{1}{\sqrt{2}}\phi_\alpha\phi_\beta L^{\alpha\beta} + \frac{1}{\sqrt{2}}\phi_\gamma\phi^{\alpha\beta}K^\gamma_{\alpha\beta} - \frac{1}{8\sqrt{6}}\varepsilon_{\mu\nu\alpha\beta\gamma}\phi^{\mu\nu}\phi^{\alpha\beta}K^\gamma$$
$$+\frac{1}{24}\varepsilon_{\mu\nu\alpha\beta\gamma}\phi^{\mu\nu}\phi^{\sigma\rho}K^{\alpha\beta\gamma}_{\sigma\rho}\Big);$$

$$I_{144} \equiv 10 \times \overline{16} \times \overline{144} = P_i P^a A_{ia}$$

$$= -\frac{2}{\sqrt{5}}\Big(R^\alpha\phi B_\alpha + \frac{\sqrt{5}}{2}R^\alpha\phi_\beta B^\beta_\alpha + \frac{1}{4\sqrt{6}}\varepsilon_{\alpha\beta\gamma\mu\nu}R^\alpha\phi^{\beta\gamma}B^{\mu\nu} + \frac{\sqrt{5}}{4}R^\alpha\phi^{\beta\gamma}B_{\alpha[\beta\gamma]}$$
$$+\frac{\sqrt{5}}{2}R_\alpha\phi B^\alpha + \sqrt{\frac{5}{8}}R_\alpha\phi_\beta C^{\alpha\beta} + \frac{\sqrt{5}}{4}R_\alpha\phi^{\beta\gamma}B^\alpha_{\beta\gamma} + \frac{1}{4}R_\alpha\phi^{\alpha\beta}B_\beta + \frac{\sqrt{6}}{4}R_\alpha\phi_\beta B^{\alpha\beta}\Big).$$

The $SU(5)$ superfields

At the final step, the $SU(5)$ superfields are presented in terms of the MSSM superfields, for instance,

$$\phi = N; \qquad \phi_\alpha = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \widetilde{E} \\ -\widetilde{N} \end{pmatrix}; \qquad \phi^{\alpha\beta} = \begin{pmatrix} 0 & U_3 & -U_2 & \widetilde{U}^1 & \widetilde{D}^1 \\ -U_3 & 0 & U_1 & \widetilde{U}^2 & \widetilde{D}^2 \\ U_2 & -U_1 & 0 & \widetilde{U}^3 & \widetilde{D}^3 \\ -\widetilde{U}^1 & -\widetilde{U}^2 & -\widetilde{U}^3 & 0 & E \\ -\widetilde{D}^1 & -\widetilde{D}^2 & -\widetilde{D}^3 & -E & 0 \end{pmatrix}.$$

Similarly, for the Higgs superfields

$$\bar{5}^\gamma \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ H_{u1} \\ H_{u2} \end{pmatrix}; \qquad 5_\beta \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ H_{d2} \\ -H_{d1} \end{pmatrix}.$$

If the Higgs superfields come from the $SU(5)$ representation 45, then

$$45^i_{jk} \to \frac{\sqrt{3}}{2}\Big[\delta^i_j (H_d)_k - \delta^i_k (H_d)_j\Big]; \qquad 45^a_{bi} = -45^a_{ib} \to -\frac{1}{2\sqrt{3}}\delta^a_b (H_d)_i,$$

The third generation

For the third generation, let us take

$$\overline{27}\Big|_{E_6} \to 10\Big|_{SO(10)} \to 5\Big|_{SU(5)} \to D_3,\, L_3; \qquad \overline{27}\Big|_{E_6} \to \overline{16}\Big|_{SO(10)} \to \overline{10}\Big|_{SU(5)} \to Q_3,\, U_3,\, E_3;$$

$$\overline{351}'\Big|_{E_6} \to 126\Big|_{SO(10)} \to \bar{5}\Big|_{SU(5)} \to (H_u)_3; \qquad \overline{351}'\Big|_{E_6} \to \overline{144}\Big|_{SO(10)} \to 5\Big|_{SU(5)} \to (H_d)_3;$$

E_6	$Y_3 \cdot \overline{27} \times \overline{27} \times \overline{351}'$			
$SO(10)$	$\overline{16} \times \overline{16} \times 10$	$\frac{1}{4} Y_3 \cdot \overline{16} \times \overline{16} \times 126$		$\sqrt{2}\, Y_3 \cdot 10 \times \overline{16} \times \overline{144}$
$SU(5)$	$\overline{10} \times \overline{10} \times \bar{5}$	$5 \times \overline{10} \times 45$	$-\frac{1}{\sqrt{3}} Y_3 \cdot \overline{10} \times \overline{10} \times \bar{5}$	$-\frac{1}{\sqrt{10}} Y_3 \cdot 5 \times \overline{10} \times 5$
MSSM			$(Y_U)_{33} = -\frac{1}{\sqrt{3}} Y_3$	$(Y_E)_{33} = (Y_D)_{33} = -\frac{1}{\sqrt{10}} Y_3$

The result reads as

$$Y_3 = -\sqrt{3}\,(Y_U)_{33} = -\sqrt{10}\,(Y_E)_{33} = -\sqrt{10}\,(Y_D)_{33}.$$

The second generation

Similarly, for the second generation we take

$$\overline{27}\Big|_{E_6} \to \overline{16}\Big|_{SO(10)} \to 5\Big|_{SU(5)} \to D_2, L_2; \qquad \overline{27}\Big|_{E_6} \to \overline{16}\Big|_{SO(10)} \to \overline{10}\Big|_{SU(5)} \to Q_2, U_2, E_2;$$

$$\overline{351}'\Big|_{E_6} \to 10\Big|_{SO(10)} \to \bar{5}\Big|_{SU(5)} \to (H_u)_2; \qquad \overline{351}'\Big|_{E_6} \to \overline{126}\Big|_{SO(10)} \to 45\Big|_{SU(5)} \to (H_d)_2;$$

E_6	$Y_2 \cdot \overline{27} \times \overline{27} \times \overline{351}'$			
$SO(10)$	$\frac{1}{4\sqrt{5}} Y_2 \cdot \overline{16} \times \overline{16} \times 10$	$\frac{1}{4} Y_2 \cdot \overline{16} \times \overline{16} \times 126$		$10 \times \overline{16} \times \overline{144}$
$SU(5)$	$-\frac{1}{\sqrt{10}} Y_2 \cdot \overline{10} \times \overline{10} \times \bar{5}$	$2Y_2 \cdot 5 \times \overline{10} \times 45$	$\overline{10} \times \overline{10} \times \bar{5}$	$5 \times \overline{10} \times 5$
MSSM	$(Y_U)_{22} = -\frac{1}{\sqrt{10}} Y_2$	$\frac{1}{3}(Y_E)_{22} = -(Y_D)_{22} = \frac{1}{\sqrt{3}} Y_2$		

In this case we obtain

$$Y_2 = -\sqrt{10}\,(Y_U)_{22} = \frac{1}{\sqrt{3}}\,(Y_E)_{22} = -\sqrt{3}\,(Y_D)_{22}.$$

Discussion

We see that E_6 invariant $\overline{27} \times \overline{27} \times \overline{351}'$ allows deriving the Yukawa relations

$$Y_3 = -\sqrt{3}\,(Y_U)_{33} = -\sqrt{10}\,(Y_E)_{33} = -\sqrt{10}\,(Y_D)_{33};$$
$$Y_2 = -\sqrt{10}\,(Y_U)_{22} = \frac{1}{\sqrt{3}}\,(Y_E)_{22} = -\sqrt{3}\,(Y_D)_{22},$$

which are in a certain degree similar to the ones discussed earlier. However,

- Numerical values of the Yukawa ratios are different. In particular, by this method, it is very difficult to obtain Yukawa ratios which are far from 1.
- Higgs superfields are different for different generations.
- The representation $\overline{351}'$ is too large and seems to be unnatural. Possibly, it will be necessary to use smaller representations, but nonrenormalizable superpotentials of higher degrees in chiral matter superfields.

Certainly, a more accurate renormalization group analysis should be done and a possibility of constructing of a phenomenologically satisfactory model needs to be thoroughly investigated.

Conclusion

- Analysing possible relations between Yukawa couplings at the scale of Grand Unification one can use the renormalization group invariants (RGIs). The exact RGIs remain constant throughout the renormalization group evolution, while the approximate RGIs receive relatively small quantum corrections.
- It is possible to construct two exact RGIs for the MSSM. However, it is difficult to apply them for the phenomenological purposes.
- Some approximate RGIs can also be constructed for the MSSM. They can be used for estimating certain ratios of the Yukawa couplings at the scale of Grand Unification.
- The resulting ratios of the Yukawa couplings for the MSSM do not agree with the predictions of the simplest models of supersymmetric Grand Unification.
- Using complicated cubic invariants for wide gauge symmetries it is possible to derive some nontrivial Yukawa relations, but at present we did not find the invariants producing the phenomenologically satisfactory results.
- How to construct a phenomenologically satisfactory model is an open problem.

Thank you for the attention!